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NEW EXAMPLES OF COMPLETE CALABI-YAU METRICS ON $\mathbb{C}^n$ FOR $n \geq 3$

BY Ronan J. CONLON AND Frédéric ROCHON

Abstract. – For each $n \geq 3$, we construct on $\mathbb{C}^n$ examples of complete Calabi-Yau metrics of Euclidean volume growth having a tangent cone at infinity with singular cross-section.

Résumé. – Pour tout $n \geq 3$, nous construisons sur $\mathbb{C}^n$ des exemples de métriques de Calabi-Yau complètes de volume à croissance euclidienne pour lesquelles le cône tangent à l'infini a une section singulière.

1. Introduction

A complete Kähler manifold M is Calabi-Yau if it is Ricci-flat and has a nowhere vanishing parallel holomorphic volume form $\Omega \in H^0(M, K_M)$. On compact Kähler manifolds, Calabi-Yau metrics are unique in their Kähler class, a statement which fails to hold true on non-compact Calabi-Yau manifolds, even modulo scaling and diffeomorphisms, as the Taub-NUT metric and flat metric on $\mathbb{C}^2$ demonstrate. However, the Taub-NUT metric has cubic volume growth whereas the flat metric has Euclidean volume growth. Thus, one can ask whether uniqueness modulo scaling and diffeomorphisms can be obtained for Calabi-Yau metrics of a fixed volume growth in each Kähler class.

In this article, we are concerned with Calabi-Yau manifolds of Euclidean volume growth. These manifolds appear naturally as a rescaled limit at a singular point of a degenerating sequence of normalized Kähler-Einstein Fano manifolds, and as such, can be used to model the formation of such singularities. They have a tangent cone at infinity which is unique when there exists a tangent cone with a smooth cross section [10] and which is likely to be unique in general. Examples of such manifolds that converge smoothly to a smooth Calabi-Yau cone at infinity at a rate of $O(r^{-\epsilon})$—so-called asymptotically conical Calabi-Yau manifolds—have been constructed in [8, 9, 17, 12, 13], whereas examples with a unique singular tangent cone at infinity can be found in [23, 6, 11]. Simplifying the above problem by considering Calabi-Yau manifolds with Euclidean volume growth that have a unique tangent cone at infinity

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as in the aforementioned examples, one may ask whether the underlying complex manifold admits another Calabi-Yau metric of Euclidean volume growth in the same Kähler class as the initial metric but with a different tangent cone at infinity.

The simplest example to consider here is $\mathbb{C}^n$ endowed with the flat metric. On $\mathbb{C}^2$, Tian showed in [36] that every Calabi-Yau metric of Euclidean volume growth has to be flat and conjectured that the same should hold true on $\mathbb{C}^n$ for all $n \geq 3$. On $\mathbb{C}^3$, a counterexample to this conjecture was recently found by Yang Li [26]. The purpose of this paper is to provide a counterexample to this conjecture on $\mathbb{C}^n$ for all $n \geq 3$. Before stating our main theorem, we introduce some preliminaries.

For $n \geq 3$, let F_k be a Kähler-Einstein Fano manifold, defined as the zero locus of a homogeneous polynomial P_k in $\mathbb{CP}^{n-1}$ of degree k for $2 \leq k \leq n-1$ (of which there are many examples). Then the affine cone $V_{0,k} \subset \mathbb{C}^n$ over F_k, defined as the zero locus of P_k considered as an equation on $\mathbb{C}^n$, is a Calabi-Yau cone. With this, our main result takes the following form (see Theorem 6.2 and Corollary 6.3 for more details).

THEOREM 1.1. – *For each $n \geq 3$ and for each Kähler-Einstein Fano manifold F_k as above, there exists a complete Calabi-Yau metric on $\mathbb{C}^n$ of Euclidean volume growth with tangent cone at infinity $\mathbb{C} \times V_{0,k}$. In particular, these metrics are not isometric to the Euclidean metric on $\mathbb{C}^n$.*

REMARK 1.2. – After our preprint was posted, Gábor Székelyhidi posted a preprint [34] where he recovers this theorem using different methods. In particular, to obtain the mapping properties of the scalar Laplacian on suitable weighted Hölder spaces, he is directly inverting model operators at infinity and combining the inverses together using adapted cut-off functions.

Since the tangent cones of the Calabi-Yau metrics in Theorem 1.1 all have a *singular* cross-section, the question remains as to whether $\mathbb{C}^n$ (or indeed, any asymptotically conical Calabi-Yau manifold) admits another Calabi-Yau metric with Euclidean volume growth with a different tangent cone at infinity also having a *smooth* cross-section.

1.1. Outline of the proof of Theorem 1.1

The general philosophy for constructing Calabi-Yau manifolds with a prescribed tangent cone at infinity is to take an affine $\mathbb{C}^*$-equivariant deformation of the cone preserving the asymptotics at infinity followed by a Kähler crepant resolution (when the deformation is singular), then construct an asymptotic Calabi-Yau metric on the resulting Kähler manifold, and finally perturb this metric to a Calabi-Yau metric by solving the complex Monge-Ampère equation. Here we work on a deformation of the cone $\mathbb{C} \times V_{0,k}$ that yields $\mathbb{C}^n$ (so we do not have to take a Kähler crepant resolution since the deformation is already smooth) and our construction of an asymptotic Calabi-Yau metric is strongly inspired by an ansatz of Hans-Joachim Hein and Aaron Naber in their unpublished work [20].

Let $P_k(z_1, \ldots, z_n)$ be a homogeneous polynomial of degree k for $2 \leq k \leq n-1$ such that the hypersurface

$$F_k := \{[0 : z_1 : \cdots : z_n] \in \mathbb{CP}^{n-1} \subset \mathbb{CP}^n \mid P_k(z_1, \ldots, z_n) = 0\}$$

is smooth and admits a Kähler-Einstein metric. By [35, 33, 3], we can take for instance F_k to be a Fermat hypersurface or a smooth hypersurface sufficiently close to a Fermat hypersurface. On $\mathbb{C}^n$, we consider the singular affine variety

$$V_{0,k} := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid P_k(z_1, \dots, z_n) = 0\},$$

which by the Calabi ansatz admits a Calabi-Yau cone metric $g_{V_{0,k}}$. In terms of this Calabi-Yau cone, the prescribed tangent cone at infinity of our examples is the Cartesian product $(V_{0,k} \times \mathbb{C}, g_{V_{0,k}} \times g_{\mathbb{C}})$, where $g_{\mathbb{C}}$ is the Euclidean metric on $\mathbb{C}$. We then consider the smoothing $W_{\epsilon,k}$ of $V_{0,k} \times \mathbb{C}$ given by the equation

$$P_k(z_1, \dots, z_n) = \epsilon z_{n+1}$$

for $\epsilon \neq 0$. This smoothing, being the graph of the polynomial $\epsilon^{-1} P_k(z_1, \dots, z_n)$, is biholomorphic to $\mathbb{C}^n$. Our strategy then comprises first constructing examples of Kähler metrics on $W_{\epsilon,k}$ that are modeled on $g_{\mathbb{C}} \times g_{V_{0,k}}$ at infinity. To do this, we compactify $W_{\epsilon,k}$ in a suitable way by a manifold with corners $\mathcal{W}_{\epsilon,k}$ having two boundary hypersurfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ describing the two types of behavior of the metric at infinity. At $\mathcal{H}_2$, the metric behaves like an asymptotically conical metric (AC-metric for short), whereas near $\mathcal{H}_1$, it is modeled on a *warped* product of AC-metrics,

$$g_{\mathbb{C}} + |z_{n+1}|^{\frac{2(n-k)}{k(n-1)}} g_{V_{\epsilon,k}},$$

where $V_{\epsilon,k}$ is the smoothing of $V_{0,k}$ defined by the equation

$$P_k(z_1, \dots, z_n) = \epsilon$$

and $g_{V_{\epsilon,k}}$ can be chosen to be a Calabi-Yau AC-metric on $V_{\epsilon,k}$. Notice that if we instead consider a *Cartesian* product of AC-metrics, then this would correspond to a special case of the quasi-asymptotically conical metrics (QAC-metrics for short) introduced by Degeratu and Mazzeo [15]. For this reason, we call the metrics we consider in the present paper *warped* QAC-metrics. One other common point with QAC-metrics to highlight is that a warped QAC-metric is conformal to a Qb-metric as defined in [11, Definition 1.29], although with a different conformal factor, a useful fact that yields a simple coordinate free definition of warped QAC-metrics; see Definition 3.7 below.

For this class of metrics, we first construct examples that are Kähler with Ricci potential decaying at infinity. To apply the work of Tian-Yau [37] and get a Calabi-Yau metric, we need however to improve the decay of the Ricci potential at infinity. To do this, we need to derive good mapping properties of the Laplacian for these metrics on suitable weighted Hölder spaces, more precisely the same Hölder spaces as in [23, 15, 11], namely those associated to a Qb-metric, but with different weights. As in [15] for QAC-metrics, we do this by deriving estimates for the heat kernel and the Green's function using the work of Grigor′yan and Saloff-Coste [19]. With these mapping properties, we can then improve the decay of the Ricci potential at infinity, which ultimately allows us to solve a corresponding complex Monge-Ampère equation and obtain our new examples of complete Calabi-Yau metrics. When $n = 3$ and $k = 2$, this also gives a different proof of the result of Yang Li [26], whose methods rely instead on the fact that in this case the metric $g_{V_{\epsilon,k}}$ is the Stenzel metric, a Calabi-Yau metric having nice symmetries and admitting an explicit Kähler potential.