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# SMALL-TIME FLUCTUATIONS FOR THE BRIDGE OF A SUB-RIEMANNIAN DIFFUSION

BY ISMAËL BAILLEUL, LAURENT MESNAGER  
AND JAMES NORRIS

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**ABSTRACT.** – We consider small-time asymptotics for diffusion processes conditioned by their initial and final positions, under the assumption that the diffusivity has a sub-Riemannian structure, not necessarily of constant rank. We show that, if the endpoints are joined by a unique path of minimal energy, and lie outside the sub-Riemannian cut locus, then the fluctuations of the conditioned diffusion from the minimal energy path, suitably rescaled, converge to a Gaussian limit. The Gaussian limit is characterized in terms of the bicharacteristic flow, and also in terms of a second variation of the energy functional at the minimal path, the formulation of which is new in this context.

**RÉSUMÉ.** – Ce travail traite de l’asymptotique en petit temps de processus de diffusions conditionnés par leurs positions initiale et finale, sous l’hypothèse que la diffusivité a une structure sous-riemannienne, possiblement de rang non constant. On démontre que les fluctuations de la diffusion conditionnée, convenablement ré-échelonnée, convergent vers une limite gaussienne, lorsque les positions initiale et finale sont reliées par un unique chemin d’énergie minimale et sont situées hors du cut-locus sous-riemannien. La limite gaussienne est caractérisée en terme du flot bicaractéristique, ainsi qu’en terme de la variation seconde de l’énergie au voisinage d’un chemin minimal, sous une forme nouvelle dans ce contexte.

## 1. Introduction

Consider a second order differential operator on  $\mathbb{R}^d$  in Hörmander’s form <sup>(1)</sup>

$$(1) \quad \mathcal{L} = \frac{1}{2} \sum_{\ell=1}^m X_{\ell}^2 + X_0,$$

where  $X_0, X_1, \dots, X_m$  are vector fields on  $\mathbb{R}^d$ . Let us assume for now that

$$(2) \quad X_0, X_1, \dots, X_m \text{ are bounded with bounded derivatives of all orders}$$

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<sup>(1)</sup> We identify here  $X_{\ell}$  with the differential operator  $\sum_{i=1}^d X_{\ell}^i(x) \partial / \partial x^i$ .

and that  $\mathcal{L}$  satisfies the strong Hörmander condition on  $\mathbb{R}^d$ , that is to say,

$$(3) \quad \text{span}\{Y(x) : Y \in \mathcal{A}(X_1, \dots, X_m)\} = T_x \mathbb{R}^d, \quad \text{for all } x \in \mathbb{R}^d.$$

Here  $\mathcal{A}(X_1, \dots, X_m)$  denotes the smallest set of vector fields on  $\mathbb{R}^d$  containing  $X_1, \dots, X_m$  and closed under the commutator product, given by

$$[X, Y](x) = \sum_{i=1}^d X^i(x) \frac{\partial Y}{\partial x^i}(x) - Y^i(x) \frac{\partial X}{\partial x^i}(x).$$

Our main result concerns the small-time fluctuations of diffusion bridges associated to  $\mathcal{L}$ . For  $x, y \in \mathbb{R}^d$ , write  $\Omega^{x,y}$  for the set of continuous paths  $\omega : [0, 1] \rightarrow \mathbb{R}^d$  such that  $\omega_0 = x$  and  $\omega_1 = y$ . For  $\varepsilon > 0$ , denote by  $\mu_\varepsilon^{x,y}$  the law on  $\Omega^{x,y}$  of the diffusion bridge associated to  $\varepsilon \mathcal{L}$  starting from  $x$  at time 0 and ending at  $y$  at time 1.

Given an absolutely continuous path  $\omega : [0, 1] \rightarrow \mathbb{R}^d$ , it may be that there exists an absolutely continuous path  $h : [0, 1] \rightarrow \mathbb{R}^m$  such that, for almost all  $t$ ,

$$\dot{\omega}_t = \sum_{\ell=1}^m X_\ell(\omega_t) \dot{h}_t^\ell.$$

Then the energy  $I(\omega)$  may be defined by

$$(4) \quad I(\omega) = \inf \int_0^1 |\dot{h}_t|^2 dt,$$

where the infimum is taken over all such paths  $h$ . If  $\omega$  is not absolutely continuous, or there is no such path  $h$ , then we set  $I(\omega) = \infty$ .

In the case where  $x$  and  $y$  are joined by a unique path  $\gamma$  of minimal energy, we will write  $T_\gamma \Omega^{x,y}$  for the set of continuous paths  $v : [0, 1] \rightarrow T\mathbb{R}^d$  such that  $v_t \in T_{\gamma_t} \mathbb{R}^d$  for all  $t$  and  $v_0 = v_1 = 0$ . Given  $\omega \in \Omega^{x,y}$  and  $\varepsilon > 0$ , define  $\sigma_\varepsilon(\omega) \in T_\gamma \Omega^{x,y}$  by

$$\sigma_\varepsilon(\omega)_t = \frac{\omega_t - \gamma_t}{\sqrt{\varepsilon}}.$$

Then define a probability measure  $\tilde{\mu}_\varepsilon^{x,y}$  on  $T_\gamma \Omega^{x,y}$  by

$$\tilde{\mu}_\varepsilon^{x,y} = \mu_\varepsilon^{x,y} \circ \sigma_\varepsilon^{-1}.$$

The sub-Riemannian cut locus was defined by Bismut [9] in terms of the bicharacteristic flow associated to the principal symbol  $a$  of the operator  $2\mathcal{L}$ , which is given by

$$(5) \quad a(x) = \sum_{\ell=1}^m X_\ell(x) \otimes X_\ell(x).$$

This is reviewed in detail in Section 2. We can now state a version of our main result.

**THEOREM 1.1.** – *Let  $\mathcal{L}$  be a second order differential operator on  $\mathbb{R}^d$  of the form (1). Assume that  $\mathcal{L}$  satisfies conditions (2) and (3). Let  $x, y \in \mathbb{R}^d$ . Suppose that there is a unique path  $\gamma$  of minimal energy in  $\Omega^{x,y}$  and that  $(x, y)$  lies outside the cut locus. Then  $\tilde{\mu}_\varepsilon^{x,y}$  converges weakly to a Gaussian probability measure  $\mu_\gamma$  on  $T_\gamma \Omega^{x,y}$  as  $\varepsilon \rightarrow 0$ .*

The covariance of the limit measure  $\mu_\gamma$  is given in terms of the bicharacteristic flow in Section 2. Theorem 1.1 is proved in Section 3.

This theorem raises some further questions. First, once the global condition is made that  $\gamma$  has minimal energy, it is natural to hope that a suitable modification of the theorem holds under more local hypotheses. In particular, we would seek to drop the strong conditions that the underlying space is  $\mathbb{R}^d$  and that the operator coefficients are bounded with bounded derivatives of all orders. Second, in the Riemannian case, the limit Gaussian measure  $\mu_\gamma$  can be characterized in terms of the second variation of the energy function at the minimal path. This leads us to seek an analogous intrinsic object in the sub-Riemannian case. In order to address these questions, we now reset to a more general framework.

Let  $M$  be a connected  $C^\infty$  manifold of dimension  $d$  and let  $a$  be a  $C^\infty$  non-negative quadratic form on the cotangent space  $T^*M$ . We assume that  $a$  has a sub-Riemannian structure, that is to say, there exist  $m \in \mathbb{N}$  and  $C^\infty$  vector fields  $X_1, \dots, X_m$  on  $M$  such that

$$(6) \quad a(\xi, \xi) = \langle \xi, a(x)\xi \rangle = \sum_{\ell=1}^m \langle \xi, X_\ell(x) \rangle^2, \quad \xi \in T_x^*M$$

and such that

$$(7) \quad \text{span}\{Y(x) : Y \in \mathcal{A}(X_1, \dots, X_m)\} = T_x M, \quad \text{for all } x \in M.$$

There is associated to the quadratic form  $a$  an energy function  $I$  on the set of continuous paths  $\Omega = C([0, 1], M)$ . While this can be defined as in (4), the following equivalent definition makes clear that  $I$  is intrinsic to the quadratic form  $a$ . An absolutely continuous path  $\omega \in \Omega$  may have a driving path  $\xi$ , by which we mean a measurable path  $\xi$  in  $T^*M$  such that  $\xi_t \in T_{\omega_t}^*M$  and  $\dot{\omega}_t = a(\omega_t)\xi_t$  for almost all  $t$ . Then  $\omega$  has energy

$$I(\omega) = \int_0^1 \langle \xi_t, a(\omega_t)\xi_t \rangle dt.$$

If  $\omega$  is not absolutely continuous or has no driving path, then  $I(\omega) = \infty$ . Write  $H^x$  for the subset of  $\Omega$  consisting of paths of finite energy starting from  $x$ . For  $x, y \in M$ , set

$$H^{x,y} = \{\omega \in H^x : \omega_1 = y\}.$$

It is well known, under the bracket condition (7), that  $H^{x,y}$  is non-empty and that the sub-Riemannian distance

$$(8) \quad d(x, y) = \inf_{\omega \in H^{x,y}} \sqrt{I(\omega)}$$

defines a metric compatible with the topology of  $M$ .

Our main result concerns the case where  $x$  and  $y$  are chosen so that  $I$  achieves a minimum on  $H^{x,y}$  uniquely, say at  $\gamma$ . We will then construct, under a regularity condition on  $\gamma$ , a vector space  $T_\gamma H^{x,y}$  of absolutely continuous paths  $v$  in  $TM$ , with  $v_t \in T_{\gamma_t} M$  for all  $t$  and  $v_0 = v_1 = 0$ , along with an equivalence class of norms on  $T_\gamma H^{x,y}$ , each making  $T_\gamma H^{x,y}$  into a Hilbert space. The paths in  $T_\gamma H^{x,y}$  can be thought of as the infinitesimal variations of  $\gamma$  in  $H^{x,y}$ . We will further construct a continuous non-negative quadratic form  $Q$  on  $T_\gamma H^{x,y}$  such that  $Q(v)$  is the minimal second variation of  $I$  in the direction  $v$ , in a sense to be made precise. These constructions are the content of Section 5.