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REGULARITY OF ENTROPY, GEODESIC CURRENTS AND ENTROPY AT INFINITY

BY Barbara SCHAPIRA AND Samuel TAPIE

ABSTRACT. – In this work, we introduce a notion of *entropy at infinity* for the geodesic flow of negatively curved manifolds. We introduce the class of noncompact manifolds which admit a critical gap between entropy at infinity and topological entropy. We call them *strongly positively recurrent manifolds* (SPR), and provide many examples. We show that dynamically, they behave as compact manifolds. In particular, they admit a finite measure of maximal entropy.

Using the point of view of currents at infinity, we show that on these SPR manifolds the topological entropy of the geodesic flow varies in a $\mathcal{C}^1$-way along $\mathcal{C}^1$-uniform perturbations of the metric. This result generalizes former work of Katok (1982) and Katok-Knieper-Weiss (1991) in the compact case.

RÉSUMÉ. – Dans ce travail, nous introduisons une notion d'*entropie à l'infini* pour les flots géodésiques des variétés à courbure négative. Nous introduisons la classe des variétés, dites *fortement positivement récurrentes* (SPR), dont l'entropie à l'infini est strictement inférieure à l'entropie topologique. Nous donnons de nombreux exemples de telles variétés. Nous montrons que d'un point de vue dynamique, ces variétés ressemblent à des variétés compactes. En particulier, elles admettent une mesure finie maximisant l'entropie.

À l'aide du point de vue des courants à l'infini, nous montrons que sur ces variétés SPR, l'entropie topologique varie de manière $\mathcal{C}^1$ le long de perturbations $\mathcal{C}^1$-uniformes de la métrique. Ceci généralise des résultats passés de Katok (1982) et Katok-Knieper-Weiss (1991) dans le cas compact.

1. Introduction

1.1. Variation of the topological entropy: An overview

The initial motivation of this work was to answer the following simple question. Consider a hyperbolic surface of finite volume and a smooth compact perturbation of the metric. Does the topological entropy of the geodesic flow vary regularly? More generally, what happens for a smooth perturbation of the metric of a noncompact negatively curved Riemannian manifold?

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The answer has been known on compact manifolds since almost thirty years [31, 30, 21], and has been extended to the convex-cocompact case in [52]. A similar argument gives the regularity of the topological entropy for a perturbation of an Anosov flow, cf [31].

Compactness of the underlying space is crucial in the above results, and no result was known until now for manifolds with a non-compact non-wandering set. Even the case of a smooth compact perturbation of the metric of a finite volume hyperbolic surface was not accessible with their arguments. Let us recall the two main steps of their argument to understand why.

The key step is the following inequality, due to Katok in [29] for surfaces, extended in [31] to all dimensions.

THEOREM 1.1 ([29]; [31]). – *Let g_1, g_2 be Riemannian metrics with negative sectional curvature on the same compact manifold M. Then the entropies of their geodesic flows satisfy*

$$h_{\mathrm{top}}(g_1) \leq h_{\mathrm{top}}(g_2) \times \int_{S^{g_1}M} \|v\|^{g_2}\, d\overline{m}^{g_1}_{\mathrm{BM}}(v), \tag{1}$$

where $\|v\|^{g_2} = \sqrt{g_2(v,v)}$ and $\overline{m}^{g_1}_{\mathrm{BM}}$ is the normalized Bowen-Margulis measure on the g_1-unit tangent bundle $S^{g_1}M$ for the g_1-geodesic flow.

Reversing the role of g_1 and g_2 also provides a lower bound for $h_{\mathrm{top}}(g_1)$, and a first order power expansion gives the following smoothness result.

THEOREM 1.2 ([31]). – *Let $(g_\varepsilon)_{\varepsilon\in(-1,1)}$ be a $\mathcal{C}^2$-family of $\mathcal{C}^2$ Riemannian metrics with negative sectional curvature on the same compact manifold M. Then $\varepsilon \mapsto h_{\mathrm{top}}(g_\varepsilon)$ is $\mathcal{C}^1$, and its derivative is given by*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} h_{\mathrm{top}}(g_\varepsilon) = -h_{\mathrm{top}}(g_0) \times \int_{S^{g_0}M} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \|v\|^{g_\varepsilon}\, d\overline{m}^{g_0}_{\mathrm{BM}}(v), \tag{2}$$

where $\overline{m}^{g_0}_{\mathrm{BM}}$ is the normalized Bowen-Margulis measure on the g_0-unit tangent bundle $S^{g_0}M$ for the g_0-geodesic flow.

In the previously quoted works, the proofs of (1) strongly use the compactness of the non-wandering set. In the first part of our paper, we use a different approach to generalize it to the non-compact setting. This improves it even in the compact case, providing an explicit transformation rule for the entropies, equality which immediately implies (1), and has other interesting consequences.

The previously known proofs of (2) use the compactness of M for a crucial point: to ensure the finiteness and the continuity of the normalized Bowen-Margulis measures $\overline{m}^{g_\varepsilon}_{\mathrm{BM}}$ in the weak-$*$ topology as ε varies. Neither finiteness of the Bowen-Margulis measure nor its continuity under a variation of the metric can be ensured in general. Maybe the most striking fact of our work is that we introduce a new wide class of manifolds, which we call *SPR manifolds*, SPR meaning *strongly / stably positively recurrent*. The terminology *Stably positively recurrent* has been introduced by Gurevic-Savchenko [26] in the context of countable Markov shifts. Sarig [48] modified it, in the same context, into *strongly positively recurrent*, terminology which has been used later by other authors as Buzzi [8]. See also the very recent work of Velozo [53], who follows also this terminology. Both terminologies are meaningful, and had not yet been considered in our context. It turns out that the same property also appeared recently and independently in the context of geometric group theory in [3] under the name of growth gap.

The class of SPR manifolds that we define here has the remarkable property that the Bowen-Margulis measure is finite, and moreover stays finite and varies continuously along small perturbations. In particular, under $\mathcal{C}^1$-uniform variation of such SPR Riemannian metrics, the topological entropy is $\mathcal{C}^1$ and its derivative is given by (2).

These SPR manifolds include finite volume hyperbolic manifolds, and more generally almost all known examples where the geodesic flow admits a (finite) measure of maximal entropy, as geometrically finite negatively curved manifolds with spectral gap [15], Schottky product examples from [39], and unpublished examples of Ancona [2]. The class of SPR manifolds is much larger than only the above mentioned examples. We postpone the extensive study of SPR manifolds to a later paper [25]. Therefore, the second half of our paper will be devoted to the presentation of a geometrical setting, as large as possible, where this finiteness and continuity of Bowen-Margulis measures can be ensured.

Let us now present our main results with more details.

1.2. Invariant measures and change of Riemannian metrics

Let (M, g_1) be a complete Riemannian manifold, and g_2 be another Riemannian metric on M such that there exists $C > 1$ with $\frac{1}{C} g_1 \leq g_2 \leq C g_1$. We assume moreover that both g_1 and g_2 have pinched negative sectional curvatures with uniformly bounded first derivatives: this implies that g_1-geodesics are g_2-quasi-geodesics and the visual boundary of the universal cover $(\widetilde{M}, g_1)$ is canonically identified with the visual boundary of $(\widetilde{M}, g_2)$; we will denote it by $\partial\widetilde{M}$. We will use extensively this correspondance to compare the dynamics of the geodesic flows on $S^{g_1}M$ and $S^{g_2}M$.

Let $\Gamma = \pi_1(M)$ acting on the universal cover $\widetilde{M}$, let m be a locally finite measure on $S^{g_1}M$, invariant by the geodesic flow $(g_1^t)_{t\in\mathbb{R}}$, and $\widetilde{m}$ its lift to $S^{g_1}\widetilde{M}$. We write $\partial^2\widetilde{M} = (\partial\widetilde{M} \times \partial\widetilde{M})\backslash\mathrm{Diag}$. In g_1-Hopf coordinates (cf Section 2), $S^{g_1}\widetilde{M} \simeq \partial^2\widetilde{M} \times \mathbb{R}$, and $\widetilde{m}$ has a local product structure of the form $d\widetilde{m} = d\mu \times dt$, where μ is a Γ-invariant *geodesic current* on $\partial^2\widetilde{M}$. We write therefore $m = m_\mu^{g_1}$.

We can now define a measure $\widetilde{m}_\mu^{g_2}$ on $S^{g_2}\widetilde{M}$, given in g_2-Hopf parametrization by the same local product formula $\widetilde{m}_\mu^{g_2} = d\mu \times dt$: by Γ-invariance, this induces a locally finite measure $m_\mu^{g_2}$ on $S^{g_2}M$, which is invariant for the geodesic flow $(g_2^t)_{t\in\mathbb{R}}$. The ergodic properties of $(S^{g_1}M, g_1^t, m_\mu^{g_1})$ and $(S^{g_2}M, g_2^t, m_\mu^{g_2})$ are strongly related.

Well known facts imply that if $m_\mu^{g_1}$ and $m_\mu^{g_2}$ are finite then one is ergodic or conservative if and only if the other is. The reader may believe that, since $\frac{1}{C} g_1 \leq g_2 \leq C g_1$, then $m_\mu^{g_1}$ is finite if and only if $m_\mu^{g_2}$ is. We will indeed show that it is the case and relate the masses and entropies of these measures.

In this purpose, let us introduce the *instantaneous geodesic stretch* $\mathcal{E}^{g_1\to g_2} : S^{g_1}\widetilde{M} \to \mathbb{R}$ defined for all $v \in S^{g_1}\widetilde{M}$ by

$$\mathcal{E}^{g_1\to g_2}(v) = \frac{d}{dt}\bigg|_{t=0^+} \mathcal{B}^{g_2}_{v_+^{g_1}}(\pi v, \pi g_1^t v) = \frac{d}{dt}\bigg|_{t=0^+} \mathcal{B}^{g_2}_{v_+^{g_1}}(o, \pi g_1^t v),$$

where $\pi : S^{g_1}\widetilde{M} \to \widetilde{M}$ is the canonical projection, and $\mathcal{B}^{g_2}_{v_+^{g_1}}(.,.)$ is the Busemann function for g_2 based at the endpoint of the g_1-geodesic generated by v. By Γ-invariance,