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A PROOF OF JAKOBSON'S THEOREM

by

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Abstract. — We give a proof of Jakobson's theorem: with positive probability on the parameter, a real quadratic map leaves invariant an absolutely continuous ergodic invariant probability measure with positive Lyapunov exponent.

Résumé. — Nous présentons une démonstration du théorème de Jakobson: avec probabilité strictement positive sur le paramètre, un polynôme quadratique réel admet une mesure de probabilité invariante ergodique qui est absolument continue par rapport à la mesure de Lebesgue et dont l'exposant de Lyapunov est strictement positif.

1. Introduction

1.1. Statement of the theorem. — In the 1960's, Sinai, Ruelle and Bowen developed the ergodic theory of uniformly hyperbolic dynamical systems. In the simplest setting of a uniformly expanding map of a torus, one obtains a unique ergodic invariant probability measure absolutely continuous w.r.t. the Lebesgue measure. In the 1970's, a systematic study of unimodal maps of the interval was initiated. The quadratic family $P_c(x) = x^2 + c$ appeared as a central object, from the point of view of real as well as complex 1-dimensional dynamics. When the critical point escapes to infinity, the same is true for almost all orbits. When P_c has an attractive periodic orbit, it attracts almost all non escaping orbits. Does there exist, for a typical parameter c, another kind of dynamical behavior?

Jakobson **[5]** provided a positive answer:

Theorem 1.1. — *There exists a set Λ of positive Lebesgue measure such that, for $c \in \Lambda$, the quadratic polynomial P_c has an ergodic invariant absolutely continuous probability measure with positive Lyapunov exponent, supported on the interval $[P_c(0), P_c^2(0)]$. One has actually*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\mathrm{Leb}(\Lambda \cap [-2, -2+\varepsilon]) = 1.$$

After Jakobson's original paper, a number of different proofs appeared **[1]** **[10]**. Jakobson's theorem was the subject of my lectures at Collège de France in 1997–98. A first handwritten version of the following notes was produced at the time, and was made available over the years to those who asked me. It is perhaps not too late for a most systematic diffusion effort.

1.2. Some facts about quadratic polynomials. — We refer to **[2]** and **[8]** as general references for the results in this subsection, with the exception of the last paragraph.

For a complex parameter c, we denote by P_c the complex quadratic polynomial $P_c(z) = z^2 + c$. Recall that the *filled-in Julia set* $K(c)$ is the set of points in $\mathbb{C}$ which have a bounded orbit under iteration of P_c. It is a non-empty full compact subset of the complex plane invariant under P_c. Its boundary is the *Julia set* $J(c)$. When c is real, we define $K_{\mathbb{R}}(c)$ to be the intersection $K(c) \cap \mathbb{R}$. Similarly, we define $J_{\mathbb{R}}(c) := J(c) \cap \mathbb{R}$.

The *Mandelbrot set* $\mathscr{M}$ is the set of parameters c such that the critical point 0 of P_c belongs to $K(c)$. By a theorem of Douady-Hubbard, this happens iff $K(c)$ is connected. The Mandelbrot set is a non-empty full compact subset of the complex plane. When the parameter c does not belong to $\mathscr{M}$, $K(c) = J(c)$ is a Cantor set and the restriction of P_c to K_c is an expanding map conjugated to the full unilateral shift on two symbols.

In the rest of this subsection, we only consider real parameters. The intersection of $\mathscr{M}$ with the real line is equal to the interval $[-2, 1/4]$. For $c > 1/4$, the Julia set is disjoint from the real line. When $c < -2$, the Julia set is contained in the real line.

For $c = 1/4$, P_c has a single fixed point at $z = 1/2$, which is *parabolic* in the sense that $DP_c(1/2) = 1$. For $c < 1/4$, the two fixed points of P_c are real. It is customary to denote the larger one by $\beta := \frac{1}{2}(1 + \sqrt{1-4c})$ and the smaller one by $\alpha := \frac{1}{2}(1 - \sqrt{1-4c})$. The fixed point β is repulsive for all $c < 1/4$. The fixed point α is attractive for $1/4 > c > -3/4$, repulsive for $c < -3/4$, with a flip bifurcation occurring at $c = -3/4$. The real filled-in Julia set $K_{\mathbb{R}}(c)$ is equal to the interval $[-\beta, \beta]$ for $c \in [-2, 1/4]$.

The basin of any attractive periodic orbit must contain the critical point. Therefore there is at most one attractive periodic orbit. Let $\mathscr{A}$ be the set of real parameters c such that P_c has an attractive periodic orbit. It is an open subset of $(-2, 1/4)$. When $c \in \mathscr{A}$, the real Julia set $J_{\mathbb{R}}(c)$ is an expanding invariant Cantor set equal to the complement in $K_{\mathbb{R}}(c)$ of the basin of the attractive periodic orbit. Conversely, a parameter $c \in [-2, 1/4]$ such that the real Julia set is expanding belongs to $\mathscr{A}$.

A deep theorem conjectured by Fatou and proved independently by Graczyk-Swiatek (**[3]**, **[4]**) and Lyubich (**[6]**), asserts that the open set $\mathscr{A}$ is dense in $[-2, 1/4]$. Their result is posterior to Jakobson's theorem. Observe that Jakobson's theorem implies that $\mathscr{A}$ does not have full Lebesgue measure in $[-2, 1/4]$. More recently, Lyubich has shown (**[7]**) that almost all parameters in $[-2, 1/4]$ either belong to $\mathscr{A}$ or satisfy the conclusions of Jakobson's theorem.

1.3. Plan of the proof. — We describe now the content of the rest of this paper.

In Section 2, we introduce some of the main concepts for the proof of Jakobson's theorem. Denote by A the *central interval* whose endpoints are the negative fixed point α and its inverse image $-\alpha$. An interval J is *regular of order* $n > 0$ if there is a branch g_J of P_c^{-n} which is a diffeomorphism on some fixed combinatorially defined neighborhood $\widehat{A}$ of A and sends A onto J. A parameter c is *regular* if the central interval is covered by regular intervals of order $\leqslant n$, except for a set of exponentially small measure.

Regular parameters satisfy the conclusions of Jakobson's theorem. One uses the maximal regular intervals contained in the central interval to define on A a Bernoulli map T which is a return map for P (but not the first return map). It is very classical that such a map has a unique absolutely continuous invariant probability measure with analytic density. As the return time relating T to P_c is integrable, one is able to spread the T-invariant measure on A into a P-invariant measure supported on $[P_c(0), P_c^2(0)]$. This measure is still absolutely continuous. Its density w.r.t. the Lebesgue measure is integrable but not square-integrable. The Lyapunov exponent of this measure is positive.

In the last three sections of the paper, we assume that the parameter c is very close to -2 (and > -2). This amounts to saying that the return time M of the critical point in the central interval A is large. In the first part of Section 3, the first iterates of P_c for such a parameter are considered. It is shown in particular that, for $2 \leqslant n \leqslant M-2$, there are a couple of maximal regular intervals $C_n^{\pm}$ of order n contained in A. These intervals are called the *simple* regular intervals. Their union covers A except for a small symmetric interval around 0 of approximate size 2^{-M}.

To go further, we introduce the main definition of the paper: a parameter c is said to be *strongly regular* if the postcritical orbit can be decomposed into regular returns into the central interval A, and if most of these returns occur in the simple regular intervals $C_n^{\pm}$. More specifically we ask that the fraction of total time spent in non simple returns is at most $2^{-\sqrt{M}}$ (to compare with the approximate size 2^{-M} of the gap left out by the simple regular intervals). For a strongly regular parameter, the derivatives of the iterates along the postcritical orbit grow exponentially fast in a very controlled way.

In Section 4, we prove that strongly regular parameters are regular, and thus satisfy the conclusions of Jakobson's theorem. For $n > 0$, an interval J is said to be n-singular if J is contained in A, its endpoints are consecutive elements of $P_c^{-n-1}(\alpha)$, and J is not contained in a regular interval of order $\leqslant n$. One has to show that, if c is a strongly regular parameter, the union of all n-singular intervals has exponentially small Lebesgue measure. This is done by induction on n, the starting point being provided by the estimates on simple regular intervals of Section 3. We divide the n-singular intervals into several classes: *peripheral, lateral, and central.* The central ones are so close to 0 that the crudest estimate of the Lebesgue measure of their union is sufficient. On the other hand, each peripheral or lateral n-singular interval J

is dynamically related to a m-singular interval J^* with $m < n$. The control on the postcritical orbit (Section 3) allows to conclude the induction step.

In the last section, we prove that, in the parameter interval $(c^{(M)}, c^{(M-1)})$ where the first return time of 0 in A is exactly equal to M, most parameters are strongly regular. More precisely, for any $\theta < 1/2$, the set of non strongly regular parameters in $(c^{(M)}, c^{(M-1)})$ has relative Lebesgue measure $O(2^{-\theta M})$.

We first transfer to the parameter space the "puzzle" structure of the phase space. In order to do this we estimate the variation w.r.t. the parameter of the relevant inverse branches of the iterates of P_c. The next step is to transfer to the parameter space the measure estimates of Section 4 on the measure of n-singular intervals. There is a rather subtle point here: while it is easy to transfer estimates for single intervals, for sets which are union of many disjoint components, we need to control the sum of the maximal measure (w.r.t. the parameter) of the components rather than the maximal measure of the set itself. Fortunately, the combinatorial nature of the arguments of Section 4 allows this control, except for the central n-singular intervals where a rough but sufficient control of the number of components is used.

The last part of the proof is an easy and classical large deviation argument: once we know that the order of a given regular return of the postcritical orbit in A is $> n$ with exponentially small probability, it is easy to control the measure of non strongly regular parameters.

2. Regular parameters and Bernoulli maps

2.1. Regular points and regular parameters. — Consider a parameter $c \in [-2, 0)$ for the real quadratic family. The polynomial P_c has two fixed points α, β which verify $-\beta < \alpha < 0$. The critical value $c = P_c(0)$ satisfies $-\beta \leqslant P_c(0) < \alpha$.

Therefore, there exists $\alpha^{(1)} \in (-\beta, \alpha)$ such that $P_c^{-1}(-\alpha) = \{\alpha^{(1)}, -\alpha^{(1)}\}$. We define
$$A := [\alpha, -\alpha], \qquad \widehat{A} := (\alpha^{(1)}, -\alpha^{(1)}).$$

Definition 2.1. — Let c be a parameter in $[-2, 0)$, and let n be a positive integer. A point $x \in [-\beta, \beta]$ is *n-regular* if there exists an integer m, with $0 < m \leqslant n$, and an open interval $\widehat{J}$ with $x \in \widehat{J}$, such that the restriction of P_c^m to $\widehat{J}$ is a diffeomorphism onto $\widehat{A}$ and $P_c^m(x) \in A$.

Definition 2.2. — A parameter $c \in [-2, 0)$ is *regular* if there exist $\theta, C > 0$ such that, for every $n > 0$:
$$\mathrm{Leb}\{x \in A, x \text{ is not } n\text{-regular}\} \leqslant Ce^{-\theta n}.$$
The set of regular parameters is denoted by $\mathscr{R}$.

Theorem 2.3. — *The set of regular parameters has positive Lebesgue measure. More precisely, we have*
$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \mathrm{Leb}(R \cap [-2, -2 + \varepsilon]) = 1.$$