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Vincent GUEDJ & Chinh H. LU & Ahmed ZERIAHI

*The Pluripotential Cauchy-Dirichlet problem  
for complex Monge-Ampère flows*

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Annales Scientifiques de l'École Normale Supérieure,  
45, rue d'Ulm, 75230 Paris Cedex 05, France.  
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.  
Email : [annaes@ens.fr](mailto:annaes@ens.fr)

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Société Mathématique de France  
Case 916 - Luminy  
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Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51  
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# THE PLURIPOTENTIAL CAUCHY-DIRICHLET PROBLEM FOR COMPLEX MONGE-AMPÈRE FLOWS

BY VINCENT GUEDJ, CHINH H. LU AND AHMED ZERIAHI

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**ABSTRACT.** – We develop the first steps of a parabolic pluripotential theory in bounded strongly pseudo-convex domains of  $\mathbb{C}^n$ . We study certain degenerate parabolic complex Monge-Ampère equations, modeled on the Kähler-Ricci flow evolving on complex algebraic varieties with Kawamata log-terminal singularities. Under natural assumptions on the Cauchy-Dirichlet boundary data, we show that the envelope of pluripotential subsolutions is semi-concave in time and continuous in space, and provides the unique pluripotential solution with such regularity.

**RÉSUMÉ.** – Nous développons une théorie pluripotentielle parabolique sur un domaine strictement pseudo-convexe borné de  $\mathbb{C}^n$ . Nous étudions certaines équations de Monge-Ampère complexes paraboliques dégénérées, modélisées sur le flot de Kähler-Ricci sur les variétés algébriques complexes à singularités Kawamata log-terminales. Sous des hypothèses naturelles sur les données de Cauchy-Dirichlet, nous montrons que l'enveloppe des sous-solutions pluripotentielles est semi-concave en temps et continue en espace, et qu'elle est l'unique solution pluripotentielle avec une telle régularité.

## Introduction

The Ricci flow, first introduced by Hamilton [18] is the equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

evolving a Riemannian metric by its Ricci curvature. If the Ricci flow starts from a Kähler metric, the evolving metrics remain Kähler and the resulting PDE is called the Kähler-Ricci flow.

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It is expected that the Kähler-Ricci flow can be used to give a geometric classification of complex algebraic and Kähler manifolds, and produce canonical metrics at the same time. Solving the Kähler-Ricci flow boils down to solving a parabolic scalar equation modeled on

$$\det \left( \frac{\partial^2 u_t}{\partial z_j \partial \bar{z}_k} (t, z) \right) = e^{\partial_t u_t(z) + H(t, z) + \lambda u_t(z)}$$

where  $t \mapsto u_t(z) = u(t, z)$  is a smooth family of strictly plurisubharmonic functions in  $\mathbb{C}^n$ ,  $\lambda \in \mathbb{R}$  and  $g = e^H$  is a smooth and positive density.

It is important for geometric applications to study *degenerate* versions of these complex Monge-Ampère flows, where the functions  $u_t$  are no longer smooth nor strictly plurisubharmonic, and the densities may vanish or blow up (see [30, 4, 29, 11] and the references therein).

A viscosity approach has been developed recently in [10], following its elliptic counterpart [9, 19, 20]. While the viscosity theory is very robust, it requires the data to be continuous hence has a limited scope of applications. Several geometric situations encountered in the Minimal Model program (MMP) necessitate one to deal with Kawamata log-terminal (klt) singularities. The viscosity approach breaks down in these cases and a more flexible method is necessary.

There is a well established pluripotential theory of weak solutions to degenerate elliptic complex Monge-Ampère equations, following the pioneering work of Bedford-Taylor [1, 2]. This theory allows to deal with  $L^p$ -densities as established in a corner stone result of Kołodziej [25], which provides a great generalization of [32].

No similar theory has ever been developed on the parabolic side. The purpose of this article, the first of a series on this subject, is to develop a pluripotential theory for degenerate complex Monge-Ampère flows. This article settles the foundational material for this theory and focuses on solving the Cauchy-Dirichlet problem in domains of  $\mathbb{C}^n$ .

We consider the following family of Monge-Ampère flows

$$(CMAF) \quad dt \wedge (dd^c u)^n = e^{\partial_t u + F(t, z, u)} g(z) dt \wedge dV,$$

in  $\Omega_T := ]0, T[ \times \Omega$ , where  $dV$  is the Euclidean volume form on  $\mathbb{C}^n$  and

- $T > 0$  and  $\Omega \Subset \mathbb{C}^n$  is a bounded strictly pseudoconvex domain;
- $F(t, z, r)$  is continuous in  $[0, T[ \times \Omega \times \mathbb{R}$ , increasing in  $r$ , bounded in  $[0, T[ \times \Omega \times J$ , for each  $J \Subset \mathbb{R}$ ;
- $(t, r) \mapsto F(t, \cdot, r)$  is uniformly Lipschitz and semi-convex in  $(t, r)$ ;
- $g \in L^p(\Omega)$ ,  $p > 1$ , and  $g > 0$  almost everywhere ;
- $u : [0, T[ \times \Omega \rightarrow \mathbb{R}$  is the unknown function.

Here  $d = \partial + \bar{\partial}$  and  $d^c = i(\bar{\partial} - \partial)/2$  so that  $dd^c = i\partial\bar{\partial}$  and  $(dd^c u)^n$  represents the determinant of the complex Hessian of  $u$  in space (the complex Monge-Ampère operator) whenever  $u$  is  $\mathcal{C}^2$ -smooth.

For less regular functions  $u$ , the Equation (CMAF) should be understood in the weak sense of pluripotential theory as we explain in Section 2.

We let  $\mathcal{P}(\Omega_T)$  denote the set of *parabolic potentials*, i.e., those functions  $u : \Omega_T \rightarrow [-\infty, +\infty[$  defined in  $\Omega_T = ]0, T[ \times \Omega$  and satisfying the following conditions:

- for any  $t \in ]0, T[$ ,  $u(t, \cdot)$  is plurisubharmonic in  $\Omega$ ;
- the family  $\{u(\cdot, z) ; z \in \Omega\}$  is locally uniformly Lipschitz in  $]0, T[$ .

We study in Section 1 basic properties of parabolic potentials. We show in Lemma 1.6 that if  $u \in \mathcal{P}(\Omega_T)$  and is bounded from above in  $\Omega_T$  then it can be uniquely extended as an upper-semicontinuous function in  $[0, T[ \times \Omega$  such that  $u(0, \cdot)$  is plurisubharmonic in  $\Omega$ . We show that parabolic potentials satisfy approximate submean-value inequalities (Lemma 1.8) and enjoy good compactness properties (Proposition 1.17).

We show in Section 2 that parabolic complex Monge-Ampère operators are well defined on  $\mathcal{P}(\Omega_T) \cap L^\infty_{\text{loc}}(\Omega_T)$  and enjoy nice continuity properties, allowing to make sense of pluripotential sub/super/solutions to (CMAF) (see Definition 3.1). A crucial convergence property is obtained in Proposition 2.9, under a semi-concavity assumption on the family of parabolic potentials.

A *Cauchy-Dirichlet boundary data* is a function  $h$  defined on the parabolic boundary of  $\Omega_T$  denoted by

$$\partial_0 \Omega_T := ([0, T[ \times \partial\Omega) \cup (\{0\} \times \Omega),$$

such that

- the restriction of  $h$  on  $[0, T[ \times \partial\Omega$  is continuous;
- the family  $\{h(\cdot, z) ; z \in \partial\Omega\}$  is locally uniformly Lipschitz in  $]0, T[$ ;
- $h$  satisfies the following compatibility condition :  $\forall \zeta \in \partial\Omega$ ,

$$(0.1) \quad h_0 := h(0, \cdot) \in \text{PSH}(\Omega) \cap L^\infty(\Omega) \text{ and } \lim_{\Omega \ni z \rightarrow \zeta} h(0, z) = h(0, \zeta).$$

The Cauchy-Dirichlet problem for the parabolic Equation (CMAF) with Cauchy-Dirichlet boundary data  $h$  consists in finding  $u \in \mathcal{P}(\Omega_T) \cap L^\infty(\Omega_T)$  such that (CMAF) holds in the pluripotential sense in  $\Omega_T$  and the following Cauchy-Dirichlet boundary conditions are satisfied :

$$(0.2) \quad \forall (\tau, \zeta) \in [0, T[ \times \partial\Omega, \quad \lim_{\Omega_T \ni (t, z) \rightarrow (\tau, \zeta)} u(t, z) = h(\tau, \zeta),$$

$$(0.3) \quad \lim_{t \rightarrow 0^+} u_t = h_0 \text{ in } L^1(\Omega).$$

In this case we say that  $u$  is a solution to the Cauchy-Dirichlet problem for the Equation (CMAF) with boundary values  $h$ .

Observe that a solution  $u$  to the Equation (CMAF) has plurisubharmonic slices in  $\Omega$  and the Cauchy condition (0.3) implies by a classical result in pluripotential theory that  $(\limsup_{t \rightarrow 0} u_t)^* = h_0^* \in \text{PSH}(\Omega)$ , hence  $h_0 = h_0^* \in \text{PSH}(\Omega)$ . This observation shows that the Cauchy data  $h_0$  must be plurisubharmonic as it is required in the compatibility condition (0.1).

For a solution to the Cauchy-Dirichlet problem for the Equation (CMAF), the Cauchy condition (0.3) implies that

$$\forall z \in \Omega, \quad \lim_{t \rightarrow 0^+} u_t(z) = h_0(z).$$

It is possible to consider less regular initial Cauchy data  $h(0, \cdot)$  (see [28, 27]), but we will not pursue this here.