Astérisque
421, 2020, p. 7–125
doi:10.24033/ast.1126

MODULARITY OF GENERATING SERIES OF DIVISORS ON UNITARY SHIMURA VARIETIES

by

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Abstract. — We form generating series, valued in the Chow group and the arithmetic Chow group, of special divisors on the compactified integral model of a Shimura variety associated to a unitary group of signature $(n-1,1)$, and prove their modularity. The main ingredient in the proof is the calculation of vertical components appearing in the divisor of a Borcherds product on the integral model.

Résumé **(Modularité des séries génératrices de diviseurs sur les variétés de Shimura unitaires)**
Nous formons des séries génératrices, à valeurs dans le groupe de Chow et dans le groupe de Chow arithmétique, formées des diviseurs spéciaux sur le modèle intégral compact d'une variété de Shimura associée à un groupe unitaire de signature $(n-1,1)$, et prouvons leur modularité. L'ingrédient principal de la preuve est le calcul des composantes verticales apparaissantes dans le diviseur d'un produit de Borcherds sur le modèle intégral.

1. Introduction

The goal of this paper is to prove the modularity of a generating series of special divisors on the compactified integral model of a Shimura variety associated to a unitary group of signature $(n-1,1)$. The special divisors in question were first studied on the open Shimura variety in **[33, 34]**, and then on the toroidal compactification in **[24]**.

This generating series is an arithmetic analogue of the classical theta kernel used to lift modular forms from $\mathrm{U}(2)$ and $\mathrm{U}(n)$. In a similar vein, our modular generating

2010 Mathematics Subject Classification. — 14G35, 14G40, 11F55, 11F27, 11G18.
Key words and phrases. — Shimura varieties, Borcherds products.

J.B. was supported in part by DFG grant BR-2163/4-2. B.H. was supported in part by NSF grants DMS-1501583 and DMS-1801905. M.R. was supported in part by the Deutsche Forschungsgemeinschaft through the grant SFB/TR 45. S.K. was supported in part by an NSERC Discovery Grant. T.Y. was supported in part by NSF grant DMS-1500743 and DMS-1762289.

series can be used to define a lift from classical cuspidal modular forms of weight n to the codimension one Chow group of the unitary Shimura variety.

1.1. Statement of the main result. — Fix a quadratic imaginary field $\boldsymbol{k} \subset \mathbb{C}$ of odd discriminant $\mathrm{disc}(\boldsymbol{k}) = -D$. We are concerned with the arithmetic of a certain unitary Shimura variety, whose definition depends on the choices of $\boldsymbol{k}$-hermitian spaces W_0 and W of signature $(1,0)$ and $(n-1,1)$, respectively, where $n \geq 3$. We assume that W_0 and W each admit an $\mathcal{O}_{\boldsymbol{k}}$-lattice that is self-dual with respect to the hermitian form.

Attached to this data is a reductive algebraic group

$$G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W) \tag{1.1.1}$$

over $\mathbb{Q}$, defined as the subgroup on which the unitary similitude characters are equal, and a compact open subgroup $K \subset G(\mathbb{A}_f)$ depending on the above choice of self-dual lattices. As explained in § 2, there is an associated hermitian symmetric domain $\mathcal{D}$, and a Deligne-Mumford stack $\mathrm{Sh}(G, \mathcal{D})$ over $\boldsymbol{k}$ whose complex points are identified with the orbifold quotient

$$\mathrm{Sh}(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K.$$

This is the unitary Shimura variety of the title.

The stack $\mathrm{Sh}(G, \mathcal{D})$ can be interpreted as a moduli space of pairs (A_0, A) in which A_0 is an elliptic curve with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$, and A is a principally polarized abelian scheme of dimension n endowed with an $\mathcal{O}_{\boldsymbol{k}}$-action. The pair (A_0, A) is required to satisfy some additional conditions, which need not concern us in the introduction.

Using the moduli interpretation, one can construct an integral model of $\mathrm{Sh}(G, \mathcal{D})$ over $\mathcal{O}_{\boldsymbol{k}}$. In fact, following work of Pappas and Krämer, we explain in § 2.3 that there are two natural integral models related by a morphism $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$. Each integral model has a canonical toroidal compactification whose boundary is a disjoint union of smooth Cartier divisors, and the above morphism extends uniquely to a morphism

$$\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}} \tag{1.1.2}$$

of compactifications.

Each compactified integral model has its own desirable and undesirable properties. For example, $\mathcal{S}^*_{\mathrm{Kra}}$ is regular, while $\mathcal{S}^*_{\mathrm{Pap}}$ is not. On the other hand, every vertical (i.e., supported in nonzero characteristic) Weil divisor on $\mathcal{S}^*_{\mathrm{Pap}}$ has nonempty intersection with the boundary, while $\mathcal{S}^*_{\mathrm{Kra}}$ has certain *exceptional* divisors in characteristics $p \mid D$ that do not meet the boundary. An essential part of our method is to pass back and forth between these two models in order to exploit the best properties of each. For simplicity, we will state our main results in terms of the regular model $\mathcal{S}^*_{\mathrm{Kra}}$.

In § 2 we define a distinguished line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$, called the *line bundle of weight one modular forms*, and a family of Cartier divisors $\mathcal{Z}_{\mathrm{Kra}}(m)$ indexed by integers $m > 0$. These special divisors were introduced in **[33, 34]**, and studied further in **[11, 23, 24]**. For the purposes of the introduction, we note only that one should regard the divisors as arising from embeddings of smaller unitary groups into G.

Denote by

$$\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \cong \mathrm{Pic}(\mathcal{S}^*_{\mathrm{Kra}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

the Chow group of rational equivalence classes of divisors with $\mathbb{Q}$ coefficients. Each special divisor $\mathcal{Z}_{\mathrm{Kra}}(m)$ can be extended to a divisor on the toroidal compactification simply by taking its Zariski closure, denoted $\mathcal{Z}^*_{\mathrm{Kra}}(m)$. The *total special divisor* is defined as

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Z}^*_{\mathrm{Kra}}(m) + \mathcal{B}_{\mathrm{Kra}}(m) \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \tag{1.1.3}$$

where the boundary contribution is defined, as in (5.3.3), by

$$\mathcal{B}_{\mathrm{Kra}}(m) = \frac{m}{n-2} \sum_{\Phi} \#\{x \in L_0 : \langle x, x \rangle = m\} \cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi).$$

The notation here is the following: The sum is over the equivalence classes of *proper cusp label representatives* Φ as defined in § 3.1. These index the connected components $\mathcal{S}^*_{\mathrm{Kra}}(\Phi) \subset \partial \mathcal{S}^*_{\mathrm{Kra}}$ of the boundary (1). Inside the sum, $(L_0, \langle \cdot, \cdot \rangle)$ is a hermitian $\mathcal{O}_{\boldsymbol{k}}$-module of signature $(n-2, 0)$, which depends on Φ.

The line bundle of modular forms $\boldsymbol{\omega}$ admits a canonical extension to the toroidal compactification, denoted the same way. For the sake of notational uniformity, we extend (1.1.3) to $m = 0$ by setting

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \boldsymbol{\omega}^{-1} + \mathrm{Exc} \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}). \tag{1.1.4}$$

Here Exc is the exceptional divisor of Theorem 2.3.4. It is a reduced effective divisor supported in characteristics $p \mid D$, disjoint from the boundary of the compactification. The following result appears in the text as Theorem 7.1.5.

Theorem A. — *Let $\chi_{\boldsymbol{k}} : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ be the Dirichlet character determined by $\boldsymbol{k}/\mathbb{Q}$. The formal generating series*

$$\sum_{m \geq 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]]$$

is modular of weight n, level $\Gamma_0(D)$, and character $\chi^n_{\boldsymbol{k}}$ in the following sense: for every $\mathbb{Q}$-linear functional $\alpha : \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \mathbb{C}$, the series

$$\sum_{m \geq 0} \alpha(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)) \cdot q^m \in \mathbb{C}[[q]]$$

is the q-expansion of a classical modular form of the indicated weight, level, and character.

We can prove a stronger version of Theorem A. Denote by $\widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$ the Gillet-Soulé **[20]** arithmetic Chow group of rational equivalence classes of pairs $\widehat{\mathcal{Z}} = (\mathcal{Z}, \mathrm{Gr})$, where $\mathcal{Z}$ is a divisor on $\mathcal{S}^*_{\mathrm{Kra}}$ with rational coefficients, and Gr is a Green function

(1) After base change to $\mathbb{C}$, each $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$ decomposes into h connected components, where h is the class number of $\boldsymbol{k}$.

for $\mathcal{Z}$. We allow the Green function to have additional log-log singularities along the boundary, as in the more general theory developed in [**13**]. See also [**8, 24**].

In § 7.3 we use the theory of regularized theta lifts to construct Green functions for the special divisors $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$, and hence obtain arithmetic divisors

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \in \widehat{\mathrm{Ch}}^{1}_{\mathbb{Q}}(\mathcal{S}^{*}_{\mathrm{Kra}})$$

for $m > 0$. We also endow the line bundle $\boldsymbol{\omega}$ with a metric, and the resulting metrized line bundle $\widehat{\boldsymbol{\omega}}$ defines a class

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \widehat{\boldsymbol{\omega}}^{-1} + (\mathrm{Exc}, -\log(D)) \in \widehat{\mathrm{Ch}}^{1}_{\mathbb{Q}}(\mathcal{S}^{*}_{\mathrm{Kra}}),$$

where the vertical divisor Exc has been endowed with the constant Green function $-\log(D)$. The following result is Theorem 7.3.1 in the text.

Theorem B. — *The formal generating series*

$$\widehat{\phi}(\tau) = \sum_{m \geq 0} \widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^{m} \in \widehat{\mathrm{Ch}}^{1}_{\mathbb{Q}}(\mathcal{S}^{*}_{\mathrm{Kra}})[[q]]$$

is modular of weight n, level $\Gamma_0(D)$, and character $\chi^{n}_{\boldsymbol{k}}$, where modularity is understood in the same sense as Theorem A.

Remark 1.1.1. — As this article was being revised for publication, Wei Zhang announced a proof of his *arithmetic fundamental lemma*, conjectured in [**52**]. Although the statement is a purely local result concerning intersections of cycles on unitary Rapoport-Zink spaces, Zhang's proof uses global calculations on unitary Shimura varieties, and makes essential use of the modularity result of Theorem B. See [**53**].

Remark 1.1.2. — Theorem B implies that the $\mathbb{Q}$-span of the classes $\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$ is finite dimensional. See Remark 7.1.2.

Remark 1.1.3. — There is a second method of constructing Green functions for the special divisors, based on the methods of [**36**], which gives rise to a non-holomorphic variant of $\widehat{\phi}(\tau)$. It is a recent theorem of Ehlen-Sankaran [**16**] that Theorem B implies the modularity of this non-holomorphic generating series. See § 7.4.

One motivation for the modularity result of Theorem B is that it allows one to construct arithmetic theta lifts. If $g(\tau) \in S_n(\Gamma_0(D), \chi^{n}_{\boldsymbol{k}})$ is a classical scalar valued cusp form, we may form the Petersson inner product

$$\widehat{\theta}(g) \stackrel{\mathrm{def}}{=} \langle \widehat{\phi}, g \rangle_{\mathrm{Pet}} \in \widehat{\mathrm{Ch}}^{1}_{\mathbb{C}}(\mathcal{S}^{*}_{\mathrm{Kra}})$$

as in [**38**]. One expects, as in [*loc. cit.*], that the arithmetic intersection pairing of $\widehat{\theta}(g)$ against other cycle classes should be related to derivatives of L-functions, providing generalizations of the Gross-Zagier and Gross-Kohnen-Zagier theorems. Specific instances in which this expectation is fulfilled can be deduced from [**11, 23, 24**]. This will be explained in the companion paper [**10**].

As this paper is rather long, we explain in the next two subsections the main ideas that go into the proof of Theorem A. The proof of Theorem B is exactly the same, but one must keep track of Green functions.

1.2. Sketch of the proof, part I: the generic fiber. — In this subsection we sketch the proof of modularity only in the generic fiber. That is, the modularity of

$$\sum_{m\geq 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}} \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}/\boldsymbol{k}})[[q]]. \tag{1.2.1}$$

The key to the proof is the study of *Borcherds products* [**4, 5**].

A Borcherds product is a meromorphic modular form on an orthogonal Shimura variety, whose construction depends on a choice of weakly holomorphic input form, typically of negative weight. In our case the input form is any

$$f(\tau) = \sum_{m\gg -\infty} c(m) q^m \in M^{!,\infty}_{2-n}(D, \chi^{n-2}_{\boldsymbol{k}}), \tag{1.2.2}$$

where the superscripts ! and ∞ indicate that the weakly holomorphic form $f(\tau)$ of weight $2-n$ and level $\Gamma_0(D)$ is allowed to have a pole at the cusp ∞, but must be holomorphic at all other cusps. We assume also that all $c(m) \in \mathbb{Z}$.

Our Shimura variety $\mathrm{Sh}(G,\mathcal{D})$ admits a natural map to an orthogonal Shimura variety. Indeed, the $\boldsymbol{k}$-vector space

$$V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$$

admits a natural hermitian form $\langle .,. \rangle$ of signature $(n-1,1)$, induced by the hermitian forms on W_0 and W. The natural action of G on V determines an exact sequence

$$1 \to \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}} \mathbb{G}_m \to G \to \mathrm{U}(V) \to 1 \tag{1.2.3}$$

of reductive groups over $\mathbb{Q}$.

We may also view V as a $\mathbb{Q}$-vector space endowed with the quadratic form $Q(x) = \langle x, x\rangle$ of signature $(2n-2, 2)$, and so obtain a homomorphism $G \to \mathrm{SO}(V)$. This induces a map from $\mathrm{Sh}(G,\mathcal{D})$ to the Shimura variety associated with the group $\mathrm{SO}(V)$.

After possibly replacing f by a nonzero integer multiple, Borcherds constructs a meromorphic modular form on the orthogonal Shimura variety, which can be pulled back to a meromorphic modular form on $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$. The result is a meromorphic section $\boldsymbol{\psi}(f)$ of $\boldsymbol{\omega}^k$, where the weight

$$k = \sum_{r\mid D} \gamma_r \cdot c_r(0) \in \mathbb{Z} \tag{1.2.4}$$

is the integer defined in § 5.3. The constant $\gamma_r = \prod_{p\mid r} \gamma_p$ is a 4^{th} root of unity (with $\gamma_1 = 1$) and $c_r(0)$ is the constant term of f at the cusp

$$\infty_r = \frac{r}{D} \in \Gamma_0(D)\backslash \mathbb{P}^1(\mathbb{Q}),$$

in the sense of Definition 4.1.1.