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MANOLESCU'S WORK ON THE TRIANGULATION CONJECTURE

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1. INTRODUCTION

Simplicial complexes are topological spaces with a simple underlying combinatorial structure. Indeed (in the compact case) such a space can be described by a system of subsets of a finite set—for the precise definition see Section 3. The combinatorial structure allows us to define invariants in a straightforward, computable manner. In particular, simplicial homology (and cohomology) is among the nicest invariants both from the point of view of definition and computation. The local structure of a simplicial complex can be, however, rather complicated—for example, different dimensional simplices might meet at a point.

Another convenient class of topological spaces is provided by manifolds, i.e., topological spaces which near every point look like Euclidean spaces. This definition gives a good idea about the local structure of the space, but gives little information about answers to global questions like homologies, etc.

It would be optimal to know that topological spaces having simple local structures also have nice global properties. The Triangulation Conjecture asserts exactly that:

Conjecture 1.1. — *A manifold is homeomorphic to a simplicial complex.*

The question in this form has been raised in 1926 by Kneser. The answer turned out to be affirmative in dimensions at most three, and for those manifolds of any dimension which admit a smooth structure. The general case, however, stayed open for almost a century. Work of Casson—relying on groundbreaking results of Freedman regarding topological 4-manifolds—showed that in dimension four (where smooth and topological manifolds are known to be more different than in any other dimensions) Conjecture 1.1 is false. Previous experience with the oddity of this particular dimension, however, warned mathematicians to draw any conclusion about the general case.

Results of Kirby and Siebenman on piecewise linear structures on manifolds helped to put the question into perspective, while results of Galewski-Stern and Matumoto from the late 70s provided a reformulation of the problem in terms of three-manifolds and cobordism properties of those. More precisely (and rather surprisingly) they showed that every closed topological manifold of dimension at least five is triangulable (i.e., homeomorphic to a simplicial complex) if there is a three-manifold Y which is an integral homology sphere (that is, $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$), admits Rokhlin invariant $\mu(Y)$ equal to 1 (for the definition of $\mu(Y)$, see Subsection 2.1) and the connected sum $Y\#Y$ is the boundary of a smooth four-manifold W with $H_*(W;\mathbb{Z}) \cong H_*(D^4;\mathbb{Z})$. (Here S^3 denotes the three-dimensional sphere, while D^4 stands for the four-dimensional disk.)

In studying the Seiberg-Witten equations and invariants, in 2013 Ciprian Manolescu discovered a new set of invariants of three-manifolds, eventually leading him to show

Theorem 1.2 (**[23]**). — *If an integral homology three-sphere Y admits $\mu(Y) = 1$ then $Y\#Y$ does not bound an integral homology disk W.*

Appealing to further related results of Galewski-Stern, this finding then implied

Theorem 1.3. — *For every dimension $n \geq 5$ there is a closed, connected topological n-manifold which admits no triangulation, i.e., it is not homeomorphic to a simplicial complex.*

This theorem puts an end to a long-standing question; the importance of Manolescu's result, however, is not limited to his disproof of Conjecture 1.1, it also lies in the way he proved Theorem 1.2. In **[23]** he defined a version of Seiberg-Witten-Floer (or Monopole Floer) homology groups of integral homology spheres, where a further symmetry of the Seiberg-Witten equations have been taken into account. The new homology groups (admitting an integral grading) then allowed him to define new functions on the abelian group Θ_3 formed by equivalence classes of integral homology spheres (where the equivalence relation is given by integral homology cobordisms, see Section 2). This approach not only allows us to understand the group Θ_3 better, but also provides ways of using further similar theories (as Heegaard Floer homology) to see invariants from a new angle. Soon after the appearance of Manolescu's work, Francesco Lin found an extension of the invariants to any (spin) three-manifolds, opening the way to further applications.

In this paper we will review the definitions of the main concepts listed above, outline the arguments leading to the (dis)proof of the Triangulation Conjecture, and review some of the further results and constructions originating from the groundbreaking ideas of Manolescu. The papers of Ciprian Manolescu provide outstanding introductions to the construction and the application of his invariants, see [**20, 21, 22, 23**]. (1) For this reason, to avoid repetitions we will try to emphasize aspects which appeared in less detail in the literature, and will try to draw attention to the aftermath of Manolescu's work in Heegaard Floer homology.

In this spirit, in Section 2 we collect some of the most fundamental infinite Abelian groups appearing in low dimensional topology and devote a paragraph to infinite Abelian groups in general. In Section 3 we review the basic notions appearing in the Triangulation Conjecture, while in Section 4 we discuss various obstruction classes. Section 5 gives a short recollection of the reformulation of the conjecture in terms of the integral homology cobordism group. Section 6 contains a (very sketchy) outline of the theory producing the novel invariants of Manolescu, leading to the disproof of Conjecture 1.1 in Subsection 6.3. We close our discussion with Section 7, where further directions and developments inspired by Manolescu's work is given (without the aim of providing a complete picture of this dynamically changing field).

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2. ABELIAN GROUPS IN LOW DIMENSIONAL TOPOLOGY

Certain infinite groups play central role in low dimensional topology. Mapping class groups (groups of isotopy classes of orientation preserving diffeomorphisms of manifolds) are rather mysterious in most dimensions, and even for two-dimensional compact manifolds there are fundamental open questions regarding these groups—although in these cases various presentations of the groups are known. Surprisingly, there are even Abelian groups in low dimensional topology which capture important information, but we do not have a good grasp on their structure. We list some of these below.

2.1. Homology cobordism groups

The three-dimensional (oriented) cobordism group Ω_3 is trivial (which is just another way to say that any closed, oriented three-dimensional manifold is the boundary

(1) The paper [**23**] was awarded by the *Moore prize* of the American Mathematical Society in 2019, recognizing this paper as an outstanding research article.

of a compact, smooth, oriented four-manifold). In a similar manner, Ω_3^{spin} (the spin cobordism classes of spin three-manifolds) is also trivial.

The *homology cobordism* group Θ_3, however, is highly nontrivial. Indeed, consider those (oriented, closed) three-manifolds for which the first homology group (with integer coefficient) vanishes. These three-manifolds are traditionally called integral homology spheres, and the condition is obviously equivalent to the requirement that for such a three-manifold Y we have $H_*(Y;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$. The most notable nontrivial example of such a manifold is the *Poincaré homology sphere* P, given as

$$P = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^2 + z_2^3 + z_3^5 = 0,\ \|(z_1, z_2, z_3)\| = 1\}.$$

This smooth three-manifold has fundamental group $\pi_1(P)$ a perfect group of order 120, implying $H_*(P;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$.

In defining the group Θ_3, regard two integral homology three-spheres Y_1 and Y_2 equivalent if there is a smooth, oriented, compact four-manifold X with boundary $\partial X = -Y_1 \cup Y_2$ and with $H_*(X;\mathbb{Z}) = H_*(S^3 \times [0,1];\mathbb{Z})$, that is, we assume that the cobordism (up to homology) is like the trivial cobordism. The group structure is given by the connected sum $(Y_1, Y_2) \mapsto Y_1 \# Y_2$ as addition, the map $Y \mapsto -Y$ as inverse (where $-Y$ denotes the same manifold as Y, with the opposite orientation) and S^3 as the identity element. It is not hard to see that the result is an Abelian group.

There are simple variants of this construction, for example the rational homology cobordism group $\Theta_3^{\mathbb{Q}}$ is defined in a similar manner, with the exception that all homologies are required to be taken with rational coefficients. In particular, a rational homology sphere Y is a closed, oriented three-manifold with $H_*(Y;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$, which is equivalent to request $H_1(Y;\mathbb{Z})$ to be a finite group, or to ask the first Betti number $b_1(Y)$ to vanish. A further common variant of this constrution is the spin^c rational homology cobordism group $\Theta_3^{\mathbb{Q},\,\mathrm{spin}^c}$, where we consider pairs $(Y, \mathfrak{s})$ with the property that Y is a rational homology sphere as above, $\mathfrak{s}$ is a spin^c structure on Y, and two such pairs $(Y_1, \mathfrak{s}_1)$ and $(Y_2, \mathfrak{s}_2)$ are considered to be equivalent if there is a rational homology cobordism X between Y_1 and Y_2, together with a spin^c structure $\mathfrak{t}$ on X with the property that $\mathfrak{t}$ restricts to $\mathfrak{s}_1$ over $-Y_1 \subset \partial X$ and to $\mathfrak{s}_2$ over $Y_2 \subset \partial X$.

These groups come with natural maps between them: for example there is the forgetful map $\Theta_3^{\mathbb{Q},\,\mathrm{spin}^c} \to \Theta_3^{\mathbb{Q}}$, and the natural map $\Theta_3 \to \Theta_3^{\mathbb{Q}}$ induced by the fact that every integral homology sphere (and integral homology cobordism) is also a rational homology sphere (and a rational homology cobordism).

As the groups introduced above are all Abelian, one can have the impression that their structure is easy to understand (even if for some reason we might not be able to compute them). At first glance is seems possible that Θ_3 (similarly to Ω_3 and Ω_3^{spin}) is indeed trivial. The Rokhlin homomorphism $\mu\colon \Theta_3 \to \mathbb{Z}/2\mathbb{Z}$, however shows that this is not the case. For defining μ, recall that an integral homology sphere Y (carrying a unique spin structure) is the boundary of a compact spin four-manifold X (as $\Omega_3^{\text{spin}} = 0$). Simple algebra (see for example [7, Lemma 1.2.20]) shows that the signature $\sigma(X)$ of such an X is divisible by 8. Rokhlin's Theorem (stating that a *closed* spin four-manifold has signature divisible by 16) implies that the mod 2 reduction of $\frac{1}{8}\sigma(X)$ is independent of the chosen X, hence by defining $\mu(Y) \in \mathbb{Z}/2\mathbb{Z}$ as the mod 2 reduction of $\frac{1}{8}\sigma(X)$ we get an invariant of Y. This value is obviously a homology cobordism invariant and provides a homomorphism $\mu\colon \Theta_3 \to \mathbb{Z}/2\mathbb{Z}$. Simple calculation shows that $\mu(P) = 1$ for the Poincaré homology sphere P (as it is the boundary of the negative definite E_8-plumbing), hence μ is onto, consequently $|\Theta_3| \geq 2$. Indeed, for a while it seemed plausible to expect that μ is an isomorphism between Θ_3 and $\mathbb{Z}/2\mathbb{Z}$.

As one of the early applications of the gauge theoretic techniques introduced by S. Donaldson in the study of four-dimensional manifolds, Furuta showed that

THEOREM 2.1. — *The Abelian groups $\Theta_3, \Theta_3^{\mathbb{Q}}$ and $\Theta_3^{\mathbb{Q},\,\text{spin}^c}$ defined above are not finitely generated.*

Therefore, despite being Abelian, their structure might be rather intricate.

2.2. Concordance groups

Before going any further, we invoke a further similar important group, the group of concordance classes of knots. Let us consider knots in the three-space, i.e., smoothly embedded circles in S^3. We say that two knots K_1 and K_2 are *concordant*, if there is a smoothly and properly embedded annulus ($\cong S^1 \times [0,1]$) in $S^3 \times [0,1]$ intersecting the two ends in K_1 and K_2, respectively. The resulting Abelian group $\mathcal{C}$ (once again, with connected sum as addition, the mirror image as inverse and the unknot representing the identity element) is called the *smooth concordance group*.

As before, this group has a number of variants. The easiest one is when we define the equivalence relation by considering concordances in integral homology cobordisms between the two copies of S^3; the resulting group will be a quotient of $\mathcal{C}$. A slightly larger group can be defined by considering knots in integral homology spheres (and the concordances in integral homology cobordisms), or in rational homology spheres (with rational homology cobordisms between them containing the concordances) and even rational homology spheres (and rational homology cobordisms) together with appropriate spin^c structures. Once again, there are various natural maps between these constructions.

A further variant of $\mathcal{C}$ is provided by the fact that in dimension four the application of smooth or merely continuous maps provide drastically different theories. Here, when we use the term 'continuous', we really mean 'locally flat', that is, the