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BRAID GROUPS OF TYPE ADE, GARSIDE MONOIDS, AND THE CATEGORIFIED ROOT LATTICE

BY Anthony M. LICATA AND Hoel QUEFFELEC

ABSTRACT. – We study Artin-Tits braid groups $\mathbb{B}_W$ of type ADE via the action of $\mathbb{B}_W$ on the homotopy category $\mathscr{K}$ of graded projective zigzag modules (which categorifies the action of the Weyl group W on the root lattice). Following Brav-Thomas [10], we define a metric on $\mathbb{B}_W$ induced by the canonical t-structure on $\mathscr{K}$, and prove that this metric on $\mathbb{B}_W$ agrees with the word-length metric in the canonical generators of the standard positive monoid $\mathbb{B}_W^+$ of the braid group. We also define, for each choice of a Coxeter element c in W, a baric structure on $\mathscr{K}$. We use these baric structures to define metrics on the braid group, and we identify these metrics with the word-length metrics in the Birman-Ko-Lee/Bessis dual generators of the associated dual positive monoid $\mathbb{B}_{W,c}^\vee$. As consequences, we give new proofs that the standard and dual positive monoids inject into the group, give linear-algebraic solutions to the membership problem in the standard and dual positive monoids, and provide new proofs of the faithfulness of the action of $\mathbb{B}_W$ on $\mathscr{K}$. Finally, we use the compatibility of the baric and t-structures on $\mathscr{K}$ to prove a conjecture of Digne and Gobet regarding the canonical word-length of the dual simple generators of ADE braid groups.

RÉSUMÉ. – Nous étudions les groupes $\mathbb{B}_W$ des tresses d'Artin-Tits en type ADE via leur action sur $\mathscr{K}$, la catégorie d'homotopie des modules projectifs gradués sur l'algèbre zig-zag (qui catégorifie l'action du groupe de Weyl W sur le réseau des racines). En suivant Brav-Thomas [10], nous définissons une métrique sur $\mathbb{B}_W$ induite par la t-structure canonique sur $\mathscr{K}$, et nous prouvons que cette métrique sur $\mathbb{B}_W$ recouvre la longueur-mot dans les générateurs canoniques du monoïde positif standard $\mathbb{B}_W^+$. Nous définissons également, pour chaque choix d'un élément de Coxeter c dans W, une structure barique sur $\mathscr{K}$. Nous utilisons ces structures bariques pour définir des métriques sur le groupe des tresses, et nous identifions ces métriques avec les longueurs-mot dans les générateurs duaux de Birman-Ko-Lee/Bessis du monoïde positif dual associé $\mathbb{B}_{W,c}^\vee$. Partant, nous donnons de nouvelles preuves de l'injection des monoïdes positifs standard et dual dans le groupe, nous donnons des solutions basées sur de l'algèbre linéaire au problème d'appartenance aux monoïdes positifs standard et dual, et nous construisons une preuve nouvelle de la fidélité de l'action de $\mathbb{B}_W$ sur $\mathscr{K}$. Enfin, nous utilisons la compatibilité de la structure barique et de la t-structure sur $\mathscr{K}$ pour prouver une conjecture de Digne et Gobet sur la longueur-mot canonique des générateurs duaux simples dans les groupes de tresses de type ADE.

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Introduction

A basic tool in the combinatorial study of a Coxeter group W is the linear action of W on the root lattice. The first important point about the action of W on the root lattice is that it is faithful. As a result, one can address basic combinatorial and group-theoretic questions about Coxeter groups, such as the word problem, by using tools of finite-dimensional linear algebra.

The Artin-Tits braid groups $\mathbb{B}_W$ associated to W are much less well-understood than the Coxeter groups themselves; for example, in the generality of arbitrary Artin-Tits groups, the word problem is still open. Perhaps one feature of the difficulty in studying $\mathbb{B}_W$ is the lack of a clear substitute for the root lattice: the action of W on the root lattice can be q-deformed to the Burau representation of $\mathbb{B}_W$, but the Burau representation fails to be faithful outside of small rank.

In fact, there is a good candidate $\mathcal{K}$ for a "root lattice" for $\mathbb{B}_W$, though it is not a lattice but rather a triangulated category. This triangulated category, which we refer to as the categorified root lattice, has a number of explicit realizations; for example, when W is simply-laced, one can take for $\mathcal{K}$ the derived category of modules over the associated preprojective algebra. (For more general Coxeter groups, the categorified root lattice may be constructed as a quotient of the homotopy category of Soergel bimodules.) At present, the question of whether or not the braid group $\mathbb{B}_W$ acts faithfully on the categorified root lattice is still open. However, when the Coxeter group W is finite, faithfulness is known: this was proven in type A_2 by Rouquier-Zimmermann [29], in type A_n by Khovanov-Seidel [24], in type ADE by Brav-Thomas [10], and for arbitrary finite W by Jensen [23]. Faithfulness is also known in affine type A by work of Riche [28], Ishii-Ueda-Uehara [22] and Gadbled-Thiel-Wagner [18], as well as in the case of the free group by work of the first author [26]. Thus, at least in these cases, one can attempt to study $\mathbb{B}_W$ via its action on the triangulated category $\mathcal{K}$ somewhat analogously to the way one studies W via its action on the root lattice Λ.

The aim of the present paper is to take up such a study when W is a finite Weyl group of type ADE. We take as our model for $\mathcal{K}$ the homotopy category of graded modules over the zigzag algebra of the Dynkin diagram Γ. Our main goal is to explain how the homological algebra of $\mathcal{K}$ can be used to define metrics on $\mathbb{B}_W$, and then to combinatorially describe these metrics. We consider two kinds of homological decompositions of $\mathcal{K}$ into positive and negative pieces: t-structures and baric structures. These decompositions turn out to be closely related to the standard and dual Garside monoids inside $\mathbb{B}_W$.

If a group G acts on a triangulated category $\mathcal{T}$ by triangulated auto-equivalences, then one way to produce a pseudo-length function on G is as follows: first, fix a t-structure $(\mathcal{T}^{\geq 0}, \mathcal{T}^{\leq 0})$ with heart $\mathcal{T}^{0}$; given $g \in G$, there is a smallest closed interval $[a,b]$, $a,b \in \mathbb{R} \cup \{\pm\infty\}$ such that $g(\mathcal{T}^{0}) \subset \mathcal{T}^{[a,b]}$. We then define the length $l(g)$ of g to be the length of the interval $[a,b]$. In good situations, the function $d(g,h) = l(h^{-1}g)$ will be a metric on g, though if the action of G on $\mathcal{T}$ is not faithful, it will at best be a pseudo-metric. Analogous definitions give rise to metrics on G using other decompositions—such as baric structures—on $\mathcal{T}$, rather than t-structures. For groups with interesting 2-representation theory (such as Artin-Tits groups and mapping class groups of surfaces), such metrics should carry interesting geometric information about the group.

In particular, in Theorem 4.12 we identify the metric on $\mathbb{B}_W$ induced from the canonical t-structure on $\mathcal{K}$ with the canonical word-length metric on $\mathbb{B}_W$ coming from the positive lifts from the Weyl group (which are the canonical generators of the standard Garside monoid). Though we give complete independent proofs of all the statements required for the proof of Theorem 4.12, this theorem is inspired by—and is in some sense largely a rederivation of—the foundational work of Brav-Thomas on braid group actions on the derived category of a resolved Kleinian singularity [10].

Our Theorem 4.1, on the other hand, begins by choosing a Coxeter element $c \in W$, and using this choice to define a baric structure (rather than a t-structure) on $\mathcal{K}$. We use this baric structure to define a metric on $\mathbb{B}_W$; in Theorem 4.1 we identify this metric with the word-length metric on $\mathbb{B}_W$ coming from the generators of the dual Garside monoid on $\mathbb{B}_W$. Taken together, Theorems 4.1 and 4.12 explain how both Garside structures on $\mathbb{B}_W$ can be studied in a parallel fashion via the action of $\mathbb{B}_W$ on the categorified root lattice.

Theorems 4.1 and 4.12 and the constructions that precede them have several important consequences. For one, we obtain new proofs of the faithfulness of the action of $\mathbb{B}_W$ on $\mathcal{K}$ (Corollary 3.6). As another consequence, we obtain new proofs of the injectivity of the canonical map from both the standard and dual Garside monoids $\mathbb{B}_W^+$, resp. $\mathbb{B}_c^{\vee+}$ into the group $\mathbb{B}_W$ (Corollaries 3.3 and 4.7). For the standard positive monoid, this injectivity was first established by Deligne [15] and Brieskorn-Saito [11], while for the dual positive monoid it is a theorem of Birman-Ko-Lee [7] in type A and by Bessis [4] and Brady-Watt [9] more generally.

We also give a solution to the membership problem in these monoids, by showing that

— the canonical t-structure $(\mathcal{K}^{\geq 0}, \mathcal{K}^{\leq 0})$ on $\mathcal{K}$ has the property that
$$\beta(\mathcal{K}^{\geq 0}) \subset \mathcal{K}^{\geq 0} \iff \beta \in \mathbb{B}_W^+, \text{ and}$$

— the baric structure $(\mathcal{K}_{\geq 0}, \mathcal{K}_{\leq 0})$ on $\mathcal{K}$ associated to a Coxeter element $c \in W$ has the property that
$$\beta(\mathcal{K}_{\geq 0}) \subset \mathcal{K}_{\geq 0} \iff \beta \in \mathbb{B}_c^{\vee+}.$$

To check each of the conditions $\beta(\mathcal{K}^{\geq 0}) \subset \mathcal{K}^{\geq 0}$ and $\beta(\mathcal{K}_{\geq 0}) \subset \mathcal{K}_{\geq 0}$ requires computing the action of β on only finitely many objects of $\mathcal{K}$; in turn, computing the action of β on these finitely-many objects is equivalent to performing Gaussian elimination on a finite integer matrix (whose size depends on β). Thus the above criteria provides a finite algorithm for determining whether or not a fixed word in the Artin braid generators represents a positive or dual-positive braid. (Of course, computing the Garside or dual Garside normal form of a braid would also solve the above membership problem, so the point to emphasize here is the linear-algebraic nature of our solution.)

An important motivation for our work is to study both Garside structures on $\mathbb{B}_W$ using the same categorical action, in an effort to shed some light on the relationship between these two structures. In [16], Digne-Gobet conjecture that any dual simple generator β of the dual positive monoid can be written $\beta = xy^{-1}$, where x and y are both simple generators of the standard positive monoid. This conjecture provides a basic relationship between the natural metrics on $\mathbb{B}_W$ coming from the standard and dual Garside monoids. Digne-Gobet prove their conjecture in all irreducible type except for type D_n, with some of the proofs