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SYSTOLES IN TRANSLATION SURFACES

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Abstract. — For a translation surface, we define the relative systole to be the length of the shortest saddle connection. We give a characterization of the maxima of the systole function on a stratum and give a family of examples providing local but non-global maxima on each stratum of genus at least 3. We further study the relation between the (local) maxima of the systole function and the number of shortest saddle connections.

Résumé (*Systoles dans les surfaces de translation*). — Pour une surface de translation, nous définissons la systole relative comme étant la longueur d'une plus petite connexion de selles. Nous donnons une caractérisation des maxima de la fonction systole sur une strate et donnons une famille d'exemples qui sont des maxima locaux mais non globaux sur chaque strate de genre au moins trois. Nous étudions de plus des relations entre les maxima (locaux) de la fonction systole et le nombre de plus petites connexions de selles.

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1. Introduction

This paper deals with flat metrics defined by Abelian differentials on compact Riemann surfaces (*translation surfaces*). Such flat metrics have conical singularities of angle $(k+1)2\pi$, where k is the order of the zero of the corresponding Abelian differential. A stratum of the moduli space of the Abelian differentials corresponds to translation surfaces that share the same combinatorics of zeroes, possibly including marked points.

A saddle connection on a translation surface is a geodesic joining two singularities (possibly the same) and with no singularity in its interior. A sequence of area 1 translation surfaces in a stratum leaves any compact set, if and only if, the length of the shortest saddle connection tends to zero. The set of translation surfaces with short saddle connections and compactification issues of strata are related to dynamics and counting problems on translation surfaces and have been widely studied in the last 30 years (see, for instance, [9, 5, 4]).

In this paper, we are interested in the opposite problem: we study surfaces that are as far as possible from the boundary and that would represent the "core" of a stratum. For a translation surface, we define the *relative systole* $\mathrm{Sys}(S)$ to be the length of the shortest saddle connection of S. Our primary goal is to study global and local maxima of the function Sys when restricted to area 1 translation surfaces. Note that our definition is different from the "true systole", i.e., the shortest closed curve that was studied by Judge and Parlier in [8]. In the rest of the paper, for simplicity, if not mentioned otherwise, the term "systole" will mean "relative systole".

This kind of question also appears in other contexts. The maxima of the systole function for moduli spaces of hyperbolic surfaces, where the systole is the length of the shortest closed geodesic, have been studied by various authors, for instance, Bavard [2], Schmutz Schaller [13], or more recently Balacheff, Makover, and Parlier [1]. A related question is the maximal number of geodesics realizing the systole, the so-called kissing number, see, for instance, Schmutz Schaller [14], and Fanoni and Parlier [7].

In the context of area 1 translation surfaces, while the characterization of global maxima for Sys seems to have been known for some time in the mathematics community, the existence of local maxima was unknown. We provide explicit examples of local maxima that are not global in each stratum with genus $g = 2$ with marked points or $g \geq 3$. We also study the relation between the (locally) maximal values of the function Sys and the (locally) maximal number of shortest saddle connections.

The paper is organized as follows. In Section 2, we give some general background on translation surfaces. In Section 3, we study global maxima of the function Sys for area 1 translation surfaces. We prove the following theorem (see Theorem 3.3):

THEOREM. — *Let S be a genus $g \geq 1$ translation surface of area 1 and $r > 0$ singularities or marked points. Then,*

$$\mathrm{Sys}(S) \leq \left(\frac{\sqrt{3}}{2}(2g-2+r) \right)^{-\frac{1}{2}}.$$

The equality is obtained if and only if S is built with equilateral triangles whose sides are saddle connections of length $\mathrm{Sys}(S)$. *Such a surface exists in any connected component of any stratum.*

This result was independently proven recently by Judge and Parlier [8] for surfaces with one singularity; the authors are interested in the shortest closed curves, but their proof should work in any strata in our context.

In Section 4, we study the local maxima of the function Sys that are not global. With the help of explicit examples we prove the following result, which is Theorem 4.7 in the text.

THEOREM. — *Each stratum of area 1 surfaces with genus $g = 2$ with marked points or $g \geq 3$ contains local maxima of the function* Sys *that are not global.*

The examples are obtained by considering surfaces that decompose into equilateral triangles and regular hexagons, with some further conditions (see Theorem 4.1 for a precise statement).

In Section 5, we study the relation between (locally) maximal values of the function Sys and the (locally) maximal number of shortest saddle connections. We call a surface *rigid* if it corresponds to a local maximum of the number of shortest saddle connections. While the connection is clear for global maxima (see Proposition 5.1), the situation is more complex for the local maxima. The examples that we provide for local maxima of the function *Sys* are rigid. Even more, a surface that is a local maximum of the function *Sys* and that decomposes into equilateral triangles and regular hexagons must be rigid (Proposition 5.2). However, rigid surfaces are not necessarily local maxima (see Proposition 5.3).

2. Background

A *translation surface* is a (real, compact, connected) genus g surface S with a translation atlas, i.e., a triple $(S, \mathcal{U}, \Sigma)$, such that Σ is a finite subset of S (whose elements are called *singularities*) and $\mathcal{U} = \{(U_i, z_i)\}$ is an atlas of $S \setminus \Sigma$ whose transition maps are translations of $\mathbb{C} \simeq \mathbb{R}^2$. We will require that, for each $s \in \Sigma$, there is a neighborhood of s isometric to a Euclidean cone, whose total angle is a multiple of 2π. One can show that the holomorphic structure on $S \setminus \Sigma$ extends to S and that the holomorphic 1-form $\omega = dz_i$ extends to a holomorphic $1-$form on X, where Σ corresponds to the zeroes of ω and maybe some marked points. We usually call ω an *Abelian differential.*