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CONSERVATIVE ANOSOV DIFFEOMORPHISMS OF $\mathbb{T}^2$ WITHOUT AN ABSOLUTELY CONTINUOUS INVARIANT MEASURE

BY ZEMER KOSLOFF

ABSTRACT. – We construct examples of C^1 Anosov diffeomorphisms on $\mathbb{T}^2$ which are of Krieger type III_1 with respect to Lebesgue measure. This shows that the Gurevic Oseledec phenomena that conservative $C^{1+\alpha}$ Anosov diffeomorphisms have a smooth invariant measure does not hold true in the C^1 setting.

RÉSUMÉ. – Sur $\mathbb{T}^2$, on construit des exemples de difféomorphismes C^1 d'Anosov qui sont de type de Krieger III_1 par rapport à la mesure de Lebesgue. Ceci montre que le phénomène de Gurevic Oseledec selon lequel tout difféomorphisme conservatif d'Anosov $C^{1+\alpha}$ a une mesure invariante lisse, n'est pas valable dans le cadre C^1.

1. Introduction

This paper provides the first examples of Anosov diffeomorphisms of $\mathbb{T}^2$ which are conservative and ergodic yet there is no Lebesgue absolutely continuous invariant measure.

Let M be a compact, boundaryless smooth manifold and $f : M \to M$ be a diffeomorphism. A natural question which arises is whether f preserves a measure which is absolutely continuous with respect to the volume measure on M. In order to avoid confusion in what follows, we would like to stress out that in this paper, the term *conservative* means the definition from ergodic theory which is non existence of wandering sets of positive measure. That is f is conservative if and only if for every $W \subset M$ so that $\{f^n W\}_{n\in\mathbb{Z}}$ are disjoint (modulo the volume measure), $\mathrm{vol}(W) = 0$.

It follows from [16] that for a generic C^2 Anosov diffeomorphism there exists no absolutely continuous invariant measure (a.c.i.m.), [6, p. 72, Corollary 4.15.]. Following this result, Sinai asked whether a generic Anosov diffeomorphism will satisfy Poincare recurrence. This question was answered by Gurevic and Oseledec [9] who proved that the set of conservative (Poincare recurrent) C^2 Anosov diffeomorphism is meager in the C^2 topology (restricted to the open set of Anosov diffeomorphisms). Indeed, they have proved that if f is a conservative C^2 Anosov (hyperbolic) diffeomorphism, then f preserves a probability

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measure in the measure class of the volume measure which combined with the result of Livsic and Sinai proves the non-genericity result. The proof in [9] uses the absolute continuity of the foliations and existence of SRB measures to show that if the SRB measure for f is not equal to the SRB measure for f^{-1} then there exists a continuous function $g : M \to \mathbb{R}$ and a set $A \subset M$ of positive volume so that,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g\left(f^k(x)\right) \neq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g\left(f^{-k}(x)\right), \ \forall x \in A.$$

It is then a straightforward argument to construct a set $B \subset A$ of positive volume measure so that for almost every $x \in B$, the set $\left\{k \in \mathbb{N} : f^k x \in B\right\}$ is finite, in contradiction with Halmos Recurrence Theorem [1].

This result remains true for $C^{1+\alpha}$, $\alpha > 0$ Anosov diffeomorphisms. However, since there exist C^1 Anosov diffeomorphisms whose stable and unstable foliations are not absolutely continuous [17], this proof can not be generalized for the C^1 setting. This paper is concerned with the question whether every conservative C^1-Anosov diffeomorphism has an absolutely continuous invariant measure.

An easier version of this question was studied before in the context of smooth expanding maps. Every C^2 expanding map of a manifold has an absolutely continuous invariant measure [14]. In contrast to the higher regularity case, Avila and Bochi [4], extending previous results of Campbell and Quas [8], have shown that a generic C^1 expanding map has no a.c.i.m and a generic C^1 expanding map of the circle is not recurrent [8]. It seems natural to argue that these generic statements for expanding maps can be transferred to Anosov diffeormorphisms via the natural extension. However there are several problems with this approach which could be summarized into roughly two parts:

- The natural extension construction is an abstract theorem and in many cases it is not clear if it has an Anosov model. See [22] for constructions of smooth natural extensions.
- In order for the natural extension to be conservative, the expanding map has to be recurrent [19, Th. 4.4] and a generic C^1 expanding map is not recurrent.

Another natural approach in finding C^1 examples with a certain property is to prove that the property is generic in the C^1 topology, see for example [5]. However since by the result of Sinai and Livsic, a generic C^1 Anosov map is dissipative, it is not clear to us how to use this approach to find a conservative example without an a.c.i.m. Nonetheless we prove the following.

THEOREM 1. – *There exists a C^1-Anosov diffeomorphism of the two torus $\mathbb{T}^2$ which is ergodic, conservative and there exists no σ-finite invariant measure which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^2$.*

In fact, the ergodic type III transformations (a transformation without an a.c.i.m.) can be further decomposed into the Krieger Araki-Woods classes III_λ, $0 \leq \lambda \leq 1$ [13] , see Section 2, and our examples are of type III_1.

The examples are constructed by modifying a linear Anosov diffeomorphism to obtain a change of coordinates which takes the Lebesgue measure to a measure which is equivalent to a type III Markovian measure (on a Markov partition of the linear diffeomorphism). These examples are greatly inspired by the ideas of Bruin and Hawkins [7] where they modify the map $f(x) = 2x \mod 1$ using the push forward (with respect to the dyadic representation) of a Hamachi product measure on $\{0,1\}^{\mathbb{N}}$ to the circle. Since by embedding a horseshoe in a linear transformation one loses the explicit formula for the Radon Nykodym derivatives of the modified transformations, we couldn't use measures on a full shift space but rather measures supported on topological Markov shifts. The measures which play the role of the Hamachi measures in our construction are the type III_1 (for the shift) inhomogeneous Markov measures.

This paper is organized as follows. In Section 2 we start by introducing the definitions and background material from nonsingular ergodic theory and smooth dynamics which are used in this paper. We end this section with a discussion on the method of the construction. Section 3 presents the inductive construction of the type III_1 Markov shift examples. In Section 4 we show how to use the one sided Markov measures from the previous section to obtain a modification of the golden mean shift. In Section 5 we show how to embed and modify the one dimensional perturbations of the previous sections to obtain homeomorphisms of the two torus, which when applied as conjugation to a certain total automorphism (the natural extension of the golden mean shift) are examples of type III_1 Anosov diffeomorphisms. Finally in the appendix we prove that these Markovian measures satisfy the aforementioned properties (ergodic, conservative and type III_1).

2. Preliminary definitions and a discussion on the method of construction

2.1. Basics of nonsingular ergodic theory

This subsection is a very short introduction to nonsingular ergodic theory. For more details and explanations please see [1].

Let $(X, \mathcal{B}, \mu)$ be a standard probability space. In what follows equalities (and inclusions) of sets are modulo the measure μ on the space. A measurable map $T : X \to X$ is *nonsingular* if $T_*\mu := \mu \circ T^{-1}$ is equivalent to μ meaning that they have the same collection of negligible sets. If T is invertible one has the Radon Nykodym derivatives

$$(T^n)'(x) := \frac{d\mu \circ T^n}{d\mu}(x) : X \to \mathbb{R}_+.$$

A set $W \subset X$ is *wandering* if $\{T^n W\}_{n\in\mathbb{Z}}$ are pairwise disjoint and as was stated before we say that T is *conservative* if there exists no wandering set of positive measure. By the Halmos' Recurrence Theorem a transformation is conservative if and only if it satisfies Poincare recurrence, that is given a set of positive measure $A \in \mathcal{B}$, almost every $x \in A$ returns to itself infinitely often. A transformation T is *ergodic* if there are no non trivial T invariant sets. That is $T^{-1}A = A$ implies $A \in \{\emptyset, X\}$.

We end this subsection with the definition of the *Krieger ratio set* $R(T)$. We say that $r \geq 0$ is in $R(T)$ if for every $A \in \mathcal{B}$ of positive μ measure and for every $\epsilon > 0$ there exists an $n \in \mathbb{Z}$ such that

$$\mu\left(A \cap T^{-n}A \cap \left\{x \in X : \left|(T^n)'(x) - r\right| < \epsilon\right\}\right) > 0.$$

The ratio set of an ergodic measure preserving transformation is a closed multiplicative subgroup of $[0,\infty)$ and hence it is of the form $\{0\}, \{1\}, \{0,1\}, \{0\} \cup \{\lambda^n : n \in \mathbb{Z}\}$ for $0 < \lambda < 1$