Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 149
Fascicule 1

2021

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 1-13

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 1, tome 149, mars 2021

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)

Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
149 (1), 2021, p. 1-13

THE BOGOMOLOV–BEAUVILLE–YAU DECOMPOSITION FOR KLT PROJECTIVE VARIETIES WITH TRIVIAL FIRST CHERN CLASS – WITHOUT TEARS

by Frédéric Campana

Abstract. — We give a simplified proof (in characteristic zero) of the decomposition theorem for connected complex projective varieties with klt singularities and a numerically trivial canonical bundle. The proof mainly consists in reorganizing some of the partial results obtained by many authors and used in the previous proof but avoids those in positive characteristic by S. Druel. The single, to some extent new, contribution is an algebraicity and bimeromorphic splitting result for generically locally trivial fibrations with fibers without holomorphic vector fields. We first give the proof in the easier smooth case, following the same steps as in the general case, treated next. The last two words of the title are plagiarized from [4].

Résumé (*La décomposition de Bogomolov-Beauville-Yau des variétés projectives klt à première classe de Chern triviale – sans larmes*). — Nous donnons une preuve simplifiée (en caractéristique zéro) du théorème de décomposition des variétés connexes et projectives complexes à singularités klt et fibré canonique numériquement trivial.Cette preuve consiste essentiellement en une réorganisation de la preuve originale basée sur des résultats partiels obtenus par divers auteurs, mais évite d'utiliser ceux de caractéristique positive obtenus par S. Druel. Le seul résultat nouveau, dans une certaine mesure, établit l'algébricité et le scindage méromorphe pour les fibrations génériquement localement triviales dont les fibres n'ont pas de champ de vecteur holomorphe non nul. Nous donnons tout d'abord la preuve dans le cas lisse, plus simple, suivant les mêmes étapes que dans le cas général, traité ensuite. Les deux derniers mots du titre plagient [4].

Texte reçu le 17 avril 2020, modifié le 8 juillet 2020, accepté le 21 octobre 2020.

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Mathematical subject classification (2010). — 14J32, 14E99, 32J25, 32Q20, 32Q25.

Key words and phrases. — Kähler–Einstein metrics, First Chern class, Hyperkähler varieties, Calabi–Yau varieties, Holonomy, Algebraic foliations, Fundamental group.

 0037-9484/2021/1/$ 5.00
 doi:10.24033/bsmf.2823

1. Introduction

When X is smooth, connected, compact Kähler, with $c_1(X) = 0$, the classical, metric, proof of the Bogomolov–Beauville–(Yau) decomposition theorem, given in [2] (the arguments of [6] being Hodge-theoretic), starts with a Ricci-flat Kähler metric ([26]) and then decomposes the universal cover X' of X according to De Rham theorem, in its holonomy factors. The Cheeger–Gromoll theorem then distinguishes the flat Euclidian factor $\mathbb{C}^s$ of X' from the (simply-connected) product P of the others (which are compact and with holonomy either $SU(m)$ or $Sp(k)$). The compactness of P combined with Bieberbach's theorem now imply that a finite étale cover of X is the product of a complex torus $\mathbb{C}^s/\Gamma$ with P.

We shall first give a different proof, but only for X smooth projective, of this product decomposition, weaker in the sense that P is not shown to be simply connected (see Theorem 2.1 below). Indeed, the proof does not go through the universal cover and uses neither the De Rham nor the Cheeger–Gromoll theorems.

This allows for its extension (given next) to the singular case obtained in [21], which uses many other partial results, among which are those of [18] and [14] (which plays a rôle analogous to that played by the Cheeger–Gromoll theorem). Our proof makes the step involving the delicate positive characteristic arguments of [14] superfluous. We, indeed, deduce the algebraicity of the foliation given by the flat factor of the holonomy from the splitting result (see Theorem 3.4) below, instead of using the Albanese map. This splitting result can be applied once the algebraicity of the leaves of the foliations given by the nonflat factors of the holonomy have been shown to be algebraic and without nonzero vector fields.

The author thanks Benoît Claudon and Mihai Păun for their help in reading the text and several discussions. After this text was posted on arXiv, the author received useful comments by S. Druel and H. Guenancia and thanks both of them too. He also thanks the referee for his careful reading and suggestions for making some statements more precise.

2. The smooth case

We treat this case first in order to show the steps in the general case in a simpler context.

THEOREM 2.1. — *Let X be a smooth connected complex projective manifold with $c_1(X) = 0$. There exists a finite étale cover of X, which is a product of an abelian variety with projective manifolds that are either irreducible symplectic or Calabi–Yau.*

REMARK 2.2. — The notions of irreducible symplectic and Calabi–Yau manifolds are defined as in [2]: either by the values of $h^{p,0}$, or by the holonomy of any Ricci-flat Kähler metric. We need the projectivity of X, because the Kähler version of [13] is not known. Our proof also does not show the finiteness of the fundamental groups of symplectic or Calabi–Yau manifolds. A partial solution to this finiteness property is given in Proposition 2.7 below, based on more general L^2-methods. A complete solution is also given in Proposition 2.9, but it does not (in an obvious way) extend to the singular case.

Proof of Theorem 2.1. — We equip X with any Ricci-flat Kähler metric ([26]). Let Hol^0 (or Hol) be its restricted holonomy (or holonomy) representation and $T_X = F \oplus (\bigoplus_i T_i)$ be a (local near any given point of X) splitting of the tangent bundle of X into factors that are irreducible for the action of Hol^0. These local factors also correspond to a local splitting of X into a direct product of Kähler submanifolds. In particular, these local products are regular holomorphic foliations. Here, F is the "flat" factor consisting of restricted holonomy-invariant tangent vectors. Now, Hol^0 is a normal subgroup of Hol, and Hol/Hol^0 acts by permutation on the factors of the restricted holonomy decomposition. Because the action of Hol/Hol^0 is induced by a representation $\pi_1(X) \to Hol/Hol^0$, the local holonomy decomposition of T_X above holds globally on a suitable finite étale cover of X.

We now replace X by such a finite étale cover and obtain a global product decomposition $T_X = F \oplus (\bigoplus_i T_i)$ by regular holomorphic foliations, the restricted holonomy of F being trivial, while the ones of T_i are irreducible and of the form $SU(m_i)$ or $Sp(k_i)$.

LEMMA 2.3. — *Let $T_X = \bigoplus_j E_j$ be a direct sum decomposition by foliations E_j, with $c_1(X) = 0$. Then, $c_1(E_j) = 0, \forall j$.*

Proof. — Assume not and let H be a polarization on X, with $n := dim(X)$. Then, $c_1(E_j).H^{n-1} \neq 0$, for some j. Since $\sum_j c_1(E_j).H^{n-1} = 0$, we get $c_1(E_h).H^{n-1} > 0$ for some h. It then follows from [13], Lemma 4.10, that E_h contains a subfoliation G with $\mu_{H,min}(G) > 0$ and by [13], Theorem 4.1, that K_X is not pseudo-effective, contrary to the hypothesis $c_1(T_X) = 0$.

A second, shorter, proof (suggested by the referee) consists in invoking the semistability of T_X with respect to any polarization, so that $c_1(E_j).H^{n-1} \leq 0, \forall j$. □

From the preceding Lemma 2.3, if $T_X = F \oplus (\bigoplus_i T_i)$ is the holonomy decomposition of T_X considered above for X smooth projective with $c_1(X) = 0$, we get that $c_1(F) = c_1(T_i) = 0, \forall i$.

LEMMA 2.4. — *The dual $T_i^\star$ of each T_i is not pseudo-effective (which means that for any polarization H and any given $k > 0$, $h^0(X, Sym^m(T_i^\star) \otimes H^k) = \{0\}$ for $m \geq m(k)$).*