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W-EXPONENTIALS, SCHUR ELEMENTS, AND THE SUPPORT OF THE SPHERICAL REPRESENTATION OF THE RATIONAL CHEREDNIK ALGEBRA

BY STEPHEN GRIFFETH AND DANIEL JUTEAU

ABSTRACT. – Given a complex reflection group W we compute the support of the spherical irreducible module $L_c(\mathbf{1})$ of the rational Cherednik algebra $H_c(W)$ in terms of the simultaneous eigenfunction of the Dunkl operators and Schur elements for finite Hecke algebras.

RÉSUMÉ. – Pour un groupe de réflexions complexe donné W, nous calculons le support du module sphérique irréductible $L_c(\mathbf{1})$ de l'algèbre de Cherednik rationnelle $H_c(W)$ en termes des fonctions propres communes des opérateurs de Dunkl et des éléments de Schur des algèbres de Hecke finies.

1. Introduction

The rational Cherednik algebra is a certain infinite-dimensional associative algebra attached to a complex reflection group. It is, roughly speaking, analogous to the enveloping algebra of a Kac-Moody Lie algebra, and its representation theory is closely related to the representation theory of the finite Hecke algebra of the complex reflection group. One of the central problems in the theory is the classification of the finite dimensional irreducible representations, which should be thought of as the cuspidal representations in a Harish-Chandra classification scheme.

In this paper we compute the support of the unique simple quotient $L_c(\mathbf{1})$ of the polynomial representation $\mathbf{C}[V] = \Delta_c(\mathbf{1})$ of the rational Cherednik algebra of a complex reflection

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group W, and in particular determine when it is finite dimensional, in terms of the joint eigenfunction for the Dunkl operators (the "W-exponential function") and Schur elements for finite Hecke algebras, which arise in this context because they compute the endomorphism of the identity functor produced by an induction-restriction biadjunction.

In order to keep this introduction as self-contained as possible, we now introduce the definitions we need in order to state our main theorems. Let V be a finite dimensional $\mathbf{C}$-vector space and let $W \subseteq \mathrm{GL}(V)$ be a *complex reflection group*: a finite group of linear transformations of V that is generated by the set

$$R = \{r \in W \mid \mathrm{codim}_V(\mathrm{fix}(r)) = 1\}$$

of *reflections* it contains. We write

$$\mathscr{A} = \{\mathrm{fix}(r) \mid r \in R\}$$

for the set of *reflecting hyperplanes* for W, and given $H \in \mathscr{A}$ we fix a linear form $\alpha_H \in V^*$ with zero set H:

$$\{v \in V \mid \alpha_H(v) = 0\} = H.$$

We will write W_H for the (cyclic) group of elements of W fixing H pointwise, let $n_H = |W_H|$ be its cardinality, and given a linear character $\chi \in W_H^\vee$ of W_H we write $e_{H,\chi} \in \mathbf{C}W_H$ for the corresponding idempotent,

$$e_{H,\chi} = \frac{1}{n_H} \sum_{r \in W_H} \chi(r^{-1}) r.$$

We now make the central definition of the paper. We fix a W-equivariant collection $c_{H,\chi}$ of parameters indexed by pairs (H, χ) consisting of a reflecting hyperplane H for W and a linear character $\chi \in W_H^\vee$, with the property that $c_{H,\chi} = 0$ if $\chi = 1$ is trivial. Each $y \in V$ determines a *Dunkl operator*, also written y, defined by giving its action on each polynomial function $f \in \mathbf{C}[V]$

$$y(f) = \partial_y(f) - \sum_{H \in \mathscr{A}} \frac{\alpha_H(y)}{\alpha_H} \sum_{\chi \in W_H^\vee} c_{H,\chi} n_H e_{H,\chi}(f).$$

The *rational Cherednik algebra* $H_c(W, V)$ is the algebra of operators on $\mathbf{C}[V]$ generated by $\mathbf{C}[V]$ (acting on itself by multiplication), the group W, and for each $y \in V$, the corresponding Dunkl operator. As a $H_c(W, V)$-module, $\mathbf{C}[V]$ has a unique maximal submodule I (which is an ideal in $\mathbf{C}[V]$). We define $L_c(\mathbf{1}) = \mathbf{C}[V]/I$, and we will compute the zero set $V(I)$, which is the *support* of $L_c(\mathbf{1})$.

Given a complex reflection group W acting in a vector space V, one obtains a stratification of V whose strata are the equivalence classes for the equivalence relation $p \equiv q$ if the stabilizer groups are equal, $W_p = W_q$. Given a stratum S we will write $W_S = W_p$ for any $p \in S$. The support of $L_c(\mathbf{1})$ is the closure of a single W-orbit of strata ([4, Section 3.8]), and in particular $L_c(\mathbf{1})$ is finite dimensional if and only if its support is the single point $\{0\}$.

In order to state our first characterization of the support of $L_c(\mathbf{1})$, we now introduce the W-exponential function $e(x, \lambda, c)$, which is a normalized joint eigenfunction for the Dunkl operators with eigenvalue $\lambda \in V^*$. More precisely, for λ fixed and c generic, $e(x, \lambda, c)$ is the entire function of x characterized by

$$y \cdot e(x, \lambda, c) = \lambda(y) e(x, \lambda, c) \quad \text{and} \quad e(0, \lambda, c) = 1 \quad \text{for all } y \in V.$$

For λ fixed, there is an entire holomorphic renormalization function $F_S(c)$ with zero set depending only on the stratum S of λ with the properties that $F_S(c)e(x,\lambda,c)$ is entire as a function of c and, for each fixed c, non-zero as a function of x. A *positive hyperplane* in $\mathcal{C}$ is a hyperplane $C \subseteq \mathcal{C}$ of the form

$$C = \left\{\sum a_{H,\chi} c_{H,\chi} = a\right\} \quad \text{with real } a_{H,\chi} \geq 0 \text{ and } a > 0.$$

The zero set of $F_S(c)$ is a certain set of positive hyperplanes. We have:

THEOREM 1.1. – *A stratum S is in the support of $L_c(\mathbf{1})$ if and only if $F_S(c) \neq 0$.*

This theorem reduces the problem of calculating the support of $L_c(\mathbf{1})$ to that of deciding, for each stratum S, which positive hyperplanes lie in the zero set $F_S(c)$. In order to do this we use the finite Hecke algebra $\mathcal{H}$ of W, and given a stratum S, the finite Hecke algebra $\mathcal{H}_S$ of the parabolic subgroup W_S of W. This algebra depends on a parameter $q_c = (q_{H,\chi})$ which is a function of c,

$$q_{H,\chi} = e^{2\pi i c_{H,\chi}}.$$

Combining recent work of Bezrukavnikov-Etingof [4], Losev [20], Marin-Pfeiffer [28], and Shan [33] shows that induction and restriction for the pair $\mathcal{H}$ and $\mathcal{H}_S$ are biadjoint, and the biadjunction produces a q-analog of the index $s(q_c) = |W : W_S|_{q_c}$, whose zeros detect precisely when the trivial representation is not a summand of its restriction-induction Ind(Res($\mathbf{1}$)), that is, is not relatively $\mathcal{H}_S$-projective.

Our second main theorem is then an analog of the fact that the trivial representation of a finite group in characteristic p is projective relative to a subgroup if and only if p does not divide the index.

THEOREM 1.2. – *A stratum S is contained in the support of $L_c(\mathbf{1})$ if and only if c is not contained in any positive hyperplane C such that $s = |W : W_S|_{q_c}$ vanishes on its intersection with the positive cone.*

Our proof of Theorem 1.2 received some inspiration from Theorem 4.5 of [1] (see also Theorem 4.2 of [15]) and combines Theorem 1.1 with an analysis of the support of $L_c(\mathbf{1})$ in the positive cone $c_{H,\chi} \geq 0$. The relevance of this cone is that $\Delta_c(\mathbf{1})$ is projective for all such non-negative parameters, allowing us to transfer the problem to the finite Hecke algebra (using that the KZ functor is fully faithful on projective objects: see 2.7). Here the Schur elements play the decisive role. Of course, one expects that the biadjunction may be chosen so that s is a Laurent polynomial in the parameters, so that in practice the Schur element is zero on the whole hyperplane C. This is known to hold at least for the infinite family $G(r,p,n)$, all real reflection groups, and the groups G_n in Shephard-Todd's notation for n belonging to $\{4,5,6,7,8,12,13,22,24\}$ (see [5], [6], and [27]), for which s may be calculated using a symmetrizing form. Moreover, it is conjectured that for all complex reflection groups a symmetrizing form with properties allowing explicit calculation exists; we have used the computer algebra package CHEVIE [29] of GAP3 [32] to tabulate results in all cases assuming these conjectures, and the latest version of CHEVIE contains a function `SphericalCriterion` that implements our results. At least in the cases just mentioned, the q-index $|W : W_S|_q$ is the quotient $|W|_q/|W_S|_q$ of the principal Schur elements of W and W_S;