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BOUNDEDNESS RESULTS FOR SINGULAR FANO VARIETIES, AND APPLICATIONS TO CREMONA GROUPS
[following Birkar and Prokhorov-Shramov]

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1. MAIN RESULTS

Throughout this paper, we work over the field of complex numbers.

1.1. Boundedness of singular Fano varieties

A normal, projective variety X is called *Fano* if a negative multiple of its canonical divisor class is Cartier and if the associated line bundle is ample. Fano varieties appear throughout geometry and have been studied intensely, in many contexts. For the purposes of this talk, we remark that Fanos with sufficiently mild singularities constitute one of the fundamental variety classes in birational geometry. In fact, given any projective manifold X, the Minimal Model Program (MMP) predicts the existence of a sequence of rather special birational transformations, known as "divisorial contractions" and "flips," as follows,

$$X = X^{(0)} \underset{\text{birational}}{\overset{\alpha^{(1)}}{\dashrightarrow}} X^{(1)} \underset{\text{birational}}{\overset{\alpha^{(2)}}{\dashrightarrow}} \cdots \underset{\text{birational}}{\overset{\alpha^{(n)}}{\dashrightarrow}} X^{(n)}.$$

The resulting variety $X^{(n)}$ is either canonically polarized (which is to say that a suitable power of its canonical sheaf is ample), or it has the structure of a fiber space whose general fibers are either Fano or have numerically trivial canonical class. The study of (families of) Fano varieties is thus one of the most fundamental problems in birational geometry.

Remark 1.1 (Singularities). — Even though the starting variety X is a manifold by assumption, it is well understood that we cannot expect the varieties $X^{(\bullet)}$ to be

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smooth. Instead, they exhibit mild singularities, known as "terminal" or "canonical"— we refer the reader to [**33**, Sect. 2.3] or [**31**, Sect. 2] for a discussion and for references. If $X^{(n)}$ admits the structure of a fiber space, its general fibers will also have terminal or canonical singularities. Even if one is primarily interested in the geometry of *manifolds*, it is therefore necessary to include families of *singular* Fanos in the discussion.

In a series of two fundamental papers, [**6, 8**], Birkar confirmed a long-standing conjecture of Alexeev and Borisov-Borisov, [**1, 12**], asserting that for every $d \in \mathbb{N}$, the family of d-dimensional Fano varieties with terminal singularities is bounded: there exists a proper morphism of quasi-projective schemes over the complex numbers, $u : \mathbb{X} \to Y$, and for every d-dimensional Fano X with terminal singularities a closed point $y \in Y$ such that X is isomorphic to the fiber $\mathbb{X}_y$. In fact, a much more general statement holds true.

THEOREM 1.2 (Boundedness of ε-lc Fanos, [**8**, Thm. 1.1]). — *Given $d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, let $\mathscr{X}_{d,\varepsilon}$ be the family of projective varieties X with dimension $\dim_{\mathbb{C}} X = d$ that admit an $\mathbb{R}$-divisor $B \in \mathbb{R}\operatorname{Div}(X)$ such that the following holds true.*

(1.2.1) *The tuple (X, B) forms a pair. In other words: X is normal, the coefficients of B are contained in the interval $[0, 1]$ and $K_X + B$ is $\mathbb{R}$-Cartier.*

(1.2.2) *The pair (X, B) is ε-lc. In other words, the total log discrepancy of (X, B) is greater than or equal to ε.*

(1.2.3) *The $\mathbb{R}$-Cartier divisor $-(K_X + B)$ is nef and big.*

Then, the family $\mathscr{X}_{d,\varepsilon}$ is bounded.

Remark 1.3 (Terminal singularities). — If X has terminal singularities, then $(X, 0)$ is 1-lc. We refer to Section 2.3, to Birkar's original papers, or to [**22**, Sect. 3.1] for the relevant definitions concerning more general classes of singularities.

For his proof of the boundedness of Fano varieties and for his contributions to the Minimal Model Program, Caucher Birkar was awarded with the Fields Medal at the ICM 2018 in Rio de Janeiro.

1.1.1. *Where does boundedness come from?*— The brief answer is: "From boundedness of volumes!" In fact, if $(X_t, A_t)_{t\in T}$ is a family of tuples where the X_t are normal, projective varieties of fixed dimension d and $A_t \in \operatorname{Div}(X_t)$ are very ample, and if there exists a number $v \in \mathbb{N}$ such that

$$\operatorname{vol}(A_t) := \limsup_{n\to\infty} \frac{d! \cdot h^0\big(X_t,\ \mathscr{O}_{X_t}(n \cdot A_t)\big)}{n^d} < v$$

for all $t \in T$, then elementary arguments using Hilbert schemes show that the family $(X_t, A_t)_{t\in T}$ is bounded.

For the application that we have in mind, the varieties X_t are the Fano varieties whose boundedness we would like to show and the divisors A_t will be chosen as fixed multiples of their anticanonical classes. To obtain boundedness results in this setting, Birkar needs to show that there exists one number m that makes all $A_t := -m \cdot K_{X_t}$ very ample, or (more modestly) ensures that the linear systems $|-m \cdot K_{X_t}|$ define birational maps. Volume bounds for these divisors need to be established, and the singularities of the linear systems need to be controlled.

1.1.2. *Earlier results, related results.* — Boundedness results have a long history, which we cannot cover with any pretense of completeness. Boundedness of smooth Fano surfaces and threefolds follows from their classification. Boundedness of Fano *manifolds* of arbitrary dimension was shown in the early 1990s, in an influential paper of Kollár, Miyaoka and Mori, [**32**], by studying their geometry as rationally connected manifolds. Around the same time, Borisov-Borisov were able to handle the toric case using combinatorial methods, [**12**]. For (singular) surfaces, Theorem 1.2 is due to Alexeev, [**1**].

Among the newer results, we will only mention the work of Hacon-McKernan-Xu. Using methods that are similar to those discussed here, but without the results on "boundedness of complements" (cf. Section 4), they are able to bound the volumes of klt pairs (X, Δ), where X is projective of fixed dimension, $K_X + \Delta$ is numerically trivial and the coefficients of Δ come from a fixed DCC set, [**22**, Thm. B]. Boundedness of Fanos with klt singularities and fixed Cartier index follows, [**22**, Cor. 1.8]. In a subsequent paper [**24**] these results are extended to give the boundedness result that we quote in Theorem 4.6, and that Birkar builds on. We conclude with a reference to [**26, 14**] for current results involving K-stability and α-invariants. The surveys [**20, 23**] give a more complete overview.

1.1.3. *Positive characteristic.* — Apart from the above-mentioned results of Alexeev, [**1**], which hold over algebraically closed field of arbitrary characteristic, little is known in case where the characteristic of the base field is positive.

1.2. Applications

As we will see in Section 8 below, boundedness of Fanos can be used to prove the existence of fixed points for actions of finite groups on Fanos, or more generally rationally connected varieties. Recall that a variety X is *rationally connected* if every two points are connected by an irreducible, rational curve contained in X. This allows us to apply Theorem 1.2 in the study of finite subgroups of birational automorphism groups.

1.2.1. *The Jordan property of Cremona groups.* — Even before Theorem 1.2 was known, it had been realized by Prokhorov and Shramov, [**42**], that boundedness of Fano varieties with terminal singularities would imply that the birational automorphism groups of projective spaces (= Cremona groups, $\operatorname{Bir}(\mathbb{P}^d)$) satisfy the *Jordan property*. Recall that a group Γ is said to *have the Jordan property* if there exists a number $j \in \mathbb{N}$ such that every finite subgroup $G \subset \Gamma$ contains a normal, Abelian subgroup $A \subset G$ of index $|G : A| \leq j$. In fact, a stronger result holds.

THEOREM 1.4 (Jordan property of Cremona groups, [**8**, Cor. 1.3], [**42**, Thm. 1.8])

Given any number $d \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that for every complex, projective, rationally connected variety X of dimension $\dim_{\mathbb{C}} X = d$, every finite subgroup $G \subset \operatorname{Bir}(X)$ contains a normal, Abelian subgroup $A \subseteq G$ of index $|G : A| \leq j$.

Remark 1.5. — Theorem 1.4 answers a question of Serre [**43**, 6.1] in the positive. A more detailed analysis establishes the Jordan property more generally for all varieties of vanishing irregularity, [**41**, Thm. 1.8].

Theorem 1.4 ties in with the general philosophy that finite subgroups of $\operatorname{Bir}(\mathbb{P}^d)$ should in many ways be similar to finite linear groups, where the property has been established by Jordan more then a century ago.

THEOREM 1.6 (Jordan property of linear groups, [**27**]). — *Given any number $d \in \mathbb{N}$, there exists $j_d^{\text{Jordan}} \in \mathbb{N}$ such that every finite subgroup $G \subset \mathrm{GL}_d(\mathbb{C})$ contains a normal, Abelian subgroup $A \subseteq G$ of index $|G : A| \leq j_d^{\text{Jordan}}$.* □

Remark 1.7 (Related results). — For further information on Cremona groups and their subgroups, we refer the reader to the surveys [**36, 13**] and to the recent research paper [**38**]. For the maximally connected components of automorphism groups of projective varieties (rather than the full group of birational automorphisms), the Jordan property has recently been established by Meng and Zhang without any assumption on the nature of the varieties, [**35**, Thm. 1.4]; their proof uses group-theoretic methods rather than birational geometry. For related results (also in positive characteristic), see [**25, 37, 48**] and references there.

1.2.2. *Boundedness of finite subgroups in birational transformation groups.* — Following similar lines of thought, Prokhorov and Shramov also deduce boundedness of finite subgroups in birational transformation groups, for arbitrary varieties defined over a finite field extension of $\mathbb{Q}$.

THEOREM 1.8 (Bounds for finite groups of birational transformation, [**41**, Thm. 1.4])
Let k be a finitely generated field over $\mathbb{Q}$. Let X be a variety over k, and let $\operatorname{Bir}(X)$ denote the group of birational automorphisms of X over $\operatorname{Spec} k$. Then, there exists $b \in \mathbb{N}$ such that any finite subgroup $G \subset \operatorname{Bir}(X)$ has order $|G| \leq b$.

As an immediate corollary, they answer another question of Serre [(1)], pertaining to finite subgroups in the automorphism group of a field.

COROLLARY 1.9 (Boundedness for finite groups of field automorphisms, [**41**, Cor. 1.5])
Let k be a finitely generated field over $\mathbb{Q}$. Then, there exists $b \in \mathbb{N}$ such that any finite subgroup $G \subset \operatorname{Aut}(k)$ has order $|G| \leq b$.

1.2.3. *Boundedness of links, quotients of the Cremona group.* — Birkar's result has further applications within birational geometry. Combined with work of Choi-Shokurov, it implies the boundedness of Sarkisov links in any given dimension, cf. [**15**, Cor. 7.1]. In [**11**], Blanc-Lamy-Zimmermann use Birkar's result to prove the existence of many quotients of the Cremona groups of dimension three or more. In particular, they show that these groups are not perfect and thus not simple.

1.3. Outline of this paper

Paraphrasing [**6**, p. 6], the main tools used in Birkar's work include the Minimal Model Program [**33, 9**], the theory of complements [**39, 40, 47**], the technique of creating families of non-klt centers using volumes [**22, 21**] and [**29**, Sect. 6], and the theory of generalized polarized pairs [**10**]. In fact, given the scope and difficulty of Birkar's work, and given the large number of technical concepts involved, it does not seem realistic to give more than a panoramic presentation of Birkar's proof here. Largely ignoring all technicalities, Sections 4–7 highlight four core results, each of independent interest. We explain the statements in brief, sketch some ideas of proof and indicate how the results might fit together to give the desired boundedness result. Finally, Section 8 discusses the application to the Jordan property in some detail.

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(1) Unpublished problem list from the workshop "Subgroups of Cremona groups: classification," 29–30 March 2010, ICMS, Edinburgh. Available at `http://www.mi.ras.ru/~prokhoro/preprints/edi.pdf`. Serre's question is found on page 7.