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APPLICATIONS OF THE MORAVA K-THEORY TO ALGEBRAIC GROUPS

BY PAVEL SECHIN AND NIKITA SEMENOV

ABSTRACT. – In this article we discuss an approach to cohomological invariants of algebraic groups based on the Morava K-theories.

We show that the second Morava K-theory detects the triviality of the Rost invariant and, more generally, relate the triviality of cohomological invariants and the splitting of Morava motives.

We compute the Morava K-theory of generalized Rost motives and of some affine varieties and characterize the powers of the fundamental ideal of the Witt ring with the help of the Morava K-theory. Besides, we obtain new estimates on torsion in Chow groups of quadrics and investigate torsion in Chow groups of $K(n)$-split varieties. An important role in the proofs is played by the gamma filtration on Morava K-theories, which gives a conceptual explanation of the nature of the torsion.

Furthermore, we show that under some conditions if the $K(n)$-motive of a smooth projective variety splits, then its $K(m)$-motive splits for all $m \leq n$.

RÉSUMÉ. – Dans cet article nous présentons une approche des invariants cohomologiques des groupes algébriques basée sur les K-théories de Morava.

Nous montrons que la deuxième K-théorie de Morava détecte la trivialité de l'invariant de Rost et, plus généralement, établissons un rapport entre la trivialité des invariants cohomologiques et le déploiement des motifs de Morava.

Nous calculons la K-théorie de Morava des motifs généralisés de Rost et de quelques variétés affines et caractérisons les puissances de l'idéal fondamental de l'anneau de Witt à l'aide de la K-théorie de Morava. Par ailleurs, nous obtenons de nouvelles estimations de la torsion dans les groupes de Chow des quadriques et étudions la torsion dans les groupes de Chow des variétés $K(n)$-déployées. La gamma-filtration de K-théorie de Morava joue un rôle important dans les preuves, et fournit une explication conceptuelle de la nature de la torsion.

De plus, nous montrons que sous certaines conditions, si le $K(n)$-motif d'une variété projective lisse est déployé, alors son $K(m)$-motif est déployé pour tout $m \leq n$.

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1. Introduction

The present article is devoted to applications of the Morava K-theory to cohomological invariants of algebraic groups and to computations of the Chow groups of quadrics.

1.1. Cohomological invariants

In his celebrated article on irreducible representations [55] Jacques Tits introduced the notion of a Tits algebra, which is an example of cohomological invariants of algebraic groups of degree 2. This invariant of a linear algebraic group G plays a crucial role in the computation of the K-theory of twisted flag varieties by Panin [39] and in the index reduction formulas by Merkurjev, Panin and Wadsworth [34]. It has important applications to the classification of linear algebraic groups and to the study of associated homogeneous varieties.

The idea to use cohomological invariants in the classification of algebraic groups goes back to Jean-Pierre Serre. In particular, Serre conjectured the existence of an invariant of degree 3 for groups of type F_4 and E_8. This invariant was later constructed by Markus Rost for all G-torsors, where G is a simple simply-connected algebraic group, and is now called the Rost invariant (see [9]).

Moreover, the Serre-Rost conjecture for groups of type F_4 says that the map

$$H^1_{\mathrm{et}}(F,\mathrm{F}_4) \hookrightarrow H^3_{\mathrm{et}}(F,\mathbb{Z}/2) \oplus H^3_{\mathrm{et}}(F,\mathbb{Z}/3) \oplus H^5_{\mathrm{et}}(F,\mathbb{Z}/2)$$

induced by the invariants f_3, g_3 and f_5 described in [23, §40] (f_3 and g_3 are the modulo 2 and modulo 3 components of the Rost invariant), is injective. The validity of the Serre-Rost conjecture would imply that one can exchange the study of the *set* $H^1_{\mathrm{et}}(F,\mathrm{F}_4)$ of isomorphism classes of groups of type F_4 over F (equivalently of isomorphism classes of F_4-torsors or of isomorphism classes of Albert algebras) by the study of the *abelian group* $H^3_{\mathrm{et}}(F,\mathbb{Z}/2) \oplus H^3_{\mathrm{et}}(F,\mathbb{Z}/3) \oplus H^5_{\mathrm{et}}(F,\mathbb{Z}/2)$.

In the same spirit one can formulate the Serre Conjecture II, saying in particular that $H^1_{\mathrm{et}}(F,\mathrm{E}_8) = 1$ if the field F has cohomological dimension 2. Namely, for such fields $H^n_{\mathrm{et}}(F,M) = 0$ for all $n \geq 3$ and all torsion modules M. In particular, for groups over F there are no invariants of degree ≥ 3, and the Serre Conjecture II predicts that the groups of type E_8 over F themselves are split.

Furthermore, the Milnor conjecture on quadratic forms (proven by Orlov, Vishik and Voevodsky) together with the Milnor conjecture on the étale cohomology (proven by Voevodsky) provides a classification of quadratic forms over fields in terms of the Galois cohomology, i.e., in terms of cohomological invariants.

In the present article we will relate the Morava K-theory with some cohomological invariants of algebraic groups.

1.2. Morava K-theory and Morava motives

Let n be a positive integer and let p be a prime. The Morava K-theory $K(n)^*$ is a free oriented cohomology theory in the sense of Levine-Morel [28] whose coefficient ring is $\mathbb{Z}_{(p)}$, whose formal group law modulo p has height n, and the logarithm is of the type

$$\log_{K(n)}(x) = x + \frac{a_1}{p}x^{p^n} + \frac{a_2}{p^2}x^{p^{2n}} + \cdots$$

with $a_i \in \mathbb{Z}_{(p)}^\times$. If $n = 1$ and all a_i are equal to 1, then the theory $K(1)^*$ is isomorphic to Grothendieck's $K^0 \otimes \mathbb{Z}_{(p)}$ as a presheaf of rings. Moreover, there is some kind of analogy between Morava K-theory in general and K^0.

More conceptually, algebraic cobordism of Levine-Morel can be considered as a functor to the category of graded comodules over the Hopf algebroid $(\mathbb{L}, \mathbb{L}B)$, where $\mathbb{L}$ is the Lazard ring and $\mathbb{L}B = \mathbb{L}[b_1, b_2, \ldots]$. This Hopf algebroid parametrizes the groupoid of formal group laws with strict isomorphisms between them, and the category of comodules over it can be identified with the category of quasi-coherent sheaves over the stack of formal groups $\mathcal{M}_{\mathrm{fg}}$. This stack modulo p has a descending filtration by closed substacks $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ which classify the formal group laws of height $\geq n$. Moreover, $\mathcal{M}_{\mathrm{fg}}^{\geq n} \setminus \mathcal{M}_{\mathrm{fg}}^{\geq n+1}$ has an essentially unique geometric point which corresponds to the Morava K-theory $K(n)^* \otimes \overline{\mathbb{F}}_p$. This chromatic picture puts $K(n)^*$ into an intermediate position between K^0 and CH^*.

We remark also that Levine and Tripathi construct in [29] a higher Morava K-theory in algebraic geometry.

1.3. Morava K-theories, split motives and vanishing of cohomological invariants

There are three different types of results in this article which fit into the following guiding principle. The leading idea of this principle has been probably well understood already by Voevodsky, since he considered the Morava K-theory in his program on the proof of the Bloch-Kato conjecture in [63].

GUIDING PRINCIPLE. – *Let X be a projective homogeneous variety, let p be a prime number and let $K(n)^*$ denote the corresponding Morava K-theory.*

Then vanishing of cohomological invariants of X with p-torsion coefficients in degrees no greater than $n+1$ should correspond to the splitting of the $K(n)^*$-motive of X.

First of all, due to the Milnor conjecture the associated graded ring of the Witt ring $W(F)$ of a field F of characteristic not 2 is canonically isomorphic to the étale cohomology of the base field with $\mathbb{Z}/2$-coefficients: $H^n_{\mathrm{et}}(F, \mathbb{Z}/2) \simeq I^n/I^{n+1}$, where I denotes the fundamental ideal of $W(F)$. Therefore, the projective quadric which corresponds to a quadratic form $q \in I^n$ has a canonical cohomological invariant of degree n. The guiding principle suggests that the $K(n)^*$-motive of an even-dimensional projective quadric is split if and only if the class of the corresponding quadratic form in the Witt ring lies in the ideal I^{n+2}. Indeed, we prove this statement in Proposition 6.18.

Secondly, we relate cohomological invariants of simple algebraic groups to Morava K-theories. We show in Section 9 that for a simple simply-connected group G with trivial Tits algebras the Morava K-theory $K(2)^*$ detects the triviality of the Rost invariant of G. Note that in a similar spirit Panin showed in [39] that the Grothendieck's K^0 detects the triviality of Tits algebras. Moreover, for a group G of type E_8 the Morava K-theory $K(4)^*$ for $p = 2$ detects the splitting of the variety of Borel subgroups of G over a field extension of odd degree (Theorem 9.1). All these results agree with the guiding principle.

Thirdly, we relate the property of being split with respect to Morava K-theories $K(n)^*$ for different n. Namely, we prove in Proposition 7.10 that if a smooth projective geometrically cellular variety X over a field F of characteristic 0 satisfies the Rost nilpotence principle for