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THE RIEMANN ZETA FUNCTION IN SHORT INTERVALS
[after Najnudel, and Arguin, Belius, Bourgade, Radziwiłł and Soundararajan]

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INTRODUCTION

The Riemann zeta function $\zeta(s)$ is one of the most important and fascinating functions in mathematics. When the complex number s has $\Re(s) > 1$, we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{primes } p} \Big(1 - \frac{1}{p^s}\Big)^{-1},$$

and already from these equivalent expressions we see some of the key themes that dominate the study of $\zeta(s)$.

Firstly, since $\zeta(s)$ is given by a *Dirichlet series* over all natural numbers n, without any difficult number theoretic coefficients, we can hope to use general analytic methods to obtain information about $\zeta(s)$. For example, one could hope to approximate $\sum_{n=1}^{\infty} \frac{1}{n^s}$ or its partial sums by an integral. In this way, one can extend the definition of $\zeta(s)$ to all $\Re(s) > 0$, and with more work to the entire complex plane. It turns out that this analytic continuation of $\zeta(s)$ is meromorphic, with only a simple pole at $s = 1$. Furthermore, when $\Re(s) > 0$ the zeta function is the sum $\sum_{n \leqslant X} \frac{1}{n^s}$ plus some easily understood other terms, for suitable $X = X(s)$.

Secondly, since $\zeta(s)$ is given by an *Euler product* over all primes p, we can hope to use results about the zeta function to draw conclusions about the distribution of primes. One can also go in the reverse direction, and hope to put in information about the primes to deduce things about the zeta function (from which, perhaps, we will later deduce other number theoretic information that we didn't have before). In this article we will discuss various results of this nature.

Thirdly, note that the Euler product is absolutely convergent when $\Re(s) > 1$, and none of the individual factors $(1 - \frac{1}{p^s})^{-1}$ vanish, so we have $\zeta(s) \neq 0$ when $\Re(s) > 1$. It is well known that the zeros of the zeta function encode number theoretic information, and here one can glimpse why—if one knows that $\zeta(s)$ doesn't vanish in a certain part of the complex plane, this suggests that something like the Euler product formula persists there, which implies something about the regularity of the primes. Again there is a kind of duality, since not only does the non-vanishing of zeta imply results about primes and products, but our methods for proving non-vanishing tend to involve establishing the influence of some kind of product formula in the region under study.

The most interesting subset of the complex plane on which to study $\zeta(s)$ is the *critical line* $\Re(s) = 1/2$. Thus the *Riemann Hypothesis* (RH), possibly the most famous unsolved problem in pure mathematics, conjectures that if $\zeta(s) = 0$ then either $s = -2, -4, -6, \ldots$ (the so-called trivial zeros), or else $\Re(s) = 1/2$. This is known to be equivalent to the estimate

$$\Big|\#\{p \leqslant x : p \text{ prime}\} - \int_2^x \frac{dt}{\log t}\Big| \ll x^{1/2} \log x$$

for the counting function of the primes (RH holds if and only if this estimate holds for all large x). For any fixed $\sigma > 1/2$, it is believed (and to some extent known) that the values taken by $\zeta(\sigma + it)$ have a rather simple behavior: for example, $\zeta(\sigma + it)$ can only attain unusually large values as a result of "conspiracies" in the behavior of p^{-it} for small primes p. As we shall discuss extensively later, the situation on the critical line is very different. All the appearances of $1/2$ here reflect the fact that in a random process, the typical size of fluctuations is like the square root of the variance. The extent to which $\zeta(s)$ behaves like various random objects, especially random objects related to the Euler product, is another key theme that we are going to explore.

Our goal in this paper is to survey some recent work on the behavior of $\zeta(1/2 + it)$ in short intervals of t. In particular, we shall describe a conjecture of Fyodorov, Hiary and Keating **[10, 11]** about the size of $\max_{0 \leqslant h \leqslant 1} |\zeta(1/2 + it + ih)|$ as t varies, and we shall explain some results that have been proved in the direction of this conjecture by Najnudel **[16]** and by Arguin–Belius–Bourgade–Radziwiłł–Soundararajan **[2]**.

This paper is organized as follows. Firstly we shall set out some Basic Principles that will guide our thinking and arguments about the zeta function. Then, to illustrate the use of these principles and to compare with the later case of $\max_{0 \leqslant h \leqslant 1} |\zeta(1/2 + it + ih)|$, we shall describe what is known about the value distribution of $\zeta(1/2 + it)$ (without any maximum) and what is known about the "long range" maximum $\max_{T \leqslant t \leqslant 2T} |\zeta(1/2 + it)|$. Next we shall introduce and motivate the conjecture of Fyodorov–Hiary–Keating, primarily using our Basic Principles rather than the random matrix theory/statistical mechanics arguments originally considered by those authors (although we shall mention those briefly). And then we shall discuss the statements and proofs of the results of Najnudel and of Arguin–Belius–Bourgade–Radziwiłł–Soundararajan, again seeing how these correspond to very nice implementations of those principles.

Notation. — We shall use various notation, of a fairly standard kind, to aid in the description of estimates and limiting processes. We write $f(x) = O(g(x))$, or $f(x) \ll g(x)$, to mean that there exists some constant C such that $|f(x)| \leqslant Cg(x)$ for all x of interest (the range of x should always either be clear, or specified explicitly). We write $f(x) = \Theta(g(x))$, or $f(x) \asymp g(x)$, to mean that there exist constants $0 < c < C$ such that $cg(x) \leqslant f(x) \leqslant Cg(x)$ for all x of interest. At some points we will give rough or heuristic descriptions of arguments, and in these we will use notation such as $\lesssim, \approx$. We do not try to assign a precise meaning to these symbols—they mean that one quantity is smaller than another, or roughly the same size as another, up to terms that turn out to be unimportant in the rigorous implementation of the arguments.

Much of this paper will involve the discussion of probabilistic issues. We will write $\mathbb{P}$ to denote a probability measure, and $\mathbb{E}$ to denote expectation (i.e., averaging, or more formally integration) with respect to the measure $\mathbb{P}$.

1. BASIC PRINCIPLES

One can build up a great deal of understanding of the zeta function beginning from the following idea, which we first state in a heuristic way.

PRINCIPLE 1.1. — *As t varies, the numbers $(p^{-it})_{p \text{ prime}}$ "behave like" a sequence of independent random variables, each distributed uniformly on the complex unit circle.*

It is clear that for any given p, as t varies over an interval the quantity $p^{-it} = e^{-it\log p}$ rotates around the complex unit circle at "speed" $\log p$, behaving like a uniform random variable. Thus the interesting assertion in Principle 1.1 is that we should think of the p^{-it} as being independent. That is because the primes are multiplicatively independent, or equivalently the speeds $\log p$ are linearly independent over the rationals. Both of these statements are just ways of expressing the uniqueness of prime factorisation. So as each of the p^{-it} rotate around, there are no fixed relations between any combinations of them and so, heuristically, they shouldn't "see one another's behavior" too much (unlike if we considered $2^{-it}, 3^{-it}, 6^{-it}$, say, for which always $6^{-it} = 2^{-it}3^{-it}$).

What rigorous statements can we make that would correspond to Principle 1.1? The following result, although easily proved, turns out to be a very powerful tool.

LEMMA 1.2. — *Let $T \geqslant 1$, and let $p_1, \ldots, p_k, p_{k+1}, \ldots, p_\ell$ be any primes (not necessarily distinct). Let $(X_p)_{p \text{ prime}}$ be a sequence of independent random variables, each distributed uniformly on the complex unit circle. Then*

$$\frac{1}{T}\int_T^{2T} \prod_{j=1}^{k} p_j^{-it} \overline{\Big(\prod_{j=k+1}^{\ell} p_j^{-it}\Big)} dt = \mathbb{E}\prod_{j=1}^{k} X_{p_j} \prod_{j=k+1}^{\ell} \overline{X_{p_j}} + O\Big(\frac{\min\{\prod_{j=1}^{k} p_j, \prod_{j=k+1}^{\ell} p_j\}}{T}\Big).$$

Proof of Lemma 1.2. — We can rewrite the integral on the left as

$$\frac{1}{T}\int_T^{2T} \exp\Big\{-it\Big(\sum_{j=1}^{k}\log p_j - \sum_{j=k+1}^{\ell}\log p_j\Big)\Big\} dt.$$

So if $\sum_{j=1}^{k}\log p_j = \sum_{j=k+1}^{\ell}\log p_j$ then the integral is exactly 1. And since this is equivalent (by uniqueness of prime factorisation) to saying that the $(p_j)_{j=k+1}^{\ell}$ are just some reordering, with the same multiplicities, of the $(p_j)_{j=1}^{k}$, we see that, in this case as well, $\mathbb{E}\prod_{j=1}^{k} X_{p_j} \prod_{j=k+1}^{\ell} \overline{X_{p_j}} = 1$ since every X_p is paired with a conjugate copy.

If $\sum_{j=1}^{k}\log p_j \neq \sum_{j=k+1}^{\ell}\log p_j$, then on the right some X_p is not paired with a conjugate copy, so by independence and symmetry of the distributions of the X_p we have $\mathbb{E}\prod_{j=1}^{k} X_{p_j} \prod_{j=k+1}^{\ell} \overline{X_{p_j}} = 0$. The integral on the left may be calculated explicitly as

$$\frac{1}{T}\left[\frac{\exp\{-it(\sum_{j=1}^{k}\log p_j - \sum_{j=k+1}^{\ell}\log p_j)\}}{-i(\sum_{j=1}^{k}\log p_j - \sum_{j=k+1}^{\ell}\log p_j)}\right]_T^{2T} \ll \frac{1}{T\left|\log\Big(\frac{\prod_{j=1}^{k} p_j}{\prod_{j=k+1}^{\ell} p_j}\Big)\right|}.$$

If $\prod_{j=1}^{k} p_j < (3/4)\prod_{j=k+1}^{\ell} p_j$ or if $\prod_{j=1}^{k} p_j > (4/3)\prod_{j=k+1}^{\ell} p_j$ then the logarithmic term here is $> \log 4/3$, so we get an acceptable error term $O(1/T)$. Otherwise, we can write $\log\Big(\frac{\prod_{j=1}^{k} p_j}{\prod_{j=k+1}^{\ell} p_j}\Big) = \log\Big(1 + \frac{\prod_{j=1}^{k} p_j - \prod_{j=k+1}^{\ell} p_j}{\prod_{j=k+1}^{\ell} p_j}\Big)$ and use the Taylor expansion of the logarithm. Since we know that $\prod_{j=1}^{k} p_j - \prod_{j=k+1}^{\ell} p_j \neq 0$, in fact it is $\geqslant 1$ and we get a lower bound $\gg \frac{1}{\prod_{j=k+1}^{\ell} p_j}$ from the Taylor expansion. Since we are in the case where $\prod_{j=1}^{k} p_j$ and $\prod_{j=k+1}^{\ell} p_j$ differ at most by a multiplicative factor $4/3$, this can also be written as $\gg \frac{1}{\min\{\prod_{j=1}^{k} p_j, \prod_{j=k+1}^{\ell} p_j\}}$. □

Lemma 1.2 implies that if we examine the t-average of some polynomial expression in the p^{-it}, this will be close to the corresponding average of the genuinely random X_p provided that when we expand things out, the product of the primes involved is small compared with T. Since one can approximate quite general functions using polynomials (with the degree and coefficient size increasing as one looks for better approximations), one can hope to show rigorously that the distribution of sums of the p^{-it} is often close to the distribution of sums of the X_p. A particular instance of this is the well known *method of moments* from probability theory. For example, if

$P = P(T)$ is some large quantity, $(a_p)_{p \text{ prime}} = (a_p(T))_{p \text{ prime}}$ are complex numbers, and if one can show that for each $k \in \mathbb{N}$ one has

$$\frac{1}{T}\int_{T}^{2T}\Big(\Re\sum_{p\leqslant P} a_p p^{-it}\Big)^k dt \to \mathbb{E}N(0,1)^k \quad \text{as } T\to\infty,$$

then it follows that the distribution of $\Re\sum_{p\leqslant P} a_p p^{-it}$ converges to the standard normal distribution as $T\to\infty$. (Here we wrote $\mathbb{E}N(0,1)^k = (2\pi)^{-1/2}\int_{-\infty}^{\infty} w^k e^{-w^2/2}dw$ to denote the k-th power moment of the standard normal distribution.) In view of the above discussion, if the size of the a_p is under control then one could hope to prove such convergence (presuming it actually holds!) when $P(T) = T^{o(1)}$, so that the error terms in Lemma 1.2 don't contribute too much.

Our other basic principle is the following.

Principle 1.3. — *For many purposes (especially statistical questions not directly involving the zeta zeros), for any $\sigma \geqslant 1/2$ the Riemann zeta function $\zeta(\sigma+it)$ "behaves like" an Euler product $\prod_{primes\ p\leqslant P}(1-\frac{1}{p^{\sigma+it}})^{-1}$ of "suitable" length $P = P(\sigma,t)$.*

As discussed in the Introduction, the reason for believing that something like Principle 1.3 could prevail is that $\zeta(\sigma+it)$ is equal to an Euler product when $\sigma > 1$, and if the primes are well distributed then one expects this identity to continue to influence the behavior of the zeta function for smaller σ. Indeed, the Riemann Hypothesis is the statement that it does continue to have an influence, at least to the extent that $\zeta(\sigma+it)\neq 0$ (like a finite product of non-vanishing terms) when $\sigma > 1/2$.

It is much harder to prove rigorous statements corresponding to Principle 1.3 than it was for Principle 1.1, and we shall discuss several examples of such statements in the sequel. One also needs to think carefully about the appropriate sense of "behaves like" here, especially when $\sigma = 1/2$, since the Riemann zeta function does have infinitely many zeros on the critical line which don't reflect Euler product type behavior. But to fix ideas a little we state one nice result, which we will also come back to later.

Proposition 1.4 (Radziwiłł and Soundararajan, 2017). — *For all $T \leqslant t \leqslant 2T$, except for a set whose measure is $o(T)$ as $T\to\infty$, we have*

$$\zeta\Big(1/2+\frac{W}{\log T}+it\Big) = (1+o(1))\exp\Big\{\sum_{p^k\leqslant P}\frac{1}{kp^{k(1/2+W/\log T+it)}}\Big\},$$

where the sum is over prime powers p^k. Here $W = (\log\log\log T)^4$, and $P = T^{1/(\log\log\log T)^2}$, and the $o(1)$ term tends to 0 as $T\to\infty$.

The reader needn't be too concerned about the exact choices of W and P here, and in any event there is some flexibility in those (they are related though, as W increases one can take P smaller). Proposition 1.4 says that $\zeta(s)$ behaves like an Euler product