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LANDAU DAMPING IN DYNAMICAL LORENTZ GASES

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Abstract. — We analyze Landau damping mechanism for variants of Vlasov equations, with a time-dependent linear force term and a self-consistent potential that involves an additional memory effect. This question is directly motivated by a model describing the interaction of particles with their environment, through momentum and energy exchanges with a vibrating field. We establish the stability of homogeneous states. We highlight how the coupling influences the stability criterion, in comparison to the standard Vlasov case.

Résumé (*Amortissement Landau pour des gaz de Lorentz inélastiques*). — On analyse le mécanisme de l'amortissement Landau pour certaines variantes d'équations de Vlasov, qui impliquent un terme de force linéaire dépendant du temps et un potentiel auto-consistant comportant un effet mémoire additionnel. Cette étude est directement motivée par la description de particules en interaction avec leur environnement, à travers des échanges de moment et d'énergie avec un champ de vibrations. On établit la stabilité d'états spatialement homogènes. On met ainsi en évidence comment le couplage affecte le critère de stabilité, en comparaison avec l'équation de Vlasov usuelle.

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1. Introduction

In this work, we go back to the analysis of Landau damping mechanisms in kinetic equations. This effect was highlighted for the Vlasov equation of plasma physics in the pioneering work of L. Landau [23] and extended to gravitational models in astrophysics [25, 26], where it is thought to play a key role in the stability of galaxies. It can be interpreted as a stability statement about steady solutions, leading to a decay of the self-consistent force. A complete mathematical analysis of Landau damping for nonlinear Vlasov equations was performed in [27], and revisited later in [6, 7] (see also [21]). Similar behaviors have been revealed for the 2D Euler system [5]. The phenomena are surprising since they describe damping mechanisms, counter-intuitive for *reversible* equations that apparently do not present any dissipative process.

The starting point of this contribution comes from an original model introduced by L. Bruneau and S. De Bièvre [8] describing the motion of a *single* classical particle interacting with its environment. The particle is described by its position $t \mapsto q(t) \in \mathbb{R}^d$, while the behavior of the environment is embodied into a scalar field $(t, x, z) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^n \mapsto \psi(t, x, z)$. The dynamic is modeled by the following set of differential equations

$$\begin{cases} \ddot{q}(t) = -\nabla V(q(t)) - \displaystyle\iint_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_1(q(t) - y)\ \sigma_2(z)\ \nabla_x \Psi(t, y, z)\,\mathrm{d}y\,\mathrm{d}z, \\ \partial^2_{tt}\Psi(t, x, z) - c^2 \Delta_z \Psi(t, x, z) = -\sigma_2(z)\sigma_1(x - q(t)), \quad x \in \mathbb{R}^d,\ z \in \mathbb{R}^n. \end{cases} \tag{1}$$

It corresponds to the intuition of a particle moving through an infinite set of n-dimensional elastic membranes, one for each position $x \in \mathbb{R}^d$. The physical properties of the membranes are characterized by the wave speed $c > 0$. The coupling between the particles and the environment is governed by two form functions σ_1, σ_2, which are both nonnegative, smooth, and radially symmetric functions; they can be seen as determining the influence domain of the particle in each direction, the direction of particle's motion and the direction of wave propagation, respectively. It is, therefore, relevant to assume that both form functions have a compact support. The particle exchanges its kinetic energy with the vibrations of the membranes. These mechanisms eventually act like a friction force since the particle's energy is evacuated in the membranes, and, depending on the shape of the external potential $x \mapsto V(x)$, they determine the large time behavior of the particle. We refer the reader to [1, 11, 12, 13, 22, 29] for further studies of the system (1), which include numerical experiments and interpretation by means of random walks.

The system (1) can be generalized by considering a set of N particles going through the membranes. The mean field regime $N \to \infty$ leads to the following PDE system

$$\partial_t F + v \cdot \nabla_x F - \nabla_x (V + \Phi[\Psi]) \cdot \nabla_v F = 0, \quad t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d, \tag{2a}$$

$$\big(\partial^2_{tt} \Psi - c^2 \Delta_z \Psi\big)(t,x,z) = -\sigma_2(z) \int_{\mathbb{R}^d} \sigma_1(x-y)\rho(t,y)\,\mathrm{d}y, \qquad t \geq 0, x \in \mathbb{R}^d,\ z \in \mathbb{R}^n, \tag{2b}$$

$$\rho(t,x) = \int_{\mathbb{R}^d} F(t,x,v)\,\mathrm{d}v, \tag{2c}$$

$$\Phi[\Psi](t,x) = \iint_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_1(x-y)\sigma_2(z)\Psi(t,y,z)\,\mathrm{d}z\,\mathrm{d}y, \quad t \geq 0,\ x \in \mathbb{R}^d, \tag{2d}$$

where now $(t,x,v) \mapsto F(t,x,v)$ is interpreted as the particle distribution function in phase space, $x \in \mathbb{R}^d$ being the position variable and $v \in \mathbb{R}^d$ the velocity variable. The system (2a)–(2d) is completed by initial conditions

$$F\big|_{t=0} = F_0, \qquad (\Psi, \partial_t \Psi)\big|_{t=0} = (\Psi_0, \Psi_1). \tag{3}$$

We refer the reader to [17, 31] for the derivation of the N-particles system and the analysis of the mean field regime that leads to (2a)–(2d). The existence of solutions of (2a)–(2d) is investigated in [9]. Furthermore, asymptotic issues that reveal an unexpected connection with the *gravitational* Vlasov–Poisson equation are also discussed. This relation with another model of statistical physics can guide our intuition to analyze further mathematical properties of (2a)–(2d). In this spirit, the existence of equilibrium states and their stability is discussed in [2], adding in the kinetic model a dissipative effect with the Fokker–Planck operator, and in [10], where a variational approach is adopted for the collisionless model, following [19, 20, 34].

We wish to continue this analysis, adopting a different viewpoint. In [2, 10] the effect of a confining potential $x \mapsto V(x)$ is considered, which governs the shape of the equilibrium states. Here, we change the geometry of the problem, replacing the confining assumption on the external potential, by the assumption that particles' motion holds in the $d-$dimensional torus $\mathbb{T}^d$. In such a framework, like for the usual Vlasov–Poisson system, we can find space-homogeneous stationary solutions and we wish to investigate their stability. This question is directly reminiscent of the well-known phenomena of damping highlighted in plasma physics by L. Landau [23]: for the electrostatic Vlasov–Poisson system, it can be shown that the electric field of the linearized system decays exponentially fast. For gravitational interactions a similar discussion dates back to D. Lynden–Bell [25, 26]. In fact, Landau's analysis [23] was concerned with the linearized equation only. Of course the linearization procedure is questionable, and the nonlinear dynamics might significantly depart from the linear behavior, as pointed out in [3]. A stunning analysis of the nonlinear problem