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$\mathsf{SO}(n, n+1)$-SURFACE GROUP REPRESENTATIONS AND HIGGS BUNDLES

BY BRIAN COLLIER

ABSTRACT. – We study the character variety of representations of the fundamental group of a closed surface of genus $g \geq 2$ into the Lie group $\mathsf{SO}(n, n+1)$ using Higgs bundles. For each integer $0 < d \leq n(2g-2)$, we show there is a smooth connected component of the character variety which is diffeomorphic to the product of a certain vector bundle over a symmetric product of a Riemann surface with the vector space of holomorphic differentials of degree $2, 4, \ldots, 2n-2$. In particular, when $d = n(2g-2)$, this recovers Hitchin's parameterization of the Hitchin component. We also exhibit $2^{2g+1}-1$ additional connected components of the $\mathsf{SO}(n, n+1)$-character variety and compute their topology. Moreover, representations in all of these new components cannot be continuously deformed to representations with compact Zariski closure. Using recent work of Guichard-Wienhard on positivity, it is shown that each of the representations which define singularities (i.e., those which are not irreducible) in these $2^{2g+1}-1$ connected components are positive Anosov representations.

RÉSUMÉ. – Nous utilisons la théorie des fibrés de Higgs pour étudier la variété des caractères du groupe fondamental d'une surface S fermée de genre $g \geq 2$ dans le groupe de Lie $\mathsf{SO}(n, n+1)$. Pour tout entier $0 < d \leq n(2g-2)$, nous montrons l'existence d'une composante connexe de la variété des caractères difféomorphe au produit d'un fibré vectoriel sur le produit symétrique de S avec l'espace vectoriel des différentielles holomorphes de degré 2, 4, …, 2n-2. Pour $d = n(2g-2)$, nous retrouvons la paramétrisation de Hitchin de la composante de Hitchin. Nous montrons aussi l'existence de $2^{2g+1}-1$ nouvelles composantes connexes de la variété des caractères et décrivons leur topologie. Ces composantes ne contiennent aucune représentation dont l'adhérence de Zariski de l'image est compacte. En utilisant des résultats récents de Guichard et Wienhard sur la notion de positivité, nous montrons que les représentations réductibles de ces $2^{2g+1}-1$ composantes sont Anosov.

1. Introduction

Since Higgs bundles were introduced, they have found application in parameterizing connected components of the moduli space of reductive surface group representations into a reductive Lie group G. In particular, for a closed surface S with genus $g \geq 2$, Hitchin

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gave an explicit parameterization of all but one of the connected components of the space of conjugacy classes of reductive representations of the fundamental group of S into the Lie group $\mathsf{PSL}(2,\mathbb{R})$ [23]. Namely, he showed that each component with nonzero Euler class is diffeomorphic to the total space of a smooth vector bundle over an appropriate symmetric product of the surface. When the Euler class is maximal, this recovers a parameterization of the Teichmüller space of S as a vector space of complex dimension $3g-3$.

Hitchin later showed that for G a connected split real form, such as $\mathsf{PSL}(n,\mathbb{R})$ or $\mathsf{SO}_0(n,n+1)$, there is a connected component of this moduli space of representations which directly generalizes Teichmüller space [24]. Moreover, Hitchin parameterized this connected component, now called the Hitchin component, by a vector space of holomorphic differentials on the surface S equipped with a Riemann surface structure. In this paper, we use Higgs bundle techniques to generalize both of these results for the group $\mathsf{SO}(n,n+1)$.

Let $\Gamma = \pi_1(S)$ be the fundamental group of a closed surface S of genus $g \geq 2$. For a real reductive algebraic Lie group G, we will refer to the space of conjugacy classes of representations $\rho : \Gamma \to \mathsf{G}$ of Γ into G whose images have reductive Zariski closure as the G-*character variety*; it will be denoted by $\mathcal{X}(\mathsf{G})$. For connected reductive Lie groups, topological G-bundles on S are classified by a characteristic class $\omega \in H^2(S,\pi_1(\mathsf{G})) \cong \pi_1(\mathsf{G})$. Thus, the G-character variety decomposes as

$$\mathcal{X}(\mathsf{G}) = \bigsqcup_{\omega\in\pi_1(\mathsf{G})} \mathcal{X}^{\omega}(\mathsf{G}),$$

where the equivalence class of a reductive representation $\rho : \Gamma \to \mathsf{G}$ lies in $\mathcal{X}^{\omega}(\mathsf{G})$ if and only if the flat G-bundle determined by ρ has topological type determined by $\omega \in \pi_1(\mathsf{G})$.

For general reductive groups, $\mathcal{X}^{\omega}(\mathsf{G})$ can be empty for certain values of ω. However, the space $\mathcal{X}^{\omega}(\mathsf{G})$ is nonempty and connected for each $\omega \in \pi_1(\mathsf{G})$ when G is compact and semisimple [33] and also when G is complex and semisimple [29]. Since G is homotopic to its maximal compact subgroup, $\mathcal{X}^{\omega}(\mathsf{G})$ is connected if every representation in $\mathcal{X}^{\omega}(\mathsf{G})$ can be continuously deformed to one with compact Zariski closure. Connectedness of $\mathcal{X}^{\omega}(\mathsf{G})$ has been proven for many real forms using this technique, see [32, 29].

There are exactly two known families of Lie groups for which the space $\mathcal{X}^{\omega}(\mathsf{G})$ is not connected. When G is a split real form, the Hitchin component is not distinguished by an invariant $\omega \in \pi_1(\mathsf{G})$. Similarly, when G is a group of Hermitian type, the connected components of *maximal representations* are usually not labeled by topological invariants $\omega \in \pi_1(\mathsf{G})$. Both Hitchin representations and maximal representations define an important class of representations: they are the only known components of $\mathcal{X}(\mathsf{G})$ which consist entirely of Anosov representations [28, 6].

1.1. New components for $\mathsf{G} = \mathsf{SO}(n,n+1)$

The group $\mathsf{SO}(n,n+1)$ has two connected components, we will denote the connected component of the identity by $\mathsf{SO}_0(n,n+1)$. For $n \geq 3$, the group $\mathsf{SO}(n,n+1)$ is a split group, but not of Hermitian type. Nevertheless, we show that the $\mathsf{SO}_0(n,n+1)$-character variety has many non-Hitchin connected components which are not distinguished by a topological invariant $\omega \in \pi_1(\mathsf{SO}_0(n,n+1)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

THEOREM 1 (4.11). – Let Γ be the fundamental group of a closed surface S of genus $g \geq 2$ and let $\mathcal{X}(\mathsf{SO}(n,n+1))$ be the $\mathsf{SO}(n,n+1)$-character variety of Γ. For each integer $d \in (0, n(2g-2)]$, there is a smooth connected component $\mathcal{X}_d(\mathsf{SO}(n,n+1))$ of $\mathcal{X}(\mathsf{SO}(n,n+1))$ which does not contain representations with compact Zariski closure. Furthermore, for each choice of Riemann surface structure X on S, the space $\mathcal{X}_d(\mathsf{SO}(n,n+1))$ is diffeomorphic to the product

$$\mathcal{X}_d(\mathsf{SO}(n,n+1)) \cong \mathcal{F}_d \times \bigoplus_{j=1}^{n-1} H^0(K^{2j}),$$

where $\mathcal{F}_d$ is the total space of a rank $d + (2n-1)(g-1)$ vector bundle over the symmetric product $\mathsf{Sym}^{n(2g-2)-d}(X)$ and $H^0(K^{2j})$ is the vector space of holomorphic differentials of degree $2j$.

In fact, the representations $\rho \in \mathcal{X}_d(\mathsf{SO}(n,n+1))$ factor through the connected component of the identity $\mathsf{SO}_0(n,n+1) \subset \mathsf{SO}(n,n+1)$.

REMARK 1.1. – As a direct corollary, the connected components $\mathcal{X}_d(\mathsf{SO}(n,n+1))$ deformation retract onto the symmetric product $\mathsf{Sym}^{n(2g-2)-d}(X)$. In particular, the cohomology ring of $\mathcal{X}_d(\mathsf{SO}(n,n+1))$ is the same as the cohomology ring of the symmetric product $\mathsf{Sym}^{n(2g-2)-d}(X)$ which was computed in [30].

Using the isomorphism $\mathsf{PSL}(2,\mathbb{R}) \cong \mathsf{SO}_0(1,2)$, Theorem 4.11 recovers Hitchin's parameterization of the nonzero Euler class components of $\mathcal{X}(\mathsf{PSL}(2,\mathbb{R}))$ mentioned above. Also, when the label d in Theorem 4.11 is maximal, the vector bundle $\mathcal{F}_{n(2g-2)}$ is the rank $(4n-1)(g-1)$ vector space of holomorphic differentials of degree $2n$. Thus, we recover the parameterization of the $\mathsf{SO}(n,n+1)$-Hitchin component as a vector space of holomorphic differentials. When $n = 2$, Theorem 4.11 gives a parameterization of an $\mathsf{SO}_0(2,3) = \mathsf{PSp}(4,\mathbb{R})$-version of $\mathsf{Sp}(4,\mathbb{R})$ components discovered in [16]. For $n > 2$ and $0 < d < n(2g-2)$ the components are new.

There is also a connected component associated to $d = 0$ which has non-orbifold singularities. We briefly describe it here. Let X be a Riemann surface structure on S and let $\mathsf{Pic}(X)$ be the Picard group of holomorphic line bundles on X. Consider the space $\widetilde{\mathcal{F}}_0$ defined by

$$\widetilde{\mathcal{F}}_0 = \{(M,\mu,\nu) \mid M \in \mathsf{Pic}^0(X),\ \mu \in H^0(M^{-1}K^n),\ \nu \in H^0(MK^n)\}.$$

Recall that the group of matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}$ for $\lambda \in \mathbb{C}^*$ is isomorphic to $\mathsf{O}(2,\mathbb{C})$. There is a natural action of $\mathsf{O}(2,\mathbb{C})$ on $\widetilde{\mathcal{F}}_0$ given by:

$$g \cdot (M,\mu,\nu) = \begin{cases} (M, \lambda^{-1}\mu, \lambda\nu) & \text{if } g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \\ (M^{-1}, \lambda^{-1}\nu, \lambda\mu) & \text{if } g = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}. \end{cases}$$

THEOREM 2 (5.1). – Let Γ be the fundamental group of a closed surface S of genus $g \geq 2$ and let $\mathcal{X}(\mathsf{SO}(n,n+1))$ be the $\mathsf{SO}(n,n+1)$-character variety of Γ. For each $n \geq 2$, there is a connected component $\mathcal{X}_0(\mathsf{SO}(n,n+1))$ of $\mathcal{X}(\mathsf{SO}(n,n+1))$ which does