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A SHARP FREIMAN TYPE ESTIMATE FOR SEMISUMS IN TWO AND THREE DIMENSIONAL EUCLIDEAN SPACES

BY Alessio FIGALLI AND David JERISON

ABSTRACT. – Freiman's theorem is a classical result in additive combinatorics concerning the approximate structure of sets of integers that contain a high proportion of their internal sums. As a consequence, one can deduce an estimate for sets of real numbers: "If $A \subset \mathbb{R}$ and $\left|\frac{1}{2}(A+A)\right| - |A| \ll |A|$, then A is close to its convex hull." In this paper we prove a sharp form of the analogous result in dimensions 2 and 3.

RÉSUMÉ. – Le théorème de Freiman est un résultat classique de la combinatoire additive concernant la structure approximative des ensembles d'entiers qui contiennent une forte proportion de leurs sommes internes. En conséquence, on déduit l'estimée suivante : "Si $A \subset \mathbb{R}$ et $\left|\frac{1}{2}(A+A)\right| - |A| \ll |A|$, alors A est proche de son enveloppe convexe." Dans cet article, nous prouvons une forme optimale du résultat correspondant en dimensions 2 et 3.

1. Introduction

Given a set $A \subset \mathbb{R}^n$, define the semisum by

$$\tfrac{1}{2}(A+A) := \left\{ \tfrac{x+y}{2} \,:\, x \in A,\ y \in A \right\}.$$

Evidently, $\frac{1}{2}(A+A) \supset A$, and for convex sets K, $\frac{1}{2}(K+K) = K$. Also, $\left|\frac{1}{2}(A+A)\right| = |A| > 0$ implies that A is equal to its convex hull $\mathrm{co}(A)$ minus a set of measure zero (see [3, Théorème 6]).

The stability of this statement is a natural question that has already been extensively investigated in the one dimensional case. Indeed, by approximating sets in $\mathbb{R}$ with finite unions of intervals, one can translate the problem to $\mathbb{Z}$ and in the discrete setting the question becomes a well studied problem in additive combinatorics. More precisely, set

$$\delta(A) := \left|\tfrac{1}{2}(A+A)\right| - |A|,$$

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where $|\cdot|$ denotes the outer Lebesgue measure. The following theorem can be obtained as a corollary of a result of G. Freiman [9] about the structure of additive subsets of $\mathbb{Z}$ (see [5] for more details, and also [11] and the references therein for more recent developments on this one dimensional problem):

THEOREM 1.1. – *Let $A \subset \mathbb{R}$ be a measurable set of positive Lebesgue measure, and assume that $\delta(A) < |A|/2$. Then*

$$|\mathrm{co}(A) \setminus A| \leq 2\,\delta(A).$$

Note that the assumption $\delta(A) < |A|/2$ is necessary, as can be seen by considering the set $A = [0,1] \cup [R, R+1]$ with $R \gg 1$.

In [5, Theorem 1.2] we extended Theorem 1.1 to every dimension, but with a dimensional dependence in the exponent (see also [6] for a stability result when one considers the semisum of two different sets). Our result was as follows.

THEOREM 1.2. – *Let $n \geq 2$. There exist computable dimensional constants $\delta_n, C_n > 0$ such that if $A \subset \mathbb{R}^n$ is a measurable set of positive Lebesgue measure with $\delta(A) \leq \delta_n|A|$, then*

$$\frac{|\mathrm{co}(A) \setminus A|}{|A|} \leq C_n \left(\frac{\delta(A)}{|A|}\right)^{\alpha_n}, \qquad \textit{where } \alpha_n := \frac{1}{8 \cdot 16^{n-2} n!(n-1)!}.$$

Note that the dimensional smallness assumption on $\delta(A)$ is necessary. Indeed, consider $t = 1/2$ and the set

$$A := B_1(0) \cup \{Re_1\}, \qquad R \gg 1.$$

Then $|\mathrm{co}(A) \setminus A| \approx R$ is arbitrarily large, while $\delta(A) = \left|B_{1/2}\left(\frac{R}{2}e_1\right)\right| = 2^{-n}|A|$, hence $\delta_n \leq 2^{-n}$.

The proof in [5] is based on induction on dimension and Fubini-type arguments, and it leads to a bad estimate for the exponent α_n. In fact, we believe that $\alpha_n = 1$, which we formulate more precisely in the following conjecture.

CONJECTURE 1.3. – *Suppose that A is a measurable subset of $\mathbb{R}^n$, of positive Lebesgue measure. There exist computable constants C_n and $d_n > 0$, depending only on n, such that the following holds: if $\delta(A) \leq d_n|A|$, then*

$$|\mathrm{co}(A) \setminus A| \leq C_n\,\delta(A).$$

In this paper we introduce a completely new strategy that allows us to prove this *sharp* stability estimate in dimensions 2 and 3.

THEOREM 1.4. – *Conjecture 1.3 is valid for $n \leq 3$.*

The exponent $\alpha_n = 1$ may look surprising at first sight, as most sharp stability results for minimizers of geometric inequalities in dimension $n \geq 2$ hold with the exponent $1/2$. In particular, the best possible stability exponent for the Brunn-Minkowski inequality on convex sets is $1/2$, see [8, 7]. In contrast, our stability inequality with exponent 1 is affine invariant and additive under partitions of the set by convex tilings, and these properties are crucial to the proof. Even though we have stopped at $n = 3$, the proof is by induction on n and is organized with the hope that parts of it will ultimately apply to the case of general n. There is at least one other stability inequality in which the exponent 1 is optimal in all dimensions, namely the one proved in [4]. (Observe that the exponent 1 becomes natural when looking at critical points instead of minimizers, see for instance [2, Theorem 1.2], but this is a consequence of the different definition of the "deficit" δ.)

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2. Proof of Theorem 1.4

As the reader will see, many of the arguments for the proof of Theorem 1.4 are valid in any dimension. For this reason we shall work with a generic n for most of the proof, and we shall use some geometric considerations specific to $n = 2$ and $n = 3$ only towards the end.

Basic considerations

Since Theorem 1.4 is known for $n = 1$ (see Theorem 1.1), we can assume that $n \geq 2$ and, by induction on dimension, we can also assume that Theorem 1.4 holds in dimension $n - 1$.

Denote the convex hull of A by $K := \mathrm{co}(A)$. Since the theorem is affine invariant, after dilation we can assume, with no loss of generality, that $|A| = 1$. Assuming that $\delta(A) \ll 1$, it follows by [1] and/or [5, Theorem 1.2] that [(1)]

$$\mu := |K \setminus A| \ll 1. \tag{2.1}$$

In particular, $1 \leq |K| \leq 2$. Therefore, using the lemma of F. John [10], up to an affine transformation with Jacobian bounded from above and below by a dimensional constant, we can assume that K satisfies

$$B_{1/\sqrt{n}} \subset K \subset B_{\sqrt{n}} \tag{2.2}$$

for balls of radius $1/\sqrt{n}$ and $\sqrt{n}$ centered at the origin.

By approximation, [(2)] we can assume the set A is compact and that ∂K consists of finitely many polygonal faces. In particular, $\frac{1}{2}(A + A)$ is compact, hence measurable. Furthermore,

(1) Although this estimate can be deduced as a consequence of [1], that result does not provide computable constants, as the proof is based on a contradiction argument relying on compactness.

(2) One way to define a suitable approximation is to consider a sequence of finite sets $V_k \subset V_{k+1} \subset A$ such that the polyhedra $P_k = \mathrm{co}(V_k)$ satisfy $|P_k| \to |\mathrm{co}(A)|$ as $k \to \infty$, and a sequence of compact subsets $A'_k \subset A$ such that $|A'_k| \to |A|$ as $k \to \infty$. Then let $A_k := V_k \cup [A'_k \cap (1 - 1/k)P_k]$. Since $|A_k| \to |A|$, it suffices to prove the estimate of Theorem 1.4 for A_k and then let $k \to \infty$.