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HIGHER RANK TEICHMÜLLER THEORIES

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INTRODUCTION

Let Γ_g be the fundamental group of a compact surface S_g with negative Euler characteristic, and let G denote $\mathrm{PSL}(2,\mathbf{R})$, the group of isometries of the hyperbolic plane $\mathbf{H}^2$. Goldman observed that the Teichmüller space, the parameter space of marked hyperbolic structures on S_g, can be identified with a connected component of the character variety $\mathrm{Hom}(\Gamma_g, G)//G$, which can be selected by means of a characteristic invariant. Thanks to the work of Labourie, Burger-Iozzi-Wienhard, Fock-Goncharov and Guichard-Wienhard we now know that, surprisingly, this is a much more general phenomenon: there are also many higher rank semisimple Lie groups G admitting components of the character variety consisting only of injective homomorphisms with discrete image, the so-called *higher rank Teichmüller theories*. The richness of these theories is partially due to the fact that, as for the Teichmüller space, truly different techniques can be used to study them: bounded cohomology, Higgs bundles, positivity, harmonic maps, incidence structures, geodesic currents, real algebraic geometry, dynamics are just some of those.

In this survey, after introducing the two known families of higher rank Teichmüller theories, the Hitchin components and the maximal representations, we will describe a conjectural unifying framework, Θ-positive representations. This theory, due to Guichard-Wienhard, encompasses both families of higher rank Teichmüller theories as well as, potentially, new families associated to orthogonal groups. In Section 3, after reinterpreting Higher Teichmüller theories as moduli spaces of locally symmetric spaces, we will discuss several geometric properties of such locally symmetric spaces, highlighting analogies and differences with geometric properties of hyperbolic surfaces, points in the Teichmüller spaces. We will be particularly concerned with the (vector valued) length functions associated to these locally symmetric spaces, and with various techniques to study them, based on dynamics, as well as incidence geometry and positivity. After a short digression, in Section 4, on harmonic maps and minimal surfaces, which provide a more analytic tool to study higher rank Teichmüller theories, we will focus, in the last section of the survey, on the interplay with other geometric structures, particularly in rank 2.

This short survey is not intended to be exhaustive, but it is rather a concise description of a few of the many ideas and tools that are being developed in the study of character varieties. In particular, for lack of space, we decided not to discuss the related theory of Anosov representations nor to detail the Higgs bundles perspective on character varieties and higher rank Teichmüller theories. We will instead discuss some of the applications of the theory of Higgs bundles, emphasizing results which can be formulated purely in terms of synthetic geometry, despite the only available proofs make heavy use of the more analytic approach. We refer the reader to the surveys [**3, 39**] and references therein for an introduction to Higgs bundles and their use in the study of character varieties, to the survey [**92**] for a discussion of other aspects of higher rank Teichmüller theories, and to the surveys [**43, 54**] for an introduction to Anosov representations and their link with geometric structures.

1. TEICHMÜLLER THEORY

In this section we recall some basic facts about Teichmüller theory that will play an important role in the higher rank generalizations that we will discuss in the rest of the survey. We refer the reader to [**37**, Part 2] for an introduction to these themes close to the viewpoint we will follow here.

Let S_g be a closed oriented surface of genus $g \geq 2$. We will define the *Teichmüller space* $\mathcal{T}(S_g)$ as the space of homotopy classes of marked hyperbolic structures on S_g.[(1)] The Teichmüller space is isomorphic to $\mathbf{R}^{6g-6}$, as can be seen using Fenchel-Nielsen coordinates: the choice of a maximal collection $\{c_1, \dots, c_{3g-3}\}$ of pairwise disjoint simple closed curves decomposes the surface S_g as a union of pairs of pants $\{P_1, \dots, P_{2g-2}\}$; the parametrization of $\mathcal{T}(S_g)$ can then be obtained recording the $3g-3$ lengths of the curves c_i and how much twist is involved in the glueings; indeed any three holed sphere (pair of pants) admits a unique hyperbolic structure for each choice of boundary lengths.

Whenever we fix a hyperbolic metric h on S_g, we can identify the metric universal covering $(\widetilde{S}_g, \widetilde{h})$ with the hyperbolic plane $\mathbf{H}^2$; the identification is natural up to

(1) This is historically inaccurate, as the Teichmüller space is the space of marked conformal structures on S_g, while the space of marked hyperbolic structures should be referred to as *Fricke space*. However it is a consequence of the uniformization theorem that these two objects can be identified.

post-composition with an element in $\mathrm{PSL}_2(\mathbf{R})$, the group of orientation-preserving isometries of $\mathbf{H}^2$. Throughout the survey we will denote by Γ_g the fundamental group of the surface S_g. The action of Γ_g on $\widetilde{S}_g$ as deck transformations induces, via the identification $\widetilde{S}_g \cong \mathbf{H}^2$, a homomorphism $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbf{R})$, which is then well-defined up to conjugation in $\mathrm{PSL}(2,\mathbf{R})$. This homomorphism is called the *holonomy* of the hyperbolic structure (S_g, h).

We will denote by $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbf{R}))//\mathrm{PSL}_2(\mathbf{R})$ the *character variety*, namely the largest Hausdorff quotient of the set of homomorphisms $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbf{R})$ for the equivalence relation that identifies two homomorphisms ρ, η if there exists $g \in \mathrm{PSL}(2,\mathbf{R})$ such that for every $\gamma \in \Gamma_g$, $\rho(\gamma) = g\eta(\gamma)g^{-1}$. The choice of a finite generating set S of Γ_g allows to realize $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbf{R}))$ as a subset of $\mathrm{PSL}_2(\mathbf{R})^{|S|}$ defined by polynomial equations (induced by the relations of the group Γ_g); this also induces a natural semi-algebraic structure on the character variety [**19**].

It is a basic fact in covering theory that the homomorphisms ρ arising as holonomies of hyperbolizations are injective and have discrete image. In his thesis Goldman showed that this procedure actually gives an identification of the Teichmüller space $\mathscr{T}(S_g)$ with a connected component of the character variety $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbf{R}))//\mathrm{PSL}_2(\mathbf{R})$, which can be selected by means of a cohomological invariant, the Euler class. [(2)]

Theorem 1.1 (Goldman [**40**]). — *The Euler class* $\mathrm{eu}(\rho)$ *distinguishes connected components in* $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbf{R}))//\mathrm{PSL}_2(\mathbf{R})$ *and has values in* $\mathbf{Z} \cap [\chi(S_g), -\chi(S_g)]$. *The representations for which* $|\mathrm{eu}(\rho)|$ *is maximal correspond to holonomies of hyperbolic structures on* S_g *(resp. hyperbolic structures on* S_g *endowed with the opposite orientation).*

It could be natural to think that there are connected components of the $\mathrm{PSL}(2,\mathbf{R})$-character variety only consisting of injective homomorphisms with discrete image because the cohomological dimension of the group Γ_g equals the dimension of $\mathbf{H}^2$ and thus Γ_g can act properly discontinuously and co-compactly on $\mathbf{H}^2$. We will discuss in the rest of the survey that the existence of such components is, instead, a much more general phenomenon: there are various classes of semisimple Lie groups G for which $\mathrm{Hom}(\Gamma_g, G)//G$ has connected components only consisting of injective homomorphisms with discrete image, the so-called *Higher Teichmüller theories*.

A lot of the richness of Teichmüller theory can be tracked back to the local isogenies between semisimple Lie groups in low ranks: $\mathrm{PSL}(2,\mathbf{R})$ is isomorphic to $\mathrm{PSp}(2,\mathbf{R})$, $\mathrm{PU}(1,1)$ and $\mathrm{PO}(2,1)$. In turn these correspond to different models for the hyperbolic

(2) We will not need the definition of the Euler class in the rest of the text. The interested reader can read more about it for example in [**24**].

plane (respectively, the upper-half plane $\mathbf{H}^2 \subset \mathbf{C}$, the Poincaré disk $\mathbf{D} \subset \mathbf{CP}^1$, and the Klein model $\mathbf{K} \subset \mathbf{RP}^2$) and therefore different perspectives on the same theory. We won't have this at our disposal when dealing with a general Lie group G, but we will discuss in Section 5 how the interplay between different geometric structures associated to G can give new insight on representations in Higher Teichmüller theories.

We conclude our very short account of Teichmüller theory by discussing an important property of hyperbolizations, which will have an avatar of fundamental importance in higher rank Teichmüller theory: the existence of boundary maps. Recall that the hyperbolic plane $\mathbf{H}^2$ has a boundary $\partial_\infty \mathbf{H}^2$ isomorphic to the circle $\mathbf{S}^1$ and consisting of equivalence classes of asymptotic rays. Given any two hyperbolic structures (S_g, h_1), (S_g, h_2) on the surface S_g with holonomies ρ_i, we obtain, via the identifications $(\widetilde{S}_g, \widetilde{h}_i) \cong \mathbf{H}^2$, a continuous (ρ_1, ρ_2)-equivariant map $f_{\rho_1,\rho_2} : \mathbf{H}^2 \to \mathbf{H}^2$. This extends to a monotone, Hölder continuous map $\xi_{\rho_1,\rho_2} : \partial_\infty \mathbf{H}^2 \to \partial_\infty \mathbf{H}^2$. Here monotonicity is defined with respect to the cyclic orientation of the circle: the map ξ_{ρ_1,ρ_2} is monotone if for every positively oriented triple (x, y, z) the image $(\xi_{\rho_1,\rho_2}(x), \xi_{\rho_1,\rho_2}(y), \xi_{\rho_1,\rho_2}(z))$ is positively oriented. We fix for simplicity (3) an auxiliary hyperbolic structure on the surface S_g with holonomy ρ, and denote by $\partial_\infty \Gamma_g$ the boundary $\partial_\infty \mathbf{H}^2$ together with the action of Γ_g induced by ρ. The discussion above shows that the Hölder structure of $\partial_\infty \Gamma_g$, as well as its cyclic order, is intrinsic and doesn't depend on the choice of ρ. It is then possible to characterize holonomies of hyperbolization using boundary maps: a representation $\eta : \Gamma_g \to \mathrm{PSL}(2, \mathbf{R})$ is the holonomy of a hyperbolization if and only if there exists a monotone, Hölder continuous map $\xi_\eta : \partial_\infty \Gamma_g \to \partial_\infty \mathbf{H}^2$.

2. HIGHER RANK TEICHMÜLLER THEORIES

Let us now consider a connected, adjoint, semisimple Lie group G of non-compact type and higher rank. Natural examples that will play a role in the text are $\mathrm{PSL}(n, \mathbf{R})$, the projective classes of matrices of determinant one, $\mathrm{PSp}(2n, \mathbf{R})$, the projective classes of matrices of determinant one preserving a symplectic form on $\mathbf{R}^{2n}$ or $\mathrm{PO}_0(2, n)$, the projectivization of the connected component of the identity in the group preserving a symmetric bilinear form on $\mathbf{R}^{n+2}$ of signature $(2, n)$. In this survey we will mostly regard G as the identity component of the group $\mathrm{Isom}(\mathscr{X})$, where $\mathscr{X} = G/K$ is the Riemannian symmetric space associated to G, a non-positively curved Riemannian manifold in which the geodesic reflections about any point are induced by isometries.

(3) With basic tools of geometric group theory one can give an intrinsic definition of the boundary $\partial_\infty \Gamma_g$, but this won't be necessary for our purposes.

DEFINITION 2.1. — *A* higher rank Teichmüller theory *is a connected component of the character variety* $\mathrm{Hom}(\Gamma_g, G)//G$ *only consisting of injective homomorphisms with discrete image.*

For most of the survey we will think of such higher rank Teichmüller theories as parametrizing special classes of locally symmetric spaces $\rho(\Gamma_g)\backslash\mathscr{X}$ covered by the Riemannian symmetric space $\mathscr{X}$, and whose fundamental group is Γ_g. Observe that, being a connected component of a character variety in a reductive algebraic group, any higher rank Teichmüller theory has a natural structure of a real semi-algebraic variety, and is thus amenable to the tools of real semi-algebraic geometry [**19, 2, 38**].

2.1. Hitchin components

Let G be a real split simple Lie group, such as $\mathrm{PSL}(n, \mathbf{R})$ or $\mathrm{PSp}(2n, \mathbf{R})$. By the work of Kostant [**58**] there exists a principal homomorphism $\tau : \mathrm{PSL}(2, \mathbf{R}) \to G$: the unique homomorphism for which the image of a unipotent element is a regular unipotent element. In particular the image of any diagonalizable element under τ is diagonalizable with distinct eigenvalues. If $G = \mathrm{PSL}(n, \mathbf{R})$, the principal homomorphism is the irreducible representation $\tau : \mathrm{PSL}(2, \mathbf{R}) \to \mathrm{PSL}(n, \mathbf{R})$ induced by the natural action of $\mathrm{PSL}(2, \mathbf{R})$ on the homogeneous polynomials of degree $n - 1$.

In [**51**] Hitchin initiated the study of the connected component in $\mathrm{Hom}(\Gamma_g, G)//G$ of the composition $\tau \circ \rho$ where $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbf{R})$ is the holonomy of a hyperbolization:

DEFINITION 2.2. — *Let G be a real split simple Lie group, $\tau : \mathrm{PSL}(2, \mathbf{R}) \to G$ the principal homomorphism, $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbf{R})$ the holonomy of a hyperbolization. The* Hitchin component $\mathrm{Hit}(\Gamma_g, G)$ *is the connected component of* $[\tau \circ \rho]$ *in* $\mathrm{Hom}(\Gamma_g, G)//G$.

Using analytic techniques, and in particular the theory of Higgs bundles developed by Hitchin [**50**], Simpson [**82, 83**], Corlette [**31**] and Donaldson [**34**], Hitchin was able to show that, as in the case of Teichmüller space, the Hitchin component $\mathrm{Hit}(\Gamma_g, G)$ is homeomorphic to the Euclidean space of dimension $(2g - 2) \dim G$.

The geometric relevance of representations in $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(n, \mathbf{R}))$ was singled out by Labourie using dynamical techniques: in [**66**] Labourie introduced the notion of Anosov representation, and proved that representations in the Hitchin component are injective, have discrete image, and are purely loxodromic, which means that for every element $\gamma \in \Gamma_g$, the image $\rho(\gamma)$ is diagonalizable with distinct real eigenvalues. In particular Hitchin components form examples of higher rank Teichmüller theories, according to Definition 2.1.

An independent approach to the study of Hitchin components was developed by Fock and Goncharov [**38**], based on Lusztig's generalization [**73**] of the notion of totally positive matrices, namely matrices whose minors are all positive. Lusztig associated