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MODULARITY OF GENERATING SERIES OF DIVISORS ON UNITARY SHIMURA VARIETIES II: ARITHMETIC APPLICATIONS

by

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Abstract. — We prove two formulas in the style of the Gross-Zagier theorem, relating derivatives of L-functions to arithmetic intersection pairings on a unitary Shimura variety. We also prove a special case of Colmez's conjecture on the Faltings heights of abelian varieties with complex multiplication. These results are derived from the authors' earlier results on the modularity of generating series of divisors on unitary Shimura varieties.

Résumé **(Modularité des séries génératrices de diviseurs sur les variétés de Shimura unitaires II: applications arithmétiques)**
Nous prouvons deux formules dans le style du théorème de Gross-Zagier, reliant les dérivées des fonctions L aux accouplements d'intersection arithmétique sur une variété de Shimura unitaire. Nous prouvons également un cas particulier de la conjecture de Colmez sur les hauteurs de Faltings des variétés abéliennes à multiplication complexe. Ces résultats sont déduits des résultats antérieurs des auteurs sur la modularité des séries génératrices de diviseurs sur les variétés de Shimura unitaires.

1. Introduction

Fix an integer $n \geq 3$, and a quadratic imaginary field $\boldsymbol{k} \subset \mathbb{C}$ of odd discriminant $\mathrm{disc}(\boldsymbol{k}) = -D$. Let $\chi_{\boldsymbol{k}} : \mathbb{A}^{\times} \to \{\pm 1\}$ be the associated quadratic character, let $\mathfrak{d}_{\boldsymbol{k}} \subset \mathcal{O}_{\boldsymbol{k}}$ denote the different of $\boldsymbol{k}$, let $h_{\boldsymbol{k}}$ be the class number of $\boldsymbol{k}$, and let $w_{\boldsymbol{k}}$ be the number of roots of unity in $\boldsymbol{k}$.

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By a *hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice* we mean a projective $\mathcal{O}_{\boldsymbol{k}}$-module of finite rank endowed with a nondegenerate hermitian form.

1.1. Arithmetic theta lifts. — Suppose we are given a pair $(\mathfrak{a}_0, \mathfrak{a})$ in which

— $\mathfrak{a}_0$ is a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(1, 0)$,

— $\mathfrak{a}$ is a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(n-1, 1)$.

This pair determines hermitian $\boldsymbol{k}$-spaces $W_0 = \mathfrak{a}_{0\mathbb{Q}}$ and $W = \mathfrak{a}_{\mathbb{Q}}$.

From this data we constructed in [**6**] a smooth Deligne-Mumford stack $\mathrm{Sh}(G, \mathcal{D})$ of dimension $n-1$ over $\boldsymbol{k}$ with complex points

$$\mathrm{Sh}(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K.$$

The reductive group $G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W)$ is the largest subgroup on which the two similitude characters agree, and $K \subset G(\mathbb{A}_f)$ is the largest subgroup stabilizing the $\widehat{\mathbb{Z}}$-lattices $\widehat{\mathfrak{a}}_0 \subset W_0(\mathbb{A}_f)$ and $\widehat{\mathfrak{a}} \subset W(\mathbb{A}_f)$.

We also defined in [**6**, §2.3] an integral model

$$\mathcal{S}_{\mathrm{Kra}} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \tag{1.1.1}$$

of $\mathrm{Sh}(G, \mathcal{D})$. It is regular and flat over $\mathcal{O}_{\boldsymbol{k}}$, and admits a canonical toroidal compactification $\mathcal{S}_{\mathrm{Kra}} \hookrightarrow \mathcal{S}^*_{\mathrm{Kra}}$ whose boundary is a smooth divisor.

The main result of [**6**] is the construction of a formal generating series of arithmetic divisors

$$\widehat{\phi}(\tau) = \sum_{m \geq 0} \widehat{\mathcal{Z}}^{\mathrm{total}}_{\mathrm{Kra}}(m) \cdot q^m \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]] \tag{1.1.2}$$

valued in the Gillet-Soulé codimension one arithmetic Chow group with rational coefficients, extended to allow log-log Green functions at the boundary as in [**10, 4**], and the proof that this generating series is modular of weight n, level $\Gamma_0(D)$, and character $\chi^n_{\boldsymbol{k}}$. The modularity result implies that the coefficients span a finite-dimensional subspace of the arithmetic Chow group [**6**, Remark 7.1.2].

After passing to the arithmetic Chow group with complex coefficients, for any classical modular form

$$g \in S_n(\Gamma_0(D), \chi^n_{\boldsymbol{k}})$$

we may form the Petersson inner product

$$\langle \widehat{\phi}, g \rangle_{\mathrm{Pet}} = \int_{\Gamma_0(D)\backslash\mathcal{H}} \overline{g(\tau)} \cdot \widehat{\phi}(\tau) \, \frac{du\, dv}{v^{2-n}}$$

where $\tau = u + iv$. As in [**24**], define the *arithmetic theta lift*

$$\widehat{\theta}(g) = \langle \widehat{\phi}, g \rangle_{\mathrm{Pet}} \in \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}). \tag{1.1.3}$$

Armed with the construction of the arithmetic theta lift (1.1.3), we are now able to complete the program of [**18, 19, 7**] to prove Gross-Zagier style formulas relating arithmetic intersections to derivatives of L-functions.

The Shimura variety $\mathcal{S}^*_{\mathrm{Kra}}$ carries different families of codimension $n-1$ cycles constructed from complex multiplication points, and our results show that the arithmetic intersections of these families with arithmetic lifts are related to central derivatives of L-functions.

1.2. Central derivatives and small CM points. — In §2 we construct an étale and proper Deligne-Mumford stack $\mathcal{Y}_{\mathrm{sm}}$ over $\mathcal{O}_{\boldsymbol{k}}$, along with a morphism

$$\mathcal{Y}_{\mathrm{sm}} \to \mathcal{S}^*_{\mathrm{Kra}}.$$

This is the *small CM cycle.* Intersecting arithmetic divisors against $\mathcal{Y}_{\mathrm{sm}}$ defines a linear functional

$$[\,-:\mathcal{Y}_{\mathrm{sm}}] : \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \mathbb{C},$$

and our first main result computes the image of the arithmetic theta lift (1.1.3) under this linear functional.

The statement involves the convolution L-function $L(\tilde{g},\theta_\Lambda,s)$ of two modular forms

$$\tilde{g} \in S_n(\overline{\omega}_L), \qquad \theta_\Lambda \in M_{n-1}(\omega_\Lambda^\vee)$$

valued in finite-dimensional representations of $\mathrm{SL}_2(\mathbb{Z})$. We refer the reader to §2.3 for the precise definitions. Here we note only that $\tilde{g}$ is the image of g under an induction map

$$S_n(\Gamma_0(D),\chi_{\boldsymbol{k}}^n) \to S_n(\overline{\omega}_L) \tag{1.2.1}$$

from scalar-valued forms to vector-valued forms, that θ_Λ is the theta function attached to a quadratic space Λ over $\mathbb{Z}$ of signature $(2n-2,0)$, and that the L-function $L(\tilde{g},\theta_\Lambda,s)$ vanishes at its center of symmetry $s=0$.

Theorem A. — *The arithmetic theta lift* (1.1.3) *satisfies*

$$[\widehat{\theta}(g):\mathcal{Y}_{\mathrm{sm}}] = -\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}})\cdot \frac{d}{ds}L(\tilde{g},\theta_\Lambda,s)\big|_{s=0}.$$

Here we have defined

$$\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}) = \sum_{y\in\mathcal{Y}_{\mathrm{sm}}(\mathbb{C})} \frac{1}{|\mathrm{Aut}(y)|},$$

where the sum is over the finitely many isomorphism classes of the groupoid of complex points of $\mathcal{Y}_{\mathrm{sm}}$, viewed as an $\mathcal{O}_{\boldsymbol{k}}$-stack.

The proof is given in §2, by combining the modularity result of **[6]** with the main result of **[7]**. In §3 we provide alternative formulations of Theorem A that involve the usual convolution L-function of scalar-valued modular forms, as opposed to the vector-valued forms $\tilde{g}$ and θ_Λ. See especially Theorem 3.4.1.

1.3. Central derivatives and big CM points. — Fix a totally real field F of degree n, and define a CM field

$$E = \boldsymbol{k} \otimes_{\mathbb{Q}} F.$$

Let $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ be a CM type of signature $(n-1, 1)$, in the sense that there is a unique $\varphi^{\mathrm{sp}} \in \Phi$, called the *special embedding*, whose restriction to $\boldsymbol{k}$ agrees with the complex conjugate of the inclusion $\boldsymbol{k} \subset \mathbb{C}$. The reflex field of the pair (E, Φ) is

$$E_\Phi = \varphi^{\mathrm{sp}}(E) \subset \mathbb{C},$$

and we denote by $\mathcal{O}_\Phi \subset E_\Phi$ its ring of integers.

We define in §4.2 an étale and proper Deligne-Mumford stack $\mathcal{Y}_{\mathrm{big}}$ over $\mathcal{O}_\Phi$, along with a morphism of $\mathcal{O}_{\boldsymbol{k}}$-stacks

$$\mathcal{Y}_{\mathrm{big}} \to \mathcal{S}^*_{\mathrm{Kra}}.$$

This is the *big CM cycle*. Here we view $\mathcal{Y}_{\mathrm{big}}$ as an $\mathcal{O}_{\boldsymbol{k}}$-stack using the inclusion $\mathcal{O}_{\boldsymbol{k}} \subset \mathcal{O}_\Phi$ of subrings of $\mathbb{C}$ (which is the complex conjugate of the special embedding $\varphi^{\mathrm{sp}} : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_\Phi$). Intersecting arithmetic divisors against $\mathcal{Y}_{\mathrm{big}}$ defines a linear functional

$$[- : \mathcal{Y}_{\mathrm{big}}] : \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \mathbb{C}.$$

Our second main result relates the image of the arithmetic theta lift (1.1.3) under this linear functional to the central derivative of a generalized L-function defined as the Petersson inner product $\langle E(s), \tilde{g}\rangle_{\mathrm{Pet}}$. The modular form $\tilde{g}(\tau)$ is, once again, the image of $g(\tau)$ under the induction map (1.2.1). The modular form $E(\tau, s)$ is defined as the restriction via the diagonal embedding $\mathcal{H} \to \mathcal{H}^n$ of a weight one Hilbert modular Eisenstein series valued in the space of the contragredient representation $\omega_L^\vee$. See §4.3 for details.

Theorem B. — *Assume that the discriminants of $\boldsymbol{k}/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime. The arithmetic theta lift* (1.1.3) *satisfies*

$$[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{big}}] = \frac{-1}{n} \cdot \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) \cdot \frac{d}{ds} \langle E(s), \tilde{g}\rangle_{\mathrm{Pet}}\big|_{s=0}.$$

Here we have defined

$$\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) = \sum_{y \in \mathcal{Y}_{\mathrm{big}}(\mathbb{C})} \frac{1}{|\mathrm{Aut}(y)|},$$

where the sum is over the finitely many isomorphism classes of the groupoid of complex points of $\mathcal{Y}_{\mathrm{big}}$, viewed as an $\mathcal{O}_{\boldsymbol{k}}$-stack.

The proof is given in §4, by combining the modularity result of [**6**] with the intersection calculations of [**8, 18, 19**].

1.4. Colmez's conjecture. — Suppose E is a CM field with maximal totally real subfield F. Let D_E and D_F be the absolute discriminants of E and F, set $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and define the completed L-function

$$\Lambda(s,\chi_E) = \left|\frac{D_E}{D_F}\right|^{\frac{s}{2}} \Gamma_{\mathbb{R}}(s+1)^{[F:\mathbb{Q}]} L(s,\chi_E)$$

of the character $\chi_E : \mathbb{A}_F^\times \to \{\pm 1\}$ determined by E/F. It satisfies the functional equation $\Lambda(1-s,\chi_E) = \Lambda(s,\chi_E)$, and

$$\frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)} = \frac{L'(0,\chi_E)}{L(0,\chi_E)} + \frac{1}{2}\log\left|\frac{D_E}{D_F}\right| - \frac{[F:\mathbb{Q}]}{2}\log(4\pi e^\gamma),$$

where $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant.

Suppose A is an abelian variety over $\mathbb{C}$ with complex multiplication by $\mathcal{O}_E$ and CM type Φ. In particular A is defined over the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. It is a theorem of Colmez [**12**] that the Faltings height

$$h^{\mathrm{Falt}}_{(E,\Phi)} = h^{\mathrm{Falt}}(A)$$

depends only on the pair (E,Φ), and not on A itself. Moreover, Colmez gave a conjectural formula for this Faltings height in terms of logarithmic derivatives of Artin L-functions. In the special case where $E = \boldsymbol{k}$, Colmez's conjecture reduces to the well-known Chowla-Selberg formula

$$h^{\mathrm{Falt}}_{\boldsymbol{k}} = -\frac{1}{2}\cdot\frac{\Lambda'(0,\chi_{\boldsymbol{k}})}{\Lambda(0,\chi_{\boldsymbol{k}})} - \frac{1}{4}\cdot\log(16\pi^3 e^\gamma), \tag{1.4.1}$$

where we omit the CM type $\{\mathrm{id}\} \subset \mathrm{Hom}(\boldsymbol{k},\mathbb{C})$ from the notation.

Now suppose we are in the special case of §1.3, where

$$E = \boldsymbol{k}\otimes_{\mathbb{Q}} F$$

and $\Phi \subset \mathrm{Hom}(E,\mathbb{C})$ has signature $(n-1,1)$. In this case, Colmez's conjecture simplifies to the equality of the following theorem.

Theorem C **([29])**. — *For a pair (E,Φ) as above,*

$$h^{\mathrm{Falt}}_{(E,\Phi)} = -\frac{2}{n}\cdot\frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)} + \frac{4-n}{2}\cdot\frac{\Lambda'(0,\chi_{\boldsymbol{k}})}{\Lambda(0,\chi_{\boldsymbol{k}})} - \frac{n}{4}\cdot\log(16\pi^3 e^\gamma).$$

In [**6**, §2.4] we defined the line bundle of weight one modular forms $\boldsymbol{\omega}$ on $\mathcal{S}^*_{\mathrm{Kra}}$. It was endowed it with a hermitian metric in [**6**, §7.2], and the resulting metrized line bundle determines a class

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}).$$

The constant term of (1.1.2) is

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = -\widehat{\boldsymbol{\omega}} + (\mathrm{Exc}, -\log(D)) \tag{1.4.2}$$

where Exc is the *exceptional locus* of $\mathcal{S}^*_{\mathrm{Kra}}$ appearing in [**6**, Theorem 2.3.4]. It is a smooth effective Cartier divisor supported in characteristics dividing D, and we view