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STRONG REGULARITY

Pierre BERGER & Jean-Christophe Yoccoz

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Mots-clefs. — Hyperbolicité non-uniforme, sélection de paramètres, application unimodale, attracteur Hénon, dynamiques chaotiques, dynamiques en petite dimension, pièce de puzzle.

Keywords. — Nonuniformly hyperbolic systems, parameter selection, unimodal map, Hénon attractor, chaotic dynamics, low dimensional dynamics, puzzle piece.

STRONG REGULARITY

Pierre Berger & Jean-Christophe Yoccoz

Abstract. — The strong regularity program was initiated by Jean-Christophe Yoccoz during his first lecture at Collège de France. As explained in the first article of this volume, this program aims to show the abundance of dynamics displaying a non-uniformly hyperbolic attractor. It proposes a topological and combinatorial definition of such mappings using the formalism of puzzle pieces. Their combinatorics enable to deduce the wished analytical properties.

In 1997, this method enabled Jean-Christophe Yoccoz to give an alternative proof of the Jakobson theorem: the existence of a set of positive Lebesgue measure of parameters a such that the map $x \mapsto x^2 + a$ has an attractor which is non-uniformly hyperbolic. This proof is the second article of this volume.

In the third article, this method is generalized in dimension 2 by Pierre Berger to show the following theorem. For every C^2 -perturbation of the family of maps $(x, y) \mapsto (x^2 + a, 0)$, there exists a parameter set of positive Lebesgue measure at which these maps display a non-uniformly hyperbolic attractor. This gives in particular an alternative proof of the Benedicks-Carleson Theorem.

Résumé (Régularité forte.) — Le programme de régularité forte fut initié par Jean-Christophe Yoccoz lors de son premier cours au Collège de France. Comme expliqué et développé dans le premier article de ce volume, ce programme a pour objectif de démontrer l'abondance des dynamiques ayant un attracteur non-uniformément hyperbolique. Il propose une définition topologique et combinatoire de telles applications via le formalisme des pièces de puzzle. Leurs combinatoires permettent de déduire les propriétés analytiques désirées.

En 1997, cette méthode permet à Jean-Christophe Yoccoz de redémontrer le théorème de Jakobson : il existe un ensemble de mesure de Lebesgue positif de paramètres a tels que l'application $x \mapsto x^2 + a$ ait un attracteur non-uniformément hyperbolique. Cette preuve est le deuxième article de ce volume.

Dans le troisième article, cette méthode est généralisée en dimension deux par Pierre Berger pour démontrer le résultat suivant. Pour toute C^2 -perturbation de la famille d'applications $(x, y) \mapsto (x^2 + a, 0)$, il existe un ensemble de mesure de Lebesgue positif de paramètres a pour lesquels ces applications ont un attracteur

non-uniformément hyperbolique. Cela donne en particulier une preuve alternative du théorème de Benedicks-Carleson.

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ABSTRACTS

Introduction

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A proof of Jakobson's theorem

JEAN-CHRISTOPHE YOCOZ	15
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We give a proof of Jakobson's theorem: with positive probability on the parameter, a real quadratic map leaves invariant an absolutely continuous ergodic invariant probability measure with positive Lyapunov exponent.

Abundance of non-uniformly hyperbolic Hénon-like endomorphisms

PIERRE BERGER	53
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For every C^2 -small function B , we prove that the map $(x, y) \mapsto (x^2 + a, 0) + B(x, y, a)$ leaves invariant a physical, SRB probability measure, for a set of parameters a of positive Lebesgue measure. When the perturbation B is zero, this is the Jakobson Theorem; when the perturbation is a small constant times $(0, x)$, this is the celebrated Benedicks-Carleson theorem.

In particular, a new proof of the last theorem is given, based on a development of the combinatorial formalism of the Yoccoz puzzles. By adding new geometrical and combinatorial ingredients, and restructuring classic analytical ideas, we are able to carry out our proof in the C^2 -topology, even when the underlying dynamics are given by endomorphisms.

INTRODUCTION

by

Pierre Berger & Jean-Christophe Yoccoz

1. Uniformly hyperbolic dynamical systems

The theory of *uniformly hyperbolic dynamical systems* was constructed in the 1960's under the dual leadership of Smale in the USA and Anosov and Sinai in the Soviet Union. It is nowadays almost complete. It encompasses various examples [35]: expanding maps, horseshoes, solenoid maps, Plykin attractors, Anosov maps and DA, all of which are *basic pieces*.

We recall standard definitions. Let f be a C^1 -diffeomorphism f of a finite dimensional manifold M . A compact f -invariant subset $\Lambda \subset M$ is *uniformly hyperbolic* if the restriction to Λ of the tangent bundle TM splits into two continuous invariant subbundles

$$TM|_{\Lambda} = E^s \oplus E^u,$$

E^s being uniformly contracted and E^u being uniformly expanded.

Then for every $z \in \Lambda$, the sets

$$W^s(z) = \{z' \in M : \lim_{n \rightarrow +\infty} d(f^n(z), f^n(z')) = 0\},$$

$$W^u(z) = \{z' \in M : \lim_{n \rightarrow -\infty} d(f^n(z), f^n(z')) = 0\}$$

are called *the stable and unstable manifolds* of z . They are immersed manifolds tangent at z to respectively $E^s(z)$ and $E^u(z)$.

The ϵ -local stable manifold $W_{\epsilon}^s(z)$ of z is the connected component of z in the intersection of $W^s(z)$ with a ϵ -neighborhood of z . The ϵ -local unstable manifold $W_{\epsilon}^u(z)$ is defined likewise.

Definition 1.1. — A *basic set* is a compact, f -invariant, uniformly hyperbolic set Λ which is transitive and *locally maximal*: there exists a neighborhood N of Λ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(N)$. A basic set is an *attractor* if the neighborhood N can be chosen in

such a way that $\Lambda = \bigcap_{n \geq 0} f^n(N)$. Such a basic set contains the unstable manifolds of its points.

A diffeomorphism whose nonwandering set is a finite union of disjoint basic sets is called *uniformly hyperbolic* or *Axiom A*.

Such diffeomorphisms enjoy nice properties, which are proved in [35] and the references therein.

SRB and physical measure. — Let $\alpha > 0$, and let Λ be an attracting basic set for a $C^{1+\alpha}$ -diffeomorphism f . Then there exists a unique invariant, ergodic probability μ supported on Λ such that its conditional measures, with respect to any measurable partition of Λ into plaques of unstable manifolds, are absolutely continuous with respect to the Lebesgue measure class (on unstable manifolds). Such a probability is called *SRB* (for Sinai-Ruelle-Bowen). It turns out that a SRB -measure is *physical*: the Lebesgue measure of its basin $B(\mu)$

$$(\mathcal{B}) \quad B(\mu) = \{z \in M : \frac{1}{n} \sum_{i < n} \delta_{f^i(x)} \rightharpoonup \mu\}$$

is positive. Actually, up to a set of Lebesgue measure 0, $B(\mu)$ is equal to the topological basin of Λ , i.e the set of points attracted by Λ .

Persistence. — A basic set Λ for a C^1 -diffeomorphism f is *persistent*: every C^1 -perturbation f' of f leaves invariant a basic set Λ' which is homeomorphic to Λ , via a homeomorphism which conjugates the dynamics $f|_\Lambda$ and $f'|_{\Lambda'}$.

Coding. — A basic set Λ for a C^1 -diffeomorphism f admits a (finite) Markov partition. This implies that its dynamics is semi-conjugated with a subshift of finite type. The semi-conjugacy is 1-1 on a generic set. Its lack of injectivity is itself coded by subshifts of finite type of smaller topological entropy. This enables to study efficiently all the invariant measures of Λ , the distribution of its periodic points, the existence and uniqueness of the maximal entropy measure, and if f is $C^{1+\alpha}$, the Gibbs measures which are related to the geometry of Λ .

1.1. End of Smale's program. — Smale wished to prove the density of Axiom A in the space of C^r -diffeomorphisms. In higher dimensions, obstructions were soon discovered by Shub [33]. For surfaces Newhouse showed the non-density of Axiom A diffeomorphisms for $r \geq 2$: he constructed robust tangencies between stable and unstable manifolds of a thick horseshoe [26]. Numerical studies by Lorenz [18] and Hénon [14] explored dynamical systems with hyperbolic features that did not fit in the uniformly hyperbolic theory. In order to include many examples such as the Hénon one, the *non-uniform hyperbolic theory* is still under construction.

2. Non-uniformly hyperbolic dynamical systems

2.1. Pesin theory. — The natural setting for non-uniform hyperbolicity is Pesin theory [2, 17], from which we recall some basic concepts. We first consider the simpler settings of invertible dynamics.

Let f be a $C^{1+\alpha}$ -diffeomorphism (for some $\alpha > 0$) of a compact manifold M and let μ be an ergodic f -invariant probability measure on M . The Oseledec's multiplicative ergodic theorem produces Lyapunov exponents (w.r.t. μ) for the tangent cocycle of f , and an associated μ -a.e f -invariant splitting of the tangent bundle into characteristic subbundles.

Denote by $E^s(z)$ (resp. $E^u(z)$) the sum of the characteristic subspaces associated to the negative (resp. positive) Lyapunov exponents.

The *stable and unstable Pesin manifolds* are defined respectively for μ -a.e. z by

$$W^s(z) = \{z' \in M : \limsup_{n \rightarrow +\infty} \frac{1}{n} \log d(f^n(z), f^n(z')) < 0\},$$

$$W^u(z) = \{z' \in M : \liminf_{n \rightarrow -\infty} \frac{1}{n} \log d(f^n(z), f^n(z')) > 0\}.$$

They are immersed manifolds through z tangent respectively at z to $E^s(z)$ and $E^u(z)$.

The measure μ is *hyperbolic* if 0 is not a Lyapunov exponent w.r.t. μ . Every invariant ergodic measure, which is supported on a uniformly hyperbolic compact invariant set, is hyperbolic.

SRB, physical measures. — An invariant ergodic measure μ is *SRB* if the largest Lyapunov exponent is positive and the conditional measures of μ w.r.t. a measurable partition into plaques of unstable manifolds are μ -a.s. absolutely continuous w.r.t. the Lebesgue class (on unstable manifolds). When μ is SRB and hyperbolic, it is also *physical*: its basin has positive Lebesgue measure.

The paper [44] provides a general setting where appropriate hyperbolicity hypotheses allow to construct hyperbolic SRB measures with nice statistical properties.

Coding. — Let μ be a f -invariant ergodic hyperbolic SRB measure. Then there is a partition mod 0 of M into finitely many disjoint subsets $\Lambda_1, \dots, \Lambda_k$, which are cyclically permuted by f and such that the restriction $f|_{\Lambda_1}^k$ is metrically conjugated to a Bernoulli automorphism.

Of a rather different flavor is Sarig's recent work [32]. For a $C^{1+\alpha}$ -diffeomorphism of a compact surface of positive topological entropy and any $\chi > 0$, he constructs a countable Markov partition for an invariant set which has full measure w.r.t. any ergodic invariant measure with metric entropy $> \chi$. The semi-conjugacy associated to this Markov partition is finite-to-one.

Non-invertible dynamics. — One should distinguish between the non-uniformly expanding case and the case of general endomorphisms.

In the first setting, a SRB measure is simply an ergodic invariant measure whose Lyapunov exponents are all positive and which is absolutely continuous.

Defining appropriately unstable manifolds and SRB measures for general endomorphisms is more delicate. One has typically to introduce the inverse limit where the endomorphism becomes invertible.

2.2. Case studies. — The paradigmatic examples in low dimension can be summarized by the following table:

Uniformly hyperbolic	Non-uniformly hyperbolic
Expanding maps of the circle	Jakobson's Theorem
Conformal expanding maps of complex tori	Rees' Theorem
Attractors (Solenoid, DA, Plykin...)	Benedicks-Carleson's Theorem
Horseshoes	Non-uniformly hyperbolic horseshoes
Anosov diffeomorphisms	Standard map ?

Let us recall what these theorems state, and the correspondence given by the lines of the table.

Expanding maps of the circle may be considered as the simplest case of uniformly hyperbolic dynamics. The Chebychev quadratic polynomial $P_{-2}(x) := x^2 - 2$ on the invariant interval $[-2, 2]$ has a critical point at 0, but it is still semi-conjugated to the doubling map $\theta \mapsto 2\theta$ on the circle (through $x = 2 \cos 2\pi\theta$). For $a \in [-2, -1]$, the quadratic polynomial $P_a(x) := x^2 + a$ leaves invariant the interval $[P_a(0), P_a^2(0)]$ which contains the critical point 0.

Theorem 2.1 (Jakobson [16]). — *There exists a set $\Lambda \subset [-2, -1]$ of positive Lebesgue measure such that for every $a \in \Lambda$ the map $P(x) = x^2 + a$ leaves invariant an ergodic, hyperbolic measure which is equivalent to the Lebesgue measure on $[P_a(0), P_a^2(0)]$.*

Actually the set Λ is nowhere dense. Indeed the set of $a \in \mathbb{R}$ such that P_a is Axiom A is open and dense [13, 20].

Let L be a lattice in $\mathbb{C}$ and let c be a complex number such that $|c| > 1$ and $cL \subset L$. Then the homothety $z \mapsto cz$ induces an expanding map of the complex torus $\mathbb{C}/L$. The Weierstrass function associated to the lattice L defines a ramified covering of degree 2 from $\mathbb{C}/L$ onto the Riemann sphere which is a semi-conjugacy from this expanding map to a rational map of degree $|c|^2$ called a *Lattes map*. For any $d \geq 2$, the set Rat_d of rational maps of degree d is naturally parametrized by an open subset of $\mathbb{P}(\mathbb{C}^{2d+2})$.

Theorem 2.2 (Rees [29]). — *For every $d \geq 2$, there exists a subset $\Lambda \subset \text{Rat}_d$ of positive Lebesgue measure such that every map $R \in \Lambda$ leaves invariant an ergodic hyperbolic probability measure which is equivalent to the Lebesgue measure on the Riemann sphere.*

For rational maps in Λ , the Julia set is equal to the Riemann sphere. On the other hand, a conjecture of Fatou [23] claims that the set of rational maps which satisfy Axiom A is open and dense in Rat_d . The restriction of such maps to their Julia set is uniformly expanding. For such maps, the Hausdorff dimension of the Julia set is smaller than 2.

The (real) Hénon family is the 2-parameter family of polynomial diffeomorphisms of the plane defined for $a, b \in \mathbb{R}$, $b \neq 0$ by

$$h_{a,b}(x, y) = (x^2 + a + y, -bx).$$

Observe that $h_{a,b}$ has constant Jacobian equal to b . For small $|b|$, there exists an interval $J(b)$ close to $[-2, -1]$ such that, for $a \in J(b)$, the Hénon map $h_{a,b}$ has the following properties

- $h_{a,b}$ has two fixed points; both are hyperbolic saddle points, one, called β with positive unstable eigenvalue, the other, called α , with negative unstable eigenvalue;
- there is a trapping open region B satisfying $h_{a,b}(B) \subseteq B$ which contains α (and therefore also its unstable manifold).

Hénon [14] investigated numerically the behavior of orbits starting in B for $b = -0.3$, $a = -1.4$. Such orbits apparently converged to a “strange attractor”.

Theorem 2.3 (Benedicks-Carleson [4]). — *For every $b < 0$ close enough to 0, there exists a set $\Lambda_b \subset J(b)$ of positive Lebesgue measure, such that for every $a \in \Lambda_b$, the maximal invariant set $\bigcap_{n \geq 0} h_{a,b}^n(B)$ is equal to the closure of the unstable manifold $W^u(\alpha)$ and contains a dense orbit along which the derivatives of iterates grow exponentially fast.*

An easy topological argument ensures that this maximal invariant set is never uniformly hyperbolic. Later Benedicks-Young [6] showed that for every such parameters $a \in \Lambda_b$ the Hénon map $h_{a,b}$ leaves invariant an ergodic hyperbolic SRB measure. Such a measure is physical. Benedicks-Viana [5] actually proved that the basin of this measure has full Lebesgue measure in the trapping region B .

From [39], every $a \in \Lambda_b$ is accumulated by parameter intervals exhibiting Newhouse phenomenon: for generic parameters in these intervals, $h_{a,b}$ has infinitely many periodic sinks in B . In particular, the set Λ_b is nowhere dense.

The starting point in [28] is a smooth diffeomorphism of a surface M having a horseshoe⁽¹⁾ K . It is assumed that there exist distinct fixed points $p_s, p_u \in K$ and $q \in M$ such that $W^s(p_s)$ and $W^u(p_u)$ have at q a quadratic heteroclinic tangency which is an isolated point of $W^s(K) \cap W^u(K)$. The authors consider a one-parameter family (f_t) unfolding the tangency and study the maximal f_t -invariant set L_t in a neighborhood of the union of K with the orbit of q . Writing d_s, d_u for the transverse Hausdorff dimensions of $W^s(K)$, $W^u(K)$ respectively, it was shown previously [27] that L_t is a horseshoe for most t when $d_s + d_u < 1$. By [25] this is no longer true when

⁽¹⁾ A horseshoe is an infinite basic set of saddle type.