Astérisque
421, 2020, p. 187–297
doi:10.24033/ast.1128

ARITHMETIC OF BORCHERDS PRODUCTS

by

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Abstract. — We compute the divisors of Borcherds products on integral models of orthogonal Shimura varieties. As an application, we obtain an integral version of a theorem of Borcherds on the modularity of a generating series of special divisors.

***Résumé* (Arithmétique des produits de Borcherds).** — Nous calculons les diviseurs des produits de Borcherds sur des modèles intégraux de variétés de Shimura orthogonales. Comme application, nous obtenons une version intégrale d'un théorème de Borcherds sur la modularité d'une série génératrice de diviseurs spéciaux.

1. Introduction

In the series of papers [**4, 5, 6**], Borcherds introduced a family of meromorphic modular forms on orthogonal Shimura varieties, whose zeroes and poles are prescribed linear combinations of special divisors arising from embeddings of smaller orthogonal Shimura varieties. These meromorphic modular forms are the Borcherds products of the title.

After work of Kisin [**31**] on integral models of general Hodge and abelian type Shimura varieties, the theory of integral models of orthogonal Shimura varieties and their special divisors was developed further in [**26, 27**] and [**39, 1, 2**].

The goal of this paper is to combine the above theories to compute the divisor of a Borcherds product on the integral model of an orthogonal Shimura variety. We show that such a divisor is given as a prescribed linear combination of special divisors, exactly as in the generic fiber.

2010 Mathematics Subject Classification. — 14G35, 14G40, 11F55, 11F27, 11G18.
Key words and phrases. — Shimura varieties, Borcherds products.

B.H. was supported in part by NSF grants DMS-1501583 and DMS-1801905. K.M.P. was supported in part by NSF grants DMS-1502142 and DMS-1802169.

The first such results were obtained by Bruinier, Burgos Gil, and Kühn [9], who worked on Hilbert modular surfaces (a special type of signature $(2,2)$ orthogonal Shimura variety). Those results were later extended to more general orthogonal Shimura varieties by Hörmann [26, 27], but with some restrictions.

Our results extend Hörmann's, but with substantially weaker hypotheses. For example, our results include cases where the integral model is not smooth, cases where the divisors in question may have irreducible components supported in nonzero characteristics, and even cases where the Shimura variety is compact (so that one has no theory of q-expansions with which to analyze the arithmetic properties of Borcherds products).

1.1. Orthogonal Shimura varieties. — Given an integer $n \geq 1$ and a quadratic space (V, Q) over $\mathbb{Q}$ of signature $(n, 2)$, one can construct a Shimura datum $(G, \mathcal{D})$ with reflex field $\mathbb{Q}$.

The group $G = \mathrm{GSpin}(V)$ is a subgroup of the group of units in the Clifford algebra $C(V)$, and sits in a short exact sequence

$$1 \to \mathbb{G}_m \to G \to \mathrm{SO}(V) \to 1.$$

The hermitian symmetric domain is

$$\mathcal{D} = \{z \in V_{\mathbb{C}} : [z, z] = 0,\ [z, \overline{z}] < 0\}/\mathbb{C}^\times \subset \mathbb{P}(V_{\mathbb{C}}),$$

where the bilinear form

$$[x, y] = Q(x + y) - Q(x) - Q(y) \tag{1.1.1}$$

on V has been extended $\mathbb{C}$-bilinearly to $V_{\mathbb{C}}$.

To define a Shimura variety, fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ on which the quadratic form is $\mathbb{Z}$-valued, and a compact open subgroup $K \subset G(\mathbb{A}_f)$ such that

$$K \subset G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^\times. \tag{1.1.2}$$

Here $C(V_{\widehat{\mathbb{Z}}})$ is the Clifford algebra of the $\widehat{\mathbb{Z}}$-quadratic space $V_{\widehat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$. The canonical model of the complex orbifold

$$\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q}) \backslash \big(\mathcal{D} \times G(\mathbb{A}_f)/K\big)$$

is a smooth n-dimensional Deligne-Mumford stack

$$\mathrm{Sh}_K(G, \mathcal{D}) \to \mathrm{Spec}(\mathbb{Q}).$$

As in work of Kudla [32, 34], our Shimura variety carries a family of effective Cartier divisors

$$Z(m, \mu) \to \mathrm{Sh}_K(G, \mathcal{D})$$

indexed by positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$, and a metrized line bundle

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Pic}}(\mathrm{Sh}_K(G, \mathcal{D}))$$

of weight one modular forms. Under the complex uniformization of the Shimura variety, this line bundle pulls back to the tautological bundle on $\mathcal{D}$, with the metric defined by (4.2.3).

We say that $V_{\mathbb{Z}}$ is *maximal* if there is no proper superlattice in V on which Q takes integer values, and is *maximal at p* if the $\mathbb{Z}_p$-quadratic space $V_{\mathbb{Z}_p} = V_{\mathbb{Z}} \otimes \mathbb{Z}_p$ has the analogous property. It is clear that $V_{\mathbb{Z}}$ is maximal at every prime not dividing the discriminant $[V_{\mathbb{Z}}^\vee : V_{\mathbb{Z}}]$.

Let Ω be a finite set of rational primes containing all primes at which $V_{\mathbb{Z}}$ is not maximal, and abbreviate

$$\mathbb{Z}_\Omega = \mathbb{Z}[1/p : p \in \Omega].$$

Assume that (1.1.2) factors as $K = \prod_p K_p$, in such a way that

$$K_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^\times$$

for every prime $p \notin \Omega$. For such K there is a flat and normal integral model

$$\mathcal{S}_K(G, \mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_\Omega)$$

of $\mathrm{Sh}_K(G, \mathcal{D})$. It is a Deligne-Mumford stack of finite type over $\mathbb{Z}_\Omega$, and is a scheme if K is sufficiently small. At any prime $p \notin \Omega$, it satisfies the following properties:

1. If the lattice $V_{\mathbb{Z}}$ is self-dual at a prime p (or even *almost self-dual* in the sense of Definition 6.1.1) then the restriction of the integral model to $\mathrm{Spec}(\mathbb{Z}_{(p)})$ is smooth.
2. If p is odd and p^2 does not divide the discriminant $[V_{\mathbb{Z}}^\vee : V_{\mathbb{Z}}]$, then the restriction of the integral model to $\mathrm{Spec}(\mathbb{Z}_{(p)})$ is regular.
3. If $n \geq 6$ then the reduction mod p is geometrically normal.

The integral model carries over it a metrized line bundle

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Pic}}(\mathcal{S}_K(G, \mathcal{D}))$$

of weight one modular forms, extending the one already available in the generic fiber, and a family of effective Cartier divisors

$$\mathcal{Z}(m, \mu) \to \mathcal{S}_K(G, \mathcal{D})$$

indexed by positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$.

Remark 1.1.1. — If $V_{\mathbb{Z}}$ is itself maximal, one can take $\Omega = \emptyset$, choose

$$K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^\times$$

for the level subgroup, and obtain an integral model of $\mathrm{Sh}_K(G, \mathcal{D})$ over $\mathbb{Z}$.

1.2. Borcherds products. — In § 5.1, we recall the Weil representation

$$\rho_{V_{\mathbb{Z}}} : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_{V_{\mathbb{Z}}})$$

of the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Z})$ on the $\mathbb{C}$-vector space

$$S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}].$$

Any weakly holomorphic form

$$f(\tau) = \sum_{\substack{m\in\mathbb{Q} \\ m\gg-\infty}} c(m)\cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

valued in the complex-conjugate representation has Fourier coefficients

$$c(m) \in S_{V_{\mathbb{Z}}},$$

and we denote by $c(m,\mu)$ the value of $c(m)$ at the coset $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Fix such an f, assume that f is *integral* in the sense that $c(m,\mu)\in\mathbb{Z}$ for all m and μ.

Using the theory of regularized theta lifts, Borcherds **[5]** constructs a Green function $\Theta^{\mathrm{reg}}(f)$ for the analytic divisor

$$\sum_{\substack{m>0 \\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu)\cdot Z(m,\mu)(\mathbb{C}) \tag{1.2.1}$$

on $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$, and shows (after possibly replacing f by a suitable multiple) that some power of $\boldsymbol{\omega}^{\mathrm{an}}$ admits a meromorphic section $\psi(f)$ satisfying

$$-2\log\|\psi(f)\| = \Theta^{\mathrm{reg}}(f). \tag{1.2.2}$$

This implies that the divisor of $\psi(f)$ is (1.2.1). These meromorphic sections are the *Borcherds products* of the title.

Our main result, stated in the text as Theorem 9.1.1, asserts that the Borcherds product $\psi(f)$ is algebraic, defined over $\mathbb{Q}$, and has the expected divisor when viewed as a rational section over the integral model.

Theorem A. — *After possibly replacing f by a positive integer multiple, there is a rational section $\psi(f)$ of the line bundle $\boldsymbol{\omega}^{c(0,0)}$ on $\mathcal{S}_K(G,\mathcal{D})$ whose norm under the metric* (4.2.3) *satisfies* (1.2.2)*, and whose divisor is*

$$\mathrm{div}(\psi(f)) = \sum_{\substack{m>0 \\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu)\cdot \mathcal{Z}(m,\mu).$$

The unspecified positive integer by which one must multiply f can be made at least somewhat more explicit. For example, it depends only on the lattice $V_{\mathbb{Z}}$, and not on the form f. See the discussion of § 9.3.

As noted earlier, similar results can be found in the work of Hörmann **[26, 27]**. Hörmann only considers self-dual lattices, so that the corresponding integral model $\mathcal{S}_K(G,\mathcal{D})$ is smooth, and always assumes that the quadratic space V admits an isotropic line. This allows him to prove the flatness of $\mathrm{div}(\psi(f))$ by examining the

q-expansion of $\psi(f)$ at a cusp. As Hörmann's special divisors $\mathcal{Z}(m,\mu)$, unlike ours, are defined as the Zariski closures of their generic fibers, the equality of divisors stated in Theorem A is then a formal consequence of the same equality in the generic fiber.

In contrast, we can prove Theorem A even in cases where the divisors in question may not be flat, and in cases where V is anisotropic, so no theory of q-expansions is available.

The reader may be surprised to learn that even the descent of $\psi(f)$ to $\mathbb{Q}$ was not previously known in full generality. Indeed, there is a product formula for the Borcherds product giving its q-expansions at every cusp, and so one should be able to detect the field of definition of $\psi(f)$ from a suitable q-expansion principle.

If V is anisotropic then $\mathrm{Sh}_K(G,\mathcal{D})$ is proper over $\mathbb{Q}$, no theory of q-expansions exists, and the above strategy fails completely. But even when V is isotropic there is a serious technical obstruction to this argument. The product formula of Borcherds is not completely precise, in that the q-expansion of $\psi(f)$ at a given cusp is only specified up to multiplication by an unknown constant of absolute value 1, and there is no a priori relation between the different constants at different cusps. These constants are the $\kappa^{(a)}$ appearing in Proposition 5.4.2.

If $\mathrm{Sh}_K(G,\mathcal{D})$ admits (a toroidal compactification with) a cusp defined over $\mathbb{Q}$ there is no problem: simply rescale the Borcherds product by a constant of absolute value 1 to remove the mysterious constant at that cusp, and now $\psi(f)$ is defined over $\mathbb{Q}$. But if $\mathrm{Sh}_K(G,\mathcal{D})$ has no rational cusps, then to prove that $\psi(f)$ descends to $\mathbb{Q}$ one must compare the q-expansions of $\psi(f)$ at all points in a Galois orbit of cusps. One can rescale the Borcherds product to trivialize the constant at one cusp, but then one has no control over the constants at other cusps in the Galois orbit.

Using the q-expansion principle alone, is seems that the best one can prove is that $\psi(f)$ descends to the minimal field of definition of a cusp. Our strategy to improve on this is sketched in § 1.4 below.

Remark 1.2.1. — As in the statement and proof of [**26**, Theorem 10.4.12], there is an elementary argument using Hilbert's Theorem 90 that allows one to rescale the Borcherds product so that it descends to $\mathbb{Q}$, but in this argument one has no control over the scaling factor, and it need not have absolute value 1. In particular this rescaling may destroy the norm relation (1.2.2). Even worse, rescaling by such factors may introduce unwanted and unknown vertical components into the divisor of the Borcherds product on the integral model of the Shimura variety, and understanding what's happening on the integral model is the central concern of this work.

1.3. Modularity of generating series. — The family of special divisors determines a family of line bundles

$$\mathcal{Z}(m,\mu)\in \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D}))$$