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Duong H. PHONG & Dat T. TÔ

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

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FULLY NON-LINEAR PARABOLIC EQUATIONS ON COMPACT HERMITIAN MANIFOLDS

BY DUONG H. PHONG AND DAT T. TÔ

ABSTRACT. – A notion of parabolic C-subsolutions is introduced for parabolic equations, extending the theory of C-subsolutions recently developed by B. Guan and more specifically G. Székelyhidi for elliptic equations. The resulting parabolic theory provides a convenient unified approach for the study of many geometric flows.

RÉSUMÉ. – Une notion de C-sous-solutions paraboliques est introduite pour les équations paraboliques, étendant la théorie des C-sous-solutions développée récemment par B. Guan et plus spécifiquement G. Székelyhidi pour les équations elliptiques. La théorie parabolique qui en résulte fournit une approche unifiée et pratique pour l'étude d'un grand nombre de flots géométriques.

1. Introduction

Subsolutions play an important role in the theory of partial differential equations. Their existence can be viewed as an indication of the absence of any global obstruction. Perhaps more importantly, it can imply crucial a priori estimates, as for example in the Dirichlet problem for the complex Monge-Ampère equation [43, 17]. However, for compact manifolds without boundary, it is necessary to extend the notion of subsolution, since the standard notion may be excluded by either the maximum principle or cohomological constraints. Very recently, more flexible and compelling notions of subsolutions have been proposed by Guan [18] and Székelyhidi [50]. In particular, they show that their notions, called C-subsolution in [50], do imply the existence of solutions and estimates for a wide variety of fully non-linear elliptic equations on Hermitian manifolds. It is natural to consider also the parabolic case. This was done by Guan, Shi, and Sui in [20] for the usual notion of subsolution and for the Dirichlet problem. We now carry this out for the more general notion of C-subsolution on compact Hermitian manifolds, adapting the methods of [18] and especially [50]. As we

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shall see, the resulting parabolic theory provides a convenient unified approach to the many parabolic equations which have been studied in the literature.

Let (X, α) be a compact Hermitian manifold of dimension n, $\alpha = i\,\alpha_{\bar k j} dz^j \wedge d\bar z^k > 0$, and $\chi(z)$ be a real $(1,1)$-form,

$$\chi = i\,\chi_{\bar k j}(z) dz^j \wedge d\bar z^k.$$

If $u \in C^2(X)$, let $A[u]$ be the matrix with entries $A[u]^k{}_j = \alpha^{k\bar m}(\chi_{\bar m j} + \partial_j \partial_{\bar m} u)$. We consider the fully nonlinear parabolic equation,

$$\partial_t u = F(A[u]) - \psi(z), \tag{1.1}$$

where $F(A)$ is a smooth symmetric function $F(A) = f(\lambda[u])$ of the eigenvalues $\lambda_j[u]$, $1 \leq j \leq n$ of $A[u]$, defined on an open symmetric, convex cone $\Gamma \subset \mathbf{R}^n$ with vertex at the origin and containing the positive orthant Γ_n. We shall assume throughout the paper that f satisfies the following conditions:

(1) $f_i > 0$ for all i, and f is concave,

(2) $f(\lambda) \to -\infty$ as $\lambda \to \partial\Gamma$,

(3) For any $\sigma < \sup_\Gamma f$ and $\lambda \in \Gamma$, we have $\lim_{t\to\infty} f(t\lambda) > \sigma$.

We shall say that a C^2 function u on X is admissible if the vector of eigenvalues of the corresponding matrix A is in Γ for any $z \in X$. Fix $T \in (0, \infty]$. To alleviate the terminology, we shall also designate by the same adjective functions in $C^{2,1}(X\times[0,T))$ which are admissible for each fixed $t \in [0,T)$. The following notion of subsolution is an adaptation to the parabolic case of Székelyhidi's [50] notion in the elliptic case:

Definition 1. – *An admissible function $\underline{u} \in C^{2,1}(X \times [0,T))$ is said to be a (parabolic) C-subsolution of (1.1), if there exist constants $\delta, K > 0$, so that for any $(z,t) \in X \times [0,T)$, the condition*

$$f(\lambda[\underline{u}(z,t)] + \mu) - \partial_t \underline{u} + \tau = \psi(z), \quad \mu + \delta I \in \Gamma_n, \quad \tau > -\delta \tag{1.2}$$

implies that $|\mu| + |\tau| < K$. Here I denotes the vector $(1, \ldots, 1)$ of eigenvalues of the identity matrix.

We shall see below (§4.1) that this notion is more general than the classical notion defined by $f(\lambda([\underline{u}])) - \partial_t \underline{u}(z,t) > \psi(z)$ and studied by Guan-Shi-Sui [20]. A C-subsolution in the sense of Székelyhidi of the equation $F(A[u]) - \psi = 0$ can be viewed as a parabolic C-subsolution of the equation (1.1) which is time-independent. But more generally, to solve the equation $F(A[u]) - \psi = 0$ by say the method of continuity, we must choose a time-dependent deformation of this equation, and we would need then a C-subsolution for each time. The heat equation (1.1) and the above notion of parabolic subsolution can be viewed as a canonical choice of deformation.

To discuss our results, we need a finer classification of non-linear partial differential operators due to Trudinger [61]. Let Γ_∞ be the projection of Γ_n onto $\mathbf{R}^{n-1}$,

$$\Gamma_\infty = \{\lambda' = (\lambda_1, \ldots, \lambda_{n-1});\ \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma \text{ for some } \lambda_n\} \tag{1.3}$$

and define the function f_∞ on Γ_∞ by

$$f_\infty(\lambda') = \lim_{\lambda_n \to \infty} f(\lambda', \lambda_n). \tag{1.4}$$

It is shown in [61] that, as a consequence of the concavity of f, the limit is either finite for all $\lambda' \in \Gamma_\infty$ or infinite for all $\lambda' \in \Gamma_\infty$. We shall refer to the first case as the *bounded case*, and to the second case as the *unbounded case*. For example, Monge-Ampère flows belong to the unbounded case, while the J-flow and Hessian quotient flows belong to the bounded case. In the unbounded case, any admissible function, and in particular 0 if $\lambda[\chi] \in \Gamma$, is a C-subsolution in both the elliptic and parabolic cases. We have then:

THEOREM 1. – *Consider the flow* (1.1)*, and assume that f is in the unbounded case. Then for any admissible initial data u_0, the flow admits a smooth solution $u(z,t)$ on* $[0,\infty)$*, and its normalization $\tilde{u}$ defined by*

$$\tilde{u} := u - \frac{1}{V}\int_X u\,\alpha^n, \qquad V = \int_X \alpha^n, \tag{1.5}$$

converges in C^∞ to a function $\tilde{u}_\infty$ satisfying the following equation for some constant c,

$$F(A[\tilde{u}_\infty]) = \psi(z) + c. \tag{1.6}$$

The situation is more complicated when f belongs to the bounded case:

THEOREM 2. – *Consider the flow* (1.1)*, and assume that it admits a subsolution $\underline{u}$ on $X \times [0,\infty)$, but that f is in the bounded case. Then for any admissible data u_0, the equation admits a smooth solution $u(z,t)$ on* $(0,\infty)$. *Let $\tilde{u}$ be the normalization of the solution u, defined as before by (1.5). Assume that either one of the following two conditions holds.*

(a) *The initial data and the subsolution satisfy*

$$\partial_t \underline{u} \geq \sup_X(F(A[u_0]) - \psi); \tag{1.7}$$

(b) *or there exists a function $h(t)$ with $h'(t) \leq 0$ so that*

$$\sup_X(u(t) - h(t) - \underline{u}(t)) \geq 0 \tag{1.8}$$

and the Harnack inequality

$$\sup_X(u(t) - h(t)) \leq -C_1 \inf_X(u(t) - h(t)) + C_2 \tag{1.9}$$

holds for some constants $C_1, C_2 > 0$ independent of time.
Then $\tilde{u}$ converges in C^∞ to a function $\tilde{u}_\infty$ satisfying (1.6) for some constant c.

The essence of the above theorems resides in the a priori estimates which are established in §2. The C^1 and C^2 estimates can be adapted from the corresponding estimates for C-subsolutions in the elliptic case, but the C^0 estimate turns out to be more subtle. Following Blocki [1] and Székelyhidi [50], we obtain C^0 estimates from the Alexandrov-Bakelman-Pucci (ABP) inequality, using this time a parabolic version of ABP due to K. Tso [62]. However, it turns out that the existence of a C-subsolution gives only partial information on the oscillation of u, and what can actually be estimated has to be formulated with some care, leading to the distinction between the cases of f bounded and unbounded, as well as Theorem 2.

The conditions (a) and especially (b) in Theorem 2 may seem impractical at first sight since they involve the initial data as well as the long-time behavior of the solution. Nevertheless, as we shall discuss in greater detail in section §4, Theorems 1 and 2 can be successfully applied to a wide range of parabolic flows on Hermitian manifolds previously studied in the literature, including the Kähler-Ricci flow, the Chern-Ricci flow, the J-flow, the Hessian flows, the