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GROMOV NORM AND TURAEV-VIRO INVARIANTS OF 3-MANIFOLDS

BY Renaud DETCHERRY AND Efstratia KALFAGIANNI

Abstract. – We establish a relation between the "large r" asymptotics of the Turaev-Viro invariants TV_r and the Gromov norm of 3-manifolds. We show that for any orientable, compact 3-manifold M, with (possibly empty) toroidal boundary, $\log|TV_r(M)|$ is bounded above by $Cr||M||$ for some universal constant C. We obtain topological criteria for the growth to be exponential; that is $\log|TV_r(M)| \geqslant Br$, for some $B > 0$, and construct infinite families of hyperbolic 3-manifolds whose Turaev-Viro invariants grow exponentially. These constructions are essential for related work of the authors which makes progress on a conjecture of Andersen, Masbaum and Ueno.

We also show that, like the Gromov norm, the values of the invariants TV_r do not increase under Dehn filling. Finally we give constructions of 3-manifolds, both with zero and non-zero Gromov norm, for which the Turaev-Viro invariants determine the Gromov norm.

Résumé. – Nous relions l'asymptotique des invariants de Turaev-Viro TV_r pour r grand à la norme de Gromov. Nous montrons que pour toute variété de dimension 3 orientable compacte M, à bord vide ou torique, $\log|TV_r(M)|$ est inférieur à $Cr||M||$ où C est une constante universelle. Nous obtenons un critère topologique garantissant la croissance exponentielle; c'est-à-dire $\log|TV_r(M)| \geqslant Br$, pour un certain $B > 0$, et nous construisons des familles de variétés hyperboliques dont les invariants de Turaev-Viro croissent exponentiellement. Ces constructions sont essentielles pour des travaux des auteurs en lien avec une conjecture d'Andersen, Masbaum et Ueno.

Nous démontrons aussi que, comme pour la norme de Gromov, les invariants de Turaev-Viro décroissent par remplissage de Dehn.

Enfin, nous contruisons des variétés de dimension 3, de norme de Gromov nulle et non-nulle, dont les invariants de Turaev-Viro déterminent la norme de Gromov.

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1. Introduction

Since the discovery of the quantum 3-manifold invariants in the late 80s, it has been a major challenge to understand their relations to the topology and geometry of 3-manifolds. Open conjectures predict tight connections between quantum invariants and the geometries coming from Thurston's geometrization picture [6, 7]. However, despite compelling physics and experimental evidence, progress to these conjectures has been scarce. For instance, the volume conjecture for the colored Jones polynomial has only been verified for a handful of hyperbolic knots to date. The reader is referred to [6] for survey articles on the subject and for related conjectures. On the other hand, coarse relations between the stable coefficients of colored Jones polynomials and volume have been established for an abundance of hyperbolic knots [9, 14, 15].

In this paper we are concerned with the question of how the "large level" asymptotics of the Turaev-Viro 3-manifold invariants relate to, and interact with, the geometric decomposition of 3-manifolds. The Turaev-Viro invariants $TV_r(M)$ of a compact oriented 3-manifold M are combinatorially defined invariants that can be computed from triangulations of M [34]. They are real valued invariants, indexed by a positive integer r, called the level, and for each r they depend on a $2r$-th root of unity. We combine TQFT techniques, geometric decomposition theory of 3-manifolds and analytical estimates of $6j$-symbols to show that the r-growth of $TV_r(M)$ is bounded above by a function exponential in r that involves the Gromov norm of M.

We also obtain topological criteria for the growth to be exponential; that is to have $TV_r(M) \geqslant \exp Br$ with B a positive constant. We use these criteria to construct infinite families of hyperbolic 3-manifolds whose $SO(3)$-Turaev-Viro invariants grow exponentially. These results are used by the authors [10] to make progress on a conjecture of Andersen, Masbaum and Ueno (AMU Conjecture) about the geometric properties of surface mapping class groups detected by the quantum representations.

1.1. Upper bounds

For a compact oriented 3-manifold M, let $TV_r(M,q)$ denote the r-th Turaev-Viro invariant of M at q, where q is a $2r$-th root of unity such that q^2 is a primitive r-th root of unity. Throughout the paper we will work with $q = e^{\frac{2\pi i}{r}}$ and r an odd integer and we will often write $TV_r(M) := TV_r(M, e^{\frac{2\pi i}{r}})$. This is the theory that corresponds to the $SO(3)$ gauge group. We define

$$LTV(M) = \limsup_{r\to\infty} \frac{2\pi}{r} \log |TV_r(M))|, \tag{1}$$

and

$$lTV(M) = \liminf_{r\to\infty} \frac{2\pi}{r} \log |TV_r(M))|, \tag{2}$$

where r runs over all odd integers. Also we will use $||M||$ to denote the Gromov norm (or simplicial volume) of M. See Section 2.1 for definitions. The main result of this article is the following.

THEOREM 1.1. – *There exists a universal constant $C > 0$ such that for any compact orientable 3-manifold M with empty or toroidal boundary we have*

$$LTV(M) \leqslant C||M||.$$

If the interior of M admits a complete hyperbolic structure then, by Mostow rigidity, the hyperbolic metric is essentially unique and the volume of the metric is a topological invariant denoted by $\mathrm{vol}(M)$, that is essentially the Gromov norm. In this case, Theorem 1.1 provides a relation between hyperbolic geometry and the Turaev-Viro invariants. If M is the complement of a hyperbolic link in S^3 then we know that $lTV(M) \geqslant 0$ and in many instances the inequality is strict (Corollary 1.4).

The problem of estimating the volume of hyperbolic 3-manifolds in terms of topological quantities and quantum invariants has been studied considerably in the literature. See for example [1, 14, 16] and references in the last item. Despite progress, to the best of our knowledge, Theorem 1.1 gives the first such linear lower bound that works for all hyperbolic 3-manifolds.

In the generality that Theorem 1.1 is stated, the constant C is about 8.3581×10^9. However, within classes of 3-manifolds, one has much more effective estimates. For instance, Theorem 7.4 of this paper shows that for most (in a certain sense) hyperbolic links $L \subset S^3$ we have

$$LTV(S^3 \setminus L) \leqslant 10.5\,\mathrm{vol}(S^3 \setminus L).$$

Furthermore, given any constant E arbitrarily close to 1, one has infinite families of hyperbolic closed and cusped 3-manifolds M, with $LTV(M) \leqslant E\,\mathrm{vol}(M)$. See Section 7.2. for precise statements and more details.

We also give families of 3-manifolds with $LTV(M) = ||M||$. One such family of examples is the class of links with zero Gromov norm in S^3 or in $S^1 \times S^2$, but we also present families with non-zero norm (Section 8).

COROLLARY 1.2. – *Suppose that M is S^3 or a connected sum of copies of $S^1 \times S^2$. Then, for any link $K \subset M$ with $||M \setminus K|| = 0$, we have*

$$lTV(M) = LTV(M) = \lim_{r\to\infty} \frac{2\pi}{r} \log|TV_r(M \setminus K)| = v_3||M \setminus K|| = 0,$$

where r runs over all odd integers.

1.2. Outline of proof of Theorem 1.1

A key step in the proof is to show that $LTV(M)$ is finite for any compact oriented 3-manifold M. This is done by studying the large r asymptotic behavior of the quantum $6j$-symbols, and using the state sum formulae for the invariants TV_r. More specifically, we prove the following.

THEOREM 1.3. – *Suppose that M is a compact, oriented manifold with a triangulation consisting of t tetrahedra. Then, we have*

$$LTV(M) \leqslant 2.08\, v_8\, t,$$

where $v_8 \simeq 3.6638$ is the volume of a regular ideal octahedron.