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LARGE SCALE CONFORMAL MAPS

BY Pierre PANSU

ABSTRACT. – Roughly speaking, let us say that a map between metric spaces is *large-scale conformal* if it maps packings by large balls to large quasi-balls with limited overlaps. This quasi-isometry invariant notion makes sense for finitely generated groups. Inspired by work by Benjamini and Schramm, we show that under such maps, some kind of dimension increases: exponent of polynomial volume growth for nilpotent groups, conformal dimension of the ideal boundary for hyperbolic groups. A purely metric space notion of ℓ^p-cohomology plays a key role.

RÉSUMÉ. – Grosso modo, une application entre espaces métriques est *conforme à grande échelle* si elle envoie tout empilement de grandes boules sur une collection de grandes quasi-boules qui ne se chevauchent pas trop. Cette notion est un invariant de quasi-isométrie, elle s'étend aux groupes de type fini. En s'inspirant de travaux de Benjamini et Schramm, on montre qu'en présence d'une telle application, une sorte de dimension doit augmenter : il s'agit de l'exposant de croissance polynômiale du volume pour les groupes nilpotents, de la dimension conforme du bord pour les groupes hyperboliques. Une nouvelle définition, purement métrique, de la cohomologie ℓ^p joue un rôle important.

1. Introduction

1.1. Microscopic conformality

Examples of conformal mappings arose pretty early in history: the stereographic projection, which is used in astrolabes, was known to ancient Greece. The metric distorsion of a conformal mapping can be pretty large. For instance, the Mercator planisphere (1569) is a conformal mapping of the surface of a sphere with opposite poles removed onto an infinite cylinder. Its metric distorsion (Lipschitz constant) blows up near the poles, as everybody knows. Nevertheless, in many circumstances, it is possible to estimate metric distorsion, and

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this lead in the last century to metric space analogues of conformal mappings, known as quasi-symmetric or quasi-Möbius maps. Grosso modo, these are homeomorphisms which map balls to quasi-balls. Quasi means that the image $f(B)$ is jammed between two concentric balls, $B \subset f(B) \subset \ell B$, with a uniform ℓ, independent of location or radius of B.

1.2. Mesoscopic conformality

Some evidence that conformality may manifest itself in a discontinuous space shows up with Koebe's 1931 circle packing theorem. A circle packing of the 2-sphere is a collection of interior-disjoint disks. The incidence graph of the packing has one vertex for each circle and an edge between vertices whenever corresponding circles touch. Koebe's theorem states that every triangulation of the 2-sphere is the incidence graph of a disk packing, unique up to Möbius transformations. Thurston conjectured that triangulating a planar domain Ω with a portion of the incidence graph of the standard equilateral disk packing, and applying Koebe's theorem to it, one would get a numerical approximation to Riemann's conformal mapping of Ω to the round disk. This was proven by Rodin and Sullivan, [33], in 1987. This leads us to interpret Koebe's circle packing theorem as a mesoscopic analogue of Riemann's conformal mapping theorem.

1.3. A new class of maps

In this paper, we propose to go one step further and define a class of large-scale conformal maps. Roughly speaking, large scale means that our definitions are unaffected by local changes in metric or topology. Technically, it means that pre- or post-composition of large-scale conformal maps with quasi-isometries are again large-scale conformal. This allows to transfer some techniques and results of conformal geometry to discrete spaces like finitely generated groups, for instance.

1.4. Examples

In first approximation, a map between metric spaces is large-scale conformal if it maps every packing by sufficiently large balls to a collection of large quasi-balls which can be split into the union of boundedly many packings. We postpone till next section the rather technical formal definition. Here are a few sources of examples.

- Quasi-isometric embeddings are large-scale conformal.
- Snowflaking (i.e., replacing a metric by a power of it) is large-scale conformal.
- Power maps $z \mapsto z|z|^{K-1}$ are large-scale conformal for $K \geq 1$. They are not quasi-isometric, nor even coarse embeddings.
- Compositions of large-scale conformal maps are large-scale conformal.

For instance, every nilpotent Lie or finitely generated group can be large-scale conformally embedded in a Euclidean space of sufficiently high dimension, [1]. Every hyperbolic group can be large-scale conformally embedded in a hyperbolic space of sufficiently high dimension, [4].

1.5. Results

Our first main result is that a kind of dimension increases under large-scale conformal maps. The relevant notion depends on classes of groups.

THEOREM 1. – *If G is a finitely generated or Lie nilpotent group, set $d_1(G) = d_2(G) =$ the exponent of volume growth of G. If G is a finitely generated or Lie hyperbolic group, let $d_1(G) := \mathrm{CohDim}(G)$ be the infimal p such that the ℓ^p-cohomology of G does not vanish. Let $d_2(G) := \mathrm{ConfDim}(\partial G)$ be the Ahlfors-regular conformal dimension of the ideal boundary of G.*

Let G and G' be nilpotent or hyperbolic groups. If there exists a large-scale conformal map $G \to G'$, then $d_1(G) \leq d_2(G')$.

Theorem 1 is a large scale version of a result of Benjamini and Schramm, [3], concerning packings in $\mathbb{R}^d$. The proof follows the same general lines but differs in details. The result is not quite sharp in the hyperbolic group case, since it may happen that $d_1(G) < d_2(G)$, [8]. However equality $d_1(G) = d_2(G)$ holds for Lie groups, their lattices and also for a few other finitely generated examples.

Our second result is akin to the fact that maps between geodesic metric spaces which are uniform/coarse embeddings in both directions must be quasi-isometries.

THEOREM 2. – *Let X and X' be bounded geometry manifolds or polyhedra. Assume that X and X' have isoperimetric dimension > 1. Every homeomorphism $f : X \to X'$ such that f and f^{-1} are large-scale conformal is a quasi-isometry.*

Isoperimetric dimension is defined in Subsection 8.3. Examples of manifolds or polyhedra with isoperimetric dimension > 1 include universal coverings of compact manifolds or finite polyhedra whose fundamental group is not virtually cyclic.

This is a large scale version of the fact that every quasiconformal diffeomorphism of hyperbolic space is a quasi-isometry. This classical result, [20], generalizes to Riemannian n-manifolds whose isoperimetric dimension is $> n$, [29]. The new feature of the large scale version is that isoperimetric dimension > 1 suffices.

1.6. Proof of Theorem 1

Our main tool is a metric space avatar of energy of functions, and, more generally, norms on cocycles giving rise to L^p-cohomology. Whereas, on Riemannian n-manifolds, only n-energy $\int |\nabla u|^n$ is conformally invariant, all p-energies turn out to be large-scale conformal invariants. Again, we postpone the rather technical definitions to Section 5 and merely give a rough sketch of the arguments.

Say a locally compact metric space is p-parabolic if for all (or some) point o, there exist compactly supported functions taking value 1 at o, of arbitrarily small p-energy, [38]. For instance, a nilpotent Lie or finitely generated group G is p-parabolic iff $p \geq d_1(G)$. Non-elementary hyperbolic groups are never p-parabolic. If X has a large-scale conformal embedding into X' and X' is p-parabolic, so is X. This proves Theorem 1 for nilpotent targets.