ISSN 0012-9593

quatrième série - tome 54 fascicule 2 mars-avril 2021

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Rémi BOUTONNET & Adrian IOANA & Jesse PETERSON

Properly proximal groups and their von Neumann algebras

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 428 euros.
Abonnement avec supplément papier :
Europe : 576 €. Hors Europe : 657 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 54, 2021, p. 445 à 482

PROPERLY PROXIMAL GROUPS AND THEIR VON NEUMANN ALGEBRAS

BY Rémi BOUTONNET, Adrian IOANA AND Jesse PETERSON

ABSTRACT. – We introduce a wide class of countable groups, called properly proximal, which contains all non-amenable bi-exact groups, all non-elementary convergence groups, and all lattices in non-compact semi-simple Lie groups, but excludes all inner amenable groups. We show that crossed product II_1 factors arising from free ergodic probability measure preserving actions of groups in this class have at most one weakly compact Cartan subalgebra, up to unitary conjugacy. As an application, we obtain the first W^*-strong rigidity results for compact actions of $SL_d(\mathbb{Z})$ for $d \geq 3$.

RÉSUMÉ. – Nous introduisons une large classe de groupes, dits proprement proximaux, qui contient tous les groupes bi-exacts non-moyennables, les groupes de convergence non-élémentaires, et les réseaux dans des groupes de Lie semi-simples non-compacts, mais aucun groupe intérieurement moyennable. Nous montrons que les facteurs II_1 qui sont des produits croisés par des actions libres ergodiques préservant une mesure de probabilité de tels groupes ont au plus une seule sous-algèbre de Cartan compacte, à conjugaison près. Comme application, nous déduisons les premiers résultats de W^*-rigidité pour des actions compactes de $SL_n(\mathbb{Z})$.

1. Introduction

Countable groups and their measure preserving actions naturally give rise to von Neumann algebras, via two constructions of Murray and von Neumann [26, 25]. This work is motivated by the following general problem: prove structural results for the von Neumann algebras associated with the arithmetic groups $\mathrm{SL}_d(\mathbb{Z})$, $d \geq 3$, and their probability measure preserving (*p.m.p.*) actions. At present, relatively little is known in this direction. Thus, nearly all available results regarding the group von Neumann algebras $L(\mathrm{SL}_d(\mathbb{Z}))$, $d \geq 3$, are either direct consequences of property (T) [12], or concern inclusions $L\Lambda \subset L(\mathrm{SL}_d(\mathbb{Z}))$ for some subgroups $\Lambda < \mathrm{SL}_d(\mathbb{Z})$, rather than $L(\mathrm{SL}_d(\mathbb{Z}))$ itself [5]. Moreover, while several remarkable rigidity results for crossed product von Neumann algebras

A.I. was supported in part by NSF Career Grant DMS #1253402 and NSF FRG Grant #1854074. J.P. was supported in part by NSF Grants DMS #1500998, DMS #1801125, and NSF FRG Grant #1853989.

 doi:10.24033/asens.2462

associated to actions of $\mathrm{SL}_d(\mathbb{Z})$ have been obtained in [36, 37, 20, 4], these are restricted to specific classes of actions.

In contrast, the structure of von Neumann algebras associated with $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ and its actions is much better understood. Indeed, from the perspective of deformation/rigidity theory there has been a lot of work in this direction, starting with two seminal results obtained in the early 2000s. First, Popa used his deformation/rigidity theory to show that the crossed product von Neumann algebra $L^\infty(X) \rtimes \Gamma$ associated to any free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ has at most one Cartan subalgebra with the relative property (T) [35]. Second, Ozawa employed C*-algebraic techniques to prove that $L\Gamma$ is *solid*: the relative commutant, $A' \cap L\Gamma$, of any diffuse von Neumann subalgebra $A \subset L\Gamma$ is amenable [27]. These results have since been considerably strengthened, also in the context of Popa's deformation/rigidity theory, following two breakthroughs of Ozawa and Popa [31] and Popa and Vaes [40]:

1. $L\Gamma$ is *strongly solid*: the normalizer of any diffuse amenable von Neumann subalgebra $A \subset L\Gamma$ generates an amenable von Neumann algebra. Moreover, $L^\infty(X, \mu) \rtimes \Gamma$ admits $L^\infty(X, \mu)$ as its unique Cartan subalgebra, up to unitary conjugacy, for any free ergodic compact p.m.p. action $\Gamma \curvearrowright (X, \mu)$ (see [31]).
2. Γ is *$\mathcal{C}$-rigid*: $L^\infty(X, \mu) \rtimes \Gamma$ admits $L^\infty(X, \mu)$ as its unique Cartan subalgebra, up to unitary conjugacy, for any free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ (see [40]). In particular, $L^\infty(X) \rtimes \Gamma$ entirely remembers the orbit equivalence relation of the action $\Gamma \curvearrowright (X, \mu)$ [15].

Recall that a Cartan subalgebra of a tracial von Neumann algebra M is a maximal abelian subalgebra $A \subset M$ whose normalizer generates M. Proving uniqueness results for Cartan subalgebras of crossed product von Neumann algebras is of crucial importance as it allows one to reduce their classification, up to isomorphism, to the classification of the underlying actions, up to orbit equivalence. Indeed, as shown in [42], two free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are orbit equivalent precisely when there is an isomorphism $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ which identifies the Cartan subalgebras $L^\infty(X)$ and $L^\infty(Y)$.

In fact, in the last 15 years, a plethora of impressive structural results have been obtained for von Neumann algebras arising from large classes of countable groups Γ and their measure preserving actions (see [28, 38, 45, 22]). However, in most of these results, some negative curvature condition on Γ is needed, in the form of a geometric assumption (e.g., Γ is a hyperbolic group or a lattice in a rank one simple Lie group [27, 41]), or a cohomological assumption (e.g., Γ has positive first ℓ^2-Betti number, [33, 34, 10, 11, 46]), or an algebraic assumption (e.g., Γ is an amalgamated free product group, [23, 9, 39, 21]). In sharp contrast, lattices in higher rank simple Lie groups, such as $\mathrm{SL}_d(\mathbb{Z})$ for $d \geq 3$, do not satisfy any reasonable notion of negative curvature.

The results (1) and (2) were generalized in [11] and [41] to any group Γ which is both *weakly amenable* [13, 18] and *bi-exact* (equivalently, belongs to Ozawa's class $\mathcal{S}$) [43, 27, 29]. The proofs of statements (1) and (2) for such groups Γ split into two parts. First, one uses the weak amenability of Γ to deduce that any amenable subalgebra of $L(\Gamma)$ or $L^\infty(X) \rtimes \Gamma$ satisfies a certain weak compactness property ([31], see Definition 2.2). This fact is then combined with the bi-exactness of Γ to prove the desired conclusions. The weak amenability and bi-exactness properties are enjoyed by hyperbolic groups and lattices in simple Lie groups of rank one. However, both of these properties fail dramatically for lattices in higher rank simple Lie groups.

One of the main goals of this paper is to generalize the bi-exactness methods to a broader class of groups. The class of groups admitting proper cocycles into nonamenable representations was already considered in [32, Theorem A], and products of such groups were considered in [11, Section 4], however, the methods therein do not apply to general higher rank lattices such as $\mathrm{SL}_d(\mathbb{Z})$ for $d \geq 3$. The following is our first main result:

THEOREM 1.1. – *Let G be any connected semi-simple Lie group with finite center and let Γ be a lattice in G (e.g., take $\Gamma = \mathrm{SL}_d(\mathbb{Z})$ and $G = \mathrm{SL}_d(\mathbb{R})$, for $d \geq 2$). Then the von Neumann algebra of Γ does not admit a weakly compact Cartan subalgebra. Moreover, for any free ergodic p.m.p. action $\sigma : \Gamma \curvearrowright (X, \mu)$, the crossed product $L^\infty(X, \mu) \rtimes \Gamma$ admits a weakly compact Cartan subalgebra A if and only if σ is weakly compact and, in this case, A is unitarily conjugate to $L^\infty(X, \mu)$.*

Let us make several comments on the assumptions and conclusions of Theorem 1.1. A Cartan subalgebra A of a tracial von Neumann algebra M is called *weakly compact* if the inclusion $A \subset M$ is weakly compact in the sense of [31] (see Definition 2.2). A free ergodic p.m.p. action $\Lambda \curvearrowright (Y, \nu)$ is called weakly compact if $L^\infty(Y, \nu)$ is a weakly compact Cartan subalgebra of $L^\infty(Y, \nu) \rtimes \Lambda$. Note that the class of weakly compact actions contains all compact actions, and thus all profinite actions, and is closed under orbit equivalence, see [31, 19]. Recall that any ergodic compact p.m.p. action $\Lambda \curvearrowright (Y, \nu)$ is isomorphic to a left translation action $\Lambda \curvearrowright (K/K_0, m_{K/K_0})$, where K is a compact group which contains Λ as a dense subgroup, $K_0 < K$ is a closed subgroup, and m_{K/K_0} is the unique K-invariant probability measure of K/K_0.

Thus, the first assertion of Theorem 1.1 implies that $L\Gamma$ is not isomorphic to any crossed product $L^\infty(Y, \nu) \rtimes \Lambda$ arising from a compact p.m.p. action $\Lambda \curvearrowright (Y, \nu)$ of an arbitrary group. The moreover assertion implies that if a free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is W*-equivalent to a compact action $\Lambda \curvearrowright (Y, \nu)$ (in the sense that their crossed product von Neumann algebras are isomorphic, $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$) then these actions are actually orbit equivalent.

By combining Theorem 1.1 with orbit equivalence rigidity results from [47] or [19] we obtain the following corollary.

COROLLARY 1.2. – *Let $\sigma : \mathrm{SL}_d(\mathbb{Z}) \curvearrowright (X, \mu)$ and $\sigma' : \mathrm{SL}_{d'}(\mathbb{Z}) \curvearrowright (X', \mu')$ be free ergodic profinite p.m.p. actions, for some $d, d' \geq 3$. If $L^\infty(X) \rtimes \mathrm{SL}_d(\mathbb{Z})$ is isomorphic to $L^\infty(X') \rtimes \mathrm{SL}_{d'}(\mathbb{Z})$, then $d = d'$ and the actions σ and σ' are virtually conjugate.*

REMARK 1.3. – Let us discuss concrete examples.

1. For $d \geq 3$ and a non-empty set of primes $\mathscr{P}$, consider the left translation action of $\mathrm{SL}_d(\mathbb{Z})$ on the compact group $K_{d,\mathscr{P}} := \prod_{p \in \mathscr{P}} \mathrm{SL}_d(\mathbb{Z}_p)$ endowed with its Haar measure, where $\mathbb{Z}_p$ denotes the ring of p-adic integers. Corollary 1.2 implies that $L^\infty(K_{d,\mathscr{P}}) \rtimes \mathrm{SL}_d(\mathbb{Z})$ and $L^\infty(K_{d',\mathscr{P}'}) \rtimes \mathrm{SL}_{d'}(\mathbb{Z})$ are isomorphic if and only if $(d, \mathscr{P}) = (d', \mathscr{P}')$.