Astérisque
421, 2020, p. 1–5
doi:10.24033/ast.1125

PREFACE

by

Jan H. Bruinier, Benjamin Howard, Stephen S. Kudla,
Keerthi Madapusi Pera, Michael Rapoport & Tonghai Yang

The formation of modular generating series whose coefficients are geometric cycles began with the work of Hirzebruch-Zagier [**14**], who constructed divisors on compactified Hilbert modular surfaces over $\mathbb{C}$, and showed that their cohomology classes formed the coefficients of a weight 2 modular form.

An extensive study of the modularity of generating series for cohomology classes of special cycles in Riemannian locally symmetric spaces $M = \Gamma\backslash X$ was undertaken in a series of papers [**21, 22, 23**] of Kudla and Millson. The main technical tool was a family of Siegel type theta series valued in the de Rham complex of M, from which modularity was inherited by the image in cohomology.

The special cycles used by Kudla-Millson are given by an explicit geometric construction, and so, in the cases where M is (the complex fiber of) a Shimura variety, it is natural to ask whether the analogous generating series for special cycle classes in the Chow group is likewise modular. In the case of Shimura varieties of orthogonal type, this question was raised in [**19**]. In some special cases modularity of the Chow group-valued generating series can be deduced from modularity of the cohomology-valued generating series; see [**27, 26**] for example.

The generating series for Heegner points in the Jacobian of a modular curve was proved to be modular by Gross-Kohnen-Zagier [**12**]. Motivated by their work, Borcherds [**2, 3**] proved the modularity of the generating series of Heegner (= special) divisors in the Chow groups of Shimura varieties of orthogonal type. His method depended on the miraculous construction of Borcherds products: meromorphic modular forms on orthogonal Shimura varieties, constructed via a regularized theta lift, whose explicitly known divisors provide enough relations among special divisors to prove modularity.

The three papers in this volume are concerned with similar modularity results, but now for generating series of divisors on integral models of orthogonal and unitary Shimura varieties; more precisely, of generating series with coefficients in the codimension one arithmetic Chow groups of Gillet-Soulé.

The first results in this arithmetic direction were obtained in [**20**], which dealt with arithmetic divisors on quaternionic Shimura curves (a special case of orthogonal Shimura varieties). Still in the Shimura curve setting, quite complete results on the modularity of generating series were obtained in the book [**24**]. There the case of arithmetic 0-cycles is also treated and the corresponding generating series is shown to coincide with the central derivative of a weight $3/2$ Siegel genus 2 incoherent Eisenstein series.

The Green functions used in [**20, 24**] are derived from the Kudla-Millson theta series, and a similar construction can be used to obtain Green functions for special divisors on all orthogonal Shimura varieties. On the other hand, Bruinier [**4**] generalized the regularized theta lift of Borcherds by allowing harmonic Maass forms as inputs. This provides a different construction of Green functions for special divisors, with the advantage that one can try to use the method of Borcherds to establish modularity of the corresponding generating series with coefficients in the arithmetic Chow group. In the case of Hilbert modular surfaces (once again, a special case of orthogonal Shimura varieties), this was done in [**5**].

The main obstruction to extending the method of Borcherds to integral models is that the divisor of a Borcherds product is, a priori, only known on the generic fiber of the Shimura variety. To obtain modularity of the generating series with coefficients in the codimension one arithmetic Chow group, one must compute the divisor of a Borcherds product on the integral model, where the divisor may contain vertical components.

The first paper [**6**] of this volume deals with arithmetic divisors on compactified unitary Shimura varieties of signature $(n-1,1)$, and the main result is the modularity of the corresponding generating series with coefficients in the arithmetic Chow group. The proof follows the method of Borcherds, with the essential new ingredient being the calculation of the vertical components and boundary components appearing in the divisor of a unitary Borcherds product.

The second paper [**7**] of this volume contains applications of the modularity result just stated. One can form the Petersson inner product of the generating series of arithmetic divisors against a cusp form g of the appropriate weight and level. This defines a class in the codimension one arithmetic Chow group of the unitary Shimura variety, called the arithmetic theta lift of g. On the other hand, taking Zariski closures of CM points yields cycles of dimension one on the integral model, which one can then intersect with the arithmetic theta lift. The main results show that such intersections are equal to central derivatives of (generalized) L-functions, somewhat in the spirit of the Gross-Zagier theorem [**13**] on heights of Heegner points. These results complete, in some sense, the series of papers [**16, 17, 8**], which contain the bulk of the intersection calculations.

The second paper also proves special cases of Colmez's conjecture [**10**] on the periods of CM abelian varieties. These special cases can actually be deduced from the averaged version of the conjecture [**1, 25**], but the proofs given here yield new information about the arithmetic of unitary Shimura varieties, which we hope is of independent interest.

Another application of the modularity result on unitary Shimura varieties has been found by W. Zhang [**29**], who has used it in his proof of the Arithmetic Fundamental Lemma.

The third paper [**18**] proves the modularity of generating series of arithmetic divisors on integral models of orthogonal type Shimura varieties. As in the unitary case, the new ingredient in the proof of modularity is the calculation of divisors of Borcherds products on integral models. This extends results of Hörmann [**15**], who does such calculations only after inverting all primes where the integral model has nonsmooth reduction. Hörmann must assume that the Shimura variety has cusps (so that one can study the Borcherds product using its q-expansion), an assumption that is removed here using an arithmetic version of the embedding trick of Borcherds.

With the results of this volume in hand, it is natural to ask about the modularity of generating series of arithmetic special cycles in higher codimension. Although the reader will find no such results in this volume, there is progress along these lines. The modularity of generating series of higher codimension cycles in the Chow group of the generic fiber of an orthogonal Shimura variety has been proved by Bruinier and Raum [**9**], building on the unpublished thesis of W. Zhang [**28**]. An extension of this result to cycles in the Chow groups of the integral model will appear in forthcoming work of Howard and Madapusi Pera, but extending the result further to arithmetic Chow groups remains an open problem. The recent construction of Green currents for higher codimension special cycles by Garcia-Sankaran [**11**] is a significant step in this direction.

References

[1] F. Andreatta, E. Z. Goren, B. Howard & K. Madapusi Pera – "Faltings heights of abelian varieties with complex multiplication," *Ann. of Math.* **187** (2018), p. 391–531.

[2] R. E. Borcherds – "Automorphic forms with singularities on Grassmannians," *Invent. math.* **132** (1998), p. 491–562.

[3] ———, "The Gross-Kohnen-Zagier theorem in higher dimensions," *Duke Math. J.* **97** (1999), p. 219–233.

[4] J. H. Bruinier – *Borcherds products on O(2, l) and Chern classes of Heegner divisors*, Lecture Notes in Math., vol. 1780, Springer, 2002.

[5] J. H. Bruinier, J. I. Burgos Gil & U. Kühn – "Borcherds products and arithmetic intersection theory on Hilbert modular surfaces," *Duke Math. J.* **139** (2007), p. 1–88.

[6] J. H. Bruinier, B. Howard, S. S. Kudla, M. Rapoport & T. Yang – "Modularity of generating series of divisors on unitary Shimura varieties," this volume.

[7] ———, "Modularity of generating series of divisors on unitary Shimura varieties II: Arithmetic applications," this volume.

[8] J. H. Bruinier, B. Howard & T. Yang – "Heights of Kudla-Rapoport divisors and derivatives of L-functions," *Invent. math.* **201** (2015), p. 1–95.

[9] J. H. Bruinier & M. Westerholt-Raum – "Kudla's modularity conjecture and formal Fourier-Jacobi series," *Forum Math. Pi* **3** (2015), e7, 30.

[10] P. Colmez – "Périodes des variétés abéliennes à multiplication complexe," *Ann. of Math.* **138** (1993), p. 625–683.

[11] L. E. Garcia & S. Sankaran – "Green forms and the arithmetic Siegel-Weil formula," *Invent. math.* **215** (2019), p. 863–975.

[12] B. H. Gross, W. Kohnen & D. B. Zagier – "Heegner points and derivatives of L-series. II," *Math. Ann.* **278** (1987), p. 497–562.

[13] B. H. Gross & D. B. Zagier – "Heegner points and derivatives of L-series," *Invent. math.* **84** (1986), p. 225–320.

[14] F. Hirzebruch & D. B. Zagier – "Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus," *Invent. math.* **36** (1976), p. 57–113.

[15] F. Hörmann – *The geometric and arithmetic volume of Shimura varieties of orthogonal type*, CRM Monograph Series, vol. 35, Amer. Math. Soc., 2014.

[16] B. Howard – "Complex multiplication cycles and Kudla-Rapoport divisors," *Ann. of Math.* **176** (2012), p. 1097–1171.

[17] ———, "Complex multiplication cycles and Kudla-Rapoport divisors, II," *Amer. J. Math.* **137** (2015), p. 639–698.

[18] B. Howard & K. Madapusi Pera – "Arithmetic of Borcherds products," this volume.

[19] S. S. Kudla – "Algebraic cycles on Shimura varieties of orthogonal type," *Duke Math. J.* **86** (1997), p. 39–78.

[20] ———, "Central derivatives of Eisenstein series and height pairings," *Ann. of Math.* **146** (1997), p. 545–646.

[21] S. S. Kudla & J. J. Millson – "The theta correspondence and harmonic forms. I," *Math. Ann.* **274** (1986), p. 353–378.

[22] ———, "The theta correspondence and harmonic forms. II," *Math. Ann.* **277** (1987), p. 267–314.

[23] ———, "Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables," *Inst. Hautes Études Sci. Publ. Math.* **71** (1990), p. 121–172.

[24] S. S. Kudla, M. Rapoport & T. Yang – *Modular forms and special cycles on Shimura curves*, Annals of Math. Studies, vol. 161, Princeton Univ. Press, 2006.

[25] X. Yuan & S.-W. Zhang – "On the averaged Colmez conjecture," *Ann. of Math.* **187** (2018), p. 533–638.

[26] X. Yuan, S.-W. Zhang & W. Zhang – "The Gross-Kohnen-Zagier theorem over totally real fields," *Compos. Math.* **145** (2009), p. 1147–1162.

[27] D. B. Zagier – "Modular points, modular curves, modular surfaces and modular forms," in *Workshop Bonn 1984 (Bonn, 1984)*, Lecture Notes in Math., vol. 1111, Springer, 1985, p. 225–248.

[28] W. Zhang – "Modularity of generating functions of special cycles on Shimura varieties," Ph.D. Thesis, Columbia University, 2009.

[29] ______, "Weil representation and arithmetic fundamental lemma," preprint arXiv:1909.02697.

J. Bruinier, Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, D-64289 Darmstadt, Germany • *E-mail :* bruinier@mathematik.tu-darmstadt.de

B. Howard, Department of Mathematics, Boston College, 140 Commonwealth Ave, Chestnut Hill, MA 02467, USA • *E-mail :* howardbe@bc.edu

S. Kudla, Department of Mathematics, University of Toronto, 40 St. George St., BA6290, Toronto, ON M5S 2E4, Canada • *E-mail :* skudla@math.toronto.edu

K. Madapusi Pera, Department of Mathematics, Boston College, 140 Commonwealth Ave, Chestnut Hill, MA 02467, USA • *E-mail :* madapusi@bc.edu

M. Rapoport, Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany, and Department of Mathematics, University of Maryland, College Park, MD 20742, USA • *E-mail :* rapoport@math.uni-bonn.de

T. Yang, Department of Mathematics, University of Wisconsin Madison, Van Vleck Hall, Madison, WI 53706, USA • *E-mail :* thyang@math.wisc.edu