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Abundance of non-uniformly hyperbolic Hénon-like endomorphisms

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ABUNDANCE OF NON-UNIFORMLY HYPERBOLIC HÉNON-LIKE ENDOMORPHISMS

by

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Abstract. — For every C^2-small function B, we prove that the map $(x, y) \mapsto (x^2 + a, 0) + B(x, y, a)$ leaves invariant a physical, SRB probability measure, for a set of parameters a of positive Lebesgue measure. When the perturbation B is zero, this is the Jakobson Theorem; when the perturbation is a small constant times $(0, x)$, this is the celebrated Benedicks-Carleson theorem.

In particular, a new proof of the last theorem is given, based on a development of the combinatorial formalism of the Yoccoz puzzles. By adding new geometrical and combinatorial ingredients, and restructuring classic analytical ideas, we are able to carry out our proof in the C^2-topology, even when the underlying dynamics are given by endomorphisms.

Résumé. — Pour toute petite C^2-function B, nous prouvons que pour un ensemble de paramètres a de mesure de Lebesgue positive, l'application $(x, y) \mapsto (x^2 + a, 0) + B(x, y, a)$ préserve une mesure de probabilité qui est physique et SRB. Quand l'application B est nulle, il s'agit du théorème de Jakobson ; quand la perturbation est égale à une petite constante fois $(0, x)$, on obtient le célèbre théorème de Benedicks-Carleson.

Nous donnons en particulier une nouvelle preuve de ce dernier théorème, basée sur le formalisme combinatoire des pièces de puzzle de Yoccoz. En ajutant de nouveaux ingrédiants géométriques et combinatoires, et en restructurant des idées analytiques classiques, nous arrivons à prouver notre résultat en topologie C^2, et cela, même quand la dynamique est un endomorphisme.

Our aim is to prove the existence of a non-uniformly hyperbolic attractor for a large set of parameters $a \in \mathbb{R}$, for the following family of maps:

$$f_{a\,B} : \ (x, y) \mapsto (x^2 + a, 0) + B(x, y, a),$$

where B is a fixed C^2-map of $\mathbb{R}^3$ to $\mathbb{R}^2$ close to 0. We denote by b an upper bound of the uniform C^2-norm of $B|[-3, 3]^2$ and of the determinant of $Df_{a\,B}$. For B fixed, we prove that for a large set Ω_B of parameters a, the dynamics $f_{a\,B}$ is *strongly regular*. This has many consequences, among which is the following theorem.

***Theorem 0.1* (Main).** — *For any $\eta > 0$, there exist constants $a_0 > -2$ and $b > 0$ such that the following property holds: for any B with C^2-norm less than b, there is a subset*

$\Omega_B \subset [-2, a_0]$ *of relative measure greater than* $1-\eta$*, such that for any* $a \in \Omega_B$*,* $f_{a\,B}$ *leaves invariant a physical, SRB measure.*

This answers a question of Pesin-Yurchenko for reaction-diffusion PDEs in applied mathematics [**14**]. This solves also a step of the program of Yoccoz stated at his first lecture at Collège de France in 1997 [**24**]. The present manuscript was also, following his own words [**25**, $-1'37''$], the main source of inspiration of his last lecture at Collège de France. Nevertheless the writing of our text has been deeply revised since this time.

To the author's knowledge, this is the first result showing the abundance of non-uniformly hyperbolic, surface, (non-invertible and not expanding) endomorphisms leaving invariant an SRB measure. It seems also to be the first result proving the abundance of non-uniformly hyperbolic surface maps (invertible or not) for families in a C^2-open set (all previous results need three derivatives even [**22**]).

Introduction

0.1. History. — The birth of chaotic dynamical systems goes back to Poincaré in his study of the 3-body problem. At the time, the prevaling belief was that dynamical systems are always deterministic: small perturbations do not change the long term behavior. Let us recall a (simplified version) of his famous counterexample.

The idea is to consider two massive planets of equal mass and in circular orbits around 0 in the complex plane $\mathbb{C}$. One views $\mathbb{C}$ as embedded into $\mathbb{C}\times\mathbb{S}^1$ via the inclusion $\mathbb{C}\approx\mathbb{C}\times\{1\}\hookrightarrow\mathbb{C}\times\mathbb{S}^1$. Put a planet P in $\{0\}\times\mathbb{S}$ with a vertical initial speed $v\in\{0\}\times\mathbb{R}$. On one hand, assuming P has negligible mass, the motions of both massive planets remain circular and included in $\mathbb{C}\times\{0\}$. On the other, the dynamics of the planet remains in the circle $\{0\}\times\mathbb{S}$; in this way the dynamics resemble the (time one) pendulum map f of the tangent space of the circle $T\mathbb{S}=\mathbb{S}\times\mathbb{R}$. We identify $T\mathbb{S}$ with the punctured plane in the phase diagram drawn at the left of Figure 1. It turns out that the point $M=(0,-1)\in T\mathbb{S}$ is a hyperbolic fixed point: the differential of f at M has two eigenvalues of modulus different from 1. Moreover, the stable and unstable manifolds of M are equal to a same curve $W^s(M)=W^u(M)$. This is a *homoclinic tangency.*

Contrary to the case of the pendulum, we can perturb the system in such a way that not only it holds $W^s(M)\neq W^u(M)$ but also the intersection $W^s(M)\cap W^u(M)$ contains a point $N\neq M$ where the intersection is transverse (see the second picture of Figure 1). Indeed, we can assume that the two massive planets have an elliptic orbit centered at 0 with small eccentricity $e\neq 0$. The hyperbolic point M persists, its stable and unstable manifolds $W^s(M)$ and $W^u(M)$ are no longer equal, the intersection $W^s(M)\cap W^u(M)$ contains all the iterates of N.

The global picture of $W^s(M)$ and $W^u(M)$ is then extremely complex (see the third picture of Figure 1); with it, the field of chaotic dynamical systems is born.

Even today, we do not know how to describe this picture mathematically. In fact, we do not even know if the closure of $W^s(M) \cap W^u(M)$ can have positive Lebesgue measure.

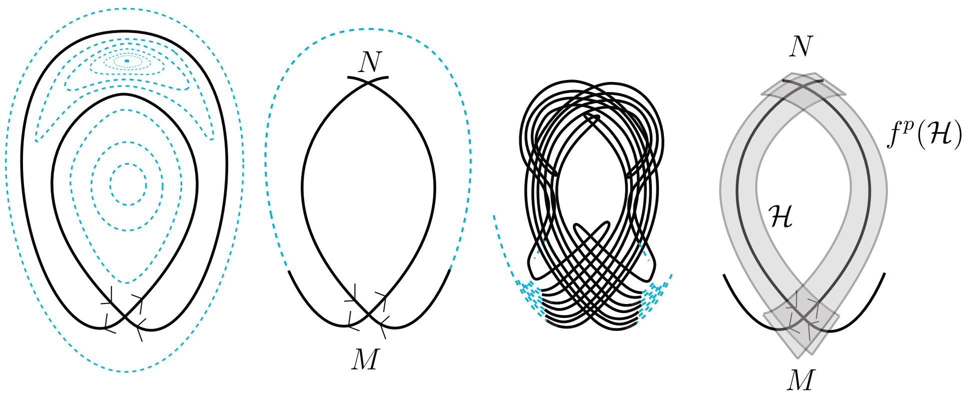

FIGURE 1. Homoclinic tangle arising from a 3-body problem.

In the sixties, Smale remarked that a thin neighborhood $\mathscr{H}$ of the segment of $W^s(M)$ containing M and N is sent by an iterate f^p onto a neighborhood of the segment of $W^u(M)$ containing M and N (see the last picture of Figure 1). If $\mathscr{H}$ is sufficiently thin, the maximal invariant $K = \bigcap_{n\in\mathbb{Z}} f^{pn}(\mathscr{H})$ is a compact hyperbolic set: the celebrated Smale Horseshoe. *Hyperbolicity* means that the space $T\mathbb{R}^2|K$ is endowed with two Df^p-invariant directions, one expanded, the other contracted by Df.

The theory of hyperbolic dynamical systems was largely developed by the schools of Smale and Sinai. It can be considered as more or less complete **[18]**. A hyperbolic set K is an attractor if it is transitive and if $K = \bigcap_{n\geq 0} f^n(V)$ for some neighborhood V of K. Hyperbolic attractors are well understood through the following properties.

Persistence. — A uniformly hyperbolic attractor is *persistent* if every C^1-perturbation f' of f leaves invariant a uniformly hyperbolic attractor K' homeomorphic to K, via a homeomorphism which conjugates the dynamics $f|K$ and $f'|K'$.

Geometry. — Every uniformly hyperbolic attractor supports a lamination whose leaves are unstable manifolds. This means that K can be covered by finitely many open sets $(U_i)_i$ whose intersection with K is homeomorphic to the product of $\mathbb{R}^d$ with a compact set T, such that $\mathbb{R}^d \times \{t\}$ corresponds to a local unstable manifold, for every $t \in T$.

SRB, physical measure. — An *SRB measure* of K is an f-invariant probability measure ν supported by K, such that the conditional measure with respect to every local unstable manifold is absolutely continuous with respect to the Lebesgue measure. Whenever the dynamics is of class $C^{1+\alpha}$, there exists a unique, ergodic, SRB measure supported by K. By Birkhoff's theorem, ν-almost every point is *ν-generic* with orbit intersecting every Borel subset U in mean with proportion $\nu(U)$. Moreover every SRB measure is *physical*: the set of ν-generic points—called *the basin of* ν—is of positive Lebesgue measure on a neighborhood of the attractor.

Coding. — Every uniformly hyperbolic attractor has a Markov partition. This provides a semi-conjugacy of the dynamics with a subshift of finite type. The conjugacy is 1–1 on a generic set supporting all the measures with large entropy. This implies the existence and uniqueness of the maximal entropy measure, it is also a key point to construct the "thermodynamic" formalism.

The persistence and the existence of an SRB measure show that deterministic dynamical systems may have (robust) statistical behaviors.

These properties enable a deep understanding of *uniformly hyperbolic dynamical systems*, or more precisely *Axiom A diffeomorphisms.* By definition, the latter are those dynamical systems whose non-wandering set is locally maximal and hyperbolic.

Many dynamical systems do not satisfy Axiom A. It is indeed easy to see that an Axiom A, conservative dynamical system is necessarily Anosov (the whole manifold is hyperbolic). Newhouse also found a (dissipative) C^2-surface diffeomorphism, robustly not Axiom A [**11**], by building an example of horseshoe of the plane whose local stable and unstable manifolds are robustly tangent.

In the meantime, Hénon [**7**] numerically exhibited a "strange attractor" for the family of maps $(x, y) \mapsto (x^2 + a + y, bx)$. Later, Benedicks-Carleson [**1**] proved its existence mathematically: for a set of parameters (a, b) of positive Lebesgue measure (with b very small), they proved that there exists a topological attractor which is not (uniformly) hyperbolic. Viana-Mora [**10**] showed that this proof can be adapted to an open set of C^3-perturbations of the Hénon family, occurring in the study of the unfolding of some homoclinic tangencies. Later Benedicks-Young showed the existence of an SRB measure for these parameters [**3**]. These works were generalized by Wang-Young [**21, 22**]; they showed further properties of the SRB measure. A recent work of Takahashi shows the existence of strange attractors for endomorphisms of the plane [**19**], nonetheless he did not show the existence of an SRB measure for these maps.

The work of [**1**] was one of the greatest achievements in dynamical systems of the last decades, especially for the analysis developed therein. Unfortunately the maps they deal with are defined by a long induction (≈ 100 pages). This makes their ideas difficult to be understood and generalized. Moreover, to state a property on such dynamics, one has to recall the whole induction process. That is why we propose a new approach to this problem based on a development of the Yoccoz puzzle pieces.