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ATTRACTED BY AN ELLIPTIC FIXED POINT

by

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À notre mentor et ami Jean-Christophe Yoccoz

Abstract. — We give examples of symplectic diffeomorphisms of $\mathbb{R}^6$ for which the origin is a non-resonant elliptic fixed point which attracts an orbit.

***Résumé* (Attiré par un point fixe elliptique).** — Nous donnons des exemples de difféomorphismes symplectiques de $\mathbb{R}^6$ pour lesquels l'origine est un point fixe elliptique non résonant qui attire une orbite.

1. Introduction

Consider a symplectic diffeomorphism of $\mathbb{R}^{2n}$ (for the canonical symplectic form) with a fixed point at the origin. We say that the fixed point is elliptic of frequency vector $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ if the linear part of the diffeomorphism at the fixed point is conjugate to the rotation map

$$S_\omega : (\mathbb{R}^2)^n \circlearrowleft, \qquad S_\omega(s_1, \dots, s_n) \coloneqq (R_{\omega_1}(s_1), \dots, R_{\omega_n}(s_n)).$$

Here, for $\beta \in \mathbb{R}$, R_β stands for the rigid rotation around the origin in $\mathbb{R}^2$ with rotation number β. We say that the frequency vector ω is non-resonant if for any $k \in \mathbb{Z}^n - \{0\}$ we have $(k, \omega) \notin \mathbb{Z}$, where $(\cdot\,, \cdot)$ stands for the Euclidean scalar product.

It is easy to construct symplectic diffeomorphisms with orbits attracted by a resonant elliptic fixed point. For instance, the time-1 map of the flow generated by the Hamiltonian function $H(x, y) = y(x^2 + y^2)$ in $\mathbb{R}^2$ has a saddle-node type fixed point, at which the linear part is zero, which attracts all the points on the positive part of the x-axis. The situation is much subtler in the non-resonant case.

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Our goal in this paper is to construct an example of symplectic diffeomorphism with an orbit converging to an elliptic non-resonant fixed point. Note that, in such an example, the inverse symplectomorphism has a Lyapunov unstable fixed point.

The Anosov-Katok construction [**1**] of ergodic diffeomorphisms by successive conjugations of periodic rotations of the disk gives examples of smooth area preserving diffeomorphisms with non-resonant elliptic fixed points at the origin that are Lyapunov unstable. The method also yields examples of ergodic symplectomorphisms with non-resonant elliptic fixed points in higher dimensions. These constructions obtained by the successive conjugation technique have totally degenerate fixed points since they are C^∞-tangent to a rotation S_ω at the origin.

In the non-degenerate case, R. Douady gave examples in [**4**] of Lyapunov unstable elliptic points for smooth symplectic diffeomorphisms for any $n \geq 2$, for which the Birkhoff normal form has non-degenerate Hessian at the fixed point but is otherwise arbitrary. Prior examples for $n = 2$ were obtained in [**5**] (note that by KAM theory, a non-resonant elliptic fixed point of a smooth area preserving surface diffeomorphism that has a non zero Birkhoff normal form is accumulated by invariant quasi-periodic smooth curves—see [**14**]—, hence, for $n = 1$, non-degeneracy implies that the point is Lyapunov stable).

In both of the above examples, there is no claim about the existence of an orbit converging to the fixed point for the forward or backward dynamics. In fact, in the Anosov-Katok examples, a sequence of iterates of the diffeomorphism converges uniformly to Identity, hence every orbit is recurrent and no forward orbit can converge to the origin, besides the origin itself. As for the non-degenerate examples of Douady and Le Calvez, their Lyapunov instability is deduced from the existence of a sequence of points that converge to the fixed point and whose orbits travel along a simple resonance away from the fixed point, not from the existence of one particular orbit.

In this paper, we will construct an example of a Gevrey diffeomorphism possessing an orbit which converges to a fixed point.Recall that, given a real $\alpha \geq 1$, Gevrey-α regularity is defined by the requirement that the partial derivatives exist at all (multi)orders ℓ and are bounded by $CM^{|\ell|}\,|\ell|!^\alpha$ for some C and M (when $\alpha = 1$, this simply means analyticity); upon fixing a real $L > 0$ which essentially stands for the inverse of the previous M, one can define a Banach algebra $\big(G^{\alpha,L}(\mathbb{R}^{2n}), \|\,.\,\|_{\alpha,L}\big)$.

We set $X := (\mathbb{R}^2)^3$ and denote by $\mathscr{U}^{\alpha,L}$ the set of all Gevrey-(α, L) symplectic diffeomorphisms of X which fix the origin and are C^∞-tangent to Id at the origin. We refer to Appendix for the precise definition of $\mathscr{U}^{\alpha,L}$ and of a distance $\operatorname{dist}(\Phi, \Psi) = \|\Phi - \Psi\|_{\alpha,L}$ which makes it a complete metric space. We will prove the following.

Theorem A. — *Fix $\alpha > 1$ and $L > 0$. For each $\gamma > 0$, there exist a non-resonant vector $\omega \in \mathbb{R}^3$, a point $z \in X$, and a diffeomorphism $\Psi \in \mathscr{U}^{\alpha,L}$ such that $\|\Psi - \mathrm{Id}\|_{\alpha,L} \leq \gamma$ and $T = \Psi \circ S_\omega$ satisfies $T^n(z) \underset{n\to+\infty}{\longrightarrow} 0$.*

We do not know how to produce real analytic examples. After the first version of the present work was completed, the first real analytic symplectomorphisms with Lyapunov unstable non-resonant elliptic fixed points were constructed in **[6]** (but with no orbits asymptotic to the fixed point). For other instances of the use of Gevrey regularity with symplectic or Hamiltonian dynamical systems, see e.g., **[15]**, **[11]**, **[12]**, **[13]**, **[10]**, **[3]**.

Our construction easily extends to the case where $X = (\mathbb{R}^2)^n$ with $n \geq 3$, however we do not know how to adapt the method to the case $n = 2$. As for the case $n = 1$, there may well be no regular examples at all. Indeed if the rotation frequency at the fixed point is Diophantine, then a theorem by Herman (see **[7]**) implies that the fixed point is surrounded by invariant quasi-periodic circles, and thus is Lyapunov stable. The same conclusion holds by Moser's KAM theorem if the Birkhoff normal form at the origin is not degenerate **[14]**. In the remaining case of a degenerate Birkhoff normal form with a Liouville frequency, there is evidence from **[2]** that the diffeomorphism should then be rigid in the neighborhood of the origin, that is, there exists a sequence of integers along which its iterates converge to Identity near the origin, which clearly precludes the convergence to the origin of an orbit.

Similar problems can be addressed where one searches for Hamiltonian diffeomorphisms (or vector fields) with orbits whose α-limit or ω-limit have large Hausdorff dimension (or positive Lebesgue measure) and in particular contain families of non-resonant invariant Lagrangian tori instead of a single non-resonant fixed point. A specific example for Hamiltonian flows on $(\mathbb{T} \times \mathbb{R})^3$ is displayed in **[9]**, while a more generic one has been announced in **[8]**. In these examples, the setting is perturbative and the Hamiltonian flow is non-degenerate in the neighborhood of the tori. The methods involved there are strongly related to Arnold diffusion and are completely different from ours.

2. Preliminaries and outline of the strategy

From now on we fix $\alpha > 1$ and $L > 0$. We also pick an auxiliary $L_1 > L$. For $z \in \mathbb{R}^2$ and $\nu > 0$, we denote by $B(z, \nu)$ the closed ball relative to $\|\,.\,\|_\infty$ centered at z with radius ν. Since $\alpha > 1$, we have

Lemma 2.1. — *There is a real $c = c(\alpha, L_1) > 0$ such that, for any $z \in \mathbb{R}^2$ and $\nu > 0$, there exists a function $f_{z,\nu} \in G^{\alpha,L_1}(\mathbb{R}^2)$ which satisfies*

(a) $0 \leq f_{z,\nu} \leq 1$,

(b) $f_{z,\nu} \equiv 1$ *on* $B(z, \nu/2)$,

(c) $f_{z,\nu} \equiv 0$ *on* $B(z, \nu)^c$,

(d) $\|f_{z,\nu}\|_{\alpha,L_1} \leq \exp(c\,\nu^{-\frac{1}{\alpha-1}})$.

Proof. — Use Lemma 3.3 of **[12]**. □

We now fix an arbitrary real $R > 0$ and pick an auxiliary function $\eta_R \in G^{\alpha,L_1}(\mathbb{R})$ which is identically 1 on the interval $[-2R, 2R]$, identically 0 outside $[-3R, 3R]$, and everywhere non-negative. We then define $g_R:\mathbb{R}^2 \to \mathbb{R}$ by the formula

$$g_R(x,y) := xy\,\eta_R(x)\,\eta_R(y). \tag{2.1}$$

The following diffeomorphisms will be of constant use in this paper:

Definition 2.1. — *For $i \neq j \in \{1,2,3\}$, $z \in \mathbb{R}^2$ and $\nu > 0$, we denote by $\Phi_{i,j,z,\nu}$ the time-one map of the Hamiltonian flow generated by the function* $\exp(-c\,\nu^{-\frac{2}{\alpha-1}}) f_{z,\nu} \otimes_{i,j} g_R$*, where $f_{z,\nu} \otimes_{i,j} g_R : X \to \mathbb{R}$ stands for the function*

$$s = (s_1, s_2, s_3) \mapsto f_{z,\nu}(s_i) g_R(s_j).$$

In the above definition, our convention for the Hamiltonian vector field generated by a function H is $X_H = \sum(-\frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i})$. Note that the Hamiltonian $\exp(-c\,\nu^{-\frac{2}{\alpha-1}}) f_{z,\nu} \otimes_{i,j} g_R$ can be viewed as a compactly supported function of s_i and s_j, hence it generates a complete vector field and Definition 2.1 makes sense. Actually, any $H \in G^{\alpha,L_1}(X)$ has bounded partial derivatives, hence X_H is always complete; the flow of X_H is made of Gevrey maps for which estimates are given in Appendix A.2. In the case of $\Phi_{i,j,z,\nu}$, for ν small enough we have

$$\Phi_{i,j,z,\nu} \in \mathscr{U}^{\alpha,L} \quad \text{and} \quad \|\Phi_{i,j,z,\nu} - \mathrm{Id}\|_{\alpha,L} \leq K \exp(-c\,\nu^{-\frac{1}{\alpha-1}}), \tag{2.2}$$

with $K := C\|g_R\|_{\alpha,L_1}$, where C is independent from i,j,z,ν and stems from (A.6). Here are the properties which make the $\Phi_{i,j,z,\nu}$'s precious. To alleviate the notations, we state them for $\Phi_{2,1,z,\nu}$ but similar properties hold for each diffeomorphism $\Phi_{i,j,z,\nu}$.

Lemma 2.2. — *Let $z \in \mathbb{R}^2$ and $\nu > 0$. Then $\Phi_{2,1,z,\nu}$ satisfies:*

(a) *For every $(s_1,s_2,s_3) \in X$ such that $s_2 \in B(z,\nu)^c$,*

$$\Phi_{2,1,z,\nu}(s_1,s_2,s_3) = (s_1,s_2,s_3).$$

(b) *For every $x_1 \in \mathbb{R}$, $s_2 \in \mathbb{R}^2$ and $s_3 \in \mathbb{R}^2$,*

$$\Phi_{2,1,z,\nu}((x_1,0),s_2,s_3) = ((\widetilde{x}_1,0),s_2,s_3) \ \text{ with } \ |\widetilde{x}_1| \leq |x_1|.$$

(c) *For every $x_1 \in [-2R,2R]$, $s_2 \in B(z,\nu/2)$ and $s_3 \in \mathbb{R}^2$,*

$$\Phi_{2,1,z,\nu}((x_1,0),s_2,s_3) = ((\widetilde{x}_1,0),s_2,s_3) \ \text{ with } \ |\widetilde{x}_1| \leq \kappa\,|x_1|,$$

where $\kappa := 1 - \frac{1}{2}\exp(-c\,\nu^{-\frac{2}{\alpha-1}})$.

Hence, a map like $\Phi_{2,1,z_2,\nu_2}$ will preserve the x_1-axis and “descend" orbits towards the origin on this axis, while keeping the other two variables frozen (item (b)). However, it is only when the second variable is inside the ball of radius ν_2 around z_2 that $\Phi_{2,1,z_2,\nu_2}$ will effectively bring down a point of the x_1-axis towards the origin (item (c)). Let us roughly summarize this by saying that $\Phi_{2,1,z_2,\nu_2}$ acts as an elevator on the first x-axis, that never goes up and that effectively goes down when the second variable is in some given ball, that we call “activating”. Moreover, if the second variable is securely outside the activating ball, then $\Phi_{2,1,z_2,\nu_2}$ is completely inactive