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A MODEL-THEORETIC APPROACH TO RIGIDITY OF STRONGLY ERGODIC, DISTAL ACTIONS

BY Tomás IBARLUCÍA AND Todor TSANKOV

Abstract. – We develop a model-theoretic framework for the study of distal factors of strongly ergodic, measure-preserving dynamical systems of countable groups. Our main result is that all such factors are contained in the (existential) algebraic closure of the empty set. This allows us to recover some rigidity results of Ioana and Tucker-Drob as well as prove some new ones: for example, that strongly ergodic, distal systems are coalescent and that every two such systems that are weakly equivalent are isomorphic. We also prove the existence of a universal distal, ergodic system that contains any other distal, ergodic system of the group as a factor.

Résumé. – Nous développons un cadre modèle-théorique pour l'étude de facteurs distaux d'actions fortement ergodiques de groupes dénombrables. Notre résultat principal est que lesdits facteurs sont contenus dans la clôture algébrique (existentielle) du vide. Cela nous permet de récupérer un théorème de rigidité de Ioana et Tucker-Drob ainsi que de démontrer de nouveaux résultats : par exemple, que les systèmes distaux fortement ergodiques sont coalescents et que si deux systèmes de ce type sont faiblement équivalents, alors ils sont isomorphes. Nous prouvons également l'existence d'un système distal ergodique universel qui contient tout système distal ergodique du groupe comme facteur.

1. Introduction

The theory of *compact* measure-preserving dynamical systems was initiated by Halmos and von Neumann, who characterized the ergodic systems of the group of integers **Z** that can be represented as translations on a compact group in terms of their spectrum. This was extended by Mackey [23] to general locally compact groups. Later, inspired by the work of Furstenberg in topological dynamics, Furstenberg [14] and Zimmer [30, 31] considered the relative notion of a *compact extension* of a system and defined a *distal* system to be one obtained via a transfinite tower of compact extensions, starting from the trivial system. The notion of a distal system was central to Furstenberg's ergodic-theoretic proof of Szemerédi's theorem, and led to the *Furstenberg-Zimmer structure theorem* for general ergodic actions of locally compact groups. In the literature, compact systems are often referred to as *isometric*

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or having *discrete* (or *pure point*) *spectrum*. Similarly, distal systems are also known as systems with *generalized discrete spectrum*. In this paper, we will use the terminology "compact" and "distal".

The notion of *weak containment* of measure-preserving systems of countable groups was introduced by Kechris [22] as a weakening of the notion of a *factor*: a system $\mathcal{X}$ is *weakly contained* in another system $\mathcal{Y}$ (notation: $\mathcal{X} \prec_w \mathcal{Y}$) if $\mathcal{X}$ is a factor of an ultrapower of $\mathcal{Y}$. Two systems $\mathcal{X}$ and $\mathcal{Y}$ are *weakly equivalent* if $\mathcal{X} \prec_w \mathcal{Y}$ and $\mathcal{Y} \prec_w \mathcal{X}$. It turns out that weak equivalence is a *smooth* equivalence relation (the set of equivalence classes is compact [1]) and that many notions in ergodic theory are invariants of weak equivalence. Free actions of infinite amenable groups are all weakly equivalent, but non-amenable groups usually admit uncountably many classes of weak equivalence [10].

By a result of Tucker-Drob [28], weak equivalence classes of free actions always contain uncountably many isomorphism classes (and they are not even classifiable by countable structures). This is why rigidity results that allow to recover the isomorphism class from the weak equivalence class of a system in special situations are particularly interesting. Two rigidity results about weak equivalence have appeared in the literature. The first one, due to Abért and Elek [2], states that if a *profinite* system $\mathcal{X}$ is weakly contained in a strongly ergodic system $\mathcal{Y}$, then $\mathcal{X}$ is a factor of $\mathcal{Y}$. This was then generalized by Ioana and Tucker-Drob [21] to distal systems. From this, they were able to conclude that for *compact*, strongly ergodic systems, weak equivalence implies isomorphism, because compact ergodic systems are *coalescent*, i.e., every endomorphism of the system is an automorphism. However, this is not true in general for distal systems (see Parry and Walters [26]). Nevertheless it turns out to be true for strongly ergodic, distal systems and we were able to prove the following.

THEOREM 1.1. – *Let Γ be a countable group.*

1. *Let $\mathcal{X}$ be a distal, strongly ergodic, probability measure-preserving Γ-system. Then $\mathcal{X}$ is coalescent and* $\mathrm{Aut}(\Gamma \curvearrowright \mathcal{X})$ *is compact.*
2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two distal, strongly ergodic, probability measure-preserving Γ-systems. If they are weakly equivalent, then they are isomorphic.*

In the course of the proof of Theorem 1.1, we also give a new proof of the rigidity result of Ioana and Tucker-Drob mentioned above. The methods we use come from continuous logic, a relatively new branch of model theory, suitable for studying metric structures. While some model-theoretic concepts, such as ultraproducts, have been used before in ergodic theory, to our knowledge, this is the first application of continuous logic to dynamics of countable groups.

The main model-theoretic notion behind the proof is that of *algebraic closure*: if M is a metric structure, $A \subseteq M$, and $b \in M$, we say that b is in the *algebraic closure* of A (notation: $b \in \mathrm{acl}^{M}(A)$) if it belongs to a compact set definable over A. Our main model-theoretic result can be stated as follows.

THEOREM 1.2. – *Let $\mathcal{X}$ be a strongly ergodic system, $\mathcal{Z}$ be a factor of $\mathcal{X}$ and $\mathcal{Y}$ be a distal extension of $\mathcal{Z}$ in $\mathcal{X}$. Then $\mathcal{Y} \subseteq \mathrm{acl}^{\mathcal{X}}(\mathcal{Z})$.*

A slight strengthening of Theorem 1.2 (replacing acl by $\mathrm{acl}_\exists$) easily implies Theorem 1.1. We provide two rather different proofs of Theorem 1.2, one based on the theory of compact extensions, and another using model-theoretic stability theory.

Another feature of our approach is that it relies, in part, on the existence of certain canonical universal systems. Those systems are usually non-separable, i.e., they cannot be realized on a standard probability space. Working on a standard space is an assumption often made in the literature, either out of habit, or for the more essential reason that many important results fail in the non-separable setting (we give several examples in this paper). We depart from this tradition and *make no separability assumptions throughout the paper unless specifically stated*. A posteriori, somewhat surprisingly, it turns out that all strongly ergodic, distal systems are separable (Corollary 5.5).

It is possible to carry out the stability-theoretic proof presented in Section 6 by only using ergodic theoretic results in a separable situation.

The following is our main result regarding universal systems.

THEOREM 1.3. – *Let Γ be a countable group and α be an ordinal. Then there exists a unique ergodic Γ-system $\mathscr{D}_\alpha(\Gamma)$ of distal rank at most α which contains every ergodic Γ-system of distal rank at most α as a factor. Moreover, $\mathscr{D}_{\omega_1}(\Gamma) = \mathscr{D}_{\omega_1+1}(\Gamma)$, that is, $\mathscr{D}_{\omega_1}(\Gamma)$ is the universal ergodic, distal Γ-system.*

An analogous result bounding the rank of a *topological* distal minimal system for $\Gamma = \mathbf{Z}$ was proved by Beleznay and Foreman [3].

For $\alpha = 1$, the existence of a universal system is well-known: the maximal ergodic, compact system of Γ is a translation on a compact group known as the *Bohr compactification of* Γ. For many groups (for example, $\mathbf{Z}$), the Bohr compactification is not metrizable, i.e., the corresponding dynamical system is not separable. However, it is a result of Wang [29] that if Γ has property (T), then its Bohr compactification is a metrizable group. We recover this result as a consequence of Theorem 1.2 and prove something more: the hierarchy of distal systems for property (T) groups collapses.

THEOREM 1.4. – *Let Γ have property (T) and $\mathscr{Z}$ be an ergodic Γ-system. Then every ergodic, distal extension of $\mathscr{Z}$ is a compact extension.*

This theorem had been previously proved by Chifan and Peterson (unpublished) but our proof is different and independent from theirs. We would like to thank them for telling us about their result.

For $\Gamma = \mathbf{Z}$, Beleznay and Foreman [4] have proved that there exist separable ergodic, distal systems of rank α for every $\alpha < \omega_1$. This implies that the tower of universal distal systems $(\mathscr{D}_\alpha(\mathbf{Z}))_{\alpha \leq \omega_1}$ is proper.

QUESTION 1.5. – Let Γ be a countable group and suppose that $\mathscr{D}_2(\Gamma)$ is not strongly ergodic. Is it true that $\mathscr{D}_\alpha(\Gamma) \subsetneq \mathscr{D}_{\alpha+1}(\Gamma)$ for all $\alpha < \omega_1$?