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LARGE SCALE CONFORMAL MAPS

BY Pierre PANSU

Abstract. – Roughly speaking, let us say that a map between metric spaces is *large-scale conformal* if it maps packings by large balls to large quasi-balls with limited overlaps. This quasi-isometry invariant notion makes sense for finitely generated groups. Inspired by work by Benjamini and Schramm, we show that under such maps, some kind of dimension increases: exponent of polynomial volume growth for nilpotent groups, conformal dimension of the ideal boundary for hyperbolic groups. A purely metric space notion of ℓ^p-cohomology plays a key role.

Résumé. – Grosso modo, une application entre espaces métriques est *conforme à grande échelle* si elle envoie tout empilement de grandes boules sur une collection de grandes quasi-boules qui ne se chevauchent pas trop. Cette notion est un invariant de quasi-isométrie, elle s'étend aux groupes de type fini. En s'inspirant de travaux de Benjamini et Schramm, on montre qu'en présence d'une telle application, une sorte de dimension doit augmenter : il s'agit de l'exposant de croissance polynômiale du volume pour les groupes nilpotents, de la dimension conforme du bord pour les groupes hyperboliques. Une nouvelle définition, purement métrique, de la cohomologie ℓ^p joue un rôle important.

1. Introduction

1.1. Microscopic conformality

Examples of conformal mappings arose pretty early in history: the stereographic projection, which is used in astrolabes, was known to ancient Greece. The metric distorsion of a conformal mapping can be pretty large. For instance, the Mercator planisphere (1569) is a conformal mapping of the surface of a sphere with opposite poles removed onto an infinite cylinder. Its metric distorsion (Lipschitz constant) blows up near the poles, as everybody knows. Nevertheless, in many circumstances, it is possible to estimate metric distorsion, and

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this lead in the last century to metric space analogues of conformal mappings, known as quasi-symmetric or quasi-Möbius maps. Grosso modo, these are homeomorphisms which map balls to quasi-balls. Quasi means that the image $f(B)$ is jammed between two concentric balls, $B \subset f(B) \subset \ell B$, with a uniform ℓ, independent of location or radius of B.

1.2. Mesoscopic conformality

Some evidence that conformality may manifest itself in a discontinuous space shows up with Koebe's 1931 circle packing theorem. A circle packing of the 2-sphere is a collection of interior-disjoint disks. The incidence graph of the packing has one vertex for each circle and an edge between vertices whenever corresponding circles touch. Koebe's theorem states that every triangulation of the 2-sphere is the incidence graph of a disk packing, unique up to Möbius transformations. Thurston conjectured that triangulating a planar domain Ω with a portion of the incidence graph of the standard equilateral disk packing, and applying Koebe's theorem to it, one would get a numerical approximation to Riemann's conformal mapping of Ω to the round disk. This was proven by Rodin and Sullivan, [33], in 1987. This leads us to interpret Koebe's circle packing theorem as a mesoscopic analogue of Riemann's conformal mapping theorem.

1.3. A new class of maps

In this paper, we propose to go one step further and define a class of large-scale conformal maps. Roughly speaking, large scale means that our definitions are unaffected by local changes in metric or topology. Technically, it means that pre- or post-composition of large-scale conformal maps with quasi-isometries are again large-scale conformal. This allows to transfer some techniques and results of conformal geometry to discrete spaces like finitely generated groups, for instance.

1.4. Examples

In first approximation, a map between metric spaces is large-scale conformal if it maps every packing by sufficiently large balls to a collection of large quasi-balls which can be split into the union of boundedly many packings. We postpone till next section the rather technical formal definition. Here are a few sources of examples.

- Quasi-isometric embeddings are large-scale conformal.
- Snowflaking (i.e., replacing a metric by a power of it) is large-scale conformal.
- Power maps $z \mapsto z|z|^{K-1}$ are large-scale conformal for $K \geq 1$. They are not quasi-isometric, nor even coarse embeddings.
- Compositions of large-scale conformal maps are large-scale conformal.

For instance, every nilpotent Lie or finitely generated group can be large-scale conformally embedded in a Euclidean space of sufficiently high dimension, [1]. Every hyperbolic group can be large-scale conformally embedded in a hyperbolic space of sufficiently high dimension, [4].

1.5. Results

Our first main result is that a kind of dimension increases under large-scale conformal maps. The relevant notion depends on classes of groups.

THEOREM 1. – *If G is a finitely generated or Lie nilpotent group, set $d_1(G) = d_2(G) =$ the exponent of volume growth of G. If G is a finitely generated or Lie hyperbolic group, let $d_1(G) := \mathrm{CohDim}(G)$ be the infimal p such that the ℓ^p-cohomology of G does not vanish. Let $d_2(G) := \mathrm{ConfDim}(\partial G)$ be the Ahlfors-regular conformal dimension of the ideal boundary of G.*

Let G and G' be nilpotent or hyperbolic groups. If there exists a large-scale conformal map $G \to G'$, then $d_1(G) \le d_2(G')$.

Theorem 1 is a large scale version of a result of Benjamini and Schramm, [3], concerning packings in $\mathbb{R}^d$. The proof follows the same general lines but differs in details. The result is not quite sharp in the hyperbolic group case, since it may happen that $d_1(G) < d_2(G)$, [8]. However equality $d_1(G) = d_2(G)$ holds for Lie groups, their lattices and also for a few other finitely generated examples.

Our second result is akin to the fact that maps between geodesic metric spaces which are uniform/coarse embeddings in both directions must be quasi-isometries.

THEOREM 2. – *Let X and X' be bounded geometry manifolds or polyhedra. Assume that X and X' have isoperimetric dimension > 1. Every homeomorphism $f : X \to X'$ such that f and f^{-1} are large-scale conformal is a quasi-isometry.*

Isoperimetric dimension is defined in Subsection 8.3. Examples of manifolds or polyhedra with isoperimetric dimension > 1 include universal coverings of compact manifolds or finite polyhedra whose fundamental group is not virtually cyclic.

This is a large scale version of the fact that every quasiconformal diffeomorphism of hyperbolic space is a quasi-isometry. This classical result, [20], generalizes to Riemannian n-manifolds whose isoperimetric dimension is $> n$, [29]. The new feature of the large scale version is that isoperimetric dimension > 1 suffices.

1.6. Proof of Theorem 1

Our main tool is a metric space avatar of energy of functions, and, more generally, norms on cocycles giving rise to L^p-cohomology. Whereas, on Riemannian n-manifolds, only n-energy $\int |\nabla u|^n$ is conformally invariant, all p-energies turn out to be large-scale conformal invariants. Again, we postpone the rather technical definitions to Section 5 and merely give a rough sketch of the arguments.

Say a locally compact metric space is p-parabolic if for all (or some) point o, there exist compactly supported functions taking value 1 at o, of arbitrarily small p-energy, [38]. For instance, a nilpotent Lie or finitely generated group G is p-parabolic iff $p \ge d_1(G)$. Non-elementary hyperbolic groups are never p-parabolic. If X has a large-scale conformal embedding into X' and X' is p-parabolic, so is X. This proves Theorem 1 for nilpotent targets.

If a metric space has vanishing L^p-cohomology, maps with finite p-energy have a limit at infinity. We show that a hyperbolic group G' admits plenty of functions with finite p-energy when $p > \mathrm{ConfDim}(G')$. First, such functions separate points of the ideal boundary $\partial G'$ (this result also appears in [7]). Second, for each ideal boundary point ξ, there exists a finite p-energy function with a pole at ξ. The first fact implies that, if $\mathrm{CohDim}(G) > \mathrm{ConfDim}(G')$, any large-scale conformal map $G \to G'$ converges to some ideal boundary point ξ, the second leads to a contradiction. This proves Theorem 1 for hyperbolic domains and targets.

In [3], Benjamini and Schramm observed that vanishing of reduced cohomology suffices for the previous argument to work, provided the domain is not p-parabolic. This proves Theorem 1 for nilpotent domains and hyperbolic targets.

In the hyperbolic to hyperbolic case, one expects CohDim to be replaced by ConfDim. For this, one could try to reconstruct the ideal boundary of a hyperbolic group merely in terms of finite p-energy functions, in the spirit of [34] and [6].

1.7. Proof of Theorem 2

Following H. Grötzsch, [20], we define 1-capacities of compact sets K in a locally compact metric space X by minimizing 1-energies of compactly supported functions taking value 1 on K. Then we minimize capacities of compact connected sets joining a given pair of points to get a pseudo-distance δ on X. This is invariant under homeomorphisms which are large-scale conformal in both directions. If X has bounded geometry, δ is finite. If X has isoperimetric dimension $d > 1$, then δ tends to infinity with distance. This shows that homeomorphisms which are large-scale conformal maps in both directions are coarse embeddings in both directions, hence quasi-isometries. It turns out that all finitely generated groups have isoperimetric dimension > 1, but virtually cyclic ones.

1.8. Larger classes

Some of our results extend to wider classes of maps. If we merely require that balls of a given range of sizes are mapped to quasi-balls which are not too small, we get the class of *uniformly conformal maps*. It is stable under precomposition with arbitrary uniform (also known as coarse) embeddings. We can show that no such map can exist between nilpotent or hyperbolic groups unless the inequalities of Theorem 1 hold.

Corollary 1. – *We keep the notation of Theorem 1. Let G and G' be hyperbolic groups. If there exists a uniform embedding $G \to G'$, then $d_1(G) \leq d_2(G')$.*

We note that special instances of this corollary have been obtained by D. Hume, J. Mackay and R. Tessera by a different method, [23]. Their results apply in particular to M. Bourdon's rich class of (isometry groups of) Fuchsian buildings, see Section 7.

If we give up the restriction on the size of the images of balls, we get the even wider class of *coarse conformal maps*. It is stable under post-composition with quasi-symmetric homeomorphisms. New examples arise, such as stereographic projections, or the Poincaré model of hyperbolic space and its generalizations to arbitrary hyperbolic groups. However, when targets are smooth, coarse conformal maps are automatically uniformly conformal, hence similar results hold.

Corollary 2. – *No coarse conformal map can exist between a finitely generated group G and a nilpotent Lie group G' equipped with a Riemannian metric unless G is itself virtually nilpotent, and $d_1(G) \leq d_1(G')$. Also, no coarse conformal map can exist from a hyperbolic group G to a bounded geometry manifold quasiisometric to a hyperbolic group G' unless $d_1(G) \leq d_2(G')$.*

1.9. Organization of the paper

Section 2 contains definitions, basic properties and examples of coarse, uniform, rough and large-scale conformal maps. In Section 3, the notion of a quasi-symmetry structure is introduced, as a tool to handle hyperbolic metric spaces: every such space has a rough conformal map onto a product of quasi-metric quasi-symmetry spaces, as shown in Section 4. The existence of this map has the effect of translating large scale problems into microscopic analytic issues. The definition of energy in Section 5 comes with moduli of curve families and parabolicity. It culminates with the proof that several families of quasi-symmetry spaces are parabolic. L^p-cohomology of metric spaces is defined in Section 6, where the main results relating parabolicity, L^p-cohomology and coarse conformal maps are proven. Section 7 draws consequences for nilpotent or hyperbolic groups, concluding the proof of Theorem 1. The material for the proof of Theorem 2 is collected in Section 8. As a byproduct, we find conditions on a pair of spaces X, X' in order that coarse conformal maps $X \to X'$ be automatically uniformly conformal, this provides the generalizations selected in Corollary 2.

1.10. Acknowledgements

The present work originates from a discussion with James Lee and Itai Benjamini on conformal changes of metrics on graphs, cf. [24], [36], during the Institut Henri Poincaré trimester on "Metric geometry, algorithms and groups" (1). It owes a lot to Benjamini and Schramm's paper [3], although Theorem 1 does not apply to sphere packings. The focus on the category of metric spaces and large-scale conformal maps was triggered by a remark by Jonas Kahn. This paper has benefitted from amazing scrutiny by an anonymous referee, I warmly thank her or him.

2. Coarse notions of conformality

A sphere packing in a metric space Y is a collection of interior-disjoint balls. The incidence graph X of the packing has one vertex for each ball and an edge between vertices whenever corresponding balls touch. A packing may be considered as a map from the vertex set X to Y, that maps the tautological packing of X (by balls of radius $1/2$) to the studied packing of Y.

We modify the notion of a sphere packing in order to make it more flexible. In the domain, we allow radii of balls to vary in some finite interval $[R, S]$. In the range, we replace collections of disjoint balls with collections of balls with bounded multiplicity (unions of boundedly many packings). We furthermore insist that ℓ-times larger concentric balls still form a bounded multiplicity packing.

(1) See `https://metric2011.wordpress.com/2011/01/24/notes-of-james-lees-lecture-nr-1/`.

The resulting notion is invariant under coarse embeddings between domains and quasi-symmetric maps between ranges. Therefore, it is a one-sided large scale concept (in terms of domain, not of range). It is reminiscent of conformality since it requires that spheres be (roughly) mapped to spheres. Whence the term “coarsely conformal”. In order to get a class which is invariant under post-composition with quasi-isometries, we shall introduce a subclass of “large-scale conformal” maps.

2.1. Coarse, uniform, rough and large-scale conformality

Let X be a metric space. A ball B in X is the data of a point $x \in X$ and a radius $r \geq 0$. For brevity, we also denote the closed ball $B(x,r)$ by B. If $\lambda \geq 0$, λB denotes $B(x,\lambda r)$. For $S \geq R \geq 0$, let $\mathcal{B}^X_{R,S}$ denote the set of balls whose radius r satisfies $R \leq r \leq S$.

Definition 3. – *Let X be a metric space. An* (ℓ,R,S)-packing *is a collection of balls $\{B_j\}$, each with radius between R and S, such that the concentric balls ℓB_j are pair-wise disjoint. An (N,ℓ,R,S)-packing is the union of at most N (ℓ,R,S)-packings.*

The balls of an (N,ℓ,R,S)-packing, $N \geq 2$, are not disjoint (I apologize for this distorted use of the word *packing*), but no more than N can contain a given point.

Definition 4. – *Let X and X' be metric spaces. Let $f : X \to X'$ be a map. Say f is* (R,S,R',S')-coarsely conformal *if there exists a map*

$$B \mapsto B', \quad \mathcal{B}^X_{R,S} \to \mathcal{B}^{X'}_{R',S'},$$

and, for all $\ell' \geq 1$, an $\ell \geq 1$ and an N' such that

1. *For all $B \in \mathcal{B}^X_{R,S}$, $f(B) \subset B'$.*
2. *If $\{B_j\}$ is a (ℓ,R,S)-packing of X, then $\{B'_j\}$ is an (N',ℓ',R',S')-packing of X'.*

Definition 5. – *Let $f : X \to X'$ be a map between metric spaces.*

1. *We say that f is* coarsely conformal *if there exists $R > 0$ such that for all finite $S \geq R$, f is $(R,S,0,\infty)$-coarsely conformal.*
2. *We say that f is* uniformly conformal *if for every $R' > 0$, there exists $R > 0$ such that for all finite $S \geq R$, f is (R,S,R',∞)-coarsely conformal.*
3. *We say that f is* roughly conformal *if there exists $R > 0$ such that f is $(R,\infty,0,\infty)$-coarsely conformal.*
4. *We say that f is* large-scale conformal *if for every $R' > 0$, there exists $R > 0$ such that f is (R,∞,R',∞)-coarsely conformal.*

Here is the motivation for these many notions. The main technical step in our theorems applies to the larger class of coarse conformal maps. To turn this into a large scale notion, one needs to forbid the occurrence of small balls, whence the slightly more restrictive uniform variant, to which all our results apply. Uniformly conformal maps do not form a category, it is the smaller class of large-scale conformal maps which does. The Poincaré models of hyperbolic metric spaces are crucial tools, but these maps are not large-scale conformal, since small balls do occur in the range, merely roughly conformal.

Proposition 6. – *The four classes enjoy the following properties*

— *Large scale conformal* $\Longrightarrow$ *roughly conformal* $\Longrightarrow$ *coarsely conformal.*

— *Large scale conformal* $\Longrightarrow$ *uniformly conformal* $\Longrightarrow$ *coarsely conformal.*

— *Let* X, X' *and* X'' *be metric spaces. Let* $f : X \to X'$ *be* (R, S, R', S')*-coarsely conformal. Let* $f' : X' \to X''$ *be* (R', S', R'', S'')*-coarsely conformal. Then* $f' \circ f : X \to X''$ *is* (R, S, R'', S'')*-coarsely conformal.*

— *Large scale conformal maps can be composed.*

— *Roughly conformal maps can be precomposed with large-scale conformal maps. Precomposing a roughly conformal map with a uniformly conformal map yields a coarsely conformal map.*

— *Uniformly conformal maps between locally compact metric spaces are automatically proper.*

Proof. – The first two points of the proposition follow from the definition. Composing maps

$$\mathcal{B}^{X}_{R,S} \to \mathcal{B}^{X'}_{R',S'} \to \mathcal{B}^{X''}_{R'',S''},$$

we get, for every ball B in X, balls B' in X' and B'' in X'' such that $f(B) \subset B'$, $f'(B') \subset B''$, hence $f' \circ f(B) \subset B''$. Furthermore, we get, for every $\ell'' \geq 1$, a scaling factor ℓ' and a multiplicity N'', and then a scaling factor ℓ and a multiplicity N'. Given an (ℓ, R, S)-packing of X, the corresponding balls can be split into at most N' (ℓ', R', S')-packings of X'. For each sub-packing, the corresponding balls in X'' can be split into at most N'' (ℓ'', R'', S'')-packings of X''. This yields a total of at most $N'N''$ (ℓ'', R'', S'')-packings of X'', i.e., a $(N'N'', \ell'', R'', S'')$-packing, as desired. The fourth and fifth points of the proposition then follow.

Properness of uniformly conformal maps is proven by contradiction. If $f : X \to X'$ is uniformly conformal but not proper, there exists a sequence $x_j \in X$ such that $f(x_j)$ has a limit $x' \in X'$. Fix $R' > 0$ and $\ell' \geq 1$, get $R > 0$, $\ell \geq 1$ and N'. One may assume that $d(x_j, x_{j'}) > 2\ell R$ for all $j' \neq j$. Then $\{B(x_j, R)\}$ is a (ℓ, R, R)-packing. There exist balls $B'_j \supset f(B(x_j, R))$ which form a (N', ℓ', R', ∞)-packing. For j large, $x' \in B'_j$, contradicting multiplicity $\leq N'$. One concludes that f is proper. □

Remark 7. – Large scale conformal maps up to *translations*, i.e., self-maps that move points a bounded distance away, constitute the morphisms of the *large-scale conformal category*.

Indeed, translations are large-scale conformal.

In the next subsections, we shall relate our large-scale conformal definitions with classical notions and collect examples.

2.2. Quasi-symmetric maps

Notions of quasi-conformal maps on metric spaces have a long history, see [22], [21], [39].

EXAMPLE 8. – By definition, a homeomorphism $f : X \to X'$ is *quasi-symmetric* if there exists a homeomorphism $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every triple x, y, z of distinct points of X,

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \le \eta\left(\frac{d(x,y)}{d(x,z)}\right).$$

PROPOSITION 9. – *Quasi-symmetric homeomorphisms are roughly (and thus coarsely) conformal.*

Proof. – When $B = B(x,r)$, we define B' to be the smallest ball centered at $f(x)$ which contains $f(B)$. Let ρ' be its radius.

Assume first that $\rho' > 0$. Let $y \in B$ be such that $d(f(x), f(y)) = \rho'$. If $z \in f^{-1}(\ell' B')$, $d(f(x), f(z)) \le \ell'\rho'$, so

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \ge \frac{\rho'}{\ell'\rho'} = \frac{1}{\ell'}.$$

By quasi-symmetry, this implies that $\eta(\frac{d(x,y)}{d(x,z)}) \ge \frac{1}{\ell'}$, and thus $d(x,z) \le \frac{1}{\eta^{-1}(\frac{1}{\ell'})} d(x,y)$. In other words, $z \in \ell B$ with $\ell = \frac{1}{\eta^{-1}(\frac{1}{\ell'})}$. Hence $f^{-1}(\ell' B') \subset \ell B$.

If $\rho' = 0$, $\ell' B' = \{f(x)\}$, $f^{-1}(\ell' B') = \{x\} \subset \ell B$ for every ℓ.

We conclude that, for every $\ell' \ge 1$, there exist $\ell \ge 1$ and a correspondance $B \mapsto B'$ such that $f(B) \subset B'$ and $f^{-1}(\ell' B') \subset \ell B$.

If B_1 and B_2 are balls in X such that ℓB_1 and ℓB_2 are disjoint, then $f^{-1}(\ell' B_1') \subset \ell B_1$ and $f^{-1}(\ell' B_2') \subset \ell B_2$ are disjoint as well, hence $\ell' B_1'$ and $\ell' B_2'$ are disjoint, thus f is $(0, \infty, 0, \infty)$-coarsely conformal. A fortiori, f is $(R, \infty, 0, \infty)$-coarsely conformal for $R > 0$, so f is roughly conformal. □

Note that R and S play no role when checking that quasi-symmetric maps are coarsely conformal, and no guarantee on radii of balls in the range is given (i.e., such maps are $(0, \infty, 0, \infty)$-coarsely conformal). In a sense, for the wider class of maps we are interested in, three of the quasi-symmetry assumptions are relaxed:

— Maps need not be homeomorphisms.

— The quasi-symmetry estimate applies only to balls in a certain range $[R, S]$ of radii.

— Centers need not be mapped to centers.

PROPOSITION 10. – *Globally defined quasiconformal mappings of Euclidean space $\mathbb{R}^n$, $n \ge 2$, are quasi-symmetric, hence roughly and coarsely conformal.*

Proof. – Although not explicitly stated, this follows from the proof of Theorem 22.3, page 78, in [40]. In $\mathbb{R}^n$, there is a uniform lower bound $h(\frac{b}{a}) > 0$ for the conformal capacity of condensers (C_0, C_1) such that C_0 connects 0 to the a-sphere and C_1 connects the b-sphere to infinity. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be K-quasiconformal and map 0 to 0.

Let $l = \min\{|f(x)|\,;\,|x| = a\}$ and $L = \max\{|f(x)|\,;\,|x| = b\}$, let $C_0 = f^{-1}(B(0,l))$ and $C_1 = f^{-1}(\mathbb{R}^n \setminus B(0,L))$. Then $\mathrm{cap}_n(C_0, C_1) \geq h(\frac{b}{a})$. On the other hand,

$$\mathrm{cap}_n(C_0, C_1) \leq K\,\mathrm{cap}_n(f(C_0, C_1)) = K\omega_{n-1}\,\log(\frac{L}{l})^{1-n},$$

this yields an upper bound on $\frac{L}{l}$ in terms of $\frac{b}{a}$, proving that f is quasi-symmetric. □

EXAMPLE 11. – For all $K > 0$, the map $z \mapsto z|z|^{K-1}$ on $\mathbb{R}^n$ is roughly conformal. If $K \geq 1$, it sends large balls to large balls, hence it is large-scale conformal.

If $n \geq 2$, Proposition 10 applies. Its 1-dimensional analogue $f : x \mapsto x|x|^{K-1}$ is quasisymmetric, and hence roughly conformal as well, and large-scale conformal if $K \geq 1$. This can also be checked directly.

2.3. Coarse quasi-symmetry

The proof of Proposition 9 suggests the following definition.

DEFINITION 12. – *Let X and X' be metric spaces. Let $f : X \to X'$ be a map. Say f is* (R, S, R', S')-coarsely quasi-symmetric *if for every $\ell' \geq 1$, there exists $\ell \geq 1$ such that for every ball $B \in \mathcal{B}^X_{R,S}$, there exists a ball $B' \in \mathcal{B}^{X'}_{R',S'}$ such that*

$$f(B) \subset B' \quad \textit{and} \quad f^{-1}(\ell' B') \subset \ell B.$$

Say that f is coarsely quasi-symmetric *if there exists $R > 0$ such that for all $S \geq R$, f is $(R, S, 0, \infty)$-coarsely quasi-symmetric.*

Note that a homeomorphism $f : X \to X'$ is quasisymmetric if and only if f and f^{-1} are $(0, \infty, 0, \infty)$ coarsely quasi-symmetric.

LEMMA 13. – *Let X and X' be metric spaces. Let $f : X \to X'$ be a map. Then the following are equivalent:*

1. *f is (R, S, R', S')-coarsely quasi-symmetric.*
2. *There exists a map*

$$B \mapsto B', \quad \mathcal{B}^X_{R,S} \to \mathcal{B}^{X'}_{R',S'},$$

such that for all $\ell' \geq 1$, there exists $\ell \geq 1$ such that

— *for all $B \in \mathcal{B}^X_{R,S}$, $f(B) \subset B'$.*

— *If $\ell B_1 \mathrm{cap} \ell B_2 = \emptyset$, then $\ell' B'_1 \mathrm{cap} \ell' B'_2 = \emptyset$.*

In particular, coarsely quasi-symmetric maps are coarsely conformal.

Proof. – The argument for (1) $\implies$ (2) appears in the proof of Proposition 9. Since (2) is coarse conformality with the extra requirement that multiplicity $N = 1$, it implies coarse conformality.

Let us prove that (2) $\implies$ (1). Fix $\ell' \geq 1$. Let ℓ be the ratio provided by assumption (2). Let $B = B(x, r)$ be a ball in X, with $R \leq r \leq S$. Let B' the corresponding ball in X'. If $z \in X$ does not belong to $2\ell B$, then $\beta = B(z, r)$ is an R, S-ball in X, and $\ell B \mathrm{cap} \ell\beta = \emptyset$. Let β' be the ball in X' containing $f(\beta)$ provided by (2). By (2) again, it follows that $\ell' B' \mathrm{cap} \ell' \beta' = \emptyset$. In particular, $f(z) \notin \ell' B$. This shows that $f^{-1}(\ell' B') \subset 2\ell B$. Hence (1) is satisfied, up to doubling the ratio ℓ. ☐

2.4. Quasi-Möbius maps

Let X be a metric space. If $a, b, c, d \in X$ are distinct, their *cross-ratio* is

$$[a, b, c, d] = \frac{d(a, c)}{d(a, d)} \frac{d(b, d)}{d(b, c)},$$

with extension to a point ∞ when X is unbounded, see [41].

An embedding $f : X \to Y$ is *quasi-Möbius* if there exists a homeomorphism $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all quadruples of distinct points $a, b, c, d \in X$,

$$[f(a), f(b), f(c), f(d)] \leq \eta([a, b, c, d]).$$

Note that if f is a homeomorphism, f^{-1} is quasi-Möbius as well.

The main examples are

— inversions $x \mapsto \frac{x}{|x|^2}$ in Banach spaces,

— the stereographic projection $\mathbb{R}^n \to S^n$,

— its complex, quaternionic and octonionic versions, sometimes known as *Cayley transforms*, where $\mathbb{R}^n$ is replaced with a Heisenberg group equipped with a Carnot-Carathéodory metric.

— the action of a hyperbolic group on its ideal boundary is (uniformly) quasi-Möbius.

According to J. Väisälä, [41], if X and Y are bounded, quasi-Möbius maps $X \to Y$ are quasi-symmetric. If X and Y are unbounded, a quasi-Möbius map $X \to Y$ that tends to infinity at infinity is quasi-symmetric. We note an other situation where a quasi-Möbius map is coarsely conformal.

LEMMA 14. – *Assume X is unbounded and Y is bounded. Let $f : X \to Y$ be a quasi-Möbius embedding. Assume that f has a limit at infinity. Then f is coarsely conformal.*

Proof. – Fix $S > 0$. Let $y = \lim_{x \to \infty} f(x)$. Fix some origin $o \in X$ and let $o' = f(o)$. We merely need to show that the ratio $\frac{d(f(a), f(c))}{d(f(b), f(c))}$ is bounded above and below in terms of $\frac{d(a,c)}{d(b,c)}$ only when a, b, c belong to an S-ball which is far from o. Then the argument in the proof of Proposition 9 shows that f is $(0, S, 0, \infty)$-coarsely conformal.

Consider a triple $a, b, c \in X$ such that $d(c, o) \geq 4S$, $d(c, a) \leq S$ and $d(c, b) \leq S$. Then $d(a, o) \geq 3S$, $d(b, o) \geq 3S$ and $d(a, b) \leq 2S$, thus

$$\frac{1}{3} \leq 1 - \frac{2S}{d(b, o)} \leq \frac{d(a, o)}{d(b, o)} \leq 1 + \frac{2S}{d(b, o)} \leq 2.$$

Let $x \in X$ satisfy $d(c,x) \geq 4S$. Then $d(x,a) \geq 3S$ and $d(x,b) \geq 3S$ as well, so

$$\frac{1}{6} \leq [a,b,x,o] \leq 6.$$

It follows that $\frac{1}{\eta(6)} \leq [f(a), f(b), f(x), f(o)] \leq \eta(6)$. As x tends to ∞, this cross-ratio tends to

$$[f(a), f(b), y, o'] = \frac{d(f(a),y)}{d(f(a),o')}\frac{d(f(b),o')}{d(f(b),y)}.$$

Since f is an embedding, when a, b, c are far from o, $f(a)$, $f(b)$, $f(c)$ are not close to o', hence $\frac{d(f(b),o')}{d(f(a),o')}$ is bounded above and away from 0, the same holds for $\frac{d(f(a),y)}{d(f(b),y)}$.

Since

$$[a,b,c,\infty] = \frac{d(a,c)}{d(b,c)}, \quad [f(a), f(b), f(c), y] = \frac{d(f(a), f(c))}{d(f(b), f(c))}\frac{d(f(b),y)}{d(f(a),y)},$$

the ratio $\frac{d(f(a),f(c))}{d(f(b),f(c))}$ is bounded above and below in terms of $\frac{d(a,c)}{d(b,c)}$ only. □

2.5. The Cayley transform

The stereographic projection (or Cayley transform) extends to all metric spaces. It is specially well suited to the class of Ahlfors regular metric spaces.

Recall that a metric space X is *Q-Ahlfors regular at scale S* if it admits a measure μ and a constant $C(S)$ such that

$$\frac{1}{C}\, r^Q \leq \mu(B(x,r)) \leq C\, r^Q$$

for all $x \in X$ and $r \leq S$. We abbreviate it in Q-Ahlfors regular if $S = \text{diameter}(X)$. Locally Q-Ahlfors regular means Q-Ahlfors regular at all scales (with constants depending on scale).

Example 15. – The set $\mathbb{R}$, intervals of $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are 1-Ahlfors regular. Carnot groups are Ahlfors regular. Snowflaking, i.e., replacing the distance d by d^α for some $0 < \alpha < 1$, turns a Q-Ahlfors regular space into a $\frac{Q}{\alpha}$-Ahlfors regular space. The product of Q-and Q'-Ahlfors regular spaces is a $Q + Q'$-Ahlfors regular space. The ideal boundary of a hyperbolic group equipped with a visual quasi-metric is Ahlfors regular, [37], [10].

Lemma 16 (compare Väisälä, [41], Theorem 1.10). – *Every metric space X has a quasi-Möbius embedding into a bounded metric space $\dot{X}$. If X is Q-Ahlfors regular, so is $\dot{X}$.*

Proof. – Here is a rough sketch of J. Väisälä's proof. Use the Kuratowski embedding $X \to L$, where $L = L^\infty(X)$. Then embed L into $L \times \mathbb{R}$ and apply an inversion. This is a quasi-Möbius map onto its image, which is bounded and homeomorphic to the one-point completion $\dot{X}$ of X.

If X is Q-Ahlfors regular with Hausdorff measure μ, let ν be the measure with density $|\dot{x}|^{2Q}$ with respect to the pushed forward measure on $\dot{X}$. Since for $r \ll |\dot{x}|$ the inverse image of $B(\dot{x}, r)$ is roughly equal to $B(x, r|\dot{x}|^{-2})$, where $|\dot{x}| = |x|^{-1}$,

$$\nu(B(\dot{x},r)) \sim |\dot{x}|^{2Q}\mu(B(x,r|\dot{x}|^{-2})) \sim \mu(B(x,r)) \sim r^Q.$$

Balls with $r \gg |\dot{x}|$ can be dealt with by a suitable annular decomposition. This shows that $\dot{X}$ is Q-Ahlfors regular as well. □

2.6. Uniform/coarse embeddings

Definition 17. – *A map $f : X \to X'$ is a* uniform *or* coarse embedding *if for every $T > 0$ there exists $\tilde{T}$ such that for every $x_1, x_2 \in X$,*

$$d(x_1, x_2) \leq T \implies d(f(x_1), f(x_2)) \leq \tilde{T},$$
$$d(f(x_1), f(x_2)) \leq T \implies d(x_1, x_2) \leq \tilde{T}.$$

Lemma 18. – *Let f be a uniform (or coarse) embedding between metric spaces. For every $R' > 0$, there exists $R > 0$ such that for every $S \geq R$ and every large enough $S' \geq R'$, f is (R, S, R', S')-coarsely quasi-symmetric for some positive and finite R and S, hence (R, S, R', S')-coarsely conformal. In particular, uniform (or coarse) embeddings are uniformly conformal.*

Proof. – Assume that $f : X \to X'$ is a coarse embedding controlled by function $T \mapsto \tilde{T}$. Pick an arbitrary $R' > 0$. Find R such that $\tilde{R} \geq R'$. Fix $S \geq R$ and $S' \geq \tilde{S}$. Given $\ell' \geq 1$, set $U = \ell' \tilde{S}$ and $\ell = \frac{\tilde{U}}{R}$.

When $B = B(x, r)$, $r \in [R, S]$, we define $B' = B(f(x), \tilde{S})$. Then B' is an R', S'-ball and $f(B) \subset B'$.

On the other hand, if $x' \in f^{-1}(\ell' B')$, then $d(f(x'), f(x)) \leq \ell' \tilde{S} = U$, thus $d(x', x) \leq \tilde{U}$. This shows that $f^{-1}(\ell' B') \subset \ell B$ with $\ell = \frac{\tilde{U}}{R}$. In other words, f is $(R, S, \tilde{S}, \tilde{S})$-coarsely quasi-symmetric. A fortiori, it is (R, S, R', S')-coarsely conformal.

Since, for every ℓ', the chosen ℓ does not depend on S', f is $(R, S, \tilde{S}, \infty)$-coarsely conformal. Thus f is uniformly conformal. $\square$

Remark 19. – Conversely, if the metric spaces are geodesic, every (R, S, R', S')-coarsely quasi-symmetric map with $0 < R \leq S < +\infty$ and $0 < R' \leq S' < +\infty$ is a coarse embedding.

We see that (R, S, R', S')-coarse quasi-symmetry does not bring anything new while $R, R' > 0$ and $S, S' < \infty$, at least in the geodesic world. Similarly, classical conformal mappings with Jacobian bounded above and below are bi-Lipschitz. Conformal geometry begins when $S' = \infty$ or $R' = 0$.

Also, the apparently minor difference ($N' = 1$) between coarse conformality and coarse quasi-symmetry seems to be significant.

Proof. – Let $f : X \to X'$ be a (R, S, R', S')-coarsely quasi-symmetric map. Then R-balls are mapped into S'-balls. If X is geodesic, this implies that

$$d(f(x_1), f(x_2)) \leq \frac{S'}{R} d(x_1, x_2) + S'.$$

Conversely, assume $T > 0$ is given. Set $\ell' = (T + 2S')/2R'$ and let ℓ be the corresponding scaling factor in the domain. Let $B_1 = B(x_1, S)$ and $B_2 = B(x_2, S)$. If $d(x_1, x_2) > 2\ell S$, then ℓB_1 and ℓB_2 are disjoint. Let $B_1' = B(x_1', r_1) \supset f(B_1)$ and $B_2' = B(x_2', r_2) \supset f(B_1)$ be the corresponding balls in X'. Since $\{B_1, B_2\}$ is a $(1, \ell, R, S)$-packing, $\{B_1', B_2'\}$ is a $(1, \ell', R', S'\}$ packing, so both $R' \leq r_i \leq S'$ and $\ell B_1' \mathrm{cap} \ell B_2' = \emptyset$, hence $d(x_1', x_2') > \ell' r_1 + \ell' r_2 \geq 2\ell' R'$. Since both $d(f(x_i), x_i') \leq S'$, $d(f(x_1), f(x_2)) > 2\ell' R' - 2S' \geq T$. $\square$

Lemma 20. – *Quasi-isometric embeddings from a geodesic metric space are large-scale conformal.*

Proof. – The assumption means that

$$\frac{1}{L}\, d(x_1, x_2) - C \le d(f(x_1), f(x_2)) \le L\, d(x_1, x_2) + C.$$

To a ball $B = B(x, r)$ in X, we attach $B' = B(f(x), Lr + C)$. Given balls $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$, assume that concentric balls ℓB_1 and ℓB_2 are disjoint. Then, looking at a minimizing path joining x_1 to x_2, one sees that $d(x_1, x_2) \ge \ell(r_1 + r_2)$. This implies that

$$d(f(x_1), f(x_2)) \ge \frac{1}{L}\ell(r_1 + r_2) - C \ge \ell'(L(r_1 + r_2) + 2C),$$

provided $r_1 + r_2 \ge 2R$ and $\ell \ge \ell' L^2 + \frac{LC(1+2\ell')}{2R}$. If so, the concentric balls $\ell' B_1'$ and $\ell' B_2'$ are disjoint. Therefore, for every $\ell' \ge 1$, there exists $\ell \ge 1$ such that f maps $(1, \ell, R, \infty)$-packings to $(1, \ell', \frac{R}{L} - C, \infty)$-packings. □

Corollary 21. – *Quasi-isometries between geodesic metric spaces are large-scale conformal.*

Indeed, quasi-isometries between geodesic metric spaces are controlled by affine functions in both directions.

Example 22. – Orbital maps of injective homomorphisms between isometry groups of locally compact metric spaces are coarse embeddings. They are rarely quasi-isometric.

For instance, the control function $T \mapsto \tilde{T}$ of the horospherical embedding of $\mathbb{R}^n$ into H^{n+1} is logarithmic. Given a Euclidean ball B of radius R, the hyperbolic ball B' whose intersection with the horosphere is B and whose projection back to the horosphere is smallest is a horoball, of infinite radius. The radius of its projection is $\frac{1}{2}(R^2 + 1)$. In order to be mapped to disjoint horoballs, two Euclidean R-balls B_1 and B_2 must have centers at distance at least $R^2 + 1$. This makes the scaling factor ℓ depend on R. Thus this embedding is not large-scale conformal.

Question. Are orbital maps of subgroups in nilpotent groups always large-scale conformal?

2.7. Assouad's embedding

Assouad's Embedding Theorem, [1] (see also [27]), states that every snowflake $X^\alpha = (X, d_X^\alpha)$, $0 < \alpha < 1$, of a doubling metric space admits a bi-Lipschitz embedding into some Euclidean space.

Proposition 23. – *Bi-Lipschitz embeddings of snowflakes are large-scale conformal. In particular, the Assouad embedding of a doubling metric space into Euclidean space is large-scale conformal.*

Proof. – The identity $X \to X^\alpha$ is large-scale conformal. Indeed, the correspondence $B \to B'$ is the identity, and any concentric ball ℓB is mapped to $\ell^\alpha B'$. So given a scaling factor $\ell' > 1$ in the range, $\ell = \ell'^{1/\alpha}$ fits as a scaling factor in the domain.

Bi-Lipschitz maps are large-scale conformal, so the composition is large-scale conformal as well (Proposition 6). □

2.8. Sphere packings

Following Benjamini and Schramm [3], we view a sphere packing in $\mathbb{R}^d$ as a map from the vertex set of a graph, the incidence graph G of the packing, to $\mathbb{R}^d$. G carries a canonical packing by balls of radius $\frac{1}{2}$ (whose incidence graph is G itself). It is a $(1, \frac{1}{2}, \frac{1}{2})$-packing, which is mapped to the given $(1, 0, \infty)$-packing.

Coarse conformality is a very strong restriction on a sphere packing. It has something to do with uniformity in the sense of [2]. Recall that a sphere packing is M-uniform if

1. the degree of the incidence graph is bounded by M,
2. the ratio of radii of adjacent spheres is $\leq M$.

But uniformity is not sufficient. For instance, let $\sigma(z) = az$ be a planar similarity. If $a = re^{i\theta}$ is suitably chosen (for instance, $\theta = \pi/10$ and $r < e^{-2\pi/100}$), there exists a circle C such that $\sigma(C)$ touches C but no iterate $\sigma^k(C)$, $k \neq 1, -1$, does. The collection of iterates $\sigma^k(C)$, $k \in \mathbb{Z}$, constitutes a uniform planar circle packing whose incidence graph is $\mathbb{Z}$ but which does not give rise to a roughly conformal map of $\mathbb{Z}$ to $\mathbb{R}^2$. Indeed, if R is large enough so that iterates $\sigma^k(C)$, $k \in \{-R, \dots, R\}$ make a full turn around the origin, any Euclidean ball B' which contains $2R + 1$ consecutive circles of the packing contains the origin. So the image of any $(N, 1, R, R)$-packing of $\mathbb{Z}$ has infinite multiplicity at the origin.

However, the same construction with $a > 0$ gives rise to a coarse conformal map, composition of an inversion with the standard isometric embedding of $\mathbb{N}$ in $\mathbb{R}^2$.

The restriction of a coarse conformal map to a subset is coarsely conformal for the induced distance. However, it need not be for the intrinsic geodesic metric. For instance, when restricting to a subgraph, coarse conformality is lost in general, unless the subgraph is metrically undistorted. This contrasts with the fact that every subgraph of the incidence graph of a sphere packing is again the incidence graph of a sphere packing.

3. Quasi-symmetry structures

This notion is needed in order to host an important example of rough conformal map, the Poincaré model of a hyperbolic metric space.

3.1. Definition

Definition 24 (compare [30]). – *A* quasi-symmetry *structure on a set X is the data of a set $\mathcal{B}$ with a family $(\Phi_\ell)_{\ell\in\mathbb{R}_+^*}$ of maps $\mathcal{B} \to$* Subsets(X) *satisfying*

$$\ell \leq \ell' \implies \forall B \in \mathcal{B},\ \Phi_\ell(B) \subset \Phi_{\ell'}(B).$$

(in the sequel, one will rarely distinguish an element of $\mathcal{B}$ from the corresponding subset $\Phi_1(B)$ of X; then $\Phi_\ell(B)$ will be denoted by ℓB). A set equipped with such a structure is called a q.s. space. *Elements of $\mathcal{B}$ (as well as the corresponding subsets of X) are called* balls.

Example 25. – Quasi-metric spaces come with a natural q.s. structure, where $\mathcal{B} = X \times (0, \infty)$, and for each $\ell \in \mathbb{R}_+^*$, a pair (x, r) is mapped to the ball $B(x, \ell r)$.

Definition 26. – *Let X be a q.s. space. An* (N,ℓ)-packing *is a collection of balls* $\{B_j\}$ *such that the collection of concentric balls* ℓB_j *has multiplicity* $\leq N$*, i.e., the collection* $\{\ell B_j\}$ *can be split into at most N sub-families, each consisting of pairwise disjoint balls.*

3.2. Coarsely and roughly conformal maps to q.s. spaces

Definition 27. – *Let X be a metric space and X' a q.s. space. Let $f : X \to X'$ be a map. Say f is* (R,S)-coarsely conformal *if there exist a map*

$$B \mapsto B', \quad \mathcal{B}^X_{R,S} \to \mathcal{B}^{X'},$$

and for all $\ell' \geq 1$, an $\ell \geq 1$ and an N' such that

1. *for all $B \in \mathcal{B}^X_{R,S}$, $f(B) \subset B'$.*
2. *If $\{B_j\}$ is a (ℓ,R,S)-packing of X, then $\{B'_j\}$ is an (N',ℓ')-packing of X'.*

We say that f is coarsely conformal *if there exists $R > 0$ such that for all finite $S \geq R$, f is (R,S)-coarsely conformal.*

We say that f is roughly conformal *if there exists $R > 0$ such that f is (R,∞)-coarsely conformal.*

It is harder to be roughly conformal than coarsely conformal.

3.3. Quasi-symmetric maps between q.s. spaces

Definition 28. – *Let X and X' be q.s. spaces. A map $f : X \to X'$ is* quasi-symmetric *if there exists a correspondence between balls*

$$B \mapsto B', \quad \mathcal{B}^X \to \mathcal{B}^{X'}$$

with the following property: for all $\ell' \geq 1$, there exist $\ell \geq 1$ such that for all $B \in \mathcal{B}^X$,

1. *$f(B) \subset B'$,*
2. *$f^{-1}(\ell' B') \subset \ell B$.*

Quasi-symmetric maps are morphisms in a category whose objects are q.s. spaces. For q.s. structures associated to metrics, isomorphisms in this category coincide with classical quasi-symmetric homeomorphisms. Indeed, bijections which are quasi-symmetric (in the sense of Definition 28) in both directions are homeomorphisms; annuli are mapped to annuli with a bound on aspect ratio, this is quasi-symmetry.

Proposition 29. – *Let X be a metric space, and X', X'' q.s. spaces. If $f : X \to X'$ is (R,S)-coarsely conformal and $f' : X' \to X''$ is quasi-symmetric, then the composition $f' \circ f$ is (R,S)-coarsely conformal.*

Proof. – Fix $\ell'' \geq 1$. Since f' is quasi-symmetric, there exists $\ell' \geq 1$ such that every ball B' in X' is mapped into a ball B'' of X'' such that $f'^{-1}(\ell'' B'') \subset \ell' B'$. If B'_1 and B'_2 are balls in X' such that $\ell' B'_1 \mathrm{cap} \ell' B'_2 = \emptyset$, then $f'^{-1}(\ell'' B''_1) \mathrm{cap} f'^{-1}(\ell'' B''_2) = \emptyset$, hence $\ell'' B''_1$ and $\ell'' B''_2$ are disjoint. Thus $f' \circ f$ is (R,S)-coarsely conformal. □

4. The Poincaré model

4.1. The 1-dimensional Poincaré model

The following example of q.s. space may be called the half hyperbolic line.

Definition 30. – *Let $\mathbb{D}$ denote the interval* $[0,1]$ *equipped with the following q.s. structure. $\mathcal{B}$ is the set of closed intervals in* $[0,1]$. *For an interval $I \subset (0,1]$ there exists a unique pair (R,t) such that $B = [e^{-R-t}, e^{R-t}]$. Then $\ell B := [e^{-\ell R-t}, \min\{1, e^{\ell R-t}\}]$. For an interval of the form $I = [0,b]$, $\ell I = I$.*

In other words, the structure is the usual metric space structure of a half real line transported by the exponential function, and extended a bit arbitrarily to a closed interval.

Remark 31. – Here is a formula for ℓI when $I = [a,b]$, $a > 0$:

$$\ell I = [a^{\frac{1+\ell}{2}} b^{\frac{1-\ell}{2}}, \min\{1, a^{\frac{1-\ell}{2}} b^{\frac{1+\ell}{2}}\}].$$

Indeed, if $[a,b] = [e^{-R-t}, e^{R-t}]$, then $R = \frac{1}{2}\log(b/a)$ and $t = -\frac{1}{2}\log(ab)$.

4.2. Warped products

There is no way to take products of q.s. spaces, so we use auxiliary quasi-metrics.

Definition 32. – *A* q.m.q.s. space *is the data of a q.s. space and a quasi-metric which defines the same balls (but possibly a different $B \mapsto \ell B$ correspondence).*

Example 33. – We keep denoting by $\mathbb{D}$ the q.m.q.s. space $[0,1]$ equipped with its usual metric but the q.s. structure of $\mathbb{D}$.

Definition 34. – *The* product *of two q.m.q.s. spaces Z_1 and Z_2 is $Z_1 \times Z_2$ equipped with the product quasi-metric*

$$d((z_1,z_2),(z_1',z_2')) = \max\{d(z_1,z_1'), d(z_2,z_2')\}$$

and, if $B = I \times \beta$ is a ball in the product, $\ell B = \ell I \times \ell\beta$.

Example 35. – Our main example is $\mathbb{D} \times Z$, where Z is a metric space with its metric q.s. structure.

4.3. The Poincaré model of a hyperbolic metric space

PROPOSITION 36. – *Let X be a geodesic hyperbolic metric space with ideal boundary ∂X. Fix an origin $o \in X$ and equip ∂X with the corresponding visual quasi-distance d_o. Assume that there exists a constant D such that for any $x \in X$ there exists a geodesic ray γ from the base point $\gamma(0) = o$ and passing near x: $d(x, \gamma) < D$. Then there is a rough conformal map π, called the* Poincaré model*, of X to the q.s. space $\mathbb{D} \times \partial X$. Its image is contained in $(0, 1] \times \partial X$. If $x \in X$ tends to $z \in \partial X$ in the natural topology of $X \cup \partial X$, then $\pi(x)$ tends to $(0, z)$. Finally, after a rescaling of ∂X, if an R-ball is mapped to $B((y, z), r)$, then $r = \tanh(R)y$.*

Proof. – For $x \in X$, pick a geodesic ray γ starting from o that passes at distance $< D$ from x. Set $\chi(x) = e^{-d(o,x)}$, $\phi(x) = \gamma(+\infty)$ and $\pi(x) = (\chi(x), \phi(x))$. π may be discontinuous, but we do not care.

Let $B = B(x, R)$ be a ball in X, and $t = d(o, x)$. Then $\phi(B)$ is contained in a ball of radius e^{R-t}, up to a multiplicative constant. Indeed, thanks to V. Shchur's Lemma, [35], Lemma 12, there exists a constant C such that, if $y \in B$ and $t' = d(o, y)$, then $|t - t'| \leq R + C + 2D$ and $t + t' + 2\log d_o(\phi(x), \phi(y)) \leq R + C + 2D$. Thus

$$2t + 2\log d_o(\phi(x), \phi(y)) \leq 2R + 2C + 4D,$$

whence $d_o(\phi(x), \phi(y)) \leq e^{R+C+2D-t}$. It turns out that $\chi(B)$ is also a ball of radius e^{R-t}, up to a multiplicative constant. Indeed, if $y' \in B$ and $t' = d(o, y')$, then $|t - t'| \leq R$, thus $\chi(y')$ belongs to the interval $[e^{-R-t}, e^{R-t}]$, centered at $s = 2e^{-t}\cosh(R)$, with radius $r = 2e^{-t}\sinh(R) = \tanh(R)s$. From now on, we decide to multiply the quasi-metric on ∂X by the constant factor $e^{-C-2D-1}$, i.e., we use the quasi-metric $d' = e^{-C-2D-1}d_o$. In this way, $\pi(B)$ is contained in the ball B' of $[0, 1] \times \partial X$ centered at $(e^{-t}\cosh(R), \phi(x))$, with radius $e^{-t}\sinh(R)$.

Conversely, if $(\tau, \eta) \in B'$, then $\eta \in \partial X$ belongs to

$$B(\phi(x), e^{-t}\sinh(R))$$

(which means that $d'(\phi(x), \eta) = e^{-C-2D-1}d_o(\phi(x), \eta) \leq e^{-t}\sinh(R)$), and $\tau \in [e^{-R-t}, e^{R-t}]$. Let t' be such that $e^{-t'} = \tau$. Then $t - R \leq t' \leq t + R$. Let x' (resp. x'') be the point of the geodesic from o to η such that $d(o, x) = t'$ (resp. $d(o, x'') = t$). Then $d(x, x'') \leq 2t + 2\log d_o(\phi(x), \eta) + C + 2D \leq 2R + 3C + 4D + 2$, and $d(z, x') = |t - t'| \leq R$. Thus $d(x, x') \leq 4R$ provided $R \geq 3C + 4D + 2$. We conclude that

$$\pi(B) \subset B', \quad \text{and} \quad B' \subset \pi(4B).$$

Let us show that

$$\pi(4\ell B) \supset \ell B'.$$

To avoid confusion, let us denote by m_ℓ the $\mathbb{R}^*$ action on intervals. If $B = B(x, R)$ and $\chi(x) = t$, then $\chi(B) = [e^{-R-t}, e^{R-t}]$ and $\chi(\ell B) = m_\ell(\chi(B))$ by definition of m_ℓ. On the other hand, $B' = \chi(B) \times \beta$ where $\phi(B) \subset \beta = B(\phi(x), e^{-t}\sinh(R))$,

$$\phi(\ell B) = B(\phi(x), e^{-t}\sinh(\ell R)) \supset B(\phi(x), e^{-t}e^{(\ell-1)R}\sinh(R)) \supset \ell\beta,$$

provided $R \geq 1$. Therefore

$$m_\ell(B') = m_\ell(\chi(\ell B)) \times \ell\beta \subset m_\ell(\chi(\ell B)) \times \phi(\ell B).$$

Since this is the ball $(\ell B)'$ corresponding to ℓB, it is contained in $\pi(4\ell B)$.

Since X is hyperbolic, there is a constant δ such that two geodesics with the same endpoints are contained in the δ-tubular neighborhood of each other. We can assume that $\delta + 2D \leq C$. It follows that if $\pi(x_1) = \pi(x_2)$, then $d(x_1, x_2) \leq \delta$. Let B_1 and B_2 be balls of radii $\geq C$. Let B_1', B_2' be the corresponding balls in $\mathbb{D} \times \partial X$. If $\ell' B_1' \mathrm{cap} \ell' B_2' \neq \emptyset$, then $\pi(4\ell' B_1) \mathrm{cap} \pi(4\ell' B_2) \neq \emptyset$. Thus $d(4\ell' B_1, 4\ell' B_2) \leq \delta + 2D \leq C$. If $B_1 = B(x_1, r_1)$, $4\ell' B_1 = B(x_1, 4\ell' r_1)$, its C-neigborhood is contained in $B(x_1, 4\ell' r_1 + C) \subset B(x_1, 5\ell' r_1) = 5\ell' B_1$. Hence $5\ell' B_1 \mathrm{cap} 5\ell' B_2 \neq \emptyset$. We may set $\ell = 5\ell'$ and conclude that π is $(3C + 4D + 2, \infty, 0, \infty)$-coarsely quasi-symmetric, hence roughly conformal. □

5. Energy

Classical analysis has made considerable use of the fact that, on n-space, the L^n norm of the gradient of functions is a conformal invariant. Benjamini and Schramm observe that only the dimension of the range matters. In fact, some coarse analogue of the L^p norm of the gradient of functions turns out to be natural under coarsely conformal mappings. This works for all p (this fact showed up in [30]).

5.1. Energy and packings

Let X be a metric space. Recall that an (ℓ, R, S)-packing is a collection of balls $\{B_j\}$, each with radius between R and S, such that the concentric balls ℓB_j are pairwise disjoint. If X is merely a q.s. space, ℓ-packings make sense.

Here is one more avatar of the definition of Sobolev spaces on metric spaces. For earlier attempts, see the surveys [21], [17].

Definition 37. – *Let X, Y be metric spaces. Let $\ell \geq 1$. Let $u : X \to Y$ be a map. Define its p-energy at parameters ℓ, R and $S \geq R$ as follows.*

$$E^p_{\ell,R,S}(u) = \sup\{\sum_j \mathrm{diameter}(u(B_j))^p \,;\, (\ell, R, S)\text{-}packings\ \{B_j\}\}.$$

Remark 38. – The definition of $E^p_{\ell,0,\infty}$ extends to X being a q.s. space, we denote it simply by

$$E^p_\ell(u) = \sup\{\sum_j \mathrm{diameter}(u(B_j))^p \,;\, \ell\text{-packings } \{B_j\}\}.$$

Our main source of functions with finite energy are Ahlfors regular metric spaces.

Proposition 39. – *Let X be a d-Ahlfors regular metric space and $u : X \to Y$ a C^α-Hölder continuous map which is constant outside a compact set. Then $E^p_{\ell,0,\infty}(u)$ is finite for all $p \geq d/\alpha$ and $\ell > 1$. In particular, $E^{d/\alpha}_{\ell,R,S}(u)$ is finite for all R, S.*

Proof. – Let μ denote Hausdorff d-dimensional measure. Assume first that $\mu(X) < \infty$. For every ball B,

$$\begin{aligned}\text{diameter}(u(B))^p &\leq \text{const. diam(B)}^{p\alpha}\\ &\leq \text{const. } \mu(B)^{p\alpha/d}\\ &\leq \text{const. } \mu(X)^{-1+p\alpha/d}\mu(B),\end{aligned}$$

thus

$$\begin{aligned}\sum_j \text{diameter}(u(B_j))^p &\leq \text{const. } \mu(X)^{-1+p\alpha/d}\sum_j \mu(B_j)\\ &\leq \text{const. } \mu(X)^{p\alpha/d} < \infty.\end{aligned}$$

If X is unbounded, we assume that u has support in a ball B_0 of radius R_0. Given $\ell > 1$, if a ball B intersects B_0, and has radius $R > 2R_0/(\ell - 1)$, then ℓB contains B_0. If such a ball arises in an ℓ-packing, it is the only element of the packing, hence an upper bound on $\sum \text{diameter}(u(B))^p \leq (\max u - \min u)^p$. Otherwise, all balls of the ℓ-packing are contained in $(1 + (2R_0/(\ell - 1))B_0$, whence the above upper bound is valid with X replaced with that ball. □

Example 40. – Let $0 < \alpha \leq 1$. Let Z be a compact Q-Ahlfors regular metric space. Let $Z^\alpha = (Z, d_Z^\alpha)$ be a snowflake of Z. Consider the q.m.q.s. space $X = \mathbb{D} \times Z^\alpha$, with its projections

$$u : X \to Z,\ u(y,z) = z, \quad \text{and} \quad v : X \to [0,1],\ v(y,z) = y.$$

Then, for all ℓ and for all $p \geq \alpha + Q$,

$$E_\ell^p(u) < +\infty, \quad E_\ell^{p/\alpha}(v) < +\infty.$$

Proof. – Since the previous argument (in the compact case) never uses concentric balls ℓB, but only the balls B themselves, the difference between $\mathbb{D} \times Z^\alpha$ and the metric space product $[0,1] \times Z^\alpha$ disappears. It is compact and $1 + \frac{Q}{\alpha}$-Ahlfors regular with Hausdorff measure $\nu = dt \otimes \mu$. For every ball $B \subset [0,1] \times Z^\alpha$,

$$\text{diameter}(u(B)) = \text{diameter}(B)^{1/\alpha} \sim \nu(B)^{\frac{1}{\alpha}\frac{1}{1+\frac{Q}{\alpha}}} = \nu(B)^{\frac{1}{\alpha+Q}},$$

thus u has finite p-energy for $p \geq \alpha + Q$. The case of v follows in a similar way. □

5.2. Dependence on radii and scaling factor

The $(\ell, 0, \infty)$-packings in the definition of $E^p_{\ell,0,\infty}$-energy are hard to handle. However, for spaces with bounded geometry, taking the supremum over (ℓ, R, S)-packings does not play such a big role, provided $R > 0$ and $S < \infty$. The precise assumption, a bit weaker than the doubling property, is called the tripling property.

Definition 41. – *A metric space X has the* tripling property *if*

1. *Balls are connected.*
2. *There is a function $N' = N'(N, \ell, R, S)$ (called the* tripling function *of X) with the following property. For every $N, \ell \geq 1$, $R > 0$, $S \geq R$, every (N, ℓ, R, S)-packing of X is an $(N', \frac{R+S}{R}, R, S)$-packing as well.*

A (N,ℓ,R,S)-packing of X is covering *if the interiors form an open covering of X.*

The tripling property is meaningful only for $\ell < \frac{R+S}{R}$. If a metric space X carries a measure μ such that for all $R>0$, the measure of R-balls is bounded above and below in terms of R only (this will be called a *metric space with controlled balls* below), then it has the tripling property.

LEMMA 42. – *Let X be a metric space which has the tripling property. Let $\{B_j\}$ be a covering (N,ℓ,R,S)-packing of X. Then for all maps $u : X \to Y$ to metric spaces,*

$$E^p_{\ell,R,S}(u) \le N'(N,\ell,R,S)^p \sum_j \mathrm{diameter}(u(B_j))^p,$$

where N' is the tripling function of X.

Proof. – Let $\{B_j\}$ be a covering (N,ℓ,R,S)-packing of X and $\{B'_k\}$ a (ℓ,R,S)-packing of X. Let B' be a ball from the second packing. Let $J(B')$ index the balls from $\{B_j\}$ whose interiors intersect B'. By assumption, $|J(B')| \le N'(N,\ell,R,S)$. Indeed, if B intersects B', then the center of B' belongs to $\frac{R+S}{R}B$. Since B' is connected, given any two points x, $x' \in B'$, there exists a chain $B_{j_1},\dots,B_{j_k}$, $j_i \in J(B')$ with $x \in B_{j_1}$, $x' \in B_{j_k}$, and each B_{j_i} intersects $B_{j_{i+1}}$. This implies that $d(u(x),u(x')) \le \sum_{i=1}^k \mathrm{diameter}(u(B_{j_i}))$, and

$$\mathrm{diameter}(u(B')) \le \sum_{j\in J(B')} \mathrm{diameter}(u(B_j)).$$

Hölder's inequality yields

$$\mathrm{diameter}(u(B'))^p \le N'^{p-1} \sum_{j\in J(B')} \mathrm{diameter}(u(B_j))^p.$$

A given ball of $\{B_j\}$ appears in as many $J(B')$ as its interior intersects balls of $\{B'_k\}$. This happens at most $N'(N,\ell,R,S)$ times. Therefore

$$\sum_k \mathrm{diameter}(u(B'_k))^p \le N'^p \sum_j \mathrm{diameter}(u(B_j))^p. \qquad \square$$

COROLLARY 43. – *Let X be a metric space which has the tripling property. Up to multiplicative constants depending only on R, S and ℓ, energies $E^p_{\ell,R,S}(u)$ do not depend on the choices of $R>0$, $S<\infty$ and $\ell \ge 1$.*

Proof. – Let us show that covering (N,ℓ,R,R)-packings exist for all $R>0$ and $\ell \ge 1$, provided N is large enough, $N = N(\ell,R)$. Let $\{B_j = B(x_j,\frac{R}{2})\}$ be a maximal collection of disjoint $\frac{R}{2}$-balls in X. In particular, $\{B_j\}$ is a $(1,\frac{R}{2},(2\ell-1)\frac{R}{2})$-packing. By the tripling property, there exists $N = N'(1,2\ell,\frac{R}{2},(2\ell-1)\frac{R}{2})$ such that $\{B_j\}$ is a $(N,2\ell,\frac{R}{2},\frac{R}{2})$-packing. Then $\{2B_j\}$ is a covering (N,ℓ,R,R)-packing.

Fix $0 < R \leq 1 \leq S < +\infty$. Let $\{B_j\}$ be a covering $(N(\ell,1),\ell,1,1)$-packing. According to Lemma 42, for all maps u to metric spaces,

$$\begin{aligned} E^p_{\ell,1,1}(u) &\leq E^p_{\ell,R,S}(u) \\ &\leq N'(1,\ell,R,S)^p \sum_j \mathrm{diameter}(u(B_j))^p \\ &\leq N'(1,\ell,R,S)^p N(\ell,1) E^p_{\ell,1,1}(u), \end{aligned}$$

so changing radii is harmless. By definition of tripling, given $\ell' \geq \ell$, every $(\ell,1,\ell')$-packing is simultaneously a $(N'',\ell'+1,1,\ell')$-packing, $N''=N'(1,\ell,1,\ell')$, hence

$$\begin{aligned} E^p_{\ell',1,1}(u) &\leq E^p_{\ell,1,1}(u) \\ &\leq E^p_{\ell,1,\ell'}(u) \\ &\leq N'' E^p_{\ell'+1,1,\ell'}(u) \\ &\leq N'' E^p_{\ell',1,\ell'}(u) \\ &\leq N'' N'(1,\ell',1,\ell')^p N(\ell',1) E^p_{\ell',1,1}(u), \end{aligned}$$

so changing scaling factor is also harmless. □

If $p=1$, even the upper bound on radii of balls does not play such a big role.

LEMMA 44. – *Let X be a geodesic metric space. For every real valued function u on X,*

$$E^1_{\ell,R,R}(u) \leq E^1_{\ell,R,\infty}(u) \leq (2\ell+2) E^1_{\ell,R,R}(u).$$

Proof. – Let $B = B(z,r)$ be a large ball. Assume u achieves its maximum on B at x and its minimum at y. Along the geodesics from x to z and from z to y, consider an array of touching R-balls B_j. Then

$$\mathrm{diameter}(u(B)) \leq \sum_j \mathrm{diameter}(u(B_j)).$$

Pick one ball every 2ℓ along the array, in order to get an (ℓ,R,R)-packing. The array is the union of $2(2\ell+1)$ such packings, whence

$$\sum_j \mathrm{diameter}(u(B_j)) \leq 2(2\ell+1) E^1_{\ell,R,R}(u).$$

Summing up over balls of an arbitrary (ℓ,R,∞)-packing yields the announced inequality. □

5.3. Functoriality of energy

LEMMA 45. – *Let X, X' and Y be metric spaces. Let $f : X \to X'$ be (R,S,R',S')-coarsely conformal. Then for all $\ell' \geq 1$, there exists $\ell \geq 1$ and N' such that for all maps $u : X' \to Y$,*

$$E^p_{\ell,R,S}(u \circ f) \leq N'\, E^p_{\ell',R',S'}(u).$$

Proof. – Let $\{B_j\}$ be an (ℓ, R, S)-packing of X. Let $\{B'_j\}$ be the corresponding (N', ℓ', R', S')-packing of X'. Split it into N' sub-collections which are $(1, \ell', R', S')$-packings. By assumption, $f(B_j) \subset B'_j$, so diameter$(u \circ f(B_j)) \leq$ diameter$(u(B'_j))$. This yields, for each sub-collection,

$$\sum_j \text{diameter}(u \circ f(B_j))^p \leq \sum_j \text{diameter}(u(B'_j))^p \leq E^p_{\ell',R',S'}(u).$$

Summing up and taking supremum, this shows that $E^p_{\ell,R,S}(u \circ f) \leq N' \, E^p_{\ell',R',S'}(u)$. □

EXAMPLE 46. – If Y is d-Ahlfors regular and compact, then the identity $Y \to Y$ has finite $E^p_{\ell',R',S'}$-energy for all $p \geq d$, $\ell' \geq 1$, $R' \geq 0$, $S' \leq \infty$, so coarsely conformal maps $X \to Y$ have finite $E^d_{\ell,R,S}$-energy themselves for suitable ℓ.

Lemma 45 generalizes to q.s. spaces, with the same proof.

LEMMA 47. – *Let X and Y be metric spaces, let X' be a q.s. space. Let $f : X \to X'$ be coarsely conformal. Then for all R, S and for all $\ell' \geq 1$, there exist $\ell \geq 1$ and N' such that for all maps $u : X' \to Y$,*

$$E^p_{\ell,R,S}(u \circ f) \leq N' \, E^p_{\ell'}(u).$$

5.4. $(1,1)$**-curves**

DEFINITION 48. – *A* $(1,1)$-curve *in a metric space X is a map $\gamma : \mathbb{N} \to X$ which is $(1,1,R,S)$-coarsely conformal for all $R > 0$ and all large enough S. When X is locally compact and equipped with a base point o, a* based $(1,1)$-curve *is a proper $(1,1)$-curve such that $\gamma(0) = o$.*

An $(\ell, 1, 1)$-packing of $\mathbb{N}$ corresponds to a subset $A \subset \mathbb{N}$ such that every ℓ-ball centered at a point of A contains at most one point of A. The packing consists of unit balls centered at points of A. Let us call such a set an *ℓ-subset* of $\mathbb{N}$. A $(1,1)$-curve in X is a sequence $(x_i)_{i\in\mathbb{N}}$ such that for all R and all sufficiently large $S \gg R$, there exists a collection of balls B_i in X with radii between R and S such that

- B_i contains $\{x_{i-1}, x_i, x_{i+1}\}$.
- For all $\ell' \geq 1$, there exist ℓ and N' such that for every ℓ-subset of $\mathbb{N}$, the collection of concentric balls $\{\ell' B_i \,;\, i \in A\}$ has multiplicity $\leq N'$.

Thus a $(1,1)$-curve is a chain of slightly overlapping balls which do not overlap too much: if radii are enlarged ℓ' times, decimating (i.e., keeping only one ball every ℓ) keeps the collection disjoint or at least bounded multiplicity.

EXAMPLE 49. – An isometric map $\mathbb{N} \to X$ is a $(1,1)$-curve. In particular, geodesic rays in Riemannian manifolds give rise to $(1,1)$-curves.

This is a special case of Lemma 18.

REMARK 50. – In locally compact metric spaces, $(1,1)$-curves are proper.

Proof. – Let $\gamma : \mathbb{N} \to X$ be a based $(1,1)$-curve. Let $\ell' = 2$, let ℓ be the corresponding scaling factor in the domain, and N' the multiplicity in the range. The covering of $\mathbb{N}$ by 1-balls $B(j,1)$ is mapped to balls B'_j of radii $\geq R > 0$, $\leq S$, with $\gamma(j) \in B'_j$. Since the covering $\{B(j,1)\}$ is the union of exactly 2ℓ ℓ-packings, $\{B'_j\}$ is the union of 2ℓ $(N', 2, R, S)$-packings, it is a $(2\ell N', 2, R, S)$-packing, with N' from Definition 48. In particular, no point of X is contained in more than $2\ell N'$ balls $2B'_j$.

If γ is not proper, there exists a sequence i_j tending to infinity such that $\gamma(i_j)$ has a limit $x \in X$. Since B'_{i_j} has radius $\geq R$, for j large enough $2B'_{i_j}$ contains x, contradicting multiplicity $\leq \ell N'$. □

Definition 51. – *Let X be a q.s. space, and $K \subset X$. A* coarse curve *in X is a coarse conformal map $\mathbb{N} \to X$. A coarse curve γ is* based at K *if $\gamma(0) \in K$.*

Example 52. – Given an isometric map $\gamma : [0,1] \to X$, set $\gamma'(j) = \gamma(\frac{1}{j+1})$. This is a coarse curve in $X \setminus \{\gamma(0)\}$. In particular, geodesic segments in punctured Riemannian manifolds give rise to coarse curves.

Since γ' is the composition of inversion $\mathbb{R} \to \mathbb{R}$ and an isometric embedding, this is a special case of Lemma 14.

Proposition 53. – *Let X be a metric space and X' a q.s. space. If $\gamma : \mathbb{N} \to X$ is a $(1,1)$-curve and $f : X \to X'$ is a coarse conformal map, then $f \circ \gamma$ is a coarse curve.*

This follows from Proposition 6.

5.5. Modulus

We need to show that certain maps with finite energy have a limit along at least one based curve. To do this, we shall use the idea, that arouse in complex analysis, of a property satisfied by almost every curve.

Definition 54. – *Let Y be a metric space. The* length *of a map $u : \mathbb{N} \to Y$ is*

$$\mathrm{length}(u) = \sum_{i=0}^{\infty} d(u(i), u(i+1)).$$

Definition 55. – *Let X be a metric space. Let Γ be a family of $(1,1)$-curves in X. The* (p,ℓ,R,S)-modulus $\mathrm{mod}_{p,\ell,R,S}(\Gamma)$ *is the infimum of $E^p_{\ell,R,S}$-energies of maps $u : X \to Y$ to metric spaces such that for every curve $\gamma \in \Gamma$,* $\mathrm{length}(u \circ \gamma) \geq 1$.

Remark 56. – The definition of $(p,\ell,0,\infty)$-modulus extends to families of coarse curves Γ in q.s. spaces X,

$$\mathrm{mod}_{p,\ell}(\Gamma) = \inf\{E^p_\ell(u)\,;\, u : X \to Y,\ \mathrm{length}(u \circ \gamma) \geq 1\ \forall \gamma \in \Gamma\}.$$

Lemma 57. – *Let X be a metric space. The union of a countable collection of $(1,1)$-curve families which have vanishing (p,ℓ,R,S)-modulus also has vanishing (p,ℓ,R,S)-modulus.*

Proof. – Fix $\epsilon > 0$. Let $u_j : X \to Y_j$ be a function with $E^p_{\ell,R,S}(u_j) \le 2^{-j}\epsilon$ such that for all curves γ in the j-th family Γ_j, $\mathrm{length}(u_j \circ \gamma) \ge 1$. Consider the ℓ^p direct product $Y = \prod_j Y_j$, i.e.,

$$d^Y((y_j),(y'_j)) = \left(\sum_j d(y_j, y'_j)^p\right)^{1/p},$$

and the product map $u = (u_j) : X \to Y$. Then

$$E^p_{\ell,R,S}(u) \le \sum_j E^p_{\ell,R,S}(u_j) \le \mathrm{const.}\epsilon,$$

whereas for all curves γ in the union curve family,

$$\mathrm{length}(u \circ \gamma) \ge \sup_j \mathrm{length}(u_j \circ \gamma) \ge 1.$$

This shows that the union curve family has vanishing (p,ℓ,R,S)-modulus. □

LEMMA 58. – *Let X be a metric space. Let Γ be a family of $(1,1)$-curves in X. Then Γ has vanishing (p,ℓ,R,S)-modulus if and only if there exists a function $u : X \to Y$ such that $E^p_{\ell,R,S}(u) < +\infty$ but $\mathrm{length}(u \circ \gamma) = +\infty$ for every $\gamma \in \Gamma$.*

Proof. – One direction is obvious. In the opposite direction, first observe that by rescaling target metric spaces, one can assume that there exist maps $u_j : X \to Y_j$ such that for all $\gamma \in \Gamma$, $\mathrm{length}(u_j \circ \gamma) \ge j$ and $E^p_{\ell,R,S}(u_j) < 2^{-j}$. Apply the ℓ^p-product construction again. Get $u = (u_j) : X \to Y$ such that $E^p_{\ell,R,S}(u) \le 1$ and $\mathrm{length}(u \circ \gamma) \ge \max_j \mathrm{length}(u_j \circ \gamma) = +\infty$. □

LEMMA 59. – *Let X be a metric space. Let Y be a complete metric space. Fix $R \le S$ and $\ell \ge 1$. Let $u : X \to Y$ be a map of finite $E^p_{\ell,R,S}$ energy. The family of $(1,1)$-curves along which u does not have a limit has vanishing (p,ℓ,R,S)-modulus.*

Proof. – If $\mathrm{length}(u \circ \gamma) < \infty$, then $u \circ \gamma$ has a limit in Y, since Y is complete. Let Γ_{nl} be the family of curves along which the length of u is infinite. It contains all curves along which u does not have a limit. By assumption, $E^p_{\ell,R,S}(u) < \infty$, but $\mathrm{length}(u \circ \gamma) = +\infty \ge 1$ for all curves $\gamma \in \Gamma_{nl}$. So $\mathrm{mod}_{p,\ell,R,S}(\Gamma_{nl}) = 0$. □

REMARK 60. – The cases $R = 0$, $S = \infty$ of Lemmata 57 to 59 extend to q.s. spaces X.

DEFINITION 61. – *Let X have a base-point. Say a property of based ($(1,1)$-or coarse) curves holds for p-almost all curves if it fails for a set of based curves of vanishing p-modulus.*

For instance, Lemma 59 states that a function of finite p-energy has a limit (depending on the curve) along p-almost every based $(1,1)$-curve.

5.6. Parabolicity

DEFINITION 62. – *Let X be a locally compact metric space. Say X is* (p,ℓ,R,S)-parabolic *if the family of all* $(1,1)$*-curves based at some point has vanishing* (p,ℓ,R,S)*-modulus.*

If X is merely a locally compact q.s. space, (p,ℓ)-parabolicity means that the (p,ℓ)-modulus of the family of proper coarse curves based at some compact set with nonempty interior vanishes.

REMARK 63. – If metric space X has the tripling property, (p,ℓ,R,S)-parabolicity does not depend on the choices of $R>0$, $S<\infty$ and $\ell\geq 1$, thanks to Corollary 43.

On the other hand, (p,ℓ)-parabolicity may depend on whether $\ell=1$ or $\ell>1$, as we shall see in the next subsections.

PROPOSITION 64. – *Let X be a metric space and X' a q.s. space. Let $f:X\to X'$ be a coarse conformal map. Let R be large enough, and $S\geq R$. Let Γ be a family of* $(1,1)$*-based curves in X. Then for all ℓ', there exist ℓ such that*

$$\mathrm{mod}_{p,\ell,R,S}(\Gamma)\leq \mathrm{mod}_{p,\ell}(f(\Gamma)).$$

Proof. – By the composition rule (Proposition 6), $f(\Gamma)$ is a family of coarse curves. Given $u:X'\to Y$ such that $\mathrm{length}(u\circ\gamma')\geq 1$ for all $\gamma'\in f(\Gamma)$, $\mathrm{length}(u\circ f\circ\gamma)\geq 1$ for all $\gamma\in\Gamma$. According to Lemma 47, for all $\ell'\geq 1$, there exist $\ell\geq 1$ and N' such that

$$E^p_{\ell,R,S}(u\circ f)\leq N'\,E^p_{\ell'}(u).$$

Taking the infimum over such maps u yields the announced inequality. □

COROLLARY 65. – *Let X be a metric space and X' a q.s. space. Let $f:X\to X'$ be a proper, coarse conformal map. Then there exists $R>0$ such that for all $S\geq R$, if X' is (p,ℓ')-parabolic for some ℓ', then X is (p,ℓ,R,S)-parabolic for a suitable ℓ.*

There is a similar statement for (R',S',R,S)-coarsely conformal maps. This shows that parabolicity does not depend on the choice of base point, provided one accepts to change parameters ℓ, R and S. Indeed, the map which is identity but for one point o which is mapped to o' is proper and (R',S',R,S)-coarsely conformal.

COROLLARY 66. – *Let X and X' be locally compact metric spaces. Let $f:X\to X'$ be a uniformly conformal map. For every $R'>0$, there exists R such that for all $S\geq R$, if X' is (p,ℓ',R',∞)-parabolic for some ℓ', then X is (p,ℓ,R,S)-parabolic for a suitable ℓ.*

Proof. – Uniformly conformal maps are proper. □

5.7. Parabolicity of Ahlfors-regular spaces

PROPOSITION 67. – *Let X be a non-compact Q-Ahlfors regular metric space with $Q > 1$. Let K be a ball. For all $\ell > 1$, there exists a finite (Q,ℓ)-energy function $w : X \to \mathbb{R}$ which has no limit along every coarse curve based on K. It follows that X is (p,ℓ)-parabolic for every $\ell > 1$ and every $p \geq Q$.*

Proof. – Let μ be a measure such that balls of radius ρ have measure ρ^Q up to multiplicative constants.

Let $m = \max\{\frac{\ell+1}{\ell-1}, e\}$. Fix an origin $o \in X$, set $r(x) = d(x,o)$, $v(r) = \log\log r$ and

$$w(x) = \begin{cases} \sin(v(r(x))) & \text{if } r \geq m^2, \\ \log\log(m^2) & \text{otherwise.} \end{cases}$$

Let $\{B_j\}$ be a ℓ-packing of X. At most one ball B is such that $o \in \ell B$, it contributes to $\sum_j \text{diameter}(w(B_j))^p$ by at most 2^p. We shall ignore it henceforth. Other balls $B = B(x,\rho)$ are such that $o \notin \ell B$, hence $r(x) = d(o,x) > \ell\rho$. For all $x' \in B$, $r(x') = d(o,x')$ satisfies $r(x) - \rho \leq r(x') \leq r(x) + \rho$, hence

$$\frac{\sup_B r}{\inf_B r} \leq \frac{r(x)+\rho}{r(x)-\rho} = \frac{\frac{r(x)}{\rho}+1}{\frac{r(x)}{\rho}-1} \leq \frac{\ell+1}{\ell-1} \leq m.$$

For $i \in \mathbb{Z}$, let $r_i = m^i$ and define $Y_i = \{x \in X\,;\, r_{i-2} \leq r(x) \leq r_i\}$ and $L_i = \text{Lip}(v_{|Y_i})$. Note that $\mu(Y_i) \leq C\, r_i^p$. By construction, each ball B of the packing is contained in at least one of the Y_i. If $i \leq 2$, w is constant on Y_i, such balls do not contribute. Let $i \geq 3$. For a ball $B \subset Y_i$,

$$\begin{aligned} \text{diameter}(w(B))^Q &\leq \text{diameter}(v(B))^Q \\ &\leq L_i^Q \text{diameter}(r(B))^Q \\ &\leq C\, L_i^Q \mu(B). \end{aligned}$$

Summing up over all balls of the packing contained in Y_i,

$$\sum_{B_j \subset Y_i} \text{diameter}(w(B_j))^Q \leq C\, L_i^Q \mu(Y_i) \leq C\,(L_i r_i)^Q.$$

Since $v'(t) = \frac{1}{t\log t}$, $L_i \leq \frac{1}{r_{i-2}\log(r_{i-2})}$, $L_i r_i \leq \frac{r_i}{r_{i-2}\log(r_{i-2})} = \frac{m^2}{(i-2)\log m}$. Summing up over $i \geq 3$ leads to

$$\sum_j \text{diameter}(B_j)^Q \leq C \sum_{i=3}^{\infty} (\frac{1}{i-2})^Q < \infty.$$

This shows that $E_\ell^p(w) < \infty$.

Let $\gamma : \mathbb{N} \to X$ be a proper coarse curve based at $K = B(o,\rho)$. Let us show that $w \circ \gamma$ has no limit. By definition, there exists $\tilde{\ell} \geq 1$ such that every $\tilde{\ell}$-packing of $\mathbb{N}$ is mapped to a ℓ-packing $\{B'_j\}$ of X. For each $i = 0, \ldots, 3\tilde{\ell}-1$, this applies to the ℓ-packing of unit balls centered at $3\tilde{\ell}\mathbb{N} + i \subset \mathbb{N}$. We know that

$$\sum_{j \in \mathbb{N}} \text{diameter}(v(B'_{3\tilde{\ell}j+i}))^p < +\infty,$$

hence, summing over i,

$$\sum_{j\in\mathbb{N}} \mathrm{diameter}(v(B'_j))^p < +\infty,$$

so these diameters tend to zero. Since B'_j contains $\gamma(j)$ and $\gamma(j+1)$, this implies that $|v\circ\gamma(j+1) - v\circ\gamma(j)|$ tends to 0. Therefore every number in the interval $[-1,1]$ is a limit of a subsequence of the sequence $w\circ\gamma$, $w\circ\gamma$ has no limit.

We conclude that the family of all proper coarse curves based at some ball has vanishing (p,ℓ)-modulus, i.e., X is (p,ℓ)-parabolic. A fortiori, the family of all proper coarse curves based at o has vanishing (p,ℓ,R,S)-modulus, i.e., X is (p,ℓ,R,S)-parabolic, for $R>0$. □

REMARK 68. – If we are merely interested in (p,ℓ,R,S)-parabolicity for some $R>0$, or (p,ℓ,R,∞)-parabolicity, a weaker form of Ahlfors-regularity is required. It suffices that balls of radius $\rho\geq R$ satisfy $c\,\rho^Q \leq \mu(B)\leq C\,\rho^Q$. Let us call this *$Q$-Ahlfors regularity in the large*.

Indeed, the argument uses only balls of radius $\geq R$.

COROLLARY 69. – *Let X be a non-compact metric space. Let $Q>1$. Assume that X is Q-Ahlfors regular in the large. Then X is (p,ℓ,R,∞)-parabolic for every $\ell>1$ and every $p\geq Q$. A fortiori, it is (p,ℓ,R,S)-parabolic for every $\ell>1$, every $0<R\leq S$ and every $p\geq Q$.*

PROPOSITION 70. – *Let X be a compact p-Ahlfors regular metric space, with $p>1$. Let $x_0\in X$. For every $\ell>1$, there exists a function $w : X\setminus\{x_0\}\to\mathbb{R}$ such that $E^p_\ell(w)<+\infty$ and w has a limit along no coarse curve converging to x. It follows that $X\setminus\{x_0\}$ is (p,ℓ)-parabolic.*

Proof. – The same as for the non-compact case, replacing function $r=d(o,x)$ with $r=1/d(\cdot,x_0)$. □

5.8. Parabolicity of $\mathbb{D}$

The half real line is 1-Ahlfors regular, a case which is not covered by Proposition 67. It is not 1-parabolic. Indeed, any function of finite 1-energy on $\mathbb{R}_+$ has a limit at infinity. However, it is p-parabolic for every $p>1$.

LEMMA 71. – *The half real line $\mathbb{R}_+$ equipped with its metric q.s. structure is p-parabolic for every $p>1$.*

Proof. – Let $\ell>1$. Denote by $m=\frac{\ell+1}{\ell-1}$. We can assume that $m\geq e$. Define

$$u(t)=\log\log|t| \text{ if } t\geq m,\quad u(t)=\log\log m \text{ otherwise.}$$

Let us show that u has finite (p,ℓ)-energy for all $p>1$ and $\ell>1$. Let $\{B_j\}$ be an ℓ-packing of $\mathbb{R}_+$. By assumption, the collection of concentric balls $\{\ell B_j\}$ consists of disjoint intervals. For simplicity, assume that 0 belongs to one of the ℓB_j's, say ℓB_0 (otherwise, translate everything). Write $B_j=[a_j,b_j]$ and assume that $a_j\geq 0$. Since ℓB_0 and ℓB_j are disjoint,

$$\frac{a_j+b_j}{2}-\ell\frac{b_j-a_j}{2}\geq 0,$$

thus

$$b_j \le \frac{\ell+1}{\ell-1} a_j = m a_j.$$

We split the sum $\sum_j |u(b_j) - u(a_j)|^p$ into sub-sums where $a_j \in [m^i, m^{i+1})$. Since intervals B_j are disjoint and u is nondecreasing,

$$\begin{aligned}\sum_{a_j\in[m^i,m^{i+1})} |u(b_j)-u(a_j)| \le u(m^{i+2}) - u(m^i) \\ &= \log((i+2)\log m) - \log(i \log m) \\ &= \log\frac{i+2}{i} \le \frac{2}{i}.\end{aligned}$$

Next, we use the general inequality, for nonnegative numbers x_j,

$$\sum x_j^p \le (\sum x_j)^p,$$

and get

$$\sum_{a_j\in[m^i,m^{i+1})} |u(b_j)-u(a_j)|^p \le (\sum_{a_j\in[m^i,m^{i+1})} |u(b_j)-u(a_j)|)^p \le (\frac{2}{i})^p.$$

This gives

$$\sum_{a_j\ge m} |u(b_j)-u(a_j)|^p \le \sum_{i=1}^{\infty} (\frac{2}{i})^p < +\infty.$$

On the remaining intervals, u is constant, except possibly on one inerval containing m. On this interval, $|u(b_j) - u(a_j)| \le u(m^2)$, so its contribution is bounded independently of the packing. We conclude that the supremum over ℓ-packings of $\sum_j |u(b_j)-u(a_j)|^p$ is bounded, i.e., u has finite p-energy. $\square$

Since, as a q.s. space, $\mathbb{D} \setminus \{0\}$ is isomorphic to the half real line, $\mathbb{D}$ is p-parabolic for all $p > 1$ as well. As a preparation for the next result, note that the isomorphism is the exponential map $t \mapsto \exp(-t) : \mathbb{R}_+ \to \mathbb{D}$. Therefore the function of finite energy on $\mathbb{D}$ is $w(y) = \sin\log|\log|\log y||$ for $y < e^{-e}$.

5.9. Parabolicity of warped products

We study the parabolicity of hyperbolic metric spaces. Thanks to the Poincaré model, they can be viewed as products $\mathbb{D} \times Z$, where the ideal boundary Z can be equipped with Q-Ahlfors-regular metrics, for every $Q > \mathrm{ConfDim}(Z)$. Note that $\mathbb{D} \times Z$ has the same balls as the direct product $[0,1] \times Z$, which is $1+Q$-Ahlfors-regular. Nevertheless, Proposition 70 does not apply directly. Indeed, the $\mathbb{R}_+^*$ action $B \mapsto \ell B$ plays a key role in the proof of Propositions 67 and 70. So a direct argument, akin to Lemma 71, is needed.

PROPOSITION 72. – *Let Z be a compact Q-Ahlfors regular metric space. Let $z_0 \in Z$. Then $\mathbb{D} \times Z \setminus \{(0, z_0)\}$ is $(1+Q, \ell)$-parabolic.*

Proof. – For $(y,z) \in [0,1] \times Z$, let $r(y,z) = \max\{y, d(z,z_0)\}$ denote the distance to $(0,z_0)$. We use the bounded function $w = \sin v(r)$, where

$$v(t) = \log|\log|\log t|| \quad \text{if} \quad t \le r_1, \quad v(t) = \log|\log|\log r_1|| \quad \text{otherwise.}$$

The constant $r_1 = r_1(\ell)$ is produced by the following lemma.

LEMMA 73. – *Let* $v(t) = \log|\log|\log t||$. *Given* $\ell > 1$, *set* $C(\ell) = 17\log(\ell)/\log(\ell/\ell-1)$. *If*

$$b \le r_1 := \min\{\frac{\ell-1}{\ell}, \ell^{-2}, \frac{e^{-e^2}}{\ell}\} \quad \textit{and} \quad \frac{b}{a} \ge \frac{\ell}{\ell-1},$$

then

$$v(a) - v(\ell b) \le C(\ell)\,(v(a) - v(b)).$$

Proof. – We use the inequalities

$$0 \le u \le 1 \implies \log(1+u) \ge \frac{1}{2}u, \quad 0 \le u \le \frac{1}{2} \implies -\log(1-u) \le 2u.$$

Set $t = \log\frac{1}{b}$ and $s' = \log\frac{1}{a} - \log\frac{1}{b} = \log\frac{b}{a} \ge s := \log\frac{\ell}{\ell-1}$. Then

$$\begin{aligned}
v(a) - v(b) &= \log\log\log\frac{1}{a} - \log\log\log\frac{1}{b} \\
&= \log\log(t+s') - \log\log(t) \\
&= \log\left(\frac{\log(t+s')}{\log(t)}\right) \\
&\ge \log\left(\frac{\log(t+s)}{\log(t)}\right) \\
&= \log\left(1 + \frac{\log(t+s) - \log(t)}{\log(t)}\right) \\
&= \log\left(1 + \frac{\log(1+(s/t))}{\log(t)}\right).
\end{aligned}$$

If $t = \log\frac{1}{b} \ge \max\{s, e^2\}$, $s/t \le 1$, $\log(t) \ge 2$, so $\frac{\log(1+(s/t))}{\log(t)} \le 1$. Also $s/t \le 1$, thus

$$\log(1+(s/t)) \ge \frac{1}{2}\frac{s}{t}, \quad \log\left(1 + \frac{\log(1+(s/t))}{\log(t)}\right) \ge \frac{1}{2}\frac{\log(1+(s/t))}{\log(t)} \ge \frac{1}{4}\frac{s}{t\log(t)}$$

and

$$v(a) - v(b) \ge \frac{s}{4t\log(t)}.$$

Conversely, setting $\sigma=\log\frac{1}{b}-\log\frac{1}{\ell b}=\log\ell$,

$$\begin{aligned}
v(b)-v(\ell b)&=\log\log\log\frac{1}{b}-\log\log\log\frac{1}{\ell b}\\
&=\log\log(t)-\log\log(t-\sigma)\\
&=-\log\left(\frac{\log(t-\sigma)}{\log(t)}\right)\\
&=-\log\left(1+\frac{\log(t-\sigma)-\log(t)}{\log(t)}\right)\\
&=-\log\left(1+\frac{\log(1-(\sigma/t))}{\log(t)}\right)\\
&\leq -2\frac{\log(1-(\sigma/t))}{\log(t)}\\
&\leq 4\frac{\sigma}{t\log(t)},
\end{aligned}$$

provided $\sigma/t\leq\frac{1}{2}$ and $-\frac{\log(1-(\sigma/t))}{\log(t)}\leq\frac{1}{2}$, which holds if $t=\log\frac{1}{b}\geq\max\{2\sigma,e^2\}$.

Combining both inequalities yields

$$\begin{aligned}
v(a)-v(\ell b)&=v(a)-v(b)+v(b)-v(\ell b)\\
&\leq v(a)-v(b)+4\frac{\sigma}{t\log(t)}\\
&\leq v(a)-v(b)+4\frac{\sigma}{s}\frac{s}{t\log(t)}\\
&\leq v(a)-v(b)+16\frac{\sigma}{s}(v(a)-v(b))\\
&\leq 17\frac{\sigma}{s}(v(a)-v(b)).\square
\end{aligned}$$

Continuation of the proof of Proposition 72. – Let $\{B_j\}$ be a ℓ-packing of $\mathbb{D}\times Z$. The packing splits into three sub-collections,

1. Balls B such that ℓB contains $(0,z_0)$.
2. Balls B such that ℓB intersects $\mathbb{R}_+\times\{z_0\}$, but does not contain $(0,z_0)$.
3. Balls B such that ℓB does not intersect $\mathbb{R}_+\times\{z_0\}$.

Fix some $p>1$

1. The first sub-collection has at most one element. Since $|w|\leq 1$, it contributes at most 2^p to energy.

2. The second sub-collection is nearly taken care of by Lemma 71. Recall the definition of $\mathbb{D}\times Z$ in Definition 34. If two balls $B=I\times\beta$ and $B'=I'\times\beta'$ in $[0,1]\times Z$ are such that ℓB and $\ell B'$ are disjoint and both intersect $[0,1]\times\{z_0\}$, then the intervals $m_\ell(I)$ and $m_\ell(I')$ are disjoint. Furthermore, if $B=I\times\beta$ and $I=[a,b]$, the radius of B is $\frac{b-a}{2}$, thus for $z\in\beta$, $d(z,z_0)\leq 2\ell\frac{b-a}{2}$, so

$$\inf_B r\geq a,\quad \sup_B r=\sup_B d(\cdot,z_0)\leq\ell(b-a)\leq\ell b.$$

The Z factor plays little role. We expect from Lemma 71 that the sum of p-th powers of diameters of images $w(B_j)$ should be bounded independently of the packing, as soon as $p > 1$.

Here comes the proof. First, the last estimate needs be sharpened. Either $\ell(b-a) \leq b$, in which case $r(B) = [a,b]$ and $v(r(B)) = [v(b), v(a)]$, or $\frac{b}{a} \geq \frac{\ell}{\ell-1}$. Lemma 73 shows that, in both cases,

$$\text{diameter}(v(r(B))) \leq C(\ell)\,(v(a) - v(b)),$$

provided b is small enough.

If an interval $I = [a,b]$ of $(0,1]$ is such that $m_\ell(I)$ does not contain 1, then $a \geq b^m$, for $m = \frac{\ell+1}{\ell-1}$. Indeed, the upper bound of $m_\ell(I)$ is $a^{\frac{1-\ell}{2}} b^{\frac{1+\ell}{2}}$. Hence if $B = I \times \beta$ is a ball of $[0,1] \times Z$ for such an I,

$$\inf_B r = a \geq b^m, \quad \sup_B r \leq \ell b \leq \ell (\inf_B r)^{1/m}.$$

With r_1 as in Lemma 73, define inductively a sequence r_i by $r_{i+1} = (\frac{r_i}{\ell})^m$ (this gives $r_i = \ell^{-m\frac{m^i-1}{m-1}}$). Also, define r_0 so that $r_1 = (\frac{r_0}{\ell})^m$. Let $\{B_j = I_j \times \beta_j\}$ be a ℓ-packing of $\mathbb{D} \times Z$. Assume that all ℓB_j intersect $[0,1] \times \{z_0\}$. At most one ℓB_j contains $(1, z_0)$, let us put it aside (it contributes at most 2^p to energy). All other intervals I_j are disjoint and each one is contained in at least one of the intervals $[r_{i+2}, r_i]$. Therefore the index set is contained in the union of subsets

$$J_i = \{j\,;\, I_j \subset [r_{i+2}, r_i]\},$$

and for all p,

$$\sum_j \text{diameter}(w(B_j))^p \leq \sum_{i=0}^{\infty} \sum_{j \in J_i} \text{diameter}(v(r(B_j)))^p.$$

We split the sub-packing into two sub-families,

1. Balls $B = I \times \beta$ such that $v(r(B)) \neq v(I)$ and $\max I \geq r_1$.
2. Balls $B = I \times \beta$ such that $v(r(B)) = v(I)$ or $\max I \leq r_1$.

Members of the first sub-family satisfy $\max I \geq \frac{\ell}{\ell-1} \min I$ and $\max I \geq r_1$. Balls that have $\max I \geq r_0$, have $\min I \geq r_1$, and v is constant on them. The other balls are contained in $[r_2, r_0]$. The number of such disjoint intervals is bounded in terms of ℓ only. Diameters $\text{diameter}(v(r(B))) \leq v(r_2)$ are bounded in terms of ℓ. Therefore the sum of p-th powers $\text{diameter}(v(r(B)))^p$ over the members of this sub-family is apriori bounded in terms of ℓ and p only.

Members of the second sub-family satisfy

$$\text{diameter}(v(r(B))) \leq C\,\text{diameter}(v(I)).$$

Since the intervals I_j are disjoint,

$$\begin{aligned}\sum_{j \in J_i} \text{diameter}(w(B_j))^p &\leq \left(\sum_{j \in J_i} \text{diameter}(v(r(B_j)))\right)^p \\ &\leq C\,|v(r_{i+2}) - v(r_i)|^p,\end{aligned}$$

$$\sum_j \text{diameter}(w(B_j))^p \leq C \sum_{i=0}^{\infty} |v(r_{i+2}) - v(r_i)|^p.$$

With our choice of $v(t) = \log|\log|\log t||$,

$$v(r_{i+2}) - v(r_i) \sim |\log(i+2) - \log(i)| \sim \frac{2}{i},$$

so the sum of p-th powers converges to a bound that depends only on ℓ and p.

3. The third sub-collection consists of balls $B = [a,b] \times \beta$ such that $z_0 \notin \ell\beta$.

Let us study how r varies along B. Note that the radius of balls B and β equals $\frac{b-a}{2}$. Let $\Delta = \max_\beta d(\cdot, z_0)$ and $\delta = \min_\beta d(\cdot, z_0)$. Then

$$\sup_B r = \max\{b, \Delta\}, \quad \inf_B r = \max\{a, \delta\}.$$

Since $z_0 \notin \ell\beta$, $\delta \geq (\ell-1)\frac{b-a}{2}$. Then $\Delta \leq \delta + b - a \leq \delta + \frac{2}{\ell-1}\delta = m\delta$. On the other hand,

— either $a < \frac{b}{2}$, $\delta \geq (\ell-1)\frac{b-a}{2} \geq (\ell-1)\frac{b}{4}$,

— or $a \geq \frac{b}{2}$.

In the first case,

$$\max\{b, \Delta\} \leq \max\{\frac{4}{\ell-1}\delta, m\delta\} \leq \max\{\frac{4}{\ell-1}, m\} \max\{a, \delta\}.$$

In the second case,

$$\max\{b, \Delta\} \leq \max\{2a, m\delta\} \leq \max\{2, m\} \max\{a, \delta\}.$$

In either case,

$$\sup_B r \leq M \inf_B r,$$

where $M = \max\{m, 2, \frac{4}{\ell-1}\}$.

Let $\nu = dt \otimes \mu$ denote $1+Q$-dimensional Hausdorff measure. Set $r_i = M^{-i}$. Let

$$Y_i = \{(y,z) \in [0,1] \times Z\,;\, r_{i+2} < r(y,z) \leq r_i\}, \quad L_i = \text{Lip}(v|_{[r_{i+2}, r_i]}).$$

Since r is 1-Lipschitz, $\text{Lip}(w_{|Y_i}) \leq L_i$. Each ball B_j of the sub-packing is entirely contained in one of the sets Y_i. Therefore the index set is contained in the union of subsets

$$J'_i = \{j\,;\, B_j \subset Y_i\},$$

and for all p,

$$\sum_j \text{diameter}(w(B_j))^p \leq \sum_{i=0}^{\infty} \sum_{j \in J'_i} \text{diameter}(w(B_j))^p.$$

From now on, $p = 1+Q$. If $j \in J'_i$,

$$\text{diameter}(w(B_j))^{1+Q} \leq L_i^{1+Q} \text{diameter}(B_j)^{1+Q} \leq C\, L_i^{1+Q} \nu(B_j).$$

Thus

$$\sum_{j \in J'_i} \text{diameter}(w(B_j))^{1+Q} \leq C\, L_i^{1+Q} \nu(Y_i) \leq C'\,(L_i r_i)^{1+Q},$$

since $Y_i \subset B((0, z_0), r_i)$.

In order to estimate the Lipschitz constant L_i, observe that

$$v'(t) = \frac{1}{t|\log t|\,\log|\log t|}$$

achieves its maximum on $[r_{i+2}, r_i]$ at $r_{i+2} = M^{-(i+2)}$, hence

$$L_i \le \frac{1}{r_{i+2}(i+2)\log(i+2)}, \quad L_i r_i \le \frac{r_i}{r_{i+2}(i+2)\log(i+2)} \le \frac{M^2}{i+2}.$$

This bounds $\sum_j \mathrm{diameter}(w(B_j))^{1+Q}$ by a quantity that depends only on ℓ and Q.

The final argument, showing that $w \circ \gamma$ has no limit for every proper coarse curve γ is the same as in Proposition 67. □

5.10. Gauges

Here comes the last trick of this section: replacing a compact Q-Ahlfors-regular metric space Z with its snowflake Z^α, which is $\frac{Q}{\alpha}$-Ahlfors-regular. Morally, the warped product $\mathbb{D} \times Z^\alpha$ is $1 + \frac{Q}{\alpha}$-dimensional. Since Z^α admits a wealth of $\frac{1}{\alpha}$-Hölder continuous functions, there are functions with finite $\alpha(1 + \frac{Q}{\alpha}) = \alpha + Q$-energy. These are functions which factor through Z^α, i.e., which are constant on segments $\mathbb{D} \times \{z\}$. It is likely that there be no more. Indeed, the $1 + \frac{Q}{\alpha}$-modulus of the family of these segments is nonzero (see [31]), so for $p < 1 + \frac{Q}{\alpha}$, every function with finite p-energy should factor through Z^α. In this mechanism, the fact that the Hausdorff α-measure of segments for $\alpha < 1$ is infinite is essential. In other words, the obstacle to the existence of finite energy $\alpha + Q$ functions with poles is a small scale phenomenon. It turns out that it disappears if small balls are avoided.

This leads us to adapt the notion of energy, by forbidding packings whose balls are too small according to a given gauge.

Definition 74. – *Let X be a quasimetric space, let $g : X \to \mathbb{R}_+$ be a nonnegative function. An (ℓ, g, S)*-packing *is a collection of balls $\{B_j(x_j, r_j)\}$, each with radius $S \ge r_j \ge g(x_j)$, such that the concentric balls ℓB_j are pairwise disjoint. An (N, ℓ, g)-packing is the union of at most N (ℓ, g)-packings.*

Example 75. – According to Proposition 36, the Poincaré model of a hyperbolic metric space X maps (ℓ, R, ∞)-packings of X to (ℓ, g, S)-packings of $\mathbb{D} \times \partial X$, with the gauge

$$g(y, z) = \tanh(R)y.$$

Note that $\tanh(R)$ tends to 1 as R tends to infinity.

Definition 76. – *Let X be a quasimetric space equipped with a gauge $g : X \to \mathbb{R}_+$. Let $\ell \ge 1$. Let $u : X \to \mathbb{R}$ be a function. Define its p-energy at parameters ℓ and g as follows.*

$$E^p_{\ell,g}(u) = \sup\{\sum_j \mathrm{diameter}(u(B_j))^p\,;\ (\ell, g)\text{-packings } \{B_j\}\}.$$

Definition 77. – *Let X be a metric space equipped with a gauge $g : X \to \mathbb{R}_+$. Let Γ be a family of coarse curves in X. The (p, ℓ, g)*-modulus $\mathrm{mod}_{p,\ell,g}(\Gamma)$ *is the infimum of $E^p_{\ell,g}$-energies of maps $u : X \to Y$ to metric spaces such that for every curve $\gamma \in \Gamma$,* $\mathrm{length}(u \circ \gamma) \ge 1$.

Definition 78. – *Let X be a locally compact noncompact quasimetric space equipped with a gauge $g : X \to \mathbb{R}_+$. Say X is* (p, ℓ, g)-parabolic *if the family of proper coarse curves based at some compact set with nonempty interior has vanishing (p, ℓ, g)-modulus.*

Definition 79. – *Let X be a metric space and X' a q.m.q.s. space equipped with a gauge $g' : X' \to \mathbb{R}$. We denote by $\mathcal{B}_{g'}^{X'}$ the set of balls $B' = B(x', r')$ of X' such that $r' \geq g'(x')$.*

Let $f : X \to X'$ be a map. Say f is (R, S, g')-coarsely conformal *if there exist a map*

$$B \mapsto B', \quad \mathcal{B}_{R,S}^{X} \to \mathcal{B}_{g'}^{X'},$$

and for all $\ell' \geq 1$, an $\ell \geq 1$ and an N' such that

1. *for all $B \in \mathcal{B}_{R,S}^{X}$, $f(B) \subset B'$.*
2. *If $\{B_j\}$ is a (ℓ, R, S)-packing of X, then $\{B_j'\}$ is an (N', ℓ', g')-packing of X'.*

We say that f is g'-coarsely conformal *if there exists $R > 0$ such that for all finite $S \geq R$, f is (R, S, g')-coarsely conformal.*

We say that f is g'-roughly conformal *if there exists $R > 0$ such that f is (R, ∞, g')-coarsely conformal.*

In other words, one merely restricts the class of balls in the range to satisfy the gauge condition $r' \geq g'(x')$.

Example 80. – Let X' be a hyperbolic metric space. Let $\pi : X' \to \mathbb{D} \times \partial X'$ be its Poincaré model, equipped with the gauge $g'(y, z) = \frac{y}{2}$. Then π is g'-roughly conformal. It follows that if X is a metric space and $f : X \to X'$ is a uniformly conformal map, then $\pi \circ f : X \to \mathbb{D} \times \partial X'$ is a g'-coarsely conformal map.

Similarly, one can define $L^p_{\ell,g}$ norms on cochains and $L^p_{\ell,g}$ cohomology. A g'-roughly conformal map $X \to X'$ induces morphisms $L^p_{\ell',g'} H^{\cdot}(X') \to L^p_{\ell} H^{\cdot}(X)$.

5.11. Parabolicity of twisted products with snowflakes

Proposition 81. – *Let Z be a compact Q-Ahlfors regular metric space. Let $0 < \alpha \leq 1$ and $\ell > 1$. Let $z_0 \in Z$. Let $g : \mathbb{D} \times Z^{\alpha} \to \mathbb{R}_+$ be the gauge defined by $g(y, z) = \frac{y}{2}$. Then*

1. *For every $\ell > 1$, functions of finite $\alpha + Q$-energy, together with the projection on the first factor, separate $\mathbb{D} \times Z^{\alpha}$.*
2. *For every $\ell > 1$, $\mathbb{D} \times Z^{\alpha} \setminus \{(0, z_0)\}$ is $(\alpha + Q, \ell, g)$-parabolic.*

Proof. – 1. When the second factor is the snowflake space Z^{α}, the discussion of Proposition 72 provides the exponent $1 + \frac{Q}{\alpha}$. This can be improved into $\alpha + Q$ for the following reason: on Z^{α}, the function

$$\rho : Z^{\alpha} \to \mathbb{R}_+, \quad \rho(z) = d_{Z^{\alpha}}(z, z_0)$$

is not merely 1-Lipschitz, it is $1/\alpha$-Hölder continuous (although with a constant that deteriorates when getting close to z_0). Indeed, if $z, z' \in Z$, with $\rho(z) \le \rho(z')$,

$$\begin{aligned}|\rho(z) - \rho(z')| &= |d_Z(z, z_0)^\alpha - d_Z(z', z_0)^\alpha| \\ &\le \alpha d_Z(z, z_0)^{\alpha-1}|d_Z(z, z_0) - d_Z(z', z_0)| \\ &\le d_Z(z, z_0)^{\alpha-1} d_Z(z, z') \\ &= \rho(z)^{\frac{\alpha-1}{\alpha}} d_{Z^\alpha}(z, z')^{1/\alpha}.\end{aligned}$$

It follows that functions like $v_{z_0,\epsilon} = \max\{0, \rho - \epsilon\}$, $z_0 \in Z$, $\epsilon > 0$ have finite $\alpha + Q$-energy. Observe that, together with $u(y, z) = y$ (which does not have finite energy), they separate points.

2. The proof closely follows the proof of Proposition 72. We first study the variation of function $r(y, z) = \max\{y, \rho(z)\}$ along balls, and then estimate the energy of $w = \sin(v) \circ r$, for $v(t) = |\log|\log|\log t|||$.

Let $B = [a, b] \times \beta$ be a ball of $\mathbb{D} \times Z^\alpha$ such that $z_0 \notin \ell\beta$. Denote again by

$$\delta = \inf_\beta \rho, \quad \Delta = \sup_\beta \rho.$$

Since the triangle inequality holds in Z^α, the bounds $\delta \ge (\ell - 1)\frac{b-a}{2}$ and $\Delta \le m\delta$ still hold. Therefore the estimate

$$\sup_B r \le M \inf_B r$$

is unaffected. On the other hand, the $1/\alpha$-Hölder character of ρ leads to

$$\begin{aligned}\Delta - \delta &\le \delta^{\frac{\alpha-1}{\alpha}}(b - a)^{1/\alpha} \\ &\le m^{\frac{1-\alpha}{\alpha}} \Delta^{\frac{\alpha-1}{\alpha}}(b - a)^{1/\alpha}.\end{aligned}$$

If B satisfies the gauge condition, i.e., $\frac{b-a}{2} \ge g(\frac{a+b}{2})$ where $g(y) = \frac{y}{2}$, then $b \le 3\frac{b-a}{2}$,

$$\frac{b-a}{2} = (\frac{b-a}{2})^{\frac{\alpha-1}{\alpha}}(\frac{b-a}{2})^{1/\alpha} \le (\frac{b}{3})^{\frac{\alpha-1}{\alpha}}(\frac{b-a}{2})^{1/\alpha}.$$

Then

$$\begin{aligned}\sup_B r - \inf_B r &\le \max\{b - a, \Delta - \delta\} \\ &\le C(\min\{b, \Delta\})^{\frac{\alpha-1}{\alpha}}(\frac{b-a}{2})^{1/\alpha} \\ &\le C(\inf_B r)^{\frac{\alpha-1}{\alpha}}(\frac{b-a}{2})^{1/\alpha} \\ &\le C'(\sup_B r)^{\frac{\alpha-1}{\alpha}}(\frac{b-a}{2})^{1/\alpha}.\end{aligned}$$

Therefore,

$$\sup_B r - \inf_B r \le C\ (\sup_B r)^{\frac{\alpha-1}{\alpha}} \text{diameter}(B)^{1/\alpha},$$

where C depends only on ℓ and α.

Given an (ℓ, g)-packing of $\mathbb{D}\times Z^\alpha$, let us split it into 3 sub-collections, according to wether concentric balls ℓB_j

— contain $(0, z_0)$;

— intersect $\mathbb{D} \times \{z_0\}$ but do not contain $(0, z_0)$;

— do not intersect $\mathbb{D} \times \{z_0\}$.

Nothing needs be changed for the first two sub-collections, since an upper bound on energy sums is obtained for any exponent $p > 1$. For the third one, the same constants can be used:

$$
\begin{aligned}
r_i &= M^{-i}, \quad Y_i = \{(y,z) \in [0,1] \times Z \,;\, r_{i+2} < r(y,z) \leq r_i\},\\
L_i &= \mathrm{Lip}(v_{|[r_{i+2},r_i]}).
\end{aligned}
$$

Again every ball in the sub-packing is contained in at least one of the Y_i. We see that a ball B contained in Y_i satisfies

$$
\begin{aligned}
\mathrm{diameter}(w(B)) &\leq \mathrm{diameter}(v(r(B)))\\
&\leq L_i(\sup_B r - \inf_B r)\\
&\leq C\, L_i\, r_i^{\frac{\alpha-1}{\alpha}}\, \mathrm{diameter}(B)^{1/\alpha}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathrm{diameter}(w(B))^{\alpha+Q} &\leq C^{\alpha+Q}\, L_i{}^{\alpha+Q}\, r_i^{\frac{(\alpha-1)(\alpha+Q)}{\alpha}} \mathrm{diameter}(B)^{1+(Q/\alpha)}\\
&\leq C'\, L_i{}^{\alpha+Q} r_i^{\frac{(\alpha-1)(\alpha+Q)}{\alpha}}\, \nu(B).
\end{aligned}
$$

Since $\nu(Y_i) \leq \mathrm{const.}\, r_i^{1+(Q/\alpha)}$, summing over all balls in the sub-packing contained in Y_i gives

$$
\begin{aligned}
\sum_{j \in J_i'} \mathrm{diameter}(w(B))^{\alpha+Q} &\leq C''\, L_i{}^{\alpha+Q} r_i^{\frac{(\alpha-1)(\alpha+Q)}{\alpha}}\, r_i^{1+(Q/\alpha)}\\
&= C''\,(L_i r_i)^{\alpha+Q}.
\end{aligned}
$$

The choice of $v(t) = \log|\log|\log t||$ yields again $L_i r_i \leq C'''/(i+2)$, and the sum is bounded above in terms of ℓ, α and Q only.

The final argument, showing that $w \circ \gamma$ has no limit for every proper coarse curve γ, is unchanged. $\square$

6. L^p-cohomology

6.1. Definition

Here is one more avatar of the definition of L^p cohomology for metric spaces. This one has the advantage that it does not require any measure. For earlier attempts, see [11], [14].

Definition 82. – *Let X be a metric space. A k-*simplex of size S *in X is a $k+1$-tuple of points belonging to some ball of radius S. A k-*cochain of size S *on X is a real valued function κ defined on the set of k-simplices of size S. Its $L^p_{\ell,R,S}$-norm is*

$$\|\kappa\|_{L^p_{\ell,R,S}} = \sup\{\sum_j \sup_{(B_j)^{k+1}} |\kappa|^p \,;\, \textit{all } (\ell,R,S)\textit{-packings } \{B_j\}\}^{1/p}.$$

Let $L^p_{\ell,R,S}C^k(X)$ denote the space of k-cochains with finite $L^p_{\ell,R,S}$-norm.

The coboundary operator d maps k-cochains to $k+1$-cochains,

$$d\kappa(x_0,\dots,x_{k+1}) = \kappa(x_1,\dots,x_{k+1}) - \kappa(x_0,x_2,\dots,x_{k+1}) + \cdots \\ + (-1)^{k+1}\kappa(x_0,\dots,x_k).$$

Denote by

$$\mathcal{L}^{q,p}_{\ell,R,S}C^k(X) = L^q_{\ell,R,S}C^k(X)\mathrm{cap}d^{-1}L^p_{\ell,R,S}C^{k+1}(X),$$

in order to turn d into a bounded operator $\mathcal{L}^{q,p}_{\ell,R,S}(X) \to L^p_{\ell,R,S}(X)$. The $L^{q,p}$-cohomology *of X is*

$$L^{q,p}_{\ell,R,S}H^k(X) = \left(\ker(d)\mathrm{cap}L^p_{\ell,R,S}C^k(X)\right)/d\left(\mathcal{L}^{q,p}_{\ell,R,S}C^{k-1}(X)\right).$$

The exact $L^{q,p}$-cohomology *of X is the kernel of the forgetful map $L^{q,p}_{\ell,R,S}H^k(X) \to H^k(X)$.*

When $p=q$, $\mathcal{L}^{p,p} = L^p$ and $L^{p,p}$-cohomology is simply called L^p-cohomology.

Remark 83. – The definition of $L^{q,p}_{\ell,0,\infty}H^k(X)$ extends to q.s. spaces X, and is simply denoted by $L^{q,p}_{\ell}H^k(X)$.

For instance, a bounded function $u : X \to \mathbb{R}$ can be viewed as a 0-cochain of infinite size, $du(x_1,x_2) = u(x_2) - u(x_1)$ is a 1-cochain of infinite size (i.e., belonging to $L^{\infty}_{\ell,0,\infty}C^1(X)$), and, for all R and S,

$$E^p_{\ell,R,S}(u) = \|du\|^p_{L^p_{\ell,R,S}}.$$

Example 84. – If X is a compact infinite d-Ahlfors regular metric space, then, for all $S>0$, $L^p_{\ell,0,S}H^1(X) \neq 0$ for $p \geq d$.

Indeed, in an infinite metric space, one can ℓ-pack infinitely many small balls. Therefore a function which is ≥ 1 has infinite $L^p_{\ell,0,S}$ norm. Since non constant Lipschitz functions on X have finite energy, and do not belong to any $L^p_{\ell,0,S}C^0(X)$, their $L^p_{\ell,0,S}$ cohomology classes do not vanish.

6.2. Link to usual L^p-cohomology

L^p-cohomology calculations on manifolds (resp. on simplicial complexes) require the classical de Rham (resp. simplicial) definition of cohomology. There is a de Rham style theorem relating Definition 82 to smooth differential forms (resp. simplicial cochains). It shows up in [15]. We shall need the more general case of $L^{q,p}$-cohomology, which appears in [13].

Proposition 85 ([32]). – *Let $1 \leq p \leq q < +\infty$. Let X be a bounded geometry simplicial complex. Assume that the cohomology of X vanishes uniformly up to degree k, i.e., for all $T>0$, there exists $\tilde{T}$ such that for all $x \in X$, the inclusion $B(x,T) \to B(x,\tilde{T})$ induces the 0 map in cohomology up to degree k.*

Then for every $R>0$ and $S<+\infty$ and for large enough $\ell \geq 1$, there is a natural isomorphism of $L^{q,p}$-cohomologies $L^{q,p}_{\ell,R,S}H^k(X) \simeq \ell^{q,p}H^k(X)$. In degree $k+1$, the isomorphism persists provided the space $\ell^{q,p}H^k(X)$ is replaced with exact cohomology*, i.e., the kernel $E\ell^{q,p}H^k(X)$ of the forgetful map $\ell^{q,p}H^k(X) \to H^k(X)$. This isomorphism is compatible with multiplicative structures.*

For degree 1 cohomology, the size limit plays no role. Indeed, a 1-cocycle of size S on a simply connected manifold or simplicial complex, say, is the differential of a function, i.e., a 0-cochain of arbitrary size, and thus uniquely extends to become a 1-cocycle without size limit. The considerations of Subsection 5.2 show that, a priori, all $L^p_{\ell,R,S}$-norms are equivalent on 1-cocycles. For higher degree cohomology, this holds only at the cohomology level, under suitable assumptions, thanks to Proposition 85.

REMARK 86. – For a bounded geometry n-manifold X with boundary, we define $\ell^{q,p}H^{\cdot}(X)$ as the $\ell^{q,p}$ cohomology of a bounded geometry simplicial complex quasiisometric to X. There is an alternative notion, defined in terms of differential forms. This leads to the same cohomology if and only if

$$\frac{1}{p}-\frac{1}{q}\leq\frac{1}{n}$$

and $p>1$ if degree $k=n$, see [32].

6.3. Functoriality of L^p-cohomology

For every $\ell'\geq 1$, (R,S,R',S')-coarse conformal map $f:X\to X'$ induces a bounded linear map

$$f^*:L^{q,p}_{\ell',R',S'}H^k(X')\to L^{q,p}_{\ell,R,S}H^k(X)$$

for suitable $\ell\geq 1$. Exact cohomology is natural as well.

Under the assumptions of Proposition 85, $\ell^{q,p}$-cohomology is natural under coarse embeddings and a quasi-isometry invariant. Indeed, it is isomorphic to $L^{q,p}_{\ell,R,S}$ cohomology which has these properties. On the other hand, it is not clear whether it is natural under large-scale conformal maps, since $L^{q,p}_{\ell,R,\infty}$ cohomology may differ from ordinary $\ell^{q,p}$-cohomology.

6.4. Vanishing of 1-cohomology and limits

DEFINITION 87. – *Let X be a metric space, Y a topological space, and $y\in Y$. Assume X is unbounded. Say a map $f:X\to Y$ tends to y at infinity if for every neighborhood V of y, there exists a bounded set $K\subset X$ such that $f(x)\in V$ when $x\notin K$.*

LEMMA 88. – *Let X be an unbounded metric space. Let $q<\infty$. Then every function $u\in L^q_{\ell,R,S}C^0(X)$ tends to 0 at infinity.*

Proof. – Fix $\epsilon>0$. Let $\{B_j\}$ be an (ℓ,R,S)-packing such that

$$\sum_j(\sup_{B_j}|u|)^q>\|u\|^q_{L^q_{\ell,R,S}}-\epsilon.$$

Pick a finite subfamily which achieves the sum minus ϵ. The union of this finite subfamily is contained in a ball K. If $d(x,\ell K)>\ell R$, add $B(x,R)$ to the finite subfamily to get a larger (ℓ,R,S)-packing. By definition of energy, $\sup_{B(x,R)}|u|^q<2\epsilon$. In particular, we have $|u(x)|<(2\epsilon)^{1/q}$ outside a bounded set. This shows that u tends to 0 at infinity. □

COROLLARY 89. – *Let X be an unbounded metric space. Let $q < \infty$. Assume that the $L^{q,p}$-cohomology of X vanishes in degree 1, i.e.,*

$$L^{q,p}_{\ell,R,S}H^1(X) = 0.$$

Let Y be a complete metric space. Then every map from X to Y with finite $E^p_{\ell,R,S}$ energy has a limit at infinity.

Proof. – Let $u : X \to Y$ have finite energy. Using a packing with only one ball, one sees that u is bounded, i.e., its image is contained in a closed ball $Z \subset Y$. For $y \in Z$, set $v_y(x) = d(u(x), y)$. Then v_y has finite $E^p_{\ell,R,S}$ energy. By assumption, there exists a 0-cochain $w \in L^q_{\ell,R,S}C^0(X)$ such that $dw = dv_y$. This implies that v_y has a finite limit $\alpha(y)$ at infinity. α belongs to the closure of the Kuratowski embedding of Z in $L^\infty(Z)$. Since Z is complete, the embedding has a closed image, so there exists a point $z \in Z$ such that $\alpha(y) = d(z, y)$ for all $y \in Z$. In particular, $d(u(x), z) = v_z(x)$ tends to $\alpha(z) = 0$. □

6.5. Vanishing of reduced 1-cohomology and limits

DEFINITION 90. – *Let X be a metric space. The* reduced $L^{q,p}$-cohomology *of X is obtained by modding out by the L^p-closure of the image of the coboundary operator d,*

$$L^{q,p}_{\ell,R,S}\bar{H}^k(X) = \left(\ker(d)\mathrm{cap}L^p_{\ell,R,S}C^k(X)\right)/\overline{d\left(\mathcal{L}^{q,p}_{\ell,R,S}C^{k-1}(X)\right)}.$$

The reduced exact $L^{q,p}$-cohomology *of X is the kernel of the forgetful map*

$$L^{q,p}_{\ell,R,S}\bar{H}^k(X) \to H^k(X).$$

LEMMA 91. – *Let X be an unbounded metric space with a base point. Then for every finite p-energy function u such that the reduced $L^{q,p}_{\ell,R,S}$-cohomology class of du vanishes, there exists $c \in \mathbb{R}$ such that u converges to c along p-almost every based* $(1,1)$*-curve.*

Proof. – Assume that $u_j \in L^q_{\ell,R,S}C^0(X)$ and

$$\|du_j - du\|_{L^p_{\ell,R,S}} \text{ tends to 0 as } j \text{ tends to } \infty.$$

For $t \in \mathbb{R}$, let $\Gamma_{t,+}$ (resp. $\Gamma_{t,-}$) be the family of based $(1,1)$-curves γ along which u has a finite limit and $\lim u \circ \gamma \geq t$ (resp. $\leq t$). Fix $s < t$. Let $v_j = \frac{2}{t-s}(u - u_j)$.

Assume that there exists a based 1, 1-curve $\gamma \in \Gamma_{t,+}$ such that, for infinitely many j, $\mathrm{length}(v_j \circ \gamma) \leq 1$. For those j's, for every $\gamma' \in \Gamma_{s,-}$, $\mathrm{length}(v_j \circ \gamma') \leq 1$. Indeed, along the bi-infinite curve obtained by concatenating γ and γ', the total variation of v_j is ≥ 2. Therefore, for infinitely many j's,

$$\mathrm{mod}_{p,\ell,R,S}(\Gamma_{s,-}) \leq E(v_j) = (\frac{2}{t-s}\|du_j - du\|_{L^p_{\ell,R,S}})^p,$$

and $\mathrm{mod}_{p,\ell,R,S}(\Gamma_{s,-}) = 0$.

Otherwise, for each $\gamma \in \Gamma_{t,+}$, for all but finitely many values of j, $\mathrm{length}(v_j \circ \gamma) \geq 1$. $\Gamma_{t,+}$ is the union of sub-families

$$\Gamma_{t,+,J} = \{\gamma \in \Gamma_{t,+}\,;\, \forall j \geq J,\ \mathrm{length}(v_j \circ \gamma) \geq 1\},$$

each of which has vanishing modulus.

By stability under countable unions, $\mathrm{mod}_{p,\ell,R,S}(\Gamma_{t,+}) = 0$.

Let c be the supremum of all $t \in \mathbb{R}$ such that $\operatorname{mod}_{p,\ell,R,S}(\Gamma_{t,+}) > 0$. By stability under countable unions, the family of based curves along which u has a finite limit $> c$ has vanishing modulus. If $c = -\infty$, for every $n \in \mathbb{Z}$, the family of based curves along which u has a finite limit $\leq n$ has vanishing modulus. Thus the family of all based curves has vanishing modulus, and the lemma is proved. Otherwise, $\operatorname{mod}_{p,\ell,R,S}(\Gamma_{s,-}) = 0$ for all $s < c$. By stability under countable unions, the family of based curves along which u has a finite limit $< c$ has vanishing modulus. Since, according to Lemma 59, u has a finite limit along p-almost every based $(1,1)$-curve, this shows that u tends to c along almost every based $(1,1)$-curve. □

Corollary 92. – *Let X be an unbounded metric space. Let Y be a complete metric space. Let $u : X \to Y$ have finite $E^p_{\ell,R,S}$ energy. Assume that the reduced $L^{q,p}$-cohomology of X vanishes, i.e., $L^{q,p}_{\ell,R,S}\bar{H}^1(X) = 0$. Then u has a common limit along p-almost every based curve.*

Proof. – For $y \in Y$, set $v_y(x) = d(u(x), y)$. Then v_y has finite $E^p_{\ell,R,S}$ energy. By assumption, dv_y belongs to the $L^p_{\ell,R,S}$-closure of $d\mathcal{L}^{q,p}_{\ell,R,S}C^0(X)$. Lemma 91 implies that v_y has a finite limit $\alpha(y)$ along almost every based curve. α belongs to the closure of the Kuratowski embedding of Y in $L^\infty(Y)$. Since Y is complete, the embedding has a closed image, so there exists a point $z \in Y$ such that $\alpha(y) = d(z, y)$ for all $y \in Y$. In particular, $d(u(x), z) = v_z(x)$ tends to $\alpha(z) = 0$ along almost every based curve. □

6.6. p-separability

Definition 93. – *Let X be a q.s. space. Let $\mathcal{E}_{p,\ell}(X)$ denote the space of continuous real-valued functions on X with finite (p,ℓ)-energy.*

Definition 94. – *Say a q.s. space X is p-separated if for every large enough ℓ, $\mathcal{E}_{p,\ell}$ separates points and for every point $x \in X$, $\mathcal{E}_{p,\ell}(X \setminus \{x\})$ contains a function which has no limit along all coarse curves converging to x. If X is non-compact, one requires further that $\mathcal{E}_{p,\ell}(X)$ contains a function which has no limit along all coarse curves tending to infinity.*

Example 95. – Q-Ahlfors-regular metric spaces are p-separated for all $p \geq Q$.

Proof. – Proposition 39 shows that Lipschitz functions with bounded support have finite energy. They separate points. Propositions 67 and 70 establish parabolicity of X and of point complements. □

Proposition 96. – *Let $1 \leq p, q < +\infty$. Let X be a locally compact unbounded metric space. Let X' be a separable locally compact p-separated q.s. space. Let $f : X \to X'$ be a coarse conformal map. Then there exists $R > 0$ such that for all $S \geq R$ and $\ell' > 1$, for all large enough ℓ,*

— *either the induced map*

$$f^* : EL^{q,p}_{\ell'}\bar{H}^1(X') \to EL^{q,p}_{\ell,R,S}\bar{H}^1(X)$$

in reduced exact $L^{q,p}$-cohomology (see Definition 90) does not vanish,

— *or X is (p,ℓ,R,S)-parabolic.*

Proof. – Let us treat first the simpler case when unreduced L^p-cohomology vanishes. Let $f : X \to X'$ be a coarse conformal map. Assume that f has distinct accumulation points x_1' and x_2' at infinity. By assumption, there exists a continuous function v on X' with finite (p, ℓ')-energy such that $v(x_1') \neq v(x_2')$. Then, for all $R \leq S$, $v \circ f$ has finite $E^p_{\ell,R,S}$ energy for suitable ℓ. If $EL^{q,p}_{\ell,R,S}H^1(X)$ vanishes, according to Corollary 89, $v \circ f$ has a limit at infinity. This should be at the same time $v(x_1')$ and $v(x_2')$, a contradiction. We conclude that f has at most one accumulation point at infinity. Hence either it has a limit x', or it tends to infinity.

In either case, there exists a finite $E^p_{\ell,R,S}$-energy function $w : X' \to \mathbb{R}$ that has no limit along p-almost every coarse curve converging to x' (resp. to infinity). Then $w \circ f$ has finite $E^p_{\ell,R,S}$ energy as well, it must have a finite limit in $\mathbb{R}$. This contradicts the assumption that the family of based $(1,1)$-curves in X has positive (p, ℓ, R, S)-modulus, by Lemma 59. We conclude that f^* does not vanish on $EL^{q,p}_{\ell}H^1(X')$.

Assume that f induces a trivial map in reduced cohomology and that X is non-(p, ℓ, R, S)-parabolic. Equip $\mathcal{E}_{p,\ell}(X')$ with the topology of uniform convergence on compact sets. Let $D \subset \mathcal{E}_{p,\ell}(X')$ be a countable dense subset. Then D still separates points. We know that for all $v \in D$, $v \circ f \circ \gamma$ has a common limit y_v for almost every based $(1,1)$-curve γ in X. For $v \in D$, let Γ_v be the family of (R, S)-based curves $\gamma \subset X$ such that $v \circ f \circ \gamma$ does not have a limit or has a limit which differs from y_v. Then $\Gamma = \bigcup_{v \in D} \Gamma_v$ has vanishing (p, ℓ, R, S)-modulus. Let Γ' be the complementary family. Since X is non-(p, ℓ, R, S)-parabolic, Γ' is non-empty. Fix two based curves $\gamma, \gamma' \in \Gamma'$. Assume that $f \circ \gamma$ and $f \circ \gamma'$ have distinct accumulation points x_1' and x_2' in X'. Let $v \in D$ be such that $v(x_1') \neq v(x_2')$. By construction, $v \circ \gamma$ and $v \circ \gamma'$ converge to y_v. Since v is continuous, $v \circ \gamma$ subconverges to $v(x_1')$ and $v \circ \gamma'$ to $v(x_2')$, a contradiction. We conclude that f has a common limit x' along all $\gamma \in \Gamma'$. The argument ends in the same manner. □

6.7. Relative p-separability

Here comes a relative version of Proposition 96, motivated by the case of warped products.

Definition 97. – *Let X be a q.m.q.s. space equipped with a gauge function $g : X \to \mathbb{R}_+$. Let $\mathcal{E}_{p,\ell,g}(X)$ denote the space of continuous real-valued functions on X with finite (p, ℓ, g)-energy.*

Definition 98. – *Let X be a q.m.q.s. Let $u : X \to Y$ be a continuous map to a topological space, let $g : X \to \mathbb{R}_+$ be a gauge. Say X is* p-separated relatively to u, g *if for every large enough ℓ,*

1. *$\mathcal{E}_{p,\ell,g} \cup \{u\}$ separates points.*
2. *Complements of points in X where $g = 0$ are (p, ℓ, g)-parabolic.*

Example 99. – Let Z be a compact Q-Ahlfors-regular metric space. Let $0 < \alpha \leq 1$. Let $Z^\alpha = (Z, d_Z^\alpha)$ be a snowflaked copy of Z. Let $X = \mathbb{D} \times Z^\alpha$. Let u be the projection to the first factor $Y = [0,1]$ and $g = \frac{u}{2}$. Then X is p-separated relative to u, g for all $p \geq \alpha + Q$.

Indeed, the projection to the second factor equipped with d_Z has finite (p, ℓ, g)-energy for all $p \geq \alpha + Q$ and all $\ell > 1$, see Example 40. Together with u, it separates points. Proposition 81 states that complements of points in X where $g = 0$ are (p, ℓ, g)-parabolic.

Proposition 100. – *Let $1 \leq p,q < +\infty$. Let X be a locally compact metric space containing at least one based $(1,1)$-curve. Let X' be a compact q.m.q.s. space which is p-separated relatively to a map $u : X' \to Y$ and a gauge $g' : X \to \mathbb{R}_+$. Let $f : X \to X'$ be a g'-coarse conformal map (see Definition 79). Assume that the map $u \circ f$ tends to some point $y \in Y$ at infinity. Then there exists $R > 0$ such that for all $S \geq R$ and $\ell' > 1$, for all large enough ℓ,*

— *either the induced map*

$$f^* : EL^{q,p}_{\ell',g'}\bar{H}^1(X') \to EL^{q,p}_{\ell,R,S}\bar{H}^1(X)$$

in reduced exact $L^{q,p}$-cohomology does not vanish,

— *or X is (p,ℓ,R,S)-parabolic.*

Proof. – By contradiction. Assume that f induces a trivial map in reduced cohomology and that X is non-(p,ℓ,R,S)-parabolic. Equip $\mathcal{E}_{p,\ell}$ with the topology of uniform convergence. Let $D \subset \mathcal{E}_{p,\ell}$ be a countable dense subset. Then $D \cup \{u\}$ still separates points. For all $v \in D$, $v \circ f \circ \gamma$ has a common limit t_v for almost every based $(1,1)$-curve γ in X. For $v \in D$, let Γ_v be the family of (R,S)-based curves $\gamma \subset X$ such that $v \circ f \circ \gamma$ does not have a limit or has a limit which differs from t_v. Then $\Gamma = \bigcup_{v\in D}\Gamma_v$ has vanishing (p,ℓ,R,S)-modulus. Let Γ' be the complementary family. Since X is non-(p,ℓ,R,S)-parabolic, Γ' is non-empty. Fix two based curves $\gamma,\gamma' \in \Gamma'$. Assume that $f \circ \gamma$ and $f \circ \gamma'$ have distinct accumulation points x'_1 and x'_2 in X'. Since $u(x'_1) = u(x'_2) = y$, there exists $v \in D$ such that $v(x'_1) \neq v(x'_2)$. By construction, $v \circ \gamma$ and $v \circ \gamma'$ converge to t_v. Since v is continuous, $v \circ \gamma$ subconverges to $v(x'_1)$ and $v \circ \gamma'$ to $v(x'_2)$, a contradiction. We conclude that f has a common limit x' along all $\gamma \in \Gamma'$.

Let $w : X' \to \mathbb{R}$ be a finite p,ℓ',g-energy function that has no limit along (p,ℓ,g)-almost every coarse curve converging to x'. Then $w \circ f$ has finite $E^p_{\ell,R,S}$ energy as well, it must have a finite limit in $\mathbb{R}$. This contradicts the fact that the family of based $(1,1)$-curves in X has positive (p,ℓ,R,S)-modulus. We conclude that either f^* does not vanish on $EL^{q,p}_{\ell}\bar{H}^1(X')$ or X is p-parabolic. □

7. Lack of coarse conformal maps

Theorem 3. – *Let $1 < p,q < +\infty$. Let X be a simplicial complex with bounded geometry and uniform vanishing of 1-cohomology. Assume that X is p-parabolic for no choices of parameters (ℓ,R,S) and that $EL^{q,p}\bar{H}^1(X) = 0$.*

1. *Let X' be a p-Ahlfors-regular metric space. Then there can be no coarse conformal maps $X \to X'$.*
2. *Let $0 < \alpha \leq 1$. Let Z be a compact $p{-}\alpha$-Ahlfors-regular metric space. Let X' be a warped product $\mathbb{D} \times Z^\alpha$, equipped with the gauge $g(y,z) = \frac{y}{2}$. For every g-coarse conformal map $X \to X'$ (see Definition 79), the projected map to the first factor $X \to [0,1]$ cannot tend to 0.*

Proof. – The first assertion follows from Example 95 (p-Ahlfors-regular spaces are p-separated) and Proposition 96 (existence of a coarse conformal map to a p-separated space implies either p-parabolicity or nonvanishing of reduced exact $L^{p,q}$ cohomology).

The second assertion follows from Example 99 (warped products of the hyperbolic half-line and snowflaked Ahlfors-regular spaces are separated relative to the projection to the first factor), Proposition 100 (existence of a coarse conformal map to a q.m.q.s. space which is p-separated relative to a map u and a gauge g implies either p-parabolicity or nonvanishing of reduced exact $L^{p,q}$ cohomology, unless u converges along the coarse conformal map). Proposition 85 is used to relate metric space $L^{p,q}$ cohomology to usual $\ell^{p,q}$ cohomology of manifolds or simplicial complexes. □

7.1. Examples

For nilpotent groups, reduced L^p-cohomology vanishes. Indeed, such groups admit unbounded central subgroups. A central element in G acts by a translation of G, i.e., moves points a bounded distance away. The corollary on page 221 of [19] applies: reduced L^p-cohomology vanishes in all degrees, in particular in degree 1.

A nilpotent group of homogeneous dimension Q is p-parabolic if and only if $p \geq Q$. (p, ℓ, R, ∞)-parabolicity for $p \geq Q$, $R > 0$ and $\ell > 1$ follows from the asymptotics of volume of balls, [28], and Remark 68. Non-(p, ℓ, R, S)-parabolicity for $p < Q$ will be proved below, in Proposition 120 and Corollary 119. Carnot groups in their Carnot-Carathéodory metrics are even (p, ℓ)-parabolic for $p \geq Q$ and $\ell > 1$, according to Proposition 67, since they are Q-Ahlfors regular.

Non-elementary hyperbolic groups, [16], have infinite isoperimetric dimension, hence they are never p-parabolic (again, this follows from Proposition 120 and Corollary 119). Their L^p-cohomology vanishes for p in an interval starting from 1, whose upper bound is denoted by CohDim ([8]).

Conformal dimension ConfDim arises as the infimal Hausdorff dimension of Ahlfors-regular metrics in the quasi-symmetric gauge of the ideal boundary. By definition, the quasi-symmetric gauge is the set of metrics which are quasi-symmetric to a visual quasi-metric (all such quasi-metrics are mutually quasi-symmetric), [8]. Quite a number of results on CohDim and ConfDim can be found in recent works by Marc Bourdon, [7], John Mackay [25] and their co-authors.

7.2. Maps to nilpotent groups

Corollary 101. – *Let G and G' be nilpotent Lie group of homogeneous dimensions Q and Q'. Assume that G' is Carnot and equipped with a homogeneous Carnot-Carathéodory metric. If there exists a coarse conformal map $G \to G'$, then $Q \leq Q'$.*

Proof. – G', a Carnot group in its Carnot-Carathéodory metric, is Q'-Ahlfors-regular. Since reduced L^p-cohomology vanishes, Theorem 3 forbids the existence of a coarse conformal map $G \to G'$ unless G is Q'-parabolic. This implies that $Q \leq Q'$. □

Note that no properness assumption was made. Also, a stronger result will be obtained by a different method in Corollary 133.

Corollary 102. – *Let G be a finitely generated group. Let G' be a nilpotent Lie or finitely generated group. If there exists a uniformly conformal map $G \to G'$, then G is itself virtually nilpotent, and $d(G) \leq d(G')$.*

Proof. – It is Q'-Ahlfors regularity of G' in the large which is used here, and Corollaries 66 and 69. Indeed, G' is p-parabolic for $p = d(G')$, therefore so is G. The combination of Lemma 118 and Proposition 126 implies that the isoperimetric dimension of G is at most p. Proposition 120 tells us that G must be virtually nilpotent and $d(G) \leq p = d(G')$. □

Corollary 103. – *Let G be a non-elementary hyperbolic group. Let G' be a Carnot group of homogeneous dimension Q' equipped with its Carnot-Carathéodory metric. If there exists a coarse conformal map $G \to G'$, then* $\mathrm{CohDim}(G) \leq Q'$.

Proof. – Since G' is Q'-Ahlfors regular and non-elementary hyperbolic groups are never p-parabolic, Theorem 3 provides this upper bound on $\mathrm{CohDim}(G)$. □

Remark 104. – Is Corollary 103 sharp? The Poincaré model of hyperbolic space $H^n \to \mathbb{D} \times \partial S^{n-1}$ is coarsely conformal, but the range is not quite $\mathbb{R}^n$. It is unlikely that there exist coarse conformal maps from hyperbolic to Carnot groups. In any case, according to Corollary 65, such a map cannot be proper.

Corollary 105. – *Let G be a non-elementary hyperbolic group. Let G' be a nilpotent group of homogeneous dimension Q'. If there exists a uniformly conformal map $G \to G'$, then* $\mathrm{CohDim}(G) \leq Q'$.

Proof. – Corollaries 66 and 69 apply, since non-elementary hyperbolic groups are never p-parabolic. □

7.3. Maps to hyperbolic groups

Corollary 106. – *Let G, G' be non-elementary hyperbolic groups. If there exists a uniformly conformal map $G \to G'$, then*

$$\mathrm{CohDim}(G) \leq \mathrm{ConfDim}(G').$$

Proof. – Let d' be an Ahlfors-regular metric in the gauge of $\partial G'$, of Hausdorff dimension Q'. According to [9], there exist a bounded geometry hyperbolic graph X, a visual quasi-metric d_o on ∂X and a bi-Lipschitz homeomorphism $q : (\partial G', d') \to (\partial X, d_o)$, arising from a quasi-isometry $q : G' \to X$. Set $Z = (\partial X, (q^{-1})^* d')$. Pick $0 < \alpha \leq 1$. Let $X' = \mathbb{D} \times Z^\alpha$. According to Proposition 36 and Example 75, the Poincaré model of X is a roughly conformal map $\pi : X \to X'$ which has the property that for R large enough, R-balls are sent to balls $B'((y,z),r)$ such that $r \geq g'(y) := \frac{y}{2}$. If $f : G \to G'$ is uniformly conformal, then $f' = \pi \circ q \circ f : G \to X'$ is g'-coarsely conformal (Proposition 6 and Example 80). Furthermore, $q \circ f$ is proper, so the projection of f' to the first factor tends to 0.

Since G is never p-parabolic, Theorem 3 asserts that $\mathrm{CohDim}(G) \leq \alpha + Q'$. Taking the infimum over $\alpha \in (0,1)$ and $Q' \geq \mathrm{ConfDim}(G)$, we get $\mathrm{CohDim}(G) \leq \mathrm{ConfDim}(G')$. □

Example 107. – Fuchsian buildings.

Right-angled Fuchsian buildings (also known as Bourdon buildings) $X_{p,q}$ are universal covers of orbihedra having one p-sided polygon, p even, with trivial face group, cyclic $\mathbb{Z}/q\mathbb{Z}$ edge groups and direct product $\mathbb{Z}/q\mathbb{Z}\times\mathbb{Z}/q\mathbb{Z}$ vertex groups. The conformal dimension and the cohomological dimension of $X_{p,q}$ are both equal to $1+\frac{\log(q-1)}{\arg\cosh(\frac{p-2}{2})}$, [5], [8]. As p and q vary, these numbers fill a dense subset of $[1,+\infty)$.

There are obvious isometric embeddings $X_{p,q}\to X_{p,q'>q}$ and a jungle of bi-Lipschitz embeddings $X_{2p-4,q}\to X_{p,q}$, $X_{3p-8,q}\to X_{p,q}$,... The only known restriction on the existence of uniform/coarse embeddings $X_{p,q}\to X_{p',q'}$ is provided by Corollary 106 or, alternatively, by D. Hume, J. Mackay and R. Tessera's p-separation estimates, [23].

Corollary 108. – *Let G be a nilpotent Lie group of homogeneous dimension Q. Let G' be a hyperbolic group. If there exists a uniformly conformal map $G\to G'$, then $Q\le \mathrm{ConfDim}(G')$.*

Proof. – Only the last paragraph of the proof of Corollary 106 needs be changed. In this case, reduced L^p-cohomology vanishes always, Theorem 3 asserts that G must be p-parabolic for $p=\alpha+Q'$, hence $Q\le\alpha+Q'$. Taking the infimum over $\alpha\in(0,1)$ and Q', we get $Q\le\mathrm{ConfDim}(G')$. □

Example 109. – This is sharp. For instance, the uniform embeddings $\mathbb{R}^{n-1}\to H^n_{\mathbb{R}}$ and $Heis^{2m-1}\to H^m_{\mathbb{C}}$ are uniformly conformal.

More generally, every Carnot group G is a subgroup of the hyperbolic Lie group $G'=\mathbb{R}\ltimes G$, where $\mathbb{R}$ acts on G through Carnot dilations. The homogeneous dimension of G is equal to the conformal dimension of G', [30]. This provides a uniformly conformal map $G\to G'$, according to Lemma 18.

7.4. Proof of Theorem 1 and Corollary 1

Theorem 1 is a combination of Corollaries 102, 105, 106 and 108 applied to the subclass of large-scale conformal maps. Corollary 1 is the special case of uniform/coarse embeddings.

8. Large scale conformal isomorphisms

8.1. Capacities

Definition 110. – *Let X be a metric space, let $K\subset X$ be a bounded set. The* (p,ℓ,R,S)-capacity *of K, $\mathrm{cap}_{p,\ell,R,S}(K)$, is the infimum of $E^p_{\ell,R,S}$-energies of functions $u:X\to[0,1]$ which take value 1 on K and have bounded support.*

Remark 111. – If $\mathrm{cap}_{p,\ell,R,S}(\{o\})=0$, then X is (p,ℓ,R,S)-parabolic.

Proof. – The capacity of the one point set $\{o\}$ bounds from above the (p,ℓ,R,S)-modulus of the family of all $(1,1)$-curves based at o. □

Note that if X is connected, $\mathrm{cap}_{p,\ell,R,S}(\{o\})=0$ implies that for every compact set K, $\mathrm{cap}_{p,\ell',R',S'}(K)=0$ for suitable constants. Indeed, if $u(o)=1$ and $E^p_{\ell,R,S}(u)\le\epsilon$, then, using the packing with only one ball $B_1=B(o,S)$, $u\ge 1-\epsilon$ on B_1, $u\ge 1-2\epsilon$ on the set B_2 of points at distance $\le S$ of B_1, and so on, $u\ge 1-N(K)\epsilon$ on K.

PROPOSITION 112. – *Let X and X' be locally compact, noncompact metric spaces. Let $f : X \to X'$ be a large-scale conformal map. Then, for every $R' > 0$, there exists $R > 0$ and for every $\ell' \geq 1$, there exist $\ell \geq 1$ and N' such that, for all compact sets $K \subset X$,*

$$\mathrm{cap}_{p,\ell,R,\infty}(K) \leq N' \,\mathrm{cap}_{p,\ell',R',\infty}(f(K)).$$

Proof. – If $u : X' \to [0,1]$ has compact support and $u(f(K)) = 1$, then $u \circ f$ has compact support and $(u \circ f)(K) \geq 1$, thus $E^p_{\ell,R,\infty}(u \circ f) \geq \mathrm{cap}_{p,\ell,R,\infty}(K)$. We know from Lemma 45 that

$$E^p_{\ell,R,\infty}(u \circ f) \leq N' \, E^p_{\ell',R',\infty}(u).$$

Taking the infimum over all such functions u,

$$\mathrm{cap}_{p,\ell,R,\infty}(K) \leq N' \,\mathrm{cap}_{p,\ell',R',\infty}(f(K)). \qquad \square$$

8.2. Non-parabolicity and $L^{q,p}$ cohomology

DEFINITION 113. – *A metric space X is* uniformly perfect in the large *if there exists a constant $c > 0$ such that, for all $x \in X$ and large enough T, $B(x,T) \setminus B(x,cT) \neq \emptyset$.*

Unbounded geodesic spaces (e.g., graphs, Riemannian manifolds), and spaces roughly isometric to such (e.g., locally compact groups) are uniformly perfect in the large. The point of this property is to ensure that R-volumes (meaning the number of disjoint R-balls that one can pack inside) of large balls are large.

LEMMA 114. – *Let X be a metric space which is uniformly perfect in the large. Fix a radius $R > 0$. Let $\mathrm{vol}_R(B)$ denote the maximal number of disjoint R-balls that can be packed in B, and $v_R(T) = \inf_{x \in X} \mathrm{vol}_R(B(x,T))$. Then $v_R(T)$ tends to infinity with T.*

Proof. – Given a ball $B = B(x_0,T)$, uniform perfectness, applied in $B(x_0, \frac{2T}{2+c})$, provides a point $x_1 \in B(x_0, \frac{2T}{2+c}) \setminus B(x, \frac{2cT}{2+c})$. Then $B(x_1, \frac{c}{2+c}T) \subset B(x,T) \setminus B(x_0, \frac{c}{2+c}T)$. Iterating the construction produces a sequence of disjoint balls $B(x_j, (\frac{c}{2+c})^{j+1}T)$ in $B(x,T)$. If $n = \lfloor \log_{c/2+c}(T/R) \rfloor$, we get n disjoint R-balls in $B(x,T)$. $\square$

LEMMA 115. – *Let X be a metric space which is uniformly perfect in the large. Fix $\ell \geq 1$ and $S \geq R > 0$. Assume that X is S-connected, i.e., any two points are connected by a chain of intersecting S-balls. If $EL^{q,p}_{\ell,R,S}H^1(X) = EL^{q,p}_{\ell,R,S}\bar{H}^1(X)$ for some finite q, then the capacity of balls tends to infinity uniformly with their radius: there exists a function $\kappa_{\ell,R}$ such that $\kappa_{\ell,R}(T)$ tends to infinity as $T \to \infty$, and such that for every ball $B(x,T)$ of radius T,*

$$\mathrm{cap}^p_{\ell,R,S}(B(x,T)) \geq \kappa_{\ell,R}(T).$$

In particular, X is non-(p,ℓ,R,S)-parabolic.

Proof. – By assumption, the coboundary

$$d : \mathcal{L}^{q,p}_{\ell,R,S}C^0(X) \to L^p_{\ell,R,S}C^1(X)$$

has a closed image. Its kernel consists of constant functions (thanks to S-connectedness), which can be modded out. d becomes a continuous isomorphism between Banach spaces. According to the isomorphism theorem, d has a bounded inverse. Thus there exists a constant C such that, for every function $u \in L^q_{\ell,R,S}C^0(X)$, there exists a constant c_u such that

$$\|u - c_u\|_{L^p_{\ell,R,S}C^0(X)} \leq C\,\|du\|_{L^p_{\ell,R,S}C^1(X)}.$$

Since $\mathrm{vol}_R(X)$ is infinite, constants do not belong to $L^q_{\ell,R,S}C^0(X)$, so $c_u = 0$. For functions $u : X \to [0,1]$ with bounded support, this translates into

$$\|u\|_{L^q_{\ell,R,S}C^0(X)} \leq C\,E^p_{\ell,R,S}(u)^{1/p}.$$

Let $B(x,T)$ be a large ball. Uniform perfectness ensures that a logarithmic number of disjoint balls of radius ℓR can be packed into $B(x,T)$. If $u = 1$ on $B(x,T)$, using this packing, we get a lower bound on $\|u\|^q_{L^q_{\ell,R,S}C^0}$ of the order of $\log(T/\ell R)$ which depends only on R, ℓ and T. This shows that the p-capacity of $B(x,T)$ tends to infinity with T. □

REMARK 116. – Euclidean spaces are examples where $H^1 \neq 0$, $\bar{H}^1 = 0$ and where the capacities of balls vanish.

8.3. $L^{q,p}$-cohomology and isoperimetric dimension

DEFINITION 117. – *Let X be a Riemannian manifold. Say that X has isoperimetric dimension $\geq d$ if compact subsets $D \subset X$ with smooth boundary and sufficiently large volume satisfy*

$$\mathrm{volume}(D) \leq C\,\mathrm{volume}(\partial D)^{\frac{d}{d-1}}.$$

If Γ is a graph, say that Γ has isoperimetric dimension $\geq d$ if all finite subsets D of vertices of Γ satisfy

$$|D| \leq C\,|\partial D|^{\frac{d}{d-1}},$$

where ∂D denotes the subset of edges of Γ with one vertex in D and one vertex outside D. Finally, define the isoperimetric dimension of a bounded geometry simplicial complex as the isoperimetric dimension of its 1*-skeleton.*

Isoperimetric dimension is a quasiisometry invariant. If a bounded geometry Riemannian manifold X is quasiisometric to a bounded geometry simplicial complex T, then X and T have the same isoperimetric dimension.

LEMMA 118. – *Let X be a Riemannian manifold or a simplicial complex with bounded geometry. If X has isoperimetric dimension $\geq d > 1$, then $E\ell^{q,p}H^1(X) = E\ell^{q,p}\bar{H}^1(X)$ for all $1 \leq p < d$ and $q < \infty$ such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{d}$.*

Proof. – Up to quasiisometry, we can assume that X is a bounded degree graph with vertex set V and edge set E.

Following a classical argument, let us check that the following ℓ^1 Sobolev inequality holds: let $d' = \frac{d}{d-1}$; for every finitely supported function u on V,

$$\|u\|_{d'} \leq C\,\|du\|_1. \tag{1}$$

Assume first that u takes its values in $\mathbb{N}$. Let u_t be the indicator function of the superlevel set $\{u > t\}$, i.e., $u_t(x) = 1$ if $u(x) > t$, $u_t(x) = 0$ otherwise. Then $u = \sum_{t\in\mathbb{N}} u_t$. For each t, $\partial\{u > t\}$ consists of edges with one vertex where $u > t$ and one where $u \leq t$. The set of such edges is the set of edges along which $du \neq 0$, and each of them is counted as many times as the value of $|du|$ on it. Therefore

$$\sum_{t\in\mathbb{N}} |\partial\{u > t\}| = \|du\|_1.$$

On the other hand, the isoperimetric inequality $|D| \leq c\,|\partial D|^{d/(d-1)}$ applies to each super-level set, yielding

$$\begin{aligned}\|u\|_{d'} &\leq \sum_{t\in\mathbb{N}} \|u_t\|_{d'}\\ &= \sum_{t\in\mathbb{N}} |\{u > t\}|^{1/d'}\\ &\leq c^{1/d'} \sum_{t\in\mathbb{N}} |\partial\{u > t\}|\\ &\leq c^{1/d'} \|du\|_1.\end{aligned}$$

Since $|d|u|| \leq |du|$, the case of integer valued functions follows. Since inequality (1) is homogeneous, the case of rational valued functions follows too, and the general case of real valued finitely supported functions as well, by density.

It follows that $\|u\|_q$ is controlled by $\|du\|_p$ provided $p < d$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{d}$. Indeed, let $r \geq 1$. For each edge $e = xy$, denote by $\max_e |u| = \max\{|u(x)|, |u(y)|\}$. Then

$$\begin{aligned}|d(|u|^r)(e)| &= ||u(y)|^r - |u(x)|^r|\\ &\leq r \max\{|u(x)|, |u(y)|\}^{r-1} ||u(y)| - |u(x)||\\ &\leq r \max_e |u|^{r-1} |du(e)|.\end{aligned}$$

Replacing $|u|$ with $|u|^r$ in inequality (1), and applying Hölder's inequality, we get

$$\begin{aligned}(\sum_{x\in V} |u(x)|^{rd'})^{1/d'} &\leq C \sum_{e\in E} r \max_e |u|^{r-1} |du|(e)\\ &\leq C r \,(\sum_{e\in E} \max_e |u|^{rd'})^{\frac{r-1}{rd'}} (\sum_{e\in E} |du|(e)^{\frac{rd'}{rd'-r+1}})^{\frac{rd'-r+1}{rd'}}.\end{aligned}$$

In the sum $\sum_{e\in E} \max_e |u|^{rd'}$, each vertex appears at most once for each edge that contains it. Hence, if the degree of X is $\leq v$,

$$\sum_{e\in E} \max_e |u|^{rd'} \leq v \sum_{x\in V} |u(x)|^{rd'} = \|u\|_{rd'}^{rd'}.$$

Therefore

$$\|u\|_{rd'} \leq C r v^{\frac{r-1}{rd'}} \|du\|_{\frac{rd'}{rd'-r+1}}.$$

If $p < d$, one can pick q such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{d}$ and $r = q/d' \geq 1$. Then

$$\|u\|_q \leq C \,\|du\|_p.$$

This says that exact reduced and unreduced $\ell^{q,p}$ cohomologies coincide. According to Proposition 85, this is equivalent to

$$EL^{q,p}_{\ell,R,S} H^1(X) = EL^{q,p}_{\ell,R,S} \bar{H}^1(X),$$

for all p, q such that $1 \leq p < d$, $\frac{1}{p} - \frac{1}{q} = \frac{1}{d}$. ☐

COROLLARY 119. – *A bounded geometry Riemannian manifold or simplicial complex which has isoperimetric dimension $\geq d > 1$ is non-(p, ℓ, R, S)-parabolic for all $1 \leq p < d$ and all large enough ℓ, R, $S \geq R$.*

Proof. – This follows from Lemmata 115 and 118. ☐

It turns out that the isoperimetric dimensions of finitely generated groups are known.

PROPOSITION 120 (Compare M. Troyanov, [38], S. Maillot, [26]).

Let G be a finitely generated group. Then the isoperimetric dimension of G is

— *either equal to 1 if G is virtually cyclic,*
— *or equal to its homogeneous dimension, an integer larger than 1 if G is virtually nilpotent but not virtually cyclic.*
— *Otherwise, it is infinite.*

If follows that a finitely generated group is p-parabolic if and only if it is virtually nilpotent of homogeneous dimension $\leq p$.

Proof. – According to T. Coulhon-L. Saloff Coste, [12], for finitely generated (or Lie) groups, volume growth provides an estimate on isoperimetric dimension. In particular, it implies that isoperimetric dimension is infinite unless volume growth is polynomial, in which case isoperimetric dimension is equal to the polynomial degree of volume growth. The only finitely generated groups of linear growth are virtually cyclic ones. That groups of polynomial growth are virtually nilpotent is M. Gromov's theorem of [18]. The isoperimetry of nilpotent groups was originally due to N. Varopoulos, [42]. ☐

8.4. Grötzsch invariant

Following [20], we use capacities to define a kind of large-scale conformally invariant distance on a metric space.

DEFINITION 121. – *Let X be a metric space. Fix parameters p, ℓ, R, S. For $x_1, x_2 \in X$, let*

$$\delta_{p,\ell,R,S}(x_1, x_2) = \inf\{\mathrm{cap}_{p,\ell,R,S}(\mathrm{im}(\gamma))\,;\, \gamma \textit{ continuous arc in } X \textit{ from } x_1 \textit{ to } x_2\}.$$

LEMMA 122. – *Let $f : X \to X'$ be a large-scale conformal map. Assume that f is a bijection and that $f^{-1} : X' \to X$ is continuous. For all $R' > 0$, there exists $R > 0$ such that for all $\ell' \geq 1$, there exists $\ell \geq 1$ and N' such that, for all $x_1, x_2 \in X$ and all $S \geq R$,*

$$\delta_{p,\ell,R,S}(x_1, x_2) \leq N'\, \delta_{p,\ell',R',\infty}(f(x_1), f(x_2)).$$

Proof. – If γ' is a continuous arc joining $f(x_1)$ to $f(x_2)$, $f^{-1}\circ\gamma'$ is a continuous arc joining x_1 to x_2, thus $\delta_{p,\ell,R,S}(x_1,x_2) \leq \mathrm{cap}_{p,\ell,R,\infty}(f^{-1}\circ\gamma')$. Therefore, according to Proposition 112,

$$\delta_{p,\ell,R,\infty}(x_1,x_2) \leq N'\mathrm{cap}_{p,\ell,R,\infty}(\gamma'),$$

and taking an infimum,

$$\delta_{p,\ell,R,S}(x_1,x_2) \leq \delta_{p,\ell,R,\infty}(x_1,x_2) \leq N'\,\delta_{p,\ell',R',\infty}(f(x_1),f(x_2)),$$

where the first inequality exploits the fact that adding constraints on packings decreases energies and capacities. □

8.5. Upper bounds on capacities

DEFINITION 123. – *Say a metric space X has* controlled balls *if there exist $R>0$ and a measure μ and continuous functions $v>0$ and $V<\infty$ on $[R,+\infty)$ such that for every $x\in X$ and every $r\geq R$,*

$$v(r) \leq \mu(B(x,r)) \leq V(r).$$

If such an estimate holds also for $r\in(0,R]$ and furthermore

$$\forall r\in(0,R],\quad v(r)\geq C\,r^Q,$$

one says that X has locally Q-controlled balls.

In a Riemannian manifold or a simplicial complex with bounded geometry, balls are automatically controlled. A Riemannian n-manifold with bounded geometry is locally n-Ahlfors regular, for arbitrarily large values of R.

LEMMA 124. – *Let X be a geodesic metric space which has controlled balls. Let $p\geq 1$ and $\ell>1$. Then $\delta_{p,\ell,R,\infty}$ is bounded above uniformly in terms of distance d. I.e. there exists a function $\Pi_{p,\ell,R}:\mathbb{R}_+\to\mathbb{R}_+$ such that, if $d(x_1,x_2)\geq R$,*

$$\delta_{p,\ell,R,\infty}(x_1,x_2) \leq \Pi_{p,\ell,R}(d(x_1,x_2)).$$

If furthermore X has locally Q-controlled balls for some $Q\leq p$, then such an upper bound still holds with $R=0$, i.e., there exists a function $\Pi_{p,\ell}$ such that

$$\delta_{p,\ell,0,\infty}(x_1,x_2) \leq \Pi_{p,\ell}(d(x_1,x_2)).$$

Proof. – Fix $r\geq R$, set $T(r)=2+\frac{8r}{\ell-1}$ and

$$C(r)=\sup_{\rho\in[R,T]}\frac{(2\rho)^p}{v(\rho)}.$$

For each $x\in X$, define a function $u_{x,r}$ as follows: $u_{x,r}=1$ on $B=B(x,r)$, vanishes outside $2B$ and is linear in the distance to x in between. Let us estimate its p-energy. Let $\{B_j\}$ be a (ℓ,R,∞)-packing of X. If a ball B_i intersects $2B$, and has radius $>4r/(\ell-1)$, then ℓB_i contains $2B$. No other ball of the packing can intersect $2B$, hence an upper bound on $\sum_j \mathrm{diameter}(u_{x,r}(B_j))^p\leq 1$. Otherwise, all balls of the ℓ-packing contributing to energy are contained in $B(x, T(r))$, for $T(r) = 2r + \frac{8r}{\ell-1}$. For each ball B_j of radius ρ_j, $\rho_j \in [R, T(r)]$ and

$$\begin{aligned}\text{diameter}(u_{x,r}(B_j))^p &\leq \text{diameter}(B_j)^p \\ &\leq (2\rho_j)^p \\ &\leq C(r)\, v(\rho_j) \\ &\leq C(r)\, \mu(B_j).\end{aligned}$$

Summing up,

$$\begin{aligned}\sum_j \text{diameter}(B_j)^p &\leq C(r) \sum_j \mu(B_j) \\ &\leq C(r)\, \mu(B(x,T)) \\ &\leq C(r) V(T(r)).\end{aligned}$$

This gives an upper bound on $\text{cap}_{p,\ell,R,\infty}(B(x,r))$ which depends on its radius r, on p, on R and on ℓ only.

If X has locally Q-controlled balls and $Q \leq p$, then

$$C'(r) = \sup_{\rho \in (0,T]} \frac{(2\rho)^p}{v(\rho)} < \infty,$$

so the argument generalizes to arbitrary $(\ell, 0, \infty)$-packings.

Balls in geodesic metric spaces contain geodesics which are continuous arcs. Thus the lower bound $\delta_{p,\ell,R,\infty}$ is bounded above by the capacity of a geodesic segment, which in turn is bounded above by the capacity of a ball, which is estimated in terms of its radius, on p and on ℓ only. □

8.6. Strong non-parabolicity

Here, we are concerned with lower bounds on Grötzsch' invariant δ.

DEFINITION 125. – *Let X be a metric space. Say that X is* strongly non-(p, ℓ, R, S)-parabolic *if $\delta_{p,\ell,R,S}(x_1, x_2)$ tends to infinity uniformly with $d(x_1, x_2)$. In other words, for every $R \leq S$ and $\ell > 1$, there exists a function $\pi_{p,\ell,R,S}$ such that $\pi_{p,\ell,R,S}(T)$ tends to infinity when $T \to \infty$, and such that*

$$\delta_{p,\ell,R,S}(x_1, x_2) \geq \pi_{p,\ell,R,S}(d(x_1, x_2)).$$

PROPOSITION 126. – *Let X be a metric space. Fix $\ell \geq 1$ and $S \geq R > 0$. If $EL^{q,p}_{\ell,R,S} H^1(X) = EL^{q,p}_{\ell,R,S} \bar{H}^1(X)$ for some finite q, then X is strongly non-(p, ℓ, R, S)-parabolic.*

Proof. – Let $x_1, x_2 \in X$. Let γ be a continuous arc joining x_1 to x_2. Fix $R > 0$ and $\ell \geq 1$. Assume that $d(x_1, x_2) \geq 2R$. For each $j = 0, \ldots, k := \lfloor d(x_1, x_2)/2\ell R \rfloor$, pick a point y_j on γ such that $d(y_j, x_1) = 2\ell R j$. Let $B_j = B(y_j, R)$. By construction, $\{B_j\}$ is a (ℓ, R, R)-packing of X. Let $u : X \to [0, 1]$ be a function of bounded support such that $u = 1$ on γ. Then $\sup_{B_j} u = 1$, thus

$$\|u\|_{L^q_{\ell,R,R}} \geq k^{1/q}.$$

As in the proof of Lemma 115, the $L^{q,p}$ cohomology assumption implies the existence of a constant C such that, for every function u of bounded support,

$$\|u\|_{L^q_{\ell,R,R}} \leq C\, E^p_{\ell,R,R}(u)^{1/p}.$$

This shows that

$$\mathrm{cap}_p(\gamma) \geq C\, k^{p/q} = C\, \lfloor \frac{d(x_1,x_2)}{2\ell R} \rfloor^{p/q},$$

this is a lower bound on $\delta_{p,\ell,R,R}(x_1,x_2)$. This yields a lower bound on $\delta_{p,\ell,R,S}(x_1,x_2)$ for any S. □

8.7. Consequences

Corollary 127. – *Let X and X' be geodesic metric spaces which are strongly non-(p,ℓ,R,S)-parabolic for some $p \geq 1$. Assume that both have controlled balls. Let $f : X \to X'$ be a homeomorphism such that both f and f^{-1} are large-scale conformal maps. Then f is a quasi-isometry.*

Proof. – By strong non-(p,ℓ,R,S)-parabolicity, δ invariants in X are bounded below,

$$\delta_{p,\ell,R,S}(x_1,x_2) \geq \pi_{p,\ell,R,S}(d(x_1,x_2)).$$

According to Lemma 124, they are bounded above in X',

$$\delta_{p,\ell,R,\infty}(x_1',x_2') \leq \Pi_{p,\ell,R}(d(x_1',x_2')).$$

If $f : X \to X'$ is a large-scale conformal homeomorphism, $N'\delta \circ f \geq \delta$ up to changes in parameters (Lemma 122),

$$N'\delta_{p,\ell',R',\infty}(f(x_1),f(x_2)) \geq \delta_{p,\ell,R,S}(x_1,x_2).$$

Combining these inequalities, we get for f,

$$N'\Pi_{p,\ell',R'}(d(f(x_1),f(x_2))) \geq \pi_{p,\ell,R,S}(d(x_1,x_2))$$

and for f^{-1},

$$N'\Pi_{p,\ell',R'}(d(f^{-1}\circ f(x_1),f^{-1}\circ f(x_2))) \geq \pi_{p,\ell,R,S}(d(f(x_1),f(x_2))),$$

hence

$$\pi_{p,\ell,R,S}(d(f(x_1),f(x_2))) \leq N'\Pi_{p,\ell',R'}(d(x_1,x_2)).$$

These inequalities show that f is a quasi-isometry. □

Corollary 128 (Proof of Theorem 2). – *Let M, M' be bounded geometry Riemannian manifolds or simplicial complexes with isoperimetric dimension > 1. Then homeomorphisms $M \to M'$ which are large-scale conformal in both directions must be quasi-isometries.*

Proof. – According to Lemma 118, M and M' satisfy the $L^{q,p}$ cohomological assumption of Proposition 126, therefore they are strongly non-(p,ℓ,R,S)-parabolic for all $p > 1$, $\ell > 1$, $R > 0$ and $S < \infty$. Bounded geometry implies controlled balls, thus Corollary 127 applies. □

REMARK 129. – This applies to Euclidean spaces of dimension ≥ 2. Note that Examples 11 are roughly conformal in both directions, but large-scale conformal in only one direction.

REMARK 130. – A natural question (Sylvain Maillot) is whether two geodesic spaces X and X' can have a large-scale conformal map $X \to X'$ and a large-scale conformal map $X' \to X$ without being quasi-isometric.

8.8. From coarse to uniformly conformal maps

PROPOSITION 131. – *Let X be a metric space which is strongly non-(p, ℓ, R, S)-parabolic for some $p \geq 1$ and all $\ell \geq 1$. Let X' be a metric space which has locally Q-controlled balls for some $Q \leq p$. Every coarsely conformal map $X \to X'$ is uniformly conformal. Every roughly conformal map $X \to X'$ is large-scale conformal.*

Proof. – Fix $S \geq R > 0$. Assume that $f : X \to X'$ is coarsely (resp. roughly) conformal. It suffices to show that for all $T' > 0$, there exists $T_0 > 0$ such that f maps no T-ball B, $T \geq T_0$, into a T'-ball B'. Given $\ell' \geq 1$, there exist $\ell \geq 1$ and N' such that if $f(B) \subset B'$, $\mathrm{cap}_{p,\ell,R,S}(B) \leq N' \mathrm{cap}_{p,\ell',0,\infty}(B')$. This upper bound fails if T is sufficiently large, $T \geq T_0$. This shows that T-balls are never mapped into T'-balls. So if f is coarsely conformal, it is in fact (T_0, S_0, T', ∞)-coarsely conformal, for large enough T_0 and for every $S_0 \geq T_0$, hence it is uniformly conformal. If f is roughly conformal, it is in fact $(T_0, \infty, T', \infty)$-coarsely conformal, for large enough T_0 thus f is large-scale conformal. □

REMARK 132. – The assumptions of Proposition 131 are satisfied for $X = \mathbb{R}^n$ provided $p < n$ and for $X' = \mathbb{R}^{n'}$ for $p \geq n'$. So Proposition 131 applies if $n' < n$, i.e., exactly when there are no coarse conformal maps $X \to X'$. In fact, the conclusion fails if $n = n'$, as Examples 11 show.

Proposition 131 allows to modify the assumptions in the corollaries of Subsections 7.2 and 7.3. For instance,

COROLLARY 133. – *Let G be a finitely generated group. Let G' be a connected nilpotent Lie group equipped with a left-invariant Riemannian metric. If there exists a coarse conformal map $G \to G'$, then G is virtually nilpotent and $d(G) \leq d(G')$.*

Proof. – If G is virtually cyclic, then it is virtually nilpotent and $d(G) = 1 \leq d(G')$. Otherwise, G has isoperimetric dimension $Q > 1$, thus it is strongly non-(p, ℓ, R, S)-parabolic for all $1 \leq p < Q$. G' is locally n'-Ahlfors regular for $n' = \mathrm{dimension}(G')$. Assume that $Q > Q'$. Since $n' \leq Q'$, one can pick p such that $n' < p < Q$. Proposition 131 asserts that a coarse conformal map $G \to G'$ is automatically uniformly conformal. Corollary 102 shows that such a map cannot exist. □

COROLLARY 134. – *Let G, G' be non-elementary hyperbolic groups. Let M' be a Riemannian manifold of bounded geometry, which is quasi-isometric to G'. If there exists a coarse conformal map $G \to M'$, then*

$$\mathrm{CohDim}(G) \leq \mathrm{ConfDim}(G').$$

Proof. – Non-elementary hyperbolic groups have infinite isoperimetric dimensions. Thus G is strongly non-(p, ℓ, R, S)-parabolic for all p. By assumption, M' is locally n'-Ahlfors regular for $n' = \text{dimension}(M')$. Choose some $p \geq n'$. Proposition 131 asserts that a coarse conformal map $G \to M'$ is automatically uniformly conformal. Composing with a quasi-isometry, we get a uniformly conformal map $G \to G'$, so Corollary 106 applies. □

Corollary 2 is a combination of Corollaries 133 and 134.

REMARK 135. – Real hyperbolic space H^n has a Poincaré model, it is a rough conformal map of H^n to $\mathbb{D} \times S^{n-1}$. Corollary 134 implies that there is no rough conformal map of H^n to a ball in $\mathbb{R}^n$ (otherwise H^n would map roughly conformally to anything). Thus $\mathbb{D} \times S^{n-1}$ should not be confused with $[0, 1] \times S^{n-1}$.

8.9. Large scale conformality in one dimension

We have been unable to extend Theorem 2 to the virtually cyclic case. Here is a partial result.

LEMMA 136. – *Let f and g be continuous maps $\mathbb{R} \to \mathbb{R}$ such that*

— *f and g are large-scale conformal;*

— *$g \circ f$ and $f \circ g$ are coarse embeddings.*

Then f is a quasi-isometry.

Proof. – Fix $R' \leq R, \ell', \ell, N'$ as given by the definition of large-scale conformality. Let $\tilde{R}$ be given by the definition of coarse embeddings: $g \circ f$ maps R-balls to $\tilde{R}$-balls. To save notation, assume that the same constants serve for g. Since we are on the real line, balls of radius R are intervals of length $2R$. In the correspondence between balls $B \mapsto B'$, one can assume that B' is a minimal interval containing $f(B)$ and of length $\geq 2R'$, i.e., $B' = f(B)$ itself if $\text{length}(f(B)) \geq 2R'$.

The balls $B_j = B(2\ell Rj, R)$ are mapped into balls B'_j forming an (N', ℓ', R', ∞)-packing. Assume that $f(B_0)$ has length $2R'_0 \geq 2R'$, in order that $B'_0 = f(B_0)$. Let $\{B''_j\}$ be an (ℓ, R, ∞)-packing of B'_0. The number of balls in this packing can be chosen to be at least $\frac{R'_0}{2\ell R}$. In turn, g maps B''_j into B'''_j, which form an (N', ℓ', R', ∞)-packing, which is the union of N' (ℓ', R', ∞)-packings. One of them has at least $\frac{R'_0}{2\ell RN'}$ elements. Every ball B'''_j contains

$$g(B''_j) \subset g(B'_0) = g \circ f(B_0) \subset \tilde{B} := B(g \circ f(0), \tilde{R}),$$

so B'''_j intersects $\tilde{B}$. At most two of these balls contain boundary points, so all others are contained in $\tilde{B}$. At least one of these balls has radius ρ no larger than

$$\frac{\tilde{R}}{\frac{R'_0}{2\ell RN'} - 2} = \frac{2\ell RN'\tilde{R}}{R'_0 - 4\ell RN'}.$$

Since $\rho \geq R'$, we obtain an upper bound on R'_0. This shows that all R-balls are mapped to R'_0-balls, i.e., f is a coarse embedding. □

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THE PLURIPOTENTIAL CAUCHY-DIRICHLET PROBLEM FOR COMPLEX MONGE-AMPÈRE FLOWS

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ABSTRACT. – We develop the first steps of a parabolic pluripotential theory in bounded strongly pseudo-convex domains of $\mathbb{C}^n$. We study certain degenerate parabolic complex Monge-Ampère equations, modeled on the Kähler-Ricci flow evolving on complex algebraic varieties with Kawamata log-terminal singularities. Under natural assumptions on the Cauchy-Dirichlet boundary data, we show that the envelope of pluripotential subsolutions is semi-concave in time and continuous in space, and provides the unique pluripotential solution with such regularity.

RÉSUMÉ. – Nous développons une théorie pluripotentielle parabolique sur un domaine strictement pseudo-convexe borné de $\mathbb{C}^n$. Nous étudions certaines équations de Monge-Ampère complexes paraboliques dégénérées, modelées sur le flot de Kähler-Ricci sur les variétés algébriques complexes à singularités Kawamata log-terminales. Sous des hypothèses naturelles sur les données de Cauchy-Dirichlet, nous montrons que l'enveloppe des sous-solutions pluripotentielles est semi-concave en temps et continue en espace, et qu'elle est l'unique solution pluripotentielle avec une telle régularité.

Introduction

The Ricci flow, first introduced by Hamilton [18] is the equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

evolving a Riemannian metric by its Ricci curvature. If the Ricci flow starts from a Kähler metric, the evolving metrics remain Kähler and the resulting PDE is called the Kähler-Ricci flow.

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It is expected that the Kähler-Ricci flow can be used to give a geometric classification of complex algebraic and Kähler manifolds, and produce canonical metrics at the same time. Solving the Kähler-Ricci flow boils down to solving a parabolic scalar equation modeled on

$$\det\left(\frac{\partial^2 u_t}{\partial z_j \partial \bar{z}_k}(t,z)\right) = e^{\partial_t u_t(z)+H(t,z)+\lambda u_t(z)}$$

where $t \mapsto u_t(z) = u(t,z)$ is a smooth family of strictly plurisubharmonic functions in $\mathbb{C}^n$, $\lambda \in \mathbb{R}$ and $g = e^H$ is a smooth and positive density.

It is important for geometric applications to study *degenerate* versions of these complex Monge-Ampère flows, where the functions u_t are no longer smooth nor strictly plurisubharmonic, and the densities may vanish or blow up (see [30, 4, 29, 11] and the references therein).

A viscosity approach has been developed recently in [10], following its elliptic counterpart [9, 19, 20]. While the viscosity theory is very robust, it requires the data to be continuous hence has a limited scope of applications. Several geometric situations encountered in the Minimal Model program (MMP) necessitate one to deal with Kawamata log-terminal (klt) singularities. The viscosity approach breaks down in these cases and a more flexible method is necessary.

There is a well established pluripotential theory of weak solutions to degenerate elliptic complex Monge-Ampère equations, following the pioneering work of Bedford-Taylor [1, 2]. This theory allows to deal with L^p-densities as established in a corner stone result of Kołodziej [25], which provides a great generalization of [32].

No similar theory has ever been developed on the parabolic side. The purpose of this article, the first of a series on this subject, is to develop a pluripotential theory for degenerate complex Monge-Ampère flows. This article settles the foundational material for this theory and focuses on solving the Cauchy-Dirichlet problem in domains of $\mathbb{C}^n$.

We consider the following family of Monge-Ampère flows

$$\text{(CMAF)} \qquad dt \wedge (dd^c u)^n = e^{\partial_t u + F(t,z,u)} g(z) dt \wedge dV,$$

in $\Omega_T :=]0,T[\times \Omega$, where dV is the Euclidean volume form on $\mathbb{C}^n$ and

— $T > 0$ and $\Omega \Subset \mathbb{C}^n$ is a bounded strictly pseudoconvex domain;

— $F(t,z,r)$ is continuous in $[0,T[\times \Omega \times \mathbb{R}$, increasing in r, bounded in $[0,T[\times \Omega \times J$, for each $J \Subset \mathbb{R}$;

— $(t,r) \mapsto F(t,\cdot,r)$ is uniformly Lipschitz and semi-convex in (t,r);

— $g \in L^p(\Omega)$, $p > 1$, and $g > 0$ almost everywhere ;

— $u : [0,T[\times \Omega \to \mathbb{R}$ is the unknown function.

Here $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)/2$ so that $dd^c = i\partial\overline{\partial}$ and $(dd^c u)^n$ represents the determinant of the complex Hessian of u in space (the complex Monge-Ampère operator) whenever u is $\mathcal{C}^2$-smooth.

For less regular functions u, the Equation (CMAF) should be understood in the weak sense of pluripotential theory as we explain in Section 2.

We let $\mathcal{P}(\Omega_T)$ denote the set of *parabolic potentials*, i.e., those functions $u : \Omega_T \to [-\infty, +\infty[$ defined in $\Omega_T =]0,T[\times \Omega$ and satisfying the following conditions:

— for any $t \in]0, T[$, $u(t, \cdot)$ is plurisubharmonic in Ω;

— the family $\{u(\cdot, z)\ ;\ z \in \Omega\}$ is locally uniformly Lipschitz in $]0, T[$.

We study in Section 1 basic properties of parabolic potentials. We show in Lemma 1.6 that if $u \in \mathscr{P}(\Omega_T)$ and is bounded from above in Ω_T then it can be uniquely extended as an upper-semicontinuous function in $[0, T[\times \Omega$ such that $u(0, \cdot)$ is plurisubharmonic in Ω. We show that parabolic potentials satisfy approximate submean-value inequalities (Lemma 1.8) and enjoy good compactness properties (Proposition 1.17).

We show in Section 2 that parabolic complex Monge-Ampère operators are well defined on $\mathscr{P}(\Omega_T) \cap L^\infty_{\mathrm{loc}}(\Omega_T)$ and enjoy nice continuity properties, allowing to make sense of pluripotential sub/super/solutions to (CMAF) (see Definition 3.1). A crucial convergence property is obtained in Proposition 2.9, under a semi-concavity assumption on the family of parabolic potentials.

A *Cauchy-Dirichlet boundary data* is a function h defined on the parabolic boundary of Ω_T denoted by

$$\partial_0 \Omega_T := ([0, T[\times \partial\Omega) \cup (\{0\} \times \Omega),$$

such that

— the restriction of h on $[0, T[\times \partial\Omega$ is continuous;

— the family $\{h(\cdot, z)\ ;\ z \in \partial\Omega\}$ is locally uniformly Lipschitz in $]0, T[$;

— h satisfies the following compatibility condition : $\forall \zeta \in \partial\Omega$,

$$h_0 := h(0, \cdot) \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega) \text{ and } \lim_{\Omega \ni z \to \zeta} h(0, z) = h(0, \zeta). \tag{0.1}$$

The Cauchy-Dirichlet problem for the parabolic Equation (CMAF) with Cauchy-Dirichlet boundary data h consists in finding $u \in \mathscr{P}(\Omega_T) \cap L^\infty(\Omega_T)$ such that (CMAF) holds in the pluripotential sense in Ω_T and the following Cauchy-Dirichlet boundary conditions are satisfied :

$$\forall (\tau, \zeta) \in [0, T[\times \partial\Omega, \quad \lim_{\Omega_T \ni (t,z) \to (\tau, \zeta)} u(t, z) = h(\tau, \zeta), \tag{0.2}$$

$$\lim_{t \to 0^+} u_t = h_0 \quad \text{in} \quad L^1(\Omega). \tag{0.3}$$

In this case we say that u *is a solution to the Cauchy-Dirichlet problem for the Equation* (CMAF) *with boundary values h.*

Observe that a solution u to the Equation (CMAF) has plurisubharmonic slices in Ω and the Cauchy condition (0.3) implies by a classical result in pluripotential theory that $(\limsup_{t \to 0} u_t)^* = h_0^* \in \mathrm{PSH}(\Omega)$, hence $h_0 = h_0^* \in \mathrm{PSH}(\Omega)$. This observation shows that the Cauchy data h_0 must be plurisubharmonic as it is required in the compatibility condition (0.1).

For a solution to the Cauchy-Dirichlet problem for the Equation (CMAF), the Cauchy condition (0.3) implies that

$$\forall z \in \Omega, \quad \lim_{t \to 0^+} u_t(z) = h_0(z).$$

It is possible to consider less regular initial Cauchy data $h(0, \cdot)$ (see [28, 27]), but we will not pursue this here.

We try and construct a solution to the Cauchy-Dirichlet problem by the Perron method, considering the upper envelope U of pluripotential subsolutions.

The technical core of the paper lies in Section 3 and Section 4. In Section 3 we construct subbarriers and controls from above to ensure that U has the right boundary values (see Theorem 3.12). In Section 4 we prove that the Perron envelope of subsolutions is locally uniformly Lipschitz and semiconcave in time.

THEOREM A. – *Assume h is a Cauchy-Dirichlet boundary data in Ω_T such that for all $0 < S < T$, and for all $(t,z) \in]0,S] \times \partial\Omega$,*

$$(\dagger) \qquad t|\partial_t h(t,z)| \leq C(S) \quad \textit{and} \quad t^2 \partial_t^2 h(t,z) \leq C(S).$$

Then the envelope $U = U_{h,g,F}$ is locally uniformly Lipschitz and locally uniformly semi-concave in $t \in]0,T[$. Moreover, U satisfies the Cauchy-Dirichlet boundary conditions (0.2) *and* (0.3).

Here $C(S)$ is a positive constant depending on S which may blow up as $S \to T$. The proof of Theorem A, which shows in particular that U satisfies (†), is given in Theorem 4.2, Theorem 4.7 and Theorem 4.8. The Lipschitz and semi-concave constants of U depend explicitly on $C(S)$.

We prove in Theorem 5.1 that the envelope U is moreover (Lipschitz) continuous in space if so are the data $(h_0, \log g, F)$.

Focusing for a while on the case of the unit ball with regular boundary data, we obtain the following parabolic analogue of Bedford and Taylor's celebrated result [1] :

THEOREM B. – *Assume $\Omega = \mathbb{B}$ is the unit ball in $\mathbb{C}^n$ and*

— *$G := \log g$ is $\mathcal{C}^{1,1}$ in $\bar{\mathbb{B}}$;*

— *h is uniformly Lipschitz in $t \in [0,T[$, satisfies $\partial_t^2 h(t,z) \leq C/t^2$, $z \in \partial\mathbb{B}$, and h is uniformly $\mathcal{C}^{1,1}$ in $z \in \bar{\mathbb{B}}$;*

— *F is Lipschitz and semi-convex in $[0,T[\times \bar{\mathbb{B}} \times J$, for each $J \Subset \mathbb{R}$.*

Then the upper envelope $U := U_{h,g,F}$ is locally uniformly $\mathcal{C}^{1,1}$ in z and locally uniformly Lipschitz in $t \in]0,T[$. For almost any $(t,z) \in \mathbb{B}_T$, we have

$$\det\left(\partial_j \bar{\partial}_k U(t,z)\right) = e^{\partial_t U(t,z) + F(t,z,U(t,z))} g(z).$$

In particular U is a pluripotential solution to the Cauchy-Dirichlet problem for the parabolic Equation (CMAF) *with boundary values h.*

This result is obtained as a combination of Theorem 5.3 and Theorem 6.1. Using an approximation and balayage process we then treat the case of more general domains Ω with less regular boundary data, obtaining the following solution to our original problem :

THEOREM C. – *Assume h is a Cauchy-Dirichlet boundary data in Ω_T such that for all $0 < S < T$, and for all $(t,z) \in]0,S] \times \partial\Omega$,*

$$(\dagger) \qquad t|\partial_t h(t,z)| \leq C(S) \quad \textit{and} \quad t^2 \partial_t^2 h(t,z) \leq C(S).$$

The envelope of all subsolutions to (CMAF) *with Cauchy-Dirichlet boundary data h is a pluripotential solution to this Cauchy-Dirichlet problem.*

The proof of this fundamental result is given in Theorem 6.5. We eventually establish a comparison principle, which shows that $U_{h,g,F}$ is unique:

THEOREM D. – *Same assumptions as in Theorem A. Let Φ be a bounded pluripotential subsolution to* (CMAF) *with boundary values h_Φ. Let Ψ be a bounded pluripotential supersolution with boundary values h_Ψ, such that Ψ is locally uniformly semi-concave in $t \in]0,T[$ and h_Φ satisfies* (†). *Then*

$$h_\Psi \geq h_\Phi \text{ on } \partial_0\Omega_T \Longrightarrow \Phi \leq \Psi \text{ in } \Omega_T.$$

In particular, there is a unique pluripotential solution to the Cauchy-Dirichlet problem for (CMAF) *with boundary data h, which is locally uniformly semi-concave in t.*

The proof of Theorem D is given in Section 6.3; it uses some ideas from [13, 6]. When all the data (h, F, g, u) are continuous, one can show that the solution U coincides with the viscosity solution constructed in [10]. We refer the reader to [16] for a detailed comparison of viscosity and pluripotential concepts.

Notations and assumptions on the data

We finish this introduction by fixing some notations that will be used throughout the paper.

The domain. – In the whole article we let dV denote the Euclidean volume form in $\mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^n$ be a strictly pseudoconvex domain : there exists a smooth function ρ in a neighborhood V of $\bar{\Omega}$ such that

$$\Omega = \{z \in V \,;\, \rho(z) < 0\},$$

where $\partial_z \rho \neq 0$ on $\partial\Omega$ and ρ is strictly plurisubharmonic in V. We set $\Omega_T :=]0,T[\times \Omega$ with $T > 0$. Most of the time we will assume that $T < +\infty$.

Recall that if a function $u : \Omega \to [-\infty, +\infty[$ is plurisubharmonic, then $dd^c u \geq 0$ is a positive current on Ω. Here $d = \partial + \overline{\partial}$ and $d^c = (i/2)(\overline{\partial} - \partial)$ are both real operators so that $dd^c = i\partial\overline{\partial}$.

We let $\mathbb{B}$ denote the Euclidean unit ball in $\mathbb{C}^n$ and $\lambda_{\mathbb{B}}$ denote the normalized Lebesgue measure on $\mathbb{B}$.

The function F. – We assume that $F : [0,T[\times \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and

— bounded in $[0,T[\times \Omega \times J$ for each $0 < S < T$, $J \Subset \mathbb{R}$;

— increasing in r: $r \mapsto F(t,x,r)$ is increasing for all $(t,x) \in \Omega_T$ fixed;

— locally uniformly Lipschitz in (t,r) : for each compact $J \Subset \mathbb{R}$ and each $0 < S < T$ there exists a constant $\kappa = \kappa(S,J) > 0$ such that for all $t, \tau \in [0,S]$, $z \in \Omega$, $r, r' \in J$,

$$|F(t,z,r) - F(\tau,z,r')| \leq \kappa(|t-\tau| + |r-r'|); \tag{0.4}$$

— locally uniformly semi-convex in (t,r) : for each compact subset $[0,S] \times J \Subset [0,T[\times \mathbb{R}$ there exists a constant $C = C(S,J) > 0$ such that, for any $z \in \Omega$, the function

$$(t,r) \mapsto F(t,z,r) + C(t^2 + r^2) \text{ is convex in } [0,S] \times J. \tag{0.5}$$

The density g. – We assume that

— $0 \leq g \in L^p(\Omega)$ for some $p > 1$ that is fixed thoughout the paper ;

— the set $\{z \in \Omega\ ;\ g(z) = 0\}$ has Lebesgue measure zero.

Boundary data h. – We assume throughout the article that

— $h : \partial_0\Omega_T \to \mathbb{R}$ is bounded, upper semi-continuous on $\partial_0\Omega_T$;

— the restriction of h on $[0, T[\times \partial\Omega$ is continuous;

— $t \mapsto h(t, z)$ is locally uniformly Lipschitz in $]0, T[$: for all $0 < S < T$ there is $C(S) > 0$ such that for all $(t, z) \in]0, S] \times \partial\Omega$,

$$t|\partial_t h(t, z)| \leq C(S);$$

— $h(0, \cdot)$ is bounded, plurisubharmonic in Ω, and satisfies

$$\lim_{\Omega \ni z \to \zeta} h(0, z) = h(0, \zeta),\ \forall \zeta \in \partial\Omega.$$

We eventually also assume that $t \mapsto h(t, z)$ is locally uniformly semi-concave in $]0, T[$: for all $0 < S < T$ there is $C(S) > 0$ such that

$$t^2 \partial_t^2 h(t, z) \leq C(S),\ \forall (t, z) \in [0, S] \times \partial\Omega.$$

The Kähler-Ricci flow. – Our assumptions on the data F, g, h are mild enough so that the results of this article can be applied to the study of the Kähler-Ricci flow on mildly singular Kähler varieties. We refer the interested reader to [15, Section 5] for more detail and geometric applications.

The constants. – We fix once and for all various uniform constants:

$$M_h := \sup_{\partial_0\Omega_T} |h|\ ,\ M_F := \sup_{\Omega_T} F(\cdot, \cdot, M_h). \tag{0.6}$$

We fix a plurisubharmonic function ρ in Ω, continuous in $\bar{\Omega}$ so that

$$(dd^c\rho)^n = g dV,\ \ \rho = 0 \text{ in } \partial\Omega, \tag{0.7}$$

in the weak sense in Ω. Such a function exists by [23, 25] (as we assumed that $p > 1$) and there is moreover a uniform a priori bound on ρ,

$$\|\rho\|_{L^\infty(\Omega)} \leq c_n \|g\|_{L^p(\Omega)}^{1/n},$$

where $c_n > 0$ is a uniform constant depending on n, Ω, p.

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1. Families of plurisubharmonic functions

Parabolic potentials form the basic objects of our study. They can be seen as a weakly regular family of plurisubharmonic functions. In this section we define them and establish their first properties.

1.1. Basic properties

1.1.1. *Parabolic potentials.* – We start with some basic definitions which will be used throughout all the paper.

Definition 1.1. – Let $u : \Omega_T :=]0,T[\times \Omega \longrightarrow [-\infty,+\infty[$ be a given function.

We say that the family $\{u(\cdot,z)\ ;\ z \in \Omega\}$ is locally uniformly Lipschitz in $]0,T[$ if for any subinterval $J \Subset]0,T[$ there exists a constant $\kappa := \kappa_J(u) > 0$ such that

$$u(t,z) \leq u(s,z) + \kappa|t-s|, \text{ for all } s,t \in J \text{ and } z \in \Omega. \tag{1.1}$$

Definition 1.2. – The set of parabolic potentials $\mathscr{P}(\Omega_T)$ is the set of functions $u : \Omega_T :=]0,T[\times \Omega \longrightarrow [-\infty,+\infty[$ such that

— for all $t \in]0,T[$, the slice $u_t : z \mapsto u(t,z)$ is plurisubharmonic in Ω;

— the family $\{u(\cdot,z)\ ;\ z \in \Omega\}$ is locally uniformly Lipschitz in $]0,T[$.

$\mathrm{PSH}(\Omega)$ embeds in $\mathscr{P}(\Omega_T)$ as the class of time independent potentials. Basic operations on plurisubharmonic functions extend naturally to parabolic potentials:

— if $u,v \in \mathscr{P}(\Omega_T)$ then $u+v \in \mathscr{P}(\Omega_T)$ and $\max(u,v) \in \mathscr{P}(\Omega_T)$;

— if $u \in \mathscr{P}(\Omega_T)$ and $t \mapsto c(t)$, $t \mapsto \lambda(t) \geq 0$ are locally Lipschitz, then $(z,t) \mapsto \lambda(t)u(z,t) + c(t)$ is also a parabolic potential.

Here is another interesting source of examples of parabolic potentials:

Example 1.3. – Consider a parabolic potential $\varphi \in \mathscr{P}(\Omega_T)$ such that $\partial_z \varphi \in L^\infty(\Omega_T)$. Let $\Theta_t : \Omega \longrightarrow \Omega$ be a family of holomorphic automorphisms of Ω depending smoothly on a real parameter $t \in]0,T[$. Then the function

$$\psi(t,z) := \varphi(t,\Theta_t(z))$$

is a parabolic potential on Ω_T. For example the function

$$\psi(t,z) := \log^+ \left| \frac{z-t}{1-tz} \right|$$

is a parabolic potential on $]0,1[\times \mathbb{D}$.

It turns out that parabolic potentials enjoy joint upper semi-continuous regularity in $]0,T[\times \Omega$, as the following result shows.

Proposition 1.4. – *Let $u \in \mathscr{P}(\Omega_T)$. Then u is upper semi-continuous in $\Omega_T :=]0,T[\times \Omega$, hence locally bounded from above in Ω_T.*

This result follows from the more general Lemma 1.5 below which will be useful later. Given a function ϕ defined on a metric space (Z,d), we define the upper semi-continuous regularization $\mathrm{usc}_Z\phi$ on Z by

$$\mathrm{usc}_Z\phi\,(z) := \limsup_{z' \to z} \phi(z') = \inf_{r>0}\big(\sup_{B(z,r)} \phi\big).$$

If ϕ is locally bounded from above in Z, then $\mathrm{usc}_Z u$ is upper semi-continuous in Z : it is the smallest upper semi-continuous function lying above u.

Fix $I \subset \mathbb{R}$ an interval, (Y, d) a metric space and $\phi : I \times Y \longrightarrow [-\infty, \infty[$ a given function. Recall that the family $\{\phi(\cdot, y); y \in Y\}$ is uniformly upper continuous at some point $t_0 \in I$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $t \in I$ with $|t - t_0| \leq \delta$ and any $y \in Y$,

$$\phi(t, y) \leq \phi(t_0, y) + \varepsilon.$$

Observe that if the family $\{\phi(\cdot, y); y \in Y\}$ is locally uniformly Lipschitz in I, then it is uniformly upper semi-continuous in I.

LEMMA 1.5. – *Let $\phi : I \times Y \longrightarrow [-\infty, +\infty[$ be a function satisfying the following conditions :*

(i) *the family $\{\phi(\cdot, y); y \in Y\}$ is uniformly upper semi-continuous in I,*

(ii) *For any $t_0 \in I$, the function $\phi_{t_0} = \phi(t_0, \cdot)$ is locally bounded from above in Y.*

Then ϕ is locally bounded from above in $I \times Y$ and for all $(t, y) \in I \times Y$,

$$(\mathrm{usc}_{I \times Y} \phi)(t, y) = (\mathrm{usc}_Y \phi_t)(y).$$

In particular if for some point $(t_0, y_0) \in J \times Y$, the function $\phi(t_0, \cdot)$ is upper semi-continuous at the point $y_0 \in Y$, then ϕ is (jointly) upper semi-continuous at the point $(t_0, y_0) \in J \times Y$.

Proof. – Fix $(t_0, y_0) \in J \times Y$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that for any $t \in J$ with $|t - t_0| \leq \delta$ and any $y \in Y$,

$$\phi(t, y) \leq \phi(t_0, y) + \varepsilon.$$

Since $\phi(t_0, \cdot)$ is bounded from above in a neighborhood of y_0, it follows that ϕ is bounded from above in a neighborhood of (t_0, y_0). Moreover taking the limsup in the previous inequality and letting $\varepsilon \to 0^+$ yields

$$\mathrm{usc}_{J \times Y}\, \phi(t_0, y_0) \leq (\mathrm{usc}_Y \phi_{t_0})(y_0) < +\infty.$$

Since the reverse inequality is obvious, the lemma is proved. □

The upper semi-continuity at $t = 0$ can be naturally obtained as follows :

LEMMA 1.6. – *Let $v : \Omega_T =]0, T[\times \Omega \longrightarrow [-\infty, +\infty[$ be a function locally bounded from above in Ω_T and satisfying the following conditions :*

(i) *for any $t \in]0, T[$ the function $v_t := v(t, \cdot)$ is plurisubharmonic in Ω;*

(ii) *for any $z \in \Omega$ the function $v(\cdot, z)$ is upper semi-continuous in $]0, T[$. Then v is upper semi-continuous in Ω_T.*

Furthermore assume that the function v is bounded from above on Ω_T and define for $z \in \Omega$,

$$v_0(z) := (\limsup_{t \to 0^+} v_t)^*(z) = \limsup_{\zeta \to z} \left(\limsup_{t \to 0^+} v_t(\zeta) \right).$$

Then v_0 is plurisubharmonic in Ω and the extension $\tilde{v} : [0, T[\times \Omega \to [-\infty, +\infty[$ of v to the vertical slice $\{0\} \times \Omega$ by $\tilde{v}(0, z) := v_0(z)$ is upper semi-continuous in $[0, T[\times \Omega$.

Proof. – Fix $(t_0, z_0) \in \Omega_T$. Since v is locally bounded from above we may assume that for some $r > 0$ and $\delta \in]0, r[$ small enough so that $\bar{B}(z_0, 2r) \Subset \Omega$ and $]t_0 - \delta, t_0 + \delta[\subset]0, T[$, we have $v \leq 0$ in $[t_0 - \delta, t_0 + \delta] \times \bar{B}(z_0, 2r)$.

Fix t such that $|t - t_0| \leq \delta$. Since $v_t \leq 0$ in $B(z, r + \delta)$, by the submean-value inequality applied to the plurisubharmonic function v_t on the ball $B(z, r+\delta)$, we have for $|z - z_0| \leq \delta$,

$$v(t, z) \leq \frac{1}{\mathrm{Vol}(B(z, r + \delta))} \int_{B(z_0, r)} v(t, \zeta) dV(\zeta). \tag{1.2}$$

It thus follows from Fatou's Lemma and assumption (ii) that

$$\limsup_{(t,z)\to(t_0,z_0)} v(t, z) \leq \frac{1}{\mathrm{Vol}(B(z_0, r + \delta))} \int_{B(z_0, r)} v(t_0, \zeta) dV(\zeta). \tag{1.3}$$

Since $v(t_0, \cdot)$ is plurisubharmonic in Ω, letting $\delta \to 0^+$ and $r \to 0^+$ we obtain

$$\limsup_{(t,z)\to(t_0,z_0)} v(t, z) \leq v(t_0, z_0),$$

which proves that v is upper semi-continuous at (t_0, z_0).

Now set $t_0 = 0$ and $z_0 \in \Omega$. Since $\{v_t \; ; \; t \in]0, T[\}$ is a family of plurisubharmonic functions in Ω which is uniformly bounded from above in Ω, it follows from a classical result of Lelong that v_0 is plurisubharmonic in Ω. Observe that the inequality (1.2) is still valid for $0 < t < \delta$ and $|z - z_0| \leq \delta$ for $\delta > 0$ small enough. Then it follows that

$$\limsup_{(t,z)\to(0,z_0)} v(t, z) \leq \frac{1}{\mathrm{Vol}(B(z_0, r + \delta))} \int_{B(z_0, r)} v_0(\zeta) dV(\zeta).$$

Since v_0 is plurisubharmonic, letting $\delta \to 0^+$ and $r \to 0^+$ we obtain

$$\limsup_{(t,z)\to(0,z_0)} v(t, z) \leq v_0(z_0) =: \tilde{v}(0, z_0),$$

which proves the semi-continuity of the extension $\tilde{v}$ at the point $(0, z_0)$. □

1.1.2. *Envelopes of parabolic potentials.* – The next result provides a parabolic analogue of a classical result of Lelong about negligible sets for plurisubharmonic functions; it will play an important role in Section 3.

LEMMA 1.7. – *Let $\mathcal{U} \subset \mathcal{P}(\Omega_T)$ be a family of functions which is locally uniformly bounded from above. Assume $U := \sup\{u \; ; \; u \in \mathcal{U}\}$ is locally uniformly Lipschitz in $t \in]0, T[$. Then*

— *the upper semi-continuous regularization U^* (in Ω_T) belongs to $\mathcal{P}(\Omega_T)$;*

— *for any $t \in]0, T[$, $U^*(t, \cdot) = (U_t)^*$ in Ω and the exceptional set*

$$E(U) := \{(t, z) \in \Omega_T \; ; \; U(t, z) < U^*(t, z)\}$$

has zero $(2n + 1)$-dimensional Lebesgue measure in $\Omega_T \subset \mathbb{R}^{2n+1}$.

The smallness of the exceptional set $E(U)$ can be made more precise: all the t-slices of $E(U)$ have zero $2n$-dimensional Lebesgue measure in Ω.

Proof. – Our assumption ensures that the function U is locally bounded from above in Ω_T. The first statement follows immediately from (1.1). Since U is locally Lipschitz in t, there is no need to regularize in the t variable : it follows from Lemma 1.5 above that for all $(t,z) \in \Omega_T$,

$$U^*(t,z) = (U_t)^*(z),$$

where the upper semi-continuous regularization in the LHS is in the (t,z)-variable, while the upper semi-continuous regularization in the RHS is in the z-variable only, t being fixed. A classical theorem of Lelong (see [17, Proposition 1.40]) ensures that $E_t = \{z \in \Omega \,;\, U_t(z) < (U_t)^*(z)\}$ has zero Lebesgue measure in $\mathbb{C}^n$. Since $E(U) = \{(t,z) \in \Omega_T \,;\; z \in E_t\}$ the second statement of the lemma follows from Fubini's theorem. □

1.1.3. *Approximate submean-value inequalities.* – Parabolic potentials satisfy approximate submean-value inequalities:

Lemma 1.8. – *Let $\Omega \subset \mathbb{C}^n$ be a domain and $u \in \mathcal{P}(\Omega_T)$. Fix $(t_0,x_0) \in \Omega_T$ and $\varepsilon_0, r_0 > 0$ so that $[t_0-\varepsilon_0, t_0+\varepsilon_0] \times \bar{B}(x_0,r_0) \Subset \Omega_T$. Then for any $0 < \varepsilon \le \varepsilon_0$, $0 < r \le r_0$,*

$$u(t_0,x_0) \le \int_{-1}^{1}\int_{\mathbb{B}} u(t_0+\varepsilon s, x_0 + r\xi)\, d\lambda_{\mathbb{B}}(\xi)\, ds/2 + \kappa_0 \varepsilon,$$

where $\kappa_0 > 0$ is the uniform Lipschitz constant of u in $[t_0-\varepsilon_0, t_0+\varepsilon_0] \times B(x_0,r)$.

Proof. – Since $u(t_0,\cdot)$ is psh in Ω, the submean-value inequality yields, for all $0 < r \le r_0$,

$$u(t_0,z_0) \le \int_{\mathbb{B}} u(t_0, z_0 + r\xi)\, d\lambda_{\mathbb{B}}(\xi).$$

The Lipschitz condition ensures that for $0 < r \le r_0$, $0 < \varepsilon \le \varepsilon_0$, and $-1 \le s \le 1$,

$$\int_{\mathbb{B}} u(t_0, z_0 + r\xi)\, d\lambda_{\mathbb{B}}(\xi) \le \int_{\mathbb{B}} u(t_0+\varepsilon s, z_0 + r\xi)\, d\lambda_{\mathbb{B}}(\xi) + \kappa_0 \varepsilon |s|.$$

Integrating in s we obtain the required inequality. □

Parabolic potentials therefore enjoy interesting integrability properties.

Corollary 1.9. – *We have $\mathcal{P}(\Omega_T) \subset L^q_{\mathrm{loc}}(\Omega_T)$ for any $q \ge 1$. Moreover if $u \in \mathcal{P}(\Omega_T)$ then for all $(t,z) \in \Omega_T$,*

$$u(t,z) = \lim_{\varepsilon,r \to 0} \int_{-1}^{1}\int_{\mathbb{B}} u(t+\varepsilon s, z + r\xi)\, d\lambda_{\mathbb{B}}(\xi)\, ds/2.$$

In particular if $u, v \in \mathcal{P}(\Omega_T)$ and $u \le v$ a.e. in Ω_T, then $u \le v$ everywhere. Here $\lambda_{\mathbb{B}}$ is the normalized Lebesgue measure on the unit ball $\mathbb{B} \subset \mathbb{C}^n$.

Proof. – Let $u \in \mathcal{P}(\Omega_T)$ and fix $K \Subset \Omega_T$ a compact subset. Then there exists a compact interval $J \Subset]0,T[$ and a compact subset $D \Subset \Omega$ such that $K \subset J \times D$.

Fix $t_0 \in J$. Since $u(t_0, \cdot)$ is plurisubharmonic in Ω we have that $u(t_0, \cdot) \in L^q(D)$. Since $u(\cdot, z)$ is uniformly Lipschitz in J we infer $u(t, z) \leq u(t_0, z) + \kappa_J |t - t_0|$ for all $t \in J$ and $z \in D$. It thus follows from Fubini's theorem that

$$\int_{J\times D} |u(t,z)|^q d\lambda_{2n}(z)dt = \int_{J\times D} |u(t,z)|^q d\lambda_{2n}(z)dt$$
$$\leq 2^{q-1} \int_{J\times D} |u(t_0,z)|^q \lambda_{2n}(z) + 2^{q-1}\kappa_J^q \mathrm{Vol}(D) \int_J |t - t_0|^q dt.$$

This proves that $u \in L^q(K)$, hence $u \in L^q_{\mathrm{loc}}(\Omega_T)$.

Fix $(t_0, z_0) \in \Omega_T$ and $\delta > 0$. Since $u(t_0, \cdot)$ is psh in Ω we have

$$u(t_0, z_0) = \lim_{r\to 0^+} \int_{\mathbb{B}} u(t_0, z_0 + r\xi) d\lambda_{\mathbb{B}}(\xi). \tag{1.4}$$

Fix $\varepsilon_0 > 0$, and $r_0 > 0$ such that $[t_0 - \varepsilon_0, t_0 + \varepsilon_0] \Subset]0, T[$, and $B(z_0, r_0) \Subset \Omega$. Let κ_0 be the uniform Lipschitz constant of u in $[t_0 - \varepsilon_0, t_0 + \varepsilon_0] \times \Omega$. Then for $\varepsilon \in]0, \varepsilon_0[$, $r \in]0, r_0[$,

$$\int_{-1}^{1}\int_{\mathbb{B}} u(t_0 + \varepsilon s, z_0 + r\xi)\, d\lambda_{\mathbb{B}}(\xi)\, ds/2 \leq \int_{\mathbb{B}} u(t_0, z_0 + r\xi)\, d\lambda_{\mathbb{B}}(\xi) + \kappa_0\varepsilon.$$

From this and (1.4) we obtain

$$u(t_0, z_0) \geq \lim_{\varepsilon, r\to 0} \int_{-1}^{1}\int_{\mathbb{B}} u(t_0 + \varepsilon s, z_0 + r\xi) d\lambda_{\mathbb{B}}(\xi) ds/2.$$

The reverse inequality was already obtained in Lemma 1.8. □

REMARK 1.10. – Let μ be a positive Borel measure s.t. $PSH(\Omega) \subset L^q_{\mathrm{loc}}(\Omega, \mu)$ for some $q \geq 1$. The previous proof shows that $\mathscr{P}(\Omega_T) \subset L^q(\Omega_T, \ell \otimes \mu)$, where ℓ is the Lebesgue measure on $]0, T[$.

Besides the Lebesgue measure λ_{2n}, another important example is $\mu = g\lambda_{2n}$, where $g \in L^p(\Omega)$ for some $p > 1$. By Hölder inequality, the measure μ satisfies the integrability condition with $q := p/(p-1)$.

1.2. Behavior on slices

We now estimate the L^1-norm on slices in terms of the global L^1-norm.

LEMMA 1.11. – *Fix $u, v \in \mathscr{P}(\Omega_T)$ and $0 < T_0 < T_1 < S < T$. Then for all $T_0 \leq t \leq T_1$,*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\Omega)} \leq 2M \max\left\{\|u - v\|^{1/2}_{L^1(\Omega_{T_1})}, \|u - v\|_{L^1(\Omega_{T_1})}\right\},$$

where $M := \max\{\sqrt{\kappa \mathrm{Vol}(\Omega)}, (S - T_1)^{-1}\}$, and κ is the uniform Lipschitz cosntant of the function $t \longmapsto \int_\Omega (|u(t,z) - v(t,z)| d\lambda_{2n}(z)$ in $[T_0, T_1]$.

This lemma quantifies the following facts : for functions in $\mathscr{P}(\Omega_T)$,

- convergence in $L^1(\Omega_T)$ implies convergence of their slices in $L^1(\Omega)$;
- boundedness in $L^1(\Omega_T)$ implies compactness of their slices in $L^1(\Omega)$.

Proof. – Assume first that u, v are bounded from below in Ω_T. Since u, v are locally uniformly Lipschitz in $t \in [T_0, S]$, we deduce that for any $T_0 \leq t \leq S$, $T_0 \leq s \leq S$, and $z \in \Omega$,

$$|u(t,z) - v(t,z)| \leq \kappa|s-t| + |u(s,z) - v(s,z)|,$$

where $\kappa := \kappa_J(u) + \kappa_J(v)$ and $\kappa_J(u)$ (resp. $\kappa_J(v)$) is the uniform Lipschitz constant of u (resp. v) on $J := [T_0, S]$. We infer

$$\int_\Omega |u(t,z) - v(t,z)| dV(z) \leq \kappa|s-t|\mathrm{Vol}(\Omega) + \int_\Omega |u(s,z) - v(s,z)| dV(z).$$

Thus the function

$$t \mapsto \theta(t) := \int_\Omega |u(t,z) - v(t,z)| dV(z)$$

is a Lipschitz function in $[T_0, S]$ with Lipschitz constant $\kappa \mathrm{Vol}(\Omega)$. The conclusion follows from the next lemma, an elementary result in one real variable.

The general case is deduced from the previous one by considering the canonical approximants $u_j := \max\{u, -j\}$ and $\max\{v, -j\}$ which have the same properties as u and v respectively. □

We have used the following inequality :

LEMMA 1.12. – *Fix $0 < S_0 < S_1 < S$ and let $\theta : [S_0, S] \longrightarrow \mathbb{R}$ be such that for all $s, \sigma \in [S_0, S]$ with $s \leq \sigma$, $\theta(s) \leq \theta(\sigma) + \kappa(\sigma - s)$. Then*

$$\max_{S_0 \leq s \leq S_1} \theta(s) \leq 2M \max\{\sqrt{\|\theta\|}, \|\theta\|\},$$

where $M := \max\{\sqrt{\kappa}, (S - S_1)^{-1}\}$ and $\|\theta\| := \|\theta\|_{L^1([S_0, S])}$.

Proof. – Fix $0 < \delta \leq S - S_1$. Then for $\sigma, s \in [S_0, S_1]$ with $s \leq \sigma$,

$$\theta(s) \leq \theta(\sigma) + \kappa(\sigma - s).$$

Fix $S_0 \leq s \leq S_1$. Integrating in σ on $[s, s+\delta] \subset [S_0, S]$, we get

$$\theta(s) \leq \frac{\kappa\,\delta}{2} + \int_s^{s+\delta} \theta(\sigma) \frac{d\sigma}{\delta} \leq \frac{\delta\kappa}{2} + \delta^{-1}\|\theta\|. \tag{1.5}$$

The minimum of $\tau \longmapsto \kappa\tau/2 + \tau^{-1}\|\theta\|$ is achieved at $\tau_0 := \sqrt{2}\|\theta\|^{1/2}/\sqrt{\kappa}$. If $2\|\theta\| \leq \kappa(S - S_1)^2$, i.e., $\tau_0 \leq S - S_1$, then $\theta(t) \leq 2\sqrt{\kappa\|\theta\|}$, for $t \in [S_0, S_1]$. If $2\|\theta\| \geq \kappa(S - S_1)^2$, applying (1.5) with $\delta = S - S_1$ yields

$$\max_{S_0 \leq t \leq S_1} \theta(t) \leq \kappa(S - S_1)/2 + \|\theta\|(S - S_1)^{-1} \leq 2\|\theta\|(S - S_1)^{-1}.$$

Alltogether we obtain

$$\max_{S_0 \leq t \leq S_1} \theta(t) \leq 2\max\{\sqrt{\kappa\|\theta\|}, \|\theta\|(S - S_1)^{-1}\}.$$ □

1.3. Time derivatives and semi-concavity

In this section we observe that a parabolic potential φ has well defined time derivatives $\partial_t \varphi$ almost everywhere.

Fix a positive Borel measure μ on Ω such that $PSH(\Omega) \subset L^1_{\text{loc}}(\Omega, \mu)$.

LEMMA 1.13. – *Let $\varphi \in \mathcal{P}(\Omega_T)$. Then there exists a Borel set $E \subset \Omega_T$ $\ell \otimes \mu$-negligible such that $\partial_t \varphi(t, z)$ exists for all $(t, z) \notin E$.*

In particular $\partial_t \varphi \in L^\infty_{\text{loc}}(\Omega_T)$ and for any continuous function $\gamma \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$, $\gamma(\partial_t \varphi)\, \ell \otimes \mu$ is a well defined Borel measure in Ω_T.

Proof. – By Remark 1.10 the set $\tilde{\Omega}_T := \{(t, z) \in \Omega_T; \varphi(t, z) > -\infty\}$ is of full $(\ell \otimes \mu)$-measure i.e., the set $\Omega_T \setminus \tilde{\Omega}_T$ has zero $\ell \otimes \mu$-measure in $\mathbb{R} \times \mathbb{C}^n \simeq \mathbb{R}^{2n+1}$.

We set, for $(t, z) \in \tilde{\Omega}_T$,

$$\partial^u_t \varphi(t, z) := \limsup_{s \to 0} \frac{\varphi(t+s, z) - \varphi(t, z)}{s} = \limsup_{\mathbb{Q}^* \ni s \to 0} \frac{\varphi(t+s, z) - \varphi(t, z)}{s},$$

and

$$\partial^l_t \varphi(t, z) := \liminf_{s \to 0} \frac{\varphi(t+s, z) - \varphi(t, z)}{s} = \liminf_{\mathbb{Q}^* \ni s \to 0} \frac{\varphi(t+s, z) - \varphi(t, z)}{s}.$$

The equalities above follow from the Lipschitz property of φ. These two functions are measurable in $(\Omega_T, \ell \otimes \mu)$, hence the set

$$E := \{(t, z) \in \tilde{\Omega}_T \; ; \; \partial^l \varphi(t, z) < \partial^u \varphi(t, z)\} \cup \{(t, z) \in \Omega_T; \varphi(t, z) = -\infty\}$$

is $\ell \otimes \mu$-measurable.

For each $(t_0, z_0) \in]0, T[\times \Omega$ such that $\varphi(t_0, z_0) > -\infty$, the function $t \mapsto \varphi(t, z_0)$ is locally Lipschitz in a neighborhood of t_0, hence differentiable almost everywhere in this neighborhood. Hence, for μ-almost all $z \in \Omega$,

$$E_z := \{t \in]0, T[\; ; \; (t, z) \in E\}$$

has zero ℓ-measure. Fubini's theorem thus ensures that $\ell \otimes \mu(E) = 0$. □

The previous lemma shows that $\partial^u_t \varphi = \partial^l_t \varphi$, $\ell \otimes \mu$-almost everywhere in Ω_T. These thus define a function which we denote by $\partial_t \varphi \in L^\infty_{\text{loc}}(\Omega_T)$.

When φ is semi-concave (or semi-convex) in t, we can improve the previous result.

DEFINITION 1.14. – We say that $\varphi : \Omega_T \longrightarrow \mathbb{R}$ is uniformly semi-concave in $]0, T[$ if for any compact $J \Subset]0, T[$, there exists $\kappa = \kappa(J, \varphi) > 0$ such that for all $z \in \Omega$, the function $t \longmapsto \varphi(t, z) - \kappa t^2$ is concave in J.

The definition of uniformly semi-convex functions is analogous. Note that such functions are automatically locally uniformly Lipschitz.

For a bounded parabolic potential φ which is locally semi-concave in t the left and right derivatives

$$\partial^+_t \varphi(t, z) = \lim_{s \to 0^+} \frac{\varphi(t+s, z) - \varphi(t, z)}{s},$$

and

$$\partial^-_t \varphi(t, z) := \lim_{s \to 0^-} \frac{\varphi(t+s, z) - \varphi(t, z)}{s}$$

exist for all $t \in]0,T[$, and $\partial_t\varphi(t,z)$ exists if $\partial_t^+\varphi(t,z) = \partial_t^-\varphi(t,z)$.

LEMMA 1.15. – *Let $\varphi : \Omega_T \longrightarrow \mathbb{R}$ be a continuous function which is uniformly semi-concave in $]0,T[$. Then $(t,z) \mapsto \partial_t^-\varphi(t,z)$ is upper semi-continuous while $(t,z) \mapsto \partial_t^+\varphi(t,z)$ is lower semi-continuous in Ω_T. In particular, there exists a Borel set $E \subset \Omega_T$ which is $\ell \otimes \mu$-negligible, such that $\partial_t^+\varphi$ and $\partial_t^-\varphi$ coincide and are continuous at each point in $\Omega_T \setminus E$.*

By replacing φ with $-\varphi$ one obtains similar conclusions for uniformly semi-convex functions.

Proof. – For simplicity we only treat the semi-convex case. It suffices to consider the case when $t \mapsto \varphi(t,z)$ is convex in $]0,T[$, for all $z \in \Omega$. In this case for all $(t,z) \in \Omega_T$, the slope function
$$s \longmapsto p_s(t,z) := \frac{\varphi(t+s,z) - \varphi(t,z)}{s}$$
is monotone increasing on each interval not containing 0. It is moreover continuous in (t,z). In particular,
$$\partial_t^+\varphi(t,z) = \lim_{s\to 0^+} p_s(t,z) = \inf_{s>0} p_s(t,z)$$
is upper semi-continuous in Ω_T and
$$\partial_t^-\varphi(t,z) = \lim_{s\to 0^-} p_s(t,z) = \sup_{s<0} p_s(t,z)$$
is lower semi-continuous in Ω_T. This proves the first part of the lemma.

The second part follows from the fact that convex functions are locally Lipschitz in their domain, and Lemma 1.13. □

1.4. Compactness properties

We introduce a natural complete metrizable topology on the convex set $\mathcal{P}(\Omega_T)$.

We recall the definition of the Sobolev space $W^{1,0}_{\infty,\mathrm{loc}}(\Omega_T)$: this is the set of functions $u \in L^1_{\mathrm{loc}}(\Omega_T)$ whose partial time derivative (in the sense of distributions) satisfies $\partial_t u \in L^\infty_{\mathrm{loc}}(\Omega_T)$. It follows from Lemma 1.13 that
$$\mathcal{P}(\Omega_T) \subset W^{1,0}_{\infty,\mathrm{loc}}(\Omega_T).$$

Let $K \subset \Omega$ be a compact subset. The local uniform Lipschitz constant of $\varphi \in \mathcal{P}(\Omega_T)$ on a compact subinterval $J \Subset]0,T[$ is given by
$$\sup_{t,s\in J, s\neq t}\ \sup_{z\in K}{}^{*} \frac{|\varphi(s,z) - \varphi(t,z)|}{|s-t|} = \|\partial_t\varphi\|_{L^\infty(J\times\Omega)}.$$

DEFINITION 1.16. – We endow $\mathcal{P}(\Omega_T)$ with the semi-norms associated to $W^{1,0}_{\infty,\mathrm{loc}}(\Omega_T)$: given a compact subset $J \Subset]0,T[$ and $u \in W^{1,0}_{\infty,\mathrm{loc}}(\Omega)$, these are
$$u \longmapsto \|\partial_t u\|_{L^\infty(J\times K)} + \int_J\int_K |u(t,z)|dV(z)dt.$$

The spaces $L^q(\Omega_T)$ are defined with respect to the $(2n+1)$-dimensional Lebesgue measure in Ω_T. For $k, \ell \in \mathbb{N}$ and $q \geq 1$, we denote by $W^{k,\ell}_{q,\mathrm{loc}}(\Omega_T)$ the Sobolev space of Lebesgue measurable functions whose partial derivatives with respect to t up to order k and partial derivatives with respect to z up to order ℓ in the sense of distributions are in $L^q_{\mathrm{loc}}(\Omega_T)$.

Parabolic potentials enjoy useful compactness properties :

PROPOSITION 1.17. – *Let* $(\varphi_j) \subset \mathcal{P}(\Omega_T)$ *be a sequence which*

— *is locally uniformly bounded from above in* Ω_T*;*

— *is locally uniformly Lipschitz in* $]0, T[$*;*

— *does not converge locally uniformly to* $-\infty$ *in* Ω_T.

Then (φ_j) *is bounded in* $L^1_{\mathrm{loc}}(\Omega_T)$ *and there exists a subsequence which converges to some function* $\varphi \in \mathcal{P}(\Omega_T)$ *in* $L^1_{\mathrm{loc}}(\Omega_T)$*-topology.*

If (φ_j) *converges weakly to* φ *in the sense of distributions in* Ω_T*, then it converges in* $L^q_{\mathrm{loc}}(\Omega_T)$ *for all* $q \geq 1$.

The proof is an extension of Hartogs' lemma for sequences of plurisubharmonic functions (see e.g., [17, Theorem 1.46]).

Proof. – We first prove that (φ_j) is bounded in $L^1_{\mathrm{loc}}(\Omega_T)$. Fix $J \Subset]0, T[$ and $K \Subset \Omega$. From the assumptions it follows that, for each $t \in J$ fixed, $\varphi_j(t, \cdot)$ does not converge locally uniformly in Ω to $-\infty$. Hence $\varphi_j(t, \cdot)$ is bounded in $L^1_{\mathrm{loc}}(\Omega, dV)$. The second condition thus ensures that $\{\varphi_j\}$ is uniformly bounded in $L^1(J \times K)$.

For each $r \in \mathbb{Q} \cap]0, T[$, there exists a subsequence of $\varphi_j(r, \cdot)$ which converges in $L^1_{\mathrm{loc}}(\Omega)$ to some plurisubharmonic function $\varphi(r, \cdot)$ in Ω. After a Cantor process we can extract a subsequence from $\{\varphi_j\}$, still denoted by $\{\varphi_j\}$, such that for each $r \in \mathbb{Q} \cap]0, T[$, the sequence $\{\varphi_j(r, \cdot)\}$ converges in $L^1_{\mathrm{loc}}(\Omega)$ to $\varphi(r, \cdot)$. Since the sequence $\{\varphi_j\}$ is locally uniformly Lipschitz in t, it follows that the function $(r, z) \mapsto \varphi(r, z)$ is also locally uniformly Lipschitz in r. The function φ therefore uniquely extends to $]0, T[\times \Omega$ by

$$\varphi(t, z) := \lim_{\mathbb{Q} \ni r \to t} \varphi(r, z).$$

Since $\{\varphi_j\}$ is uniformly Lipschitz in t it follows that $\{\varphi_j(t, \cdot)\}$ converges in $L^1_{\mathrm{loc}}(\Omega)$ to $\varphi(t, \cdot)$, for all $t \in]0, T[$ and φ is locally uniformly Lipschitz in $t \in]0, T[$. The latter then implies that $\varphi \in \mathcal{P}(\Omega_T)$. By Fubini's theorem and dominated convergence it follows that $\{\varphi_j\}$ converges in $L^1_{\mathrm{loc}}(\Omega_T)$ to φ.

We now prove the last statement, assuming that $\varphi \in \mathcal{P}(\Omega_T)$ and that the sequence $\{\varphi_j\}$ converges in the weak sense of distributions to φ. We claim that for each $t \in]0, T[$, $\{\varphi_j(t, \cdot)\}$ converges in the sense of distributions in Ω to $\varphi(t, \cdot)$. Indeed, fix $t_0 \in]0, T[$ and let $\chi : \Omega \to \mathbb{R}$ be a smooth test function in Ω. Let $\varepsilon > 0$ be a small constant and let $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}^+$ be a smooth test function which is supported in $[t_0 - \varepsilon, t_0 + \varepsilon]$ and such that $\int_{\mathbb{R}} \eta_\varepsilon(t) dt = 1$. By assumption,

$$\lim_{j \to +\infty} \int_{\Omega_T} \varphi_j(t, z) \chi(z) \eta_\varepsilon(t) dt dV(z) = \int_{\Omega_T} \varphi(t, z) \chi(z) \eta_\varepsilon(t) dt dV(z). \tag{1.6}$$

Since the sequence $\{\varphi_j\}$ is locally uniformly Lipschitz in t, there exists a constant κ_0 depending on $\varepsilon_0 := \min(t_0, T - t_0)/2$ such that

$$|\varphi_j(t,z) - \varphi_j(t_0,z)| + |\varphi(t,z) - \varphi(t_0,z)| \leq \kappa_0 |t - t_0|,$$

for all $t \in [t_0 - \varepsilon_0, t_0 + \varepsilon_0]$ and $z \in \Omega$. We infer

$$\left| \int_{\Omega_T} \varphi_j(t,z)\chi(z)\eta_\varepsilon(t)dt dV(z) - \int_{\Omega_T} \varphi_j(t_0,z)\chi(z)\eta_\varepsilon(t)dt dV(z) \right| \leq \kappa_0 \varepsilon \int_\Omega |\chi(z)| dV(z). \tag{1.7}$$

The same estimate holds for φ. Combining (1.6) and (1.7) yields

$$\lim_{j\to+\infty} \int_\Omega \varphi_j(t_0,z)\chi(z)dV(z) = \int_\Omega \varphi(t_0,z)\chi(z)dV(z) + O(\varepsilon).$$

We finally let $\varepsilon \to 0$ to conclude the proof of the claim.

Classical properties of plurisubharmonic functions now ensure that $\{\varphi_j(t_0,\cdot)\}$ converges in $L^q_{\mathrm{loc}}(\Omega)$ to $\varphi(t_0,\cdot)$. Since $\{\varphi_j\}$ is locally uniformly Lipschitz in t, we conclude as above that $\{\varphi_j\}$ converges in $L^q_{\mathrm{loc}}(\Omega_T)$ to φ. □

Corollary 1.18. – *The class $\mathscr{P}(\Omega_T)$ is a subset of $L^q_{\mathrm{loc}}(\Omega_T)$ for all $q \geq 1$, and the inclusions $\mathscr{P}(\Omega_T) \hookrightarrow L^q_{\mathrm{loc}}(\Omega_T)$ are continuous.*

The weak topology and the L^q_{loc}-topologies are thus all equivalent when restricted to the class $\mathscr{P}(\Omega_T)$. The set $\mathscr{P}(\Omega_T)$ is thus a complete metric space when endowed with any of these topologies.

Lemma 1.19. – *We have $\mathscr{P}(\Omega_T) \subset W^{1,1}_{\mathrm{loc}}(\Omega_T)$.*

Proof. – Fix $u \in \mathscr{P}(\Omega_T)$. The goal is to prove that u has partial derivative (in t and z) in $L^1_{\mathrm{loc}}(\Omega_T)$.

We first recall a basic estimate for the gradient of a plurisubharmonic function. Fix $z_0 \in \Omega$ and $r > 0$ such that the polydisk $D(z_0, 2r)$ is contained in Ω. It follows from [21, Theorem 4.1.8] (see also [17, Theorem 1.48] and its proof at page 32-33) that the derivative of any plurisubharmonic function $z \mapsto \varphi(z)$ exists in $L^p_{\mathrm{loc}}(\Omega)$ for any $p < 2$ and the uniform estimate

$$\left(\int_{D(z_0,r)} |\nabla_z \varphi|^p dV \right)^{1/p} \leq C(p,r) \int_{D(z_0,2r)} |\varphi| dV$$

holds for a positive constant $C(p,r)$ depending only on r and p.

Fix $J \times K$ a compact subset of Ω_T. Then by our previous analysis and the compactness of K there exists a constant $C > 0$ depending on K and $\mathrm{dist}(K, \partial\Omega)$ and a compact subset $K \Subset L \Subset \Omega$ such that

$$\left(\int_K |\nabla_z \varphi|^p dV \right)^{1/p} \leq C \int_L |\varphi| dV,$$

for every $\varphi \in \mathrm{PSH}(\Omega)$.

Now, for each $t \in]0, T[$ the derivative of u in z exists and belongs to $L^p_{\mathrm{loc}}(\Omega)$ for any $p < 2$ (with uniform bound). Since u is locally uniformly Lipschitz in t it follows that $\partial_t u(t,z)$ is bounded in $J \times K$ and $u \in L^1(J \times L, dt dV)$. Altogether we obtain $u \in W^{1,1}_{\mathrm{loc}}(\Omega_T)$ as desired. □

2. Parabolic Monge-Ampère operators

2.1. Parabolic Chern-Levine-Nirenberg inequalities

We assume here that $\varphi \in \mathcal{P}(\Omega_T) \cap L^\infty_{\mathrm{loc}}(\Omega_T)$. For all $t \in]0, T[$, the function

$$\Omega \ni z \mapsto \varphi_t(z) = \varphi(t, z) \in \mathbb{R}$$

is psh and locally bounded, hence the Monge-Ampère measures $(dd^c \varphi_t)^n$ are well defined Borel measures in the sense of Bedford and Taylor [1].

We now show that this family depends continuously on t :

LEMMA 2.1. – *Fix $\varphi \in \mathcal{P}(\Omega_T) \cap L^\infty_{\mathrm{loc}}(\Omega_T)$ and χ a continuous test function in Ω_T. Then the function*

$$\Gamma_\chi : t \longmapsto \int_\Omega \chi(t, \cdot)(dd^c \varphi_t)^n$$

is continuous in $]0, T[$. Moreover if $\mathrm{Supp}(\chi) \Subset E_1 \Subset E_2 \Subset \Omega_T$, then

$$\sup_{0 \le t < T} \left| \int_\Omega \chi(t, \cdot)(dd^c \varphi_t)^n \right| \le C \max_{\Omega_T} |\chi| (\max_{E_2} |\varphi|)^n, \tag{2.1}$$

where $C > 0$ is a constant depending only on (E_1, E_2, Ω_T).

In particular, $t \longmapsto (dd^c \varphi_t)^n$ is continuous, as a map from $]0, T[$ to the space of positive Radon measures in Ω endowed with the weak*-topology.

Proof. – We can reduce to the case when the support of χ is contained in a product of compact subsets $J \times K \subset E^\circ \subset]0, T[\times \Omega$. Let $L \Subset \Omega$ be a compact subset such that $K \Subset L^\circ$ and $J \times L \subset E$.

We first prove (2.1). For any fixed $t \in]0, T[$,

$$\left| \int_\Omega \chi(t, \cdot)(dd^c \varphi_t)^n \right| \le \max_{\Omega_T} |\chi| \int_K (dd^c \varphi_t)^n.$$

The classical Chern-Levine-Nirenberg inequalities (see [17, Theorem 3.9]) ensure that there exists a constant $C = C(K, L) > 0$ such that

$$\int_\Omega \chi_t (dd^c \varphi_t)^n \le C \max_\Omega |\chi| (\max_L |\varphi_t|)^n \le C \max_{\Omega_T} |\chi| (\max_L |\varphi|)^n,$$

where $C > 0$ depends only on (K, L, Ω_T). This yields (2.1).

We now prove that Γ_χ is continous in $]0, T[$. Fix compact sets $J \Subset]0, T[$, $K \Subset \Omega$ such that $\mathrm{Supp}(\chi) \subset J \times K$. The continuity of Γ_χ on $]0, T[\setminus J$ is clear. Fixing $t_0 \in J$, we have for any $t \in J$,

$$\int_\Omega |\chi(t, \cdot) - \chi(t_0, \cdot)| (dd^c \varphi_t)^n \le \left(\sup_{t \in J} \int_K (dd^c \varphi_t)^n \right) \sup_K |\chi(t, \cdot) - \chi(t_0, \cdot)|.$$

The second term on the right-hand side converges to 0 by uniform continuity of χ while the first term is finite thanks to the CLN inequality as before (see [17, Theorem 3.9]). Therefore,

$$\lim_{t \to 0} \int_\Omega |\chi(t, \cdot) - \chi(t_0, \cdot)| (dd^c \varphi_t)^n = 0.$$

Since χ is a continuous test function we also have

$$\lim_{t\to t_0}\int_\Omega \chi(t_0,\cdot)(dd^c\varphi_t)^n = \int_\Omega \chi(t_0,\cdot)(dd^c\varphi_{t_0})^n.$$

This proves the continuity of Γ_χ at t_0, finishing the proof. $\square$

DEFINITION 2.2. – Fix $\varphi \in \mathscr{P}(\Omega_T)\cap L^\infty_{\mathrm{loc}}(\Omega_T)$. The map

$$\chi \mapsto \int_{\Omega_T} \chi dt \wedge (dd^c\varphi)^n := \int_0^T dt \left(\int_\Omega \chi(t,\cdot)(dd^c\varphi_t)^n\right)$$

defines a positive distribution in Ω_T denoted by $dt\wedge(dd^c\varphi)^n$, which can be identified with a positive Radon measure in Ω_T.

PROPOSITION 2.3. – *Fix $\varphi \in \mathscr{P}(\Omega_T)\cap L^\infty_{\mathrm{loc}}(\Omega_T)$ and let (φ^j) be a monotone sequence of functions in $\mathscr{P}(\Omega_T)\cap L^\infty_{\mathrm{loc}}(\Omega_T)$ converging to φ almost everywhere in Ω_T. Then*

$$dt\wedge(dd^c\varphi^j)^n \to dt\wedge(dd^c\varphi)^n$$

in the weak sense of measures in Ω_T.

Proof. – Let χ be a continuous test function in Ω_T. By definition,

$$\int_{\Omega_T} \chi dt\wedge(dd^c\varphi^j)^n = \int_0^T dt\left(\int_\Omega \chi(t,\cdot)(dd^c\varphi^j(t,\cdot))^n\right) =: \int_0^T F_j(t)dt.$$

It follows from [2, Theorem 2.1 and Proposition 5.2] that F_j converges to F pointwise in $]0,T[$. Lemma 2.1 ensures that F_j is uniformly bounded hence the conclusion follows from Lebesgue convergence theorem. $\square$

REMARK 2.4. – The conclusion of Proposition 2.3 also holds if the sequence (φ^j) uniformly converges to $\varphi \in \mathscr{P}(\Omega_T)$.

2.2. Semi-continuity properties

It is difficult to pass to the limit in the parabolic equation, due to the time derivative. We have the following general semi-continuity property :

LEMMA 2.5. – *Let (ν_j) be positive Borel measures on a topological manifold Y which converge weakly to ν in the sense of Radon measures on Y. Let $v_j : Y \longrightarrow \mathbb{R}$ be a locally uniformly bounded sequence of measurable functions which weakly converge to a measurable function v in $L^2(Y,\nu)$.*

1. *If $\|\nu_j-\nu\|\to 0$ (total variation) then $\lim_j \int_Y v_j\nu_j = \int_Y v\nu$ and*

$$\liminf_{j\to+\infty} e^{v_j}\nu_j \geq e^v\nu$$

   in the weak sense of Radon measures in Y.

2. *If $v_j \to v$ ν-a.e. in Y and $\mathscr{M} := \{\nu_j; j\in\mathbb{N}\}\cup\{\nu\}$ is uniformly absolutely continuous with respect to a fixed positive Borel measure $\tilde{\nu}$ on Y, then for any continuous function $\theta:\mathbb{R}\to\mathbb{R}$,*

$$\theta(v_j)\nu_j \longrightarrow \theta(v)\nu,$$

   as $j\to+\infty$, in the weak sense of Radon measures in Y.

Recall that a set $\mathcal{M}$ of positive Borel measures is uniformly absolutely continuous with respect to a positive Borel measure $\tilde{\nu}$ on Y if for any $\delta > 0$ there exists $\alpha > 0$ such that $\sup_{\sigma \in \mathcal{M}} \sigma(B) \leq \delta$ whenever $B \subset Y$ is a Borel subset with $\tilde{\nu}(B) \leq \alpha$

A typical example is when $\sigma = f_\sigma \tilde{\nu}$, where $\sup_{\sigma \in \mathcal{M}} f_\sigma$ is $\tilde{\nu}$-integrable. When $\|\nu_j - \nu\| \to 0$ in the sense of total variation, then the set $\mathcal{M} := \{\nu_j \ ; \ j \in \mathbb{N}\} \cup \{\nu\}$ is uniformly absolutely continuous with respect to $\nu = \tilde{\nu}$.

Proof. – We first prove (1). Recall Young's formula which states that

$$e^t = \sup_{s>0}\{st - s\log s + s\}$$

for all $t \in \mathbb{R}$. It therefore suffices to prove that for all $s > 0$,

$$\liminf_{j\to+\infty} e^{v_j}\nu_j \geq (sv - s\log s + s)\nu$$

in the weak sense of Radon measures on Y. Now for all $s > 0$

$$e^{v_j}\nu_j = \sup_{s>0}\{(sv_j - s\log s + s)\nu_j\},$$

so it suffices to prove that $\liminf_j v_j\nu_j \geq v\nu$ in the sense of Radon measures.

Let χ be a test function on Y. Observe that

$$\int_Y \chi v_j d\nu_j - \int_Y \chi v d\nu = \int_Y \chi(v_j - v)d\nu + \int_Y \chi v_j d(\nu_j - \nu).$$

The first term converges to zero by weak convergence. Since χv_j is uniformly bounded by a constant M the absolute value of the second term is less than $M\|\nu_j - \nu\|_{\mathrm{Supp}(\chi)}$, which converges to 0.

We now prove (2). Set $f_j := \theta(v_j)$ and $f := \theta(v)$ and write

$$\int_Y \chi f_j d\nu_j - \int_Y \chi f d\nu = \int_Y \chi(f_j - f)d\nu_j + \int_Y \chi f d(\nu_j - \nu).$$

Observe that $g_j := \chi(f - f_j) \to 0$ ν a.e. in Y since $v_j \to v$ ν-a.e. in Y. It follows from Egorov's theorem that the sequence (f_j) converges $\tilde{\nu}$-quasi uniformly to f. Since the sequence (ν_j) is uniformly absolutely continuous with respect to $\tilde{\nu}$ it follows that the first term above converges to 0 as $j \to +\infty$. By Lusin's theorem, the function f is $\tilde{\nu}$-quasi continuous in Y, hence the second term also converges to 0 as $j \to +\infty$, completing the proof of the lemma. □

PROPOSITION 2.6. – *Let $J \subset \mathbb{R}$ be a bounded open interval, D a bounded open set in $\mathbb{R}^m$, $m \in \mathbb{N}^*$, and $0 \leq g \in L^p(D)$ with $p > 1$. Let (ψ_j) be a sequence of Borel functions in $J \times D$ such that $(e^{\psi_j} g)$ is uniformly bounded in $L^1(J \times D, dtdV)$. Assume that there exists $E \subset D$ with zero Lebesgue measure such that for all $z \in D \setminus E$, $\psi_j(\cdot, z)$ converge to a bounded Borel function $\psi(\cdot, z)$ in the sense of distributions on J and*

$$\sup_{j\in\mathbb{N}, z\in D\setminus E}\left|\int_J \chi(t,z)\psi_j(t,z)dt\right| < +\infty, \text{ for all } \chi \in \mathcal{C}_0^\infty(J \times D). \tag{2.2}$$

Then for any positive smooth test function $\chi \in \mathcal{C}_0^\infty(J \times D)$,

$$\int_{J\times D} \chi(t,z)e^{\psi(t,z)}g(z)dtdV \leq \liminf_{j\to+\infty}\int_{J\times D}\chi(t,z)e^{\psi_j(t,z)}g(z)dtdV. \tag{2.3}$$

Proof. – Fix $C > 0$ such that $|\psi| \leq C$ in X. Set $\varphi_j := \max(\psi_j, -C)$, $j \in \mathbb{N}$. Then e^{φ_j} is uniformly bounded in $L^1(J \times D, gdtdV)$. It follows that (φ_j) is bounded in $L^2(J \times D, g\,dt\,dV)$. Up to extracting and relabeling, it follows from Banach-Saks theorem that the arithmetic mean sequence

$$\Psi_N := \frac{1}{N} \sum_{j=0}^{N} \varphi_j$$

converges a.e. and in $L^2(J \times D, gdtdV)$ towards a function $\Psi \in L^2(J \times D, gdtdV)$.

Condition (2.2) and Lebesgue's theorem ensure that $\psi_j g$ converges in the sense of distributions on $J \times D$ to ψg. This together with the convergence of Ψ_N towards Ψ ensure that for any positive smooth test function χ in $J \times D$,

$$\begin{aligned} \int_{J \times D} \chi \Psi g dt dV &= \lim_{N \to +\infty} \int_{J \times D} \chi \Psi_N g dt dV \\ &= \lim_{N \to +\infty} \frac{1}{N} \sum_{j=1}^{N} \int_{J \times D} \chi \max(\psi_j, -C) g dt dV \\ &\geq \lim_{N \to +\infty} \frac{1}{N} \sum_{j=1}^{N} \int_{J \times D} \chi \psi_j g dt dV = \int_{J \times D} \chi \psi g dt dV. \end{aligned}$$

This implies that $\Psi g \geq \psi g$ in $L^1(J \times D)$, hence $e^{\Psi} g \geq e^{\psi} g$ in $L^1(J \times D)$.

It thus follows from Fatou's lemma that

$$\liminf_{N \to +\infty} \int_{J \times D} e^{\Psi_N} \chi g dt dV \geq \int_{J \times D} e^{\Psi} \chi g dt dV \geq \int_{J \times D} e^{\psi} \chi g dt dV.$$

It follows now from the convexity of the exponential that

$$\begin{aligned} \int_X e^{\Psi_N} \chi g dt dV &\leq \frac{1}{N} \sum_{j=1}^{N} \int_X e^{\varphi_j} \chi g d\mu \\ &\leq \frac{1}{N} \sum_{j=1}^{N} \int_X e^{\psi_j} \chi g d\mu + \int_X e^{-C} \chi g dt dV, \end{aligned}$$

hence letting $N \to +\infty$ we get

$$\int e^{\Psi} \chi g dt dV \leq \liminf_{j \to +\infty} \int_X \chi e^{\psi_j} g dt dV + e^{-C} \int_X \chi g dt dV.$$

Letting $C \to +\infty$ we obtain (2.3). □

2.3. Semi-concavity and convergence

In the sequel we need more precise convergence results which require stronger assumptions :

Definition 2.7. – A function $\gamma : I \to \mathbb{R}$ is κ-concave if $t \mapsto \gamma(t) - \kappa t^2$ is concave. It is called locally semi-concave in I if for any subinterval $J \subset I$, there exists $\kappa = \kappa(J, \gamma) > 0$ such that γ is κ-concave in J.

A family $\mathscr{A}$ of semi-concave functions in some interval $I \subset \mathbb{R}$ is called *locally uniformly semi-concave* if for any compact subinterval $J \Subset I$, there exists a constant $\kappa = \kappa(J, \mathscr{A}) > 0$ such that any $\gamma \in \mathscr{A}$ is κ-concave in J.

The following elementary lemma is useful :

LEMMA 2.8. – *Let (γ_j) be a sequence of uniformly semi-concave functions in an interval $I \subset \mathbb{R}$ which converges pointwise to a function γ. Then there exists a countable subset $S \subset I$ such that for all $t \in I \setminus S$, the derivatives $\dot{\gamma}_j(t), \dot{\gamma}(t)$ exist and $\lim_{j\to+\infty} \dot{\gamma}_j(t) = \dot{\gamma}(t)$. Moreover if $\dot{\gamma}(t_0)$ exists then*

$$\lim_{j\to+\infty} \partial_t^- \gamma_j(t_0) = \lim_{j\to+\infty} \partial_t^+ \gamma_j(t_0) = \dot{\gamma}(t_0).$$

We include a proof for the reader's convenience.

Proof. – We can assume that γ_j is concave in I for all j and $t_0 = 0$. Thus for all $j \in \mathbb{N}$ and $t < 0$,

$$t\partial_t^- \gamma_j(0) \geq \gamma_j(t) - \gamma_j(0).$$

Dividing by $t < 0$ and taking limits (first $j \to +\infty$, then $t \to 0^-$), we obtain $\partial_t^- \gamma(0) \geq \limsup_{j\to+\infty} \partial_t^- \gamma_j(0)$. Similarly $\liminf_{j\to+\infty} \partial_t^+ \gamma_j(0) \geq \partial_t^+ \gamma(0)$. Since $\partial_t^- \gamma_j(0) \geq \partial_t^+ \gamma_j(0)$ we conclude that

$$\partial_t^- \gamma(0) \geq \limsup_{j\to+\infty} \partial_t^- \gamma_j(0) \geq \liminf_{j\to+\infty} \partial_t^+ \gamma_j(0) \geq \partial_t^+ \gamma(0).$$

If $\dot{\gamma}(0)$ exists, $\partial_t^- \gamma(0) = \dot{\gamma}(0) = \partial_t^+ \gamma(0)$, hence

$$\lim_{j\to+\infty} \partial_t^- \gamma_j(0) = \lim_{j\to+\infty} \partial_t^+ \gamma_j(0) = \dot{\gamma}(0).$$

Observe now that the derivatives of a concave function $\partial_t^{\pm} \gamma(t)$ are monotone decreasing, hence continuous outside a countable subset of I. Since $\partial_t^+ \gamma(t) = \partial_t^- \gamma(t)$ almost everywhere by Lemma 1.15, it follows that they are equal outside a countable set in I. □

We now prove a convergence result that will play a key role in the sequel. We fix μ a positive Borel measure on Ω such that $PSH(\Omega) \subset L^1_{\mathrm{loc}}(\Omega, \mu)$ and let ℓ denote the Lebesgue measure on $\mathbb{R}$.

PROPOSITION 2.9. – *Let (f_j) be a sequence of positive functions converging to f in $L^1(\Omega_T, \ell \otimes \mu)$. Let (φ^j) be a sequence of functions in $\mathscr{P}(\Omega_T)$ which*

- *converges $\ell \otimes \mu$-almost everywhere in Ω_T to a function $\varphi \in \mathscr{P}(\Omega_T)$;*
- *is locally uniformly semi-concave in $]0, T[$.*

Then $\lim_{j\to+\infty} \dot{\varphi}^j(t, x) = \dot{\varphi}(t, x)$ for $\ell \otimes \mu$-almost any $(t, x) \in \Omega_T$, and

$$\theta(\dot{\varphi}^j)\, f_j\, \ell \otimes \mu \to \theta(\dot{\varphi})\, f\, \ell \otimes \mu,$$

in the weak sense of Radon measures in Ω_T, for all $\theta \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$.

Proof. – Fix a compact subinterval $J \Subset]0,T[$. By definition there exists a constant $\kappa > 0$ such that all the functions $t \longmapsto u^j(t,x) := \varphi^j(t,x) - \kappa t^2$ are concave in J. By our hypothesis there exists a μ-negligible subset $E_1 \subset \Omega_T$ such that for any $(t,x) \notin E_1$, the sequence $u^j(t,x)$ converges to $u(t,x) := \varphi(t,x) - \kappa t^2$. It follows from Lemma 1.15 and Lemma 2.8 that there exists a $\ell \otimes \mu$-negligeable subset $E_2 \subset \Omega_T$ containing E_1 such that $\dot{\varphi}^j(t,x)$ and $\dot{\varphi}(t,x)$ are well defined for all j and all $(t,x) \notin E_2$, with

$$\lim_{j \to +\infty} \dot{\varphi}^j(t,x) = \dot{\varphi}(t,x).$$

Since $f_j \to f$ in $L^1(\Omega_T, \ell \otimes \mu)$, extracting a subsequence if necessary, we can find $g \in L^1(\Omega_T, \ell \otimes \mu)$ such that $0 \leq f_j \leq g$ in Ω_T for any $j \in \mathbb{N}$ and $f_j \to f$ $\ell \otimes \mu$-almost everywhere in Ω_T. The measures $(f_j \ell \otimes \mu)$ are thus uniformly absolutely continuous with respect to the positive Borel measure $g\, \ell \otimes \mu$. Since $(\dot{\varphi}^j)$ is bounded in $L^\infty_{\mathrm{loc}}(\Omega_T, \ell \otimes \Omega)$, it follows from Lebesgue convergence theorem that $\theta(\dot{\varphi}^j)\, f_j\, \ell \otimes \mu \to \theta(\dot{\varphi})\, f\, \ell \otimes \mu$ in $L^1_{\mathrm{loc}}(\Omega_T)$. Since this is true for any converging subsequence, the conclusion follows. □

2.4. Elliptic tools

LEMMA 2.10. – *Let u, v be bounded psh functions in Ω such that*

$$(dd^c u)^n \geq e^{f_1} \mu \text{ and } (dd^c v)^n \geq e^{f_2} \mu,$$

where f_1, f_2 are bounded Borel functions in Ω and μ is a positive Radon measure with L^1 density with respect to Lebesgue measure. Then

$$(dd^c(\lambda u + (1-\lambda)v))^n \geq e^{\lambda f_1 + (1-\lambda) f_2} \mu, \text{ for all }, \lambda \in [0,1].$$

Proof. – Observe first that

$$(dd^c(\lambda u + (1-\lambda)v))^n = \sum_{k=0}^{n} a_k (dd^c u)^k \wedge (dd^c v)^{n-k},$$

where $a_k \in (0,1)$, for all k and $\sum_{k=0}^n a_k = 1$. It follows from the mixed Monge-Ampère inequalities [26] (see also [7]) that for all $k = 0, \ldots, n$,

$$(dd^c u)^k \wedge (dd^c v)^{n-k} \geq e^{(k f_1 + (n-k) f_2)/n} \mu.$$

Summing up the above inequalities and using the convexity of the exponential yields the desired inequality. □

LEMMA 2.11. – *Let u be a psh function in Ω such that $\lim_{z \to \zeta} u(z) = \phi(\zeta)$ where ϕ is a continuous function on $\partial\Omega$. There exists a decreasing sequence (u_j) of plurisubharmonic functions which are continuous on $\bar{\Omega}$ and such that $u_j = \phi$ on $\partial\Omega$ and $u_j \searrow u$ in Ω.*

This result is classical but we include a proof for the reader's convenience.

Proof. – It follows from the strictly pseudoconvex assumption on Ω that there exists a harmonic function Φ in Ω with boundary value ϕ. We first take a sequence of continuous functions $\{f_j\} \subset \mathcal{C}(\bar{\Omega})$ which decreases pointwise to u in $\bar{\Omega}$. By considering $\min(f_j, \Phi)$ we can assume that $f_j = \phi$ on $\partial\Omega$. For each j, consider the psh envelope

$$u_j := P(f_j) := \sup\{v \in \mathrm{PSH}(\Omega)\ ;\ v^* \leq f_j \text{ in } \bar{\Omega}\}.$$

Then $u \leq u_j \leq f_j$ and $u_j \downarrow u$. Hence $(u_j)_* = (u_j)^* = \phi$ on $\partial\Omega$. It thus follows from [31, Lemma 1] (see also [3, Proposition 3.2]) that u_j is continuous in $\bar{\Omega}$. $\square$

3. Boundary behavior of parabolic envelopes

Our aim is to solve the Cauchy-Dirichlet problem for (CMAF) with compatible boundary data h using the Perron method of upper envelopes. In this section we prove that, under natural assumptions, the parabolic Perron envelope has the right boundary values. We assume $T < +\infty$.

3.1. Parabolic pluripotential subsolutions

Recall that for $u \in \mathscr{P}(\Omega_T)$ the time derivative $\partial_t u$ exists a.e. in Ω_T and satisfies the local uniform bound $|\partial_t u| \leq \kappa_J(u)$ in $J \times \Omega$, for each $J \Subset]0, T[$ (see Lemma 1.13).

Definition 3.1. – Fix $u \in \mathscr{P}(\Omega_T) \cap L^\infty(\Omega_T)$. The function u is called a pluripotential subsolution to (CMAF) if it satisfies the inequality

$$dt \wedge (dd^c u)^n \geq e^{\dot{u}+F(t,x,u)} g\, dt \wedge dV$$

in the sense of measures in Ω_T. It is called a pluripotential supersolution to (CMAF) if the reverse inequality holds in the sense of measures in Ω_T.

If moreover $u^* \leq h$ in $\partial_0\Omega_T$, we say that u is a *pluripotential subsolution to the Cauchy-Dirichlet problem* for the parabolic complex Monge-Ampère Equation (CMAF) with boundary data h. Here

$$u^*(\tau,\zeta) := \limsup_{\Omega_T \ni (t,z) \to (\tau,\zeta)} u(t,z), \quad (\tau,\zeta) \in \partial_0\Omega_T.$$

Proposition 3.2. – *Fix* $u \in \mathscr{P}(\Omega_T) \cap L^\infty_{\mathrm{loc}}(\Omega_T)$.

(1) *u is a pluripotential subsolution to* (CMAF) *if and only if for a.e. t,*

$$(dd^c u_t)^n \geq e^{\partial_t u(t,\cdot)+F(t,\cdot,u_t)} g\, dV, \tag{3.1}$$

in the sense of measures in Ω.

(2) *If u is moreover locally semi-concave in t, it is a pluripotential subsolution to* (CMAF) *if and only if for all t,*

$$(dd^c u_t)^n \geq e^{\partial_t^+ u(t,\cdot)+F(t,\cdot,u_t)} g\, dV,$$

in the sense of measures in Ω.

Proof. – Recall that $\partial_t u$ makes sense almost everywhere and, in case u is semi-concave, coincides with $\partial_t^+ u$ which is well defined at every point.

Assume first that (3.1) holds for a.e. t. Let $\chi \in \mathscr{C}_0^\infty(\Omega_T)$ be a non-negative test function. Multiplying (3.1) by χ and integrating in x we obtain

$$\int_\Omega \chi(t,x)(dd^c u_t)^n \geq \int_\Omega \chi(t,x) e^{\partial_t u+F(t,x,u_t)} g(x) dV(x).$$

Integrating with respect to t, we infer

$$\int_{\Omega_T} \chi(t,x)(dd^c u_t)^n \wedge dt \geq \int_{\Omega_T} \chi(t,x) e^{\partial_t u+F(t,x,u_t)} g(x) dV(x) \wedge dt,$$

i.e., u is a pluripotential subsolution to (CMAF).

Assume now that u is a pluripotential subsolution to (CMAF). Let (θ_j) be a sequence of non-negative test functions on Ω which generates a dense subspace of the space of non-negative test functions on Ω (for the $\mathcal{C}^0$-topology) and let α be a non-negative test function on $]0,T[$.

We consider the product test function defined on $]0,T[\times\Omega$ by the formula

$$\chi(t,x)=\alpha(t)\theta_j(x).$$

It follows from Fubini theorem that

$$\int_0^T\left\{\int_\Omega \theta_j(x)(dd^cu_t)^n\right\}\alpha(t)dt \geq \int_0^T\left\{\int_\Omega \theta_j(x)e^{\partial_t u+F(t,x,u_t)}g(x)dV(x)\right\}\alpha(t)dt.$$

We infer that for all $t\in B_j\subset[0,T[$,

$$\int_\Omega \theta_j(x)(dd^cu_t)^n \geq \int_\Omega \theta_j(x)e^{\partial_t u+F(t,x,u_t)}g(x)dV(x),$$

where B_j has full measure in $[0,T[$. The set $B=\bigcap_j B_j\subset[0,T[$ has full measure and the previous inequality holds for all $t\in B$ and for all $j\in\mathbb{N}$. Approximating an arbitrary nonnegative test function $\theta\in\mathcal{C}^0(\Omega)$ by convex combinations of the θ_j's, we infer that for almost every t,

$$(dd^cu_t)^n\geq e^{\partial_t u(t,\cdot)+F(t,\cdot,u_t)}gdV.$$

When u is moreover locally semi-concave in t the function ∂_t^+u is lower semi-continuous (see Lemma 1.15), hence

$$t\mapsto\int_\Omega \chi(x)e^{\partial_t^+u(t,x)+F(t,x,u_t(x))}g(x)dV(x)$$

is lower semi-continuous by Fatou's lemma. Since $t\mapsto\int_\Omega\chi(dd^cu_t)^n$ is continuous (by Lemma 2.1), we infer that (3.1) holds for almost every t if and only if it holds for every t. □

REMARK 3.3. – Proposition 3.2 deals with subsolutions. A similar result holds for supersolutions, using the partial derivative ∂_t^-u which is upper semi-continuous when u is locally semi-concave (by Lemma 1.15 again). As a consequence, if $u\in\mathcal{P}(\Omega_T)\cap L^\infty_{\mathrm{loc}}(\Omega_T)$ solves (CMAF) and u is locally uniformly semi-concave in $t\in]0,T[$ then for almost all $t\in]0,T[$,

$$(dd^cu_t)^n=e^{\partial_t u_t+F(t,\cdot,u_t)}gdV.$$

LEMMA 3.4. – *For any $u,v\in\mathcal{P}(\Omega_T)\cap L^\infty_{\mathrm{loc}}(\Omega_T)$, we have*

$$\mathbf{1}_{\{u\geq v\}}\partial_t\max(u,v)=\mathbf{1}_{\{u\geq v\}}\partial_t u \text{ and } \mathbf{1}_{\{u>v\}}\partial_t\max(u,v)=\mathbf{1}_{\{u>v\}}\partial_t u$$

almost everywhere in Ω_T and

$$(dd^c\max(u,v))^n\wedge dt\geq\mathbf{1}_{\{u>v\}}(dd^cu)^n\wedge dt+\mathbf{1}_{\{u\leq v\}}(dd^cv)^n\wedge dt.$$

In particular the maximum of two subsolutions is again a subsolution.

Proof. – It follows from Lemma 1.19 that $\mathcal{P}(\Omega_T)\subset W^{1,1}_{\mathrm{loc}}(\Omega_T)$. The first identity is then a classical result in the theory of Sobolev spaces (see e.g., [12, Lemma 7.6 page 152]). The second inequality is a consequence of the elliptic maximum principle for psh functions (see e.g., [17, Corollary 3.28]). □

It is therefore natural to consider the Perron envelope of subsolutions :

DEFINITION 3.5. – We let $\mathcal{S}_{h,g,F}(\Omega_T)$ denote the set of $u \in \mathcal{P}(\Omega_T)$ such that

1. u is a pluripotential subsolution to (CMAF) in Ω_T;
2. $u^* \leq h$ on $\partial_0 \Omega_T$, i.e., for all $(s,\zeta) \in \partial_0 \Omega_T$,

$$\limsup_{\Omega_T \ni (t,z) \to (s,\zeta)} u(t,z) \leq h(s,\zeta).$$

We let

$$U = U_{h,g,F,\Omega_T} := \sup\{u \; ; \; u \in \mathcal{S}_{h,g,F}(\Omega_T)\}$$

denote the upper envelope of all subsolutions.

LEMMA 3.6. – *The set $\mathcal{S}_{h,g,F}(\Omega_T)$ is not empty, uniformly bounded in Ω_T, stable under finite maxima. The envelope $U := U_{h,g,F,\Omega_T}$ and its upper semi-continuous regularization U^* satisfy for all $(t,z) \in \Omega_T$,*

$$B\rho(z) - M_h \leq U(t,z) \leq U^*(t,z) \leq M_h,$$

where $B = e^{M_F/n}$. In particular

$$\|U\|_{L^\infty(\Omega_T)} \leq M_U := M_h + c_n e^{M_F} \|g\|^{1/n}_{L^p(\Omega)}. \tag{3.2}$$

Recall that

$$M_h := \sup_{\partial_0 \Omega_T} |h| \,, \; M_F := \sup_{\Omega_T} F(\cdot,\cdot,M_h).$$

Proof. – Fix $B = e^{M_F/n}$. Since $g dV = (dd^c \rho)^n$ we obtain $e^{M_F} g dV = B^n (dd^c\rho)^n$. Set, for $(t,z) \in \Omega_T$,

$$\underline{u}(t,z) := B\rho(z) - M_h.$$

Then $\underline{u} \in \mathcal{S}_{h,g,F}(\Omega_T)$, hence $\underline{u} \leq U_{h,g,F,\Omega_T}$.

Fix $u \in \mathcal{S}_{h,g,F}(\Omega_T)$ and fix $t \in]0,T[$. Then $\limsup_{z\to\zeta} u(t,z) \leq h(t,\zeta)$, for every $\zeta \in \partial\Omega$. It thus follows from the classical maximum principle for plurisubharmonic functions that $u(t,z) \leq M_h$ for every $z \in \Omega$. Thus $U(t,\cdot) \leq M_h$ for any $t \in]0,T[$.

Therefore, the upper envelope U is well defined and satisfies the uniform estimates $\underline{u} \leq U \leq M_h$, in Ω_T, hence

$$U(t,z) := \sup\{u(t,z) \; ; \; u \in \mathcal{S}_{h,g,F}(\Omega_T), \; \underline{u} \leq u \leq M_h\}.$$

The stability under finite maxima follows from Lemma 3.4. □

3.2. Construction of sub-barriers

The family $t \longmapsto h(t,z)$ $(z \in \partial\Omega)$ is uniformly Lipschitz in $]0,T[$ if there exists a constant $\kappa(h) > 0$ such that

$$|h(t,z) - h(s,z)| \leq \kappa(h)\,|t-s|, \; \forall (t,s) \in [0,T[^2, \forall z \in \partial\Omega. \tag{3.3}$$

The parabolic boundary of Ω_T consists in two different types of points. We provide barriers for each type.

3.2.1. *Sub-barriers at boundary points of Dirichlet type.* – We first construct subbarriers at Dirichlet boundary points in $[0,T[\times\partial\Omega$.

LEMMA 3.7. – *Assume h satisfies* (3.3). *Then there exists $u\in\mathcal{S}_{h,g,F}(\Omega_T)$ such that $u(\cdot,z)$ ($z\in\Omega$) is uniformly Lipschitz in* $[0,T[$ *and satisfies : for any* $(s,\zeta)\in[0,T[\times\partial\Omega$,

$$\lim_{(t,z)\to(s,\zeta)} u(t,z)=h(s,\zeta).$$

If h_0 is continuous on $\bar\Omega$ then u can be chosen to be continuous in $[0,T[\times\bar\Omega$.

Proof. – Fix $t\in[0,T[$ and set $h_t:=h(t,\cdot)\in\mathcal{C}(\partial\Omega)$. Let ϕ_t be the unique continuous plurisubharmonic function in Ω such that

$$\begin{cases}(dd^c\phi_t)^n=0 \text{ in } \Omega,\\ \lim_{z\to\zeta}\phi_t(z)=h_t(\zeta)-h_0(\zeta),\ \forall\zeta\in\partial\Omega.\end{cases}\tag{3.4}$$

The existence and continuity of ϕ_t on $\bar\Omega$ follows from classical results in pluripotential theory (see [1, 2], [17, Theorem 5.12]). Moreover, ϕ_t can be characterized as the supremum of all subsolutions to (3.4). Since $t\longmapsto h(t,z)$ ($z\in\partial\Omega$) is uniformly Lipschitz in $[0,T[$, the tautological maximum principle reveals that the family of functions $t\longmapsto\phi(t,z):=\phi_t(z)$ ($z\in\Omega$) is uniformly Lipschitz in $[0,T[$ with a Lipschitz constant $\kappa(\phi)\leq\kappa(h)$. By Lemma 1.5, $(t,z)\longmapsto\phi_t(z)$ is continuous in $[0,T[\times\bar\Omega$. Consider now, for $(t,z)\in\Omega_T$,

$$u(t,z):=\phi_t(z)+h_0(z)+A\rho(z),$$

where $A>0$ is a large constant to be chosen later, and ρ is defined in (0.7). Observe that $u\in\mathcal{P}(\Omega_T)$ and $u^*\leq h$ in $\partial_0\Omega_T$. It is clear that $t\longmapsto u(t,z)$ ($z\in\Omega$) is uniformly Lipschitz in $[0,T[$ with $\kappa(u)\leq\kappa(h)$. Moreover

$$dt\wedge(dd^cu)^n\geq A^n dt\wedge(dd^c\rho)^n\geq A^n dt\wedge g dV$$

in the weak sense of measures in Ω_T. We choose $A>0$ so that $n\log A\geq\kappa(h)+M_F$. It is then clear that $u\in\mathcal{S}_{h,g,F}(\Omega_T)$. By definition, u is continuous in $[0,T[\times\Omega$ provided that h_0 is continuous on $\bar\Omega$. □

3.2.2. *Sub-barriers at boundary points of Cauchy type.* – We now construct sub-barriers at boundary points in $\{0\}\times\Omega$.

LEMMA 3.8. – *Assume h satisfies* (3.3). *Then there exists $v\in\mathcal{S}_{h,g,F}(\Omega_T)$ such that for all $\zeta\in\bar\Omega$,*

$$\limsup_{\Omega_T\ni(t,z)\to(0,\zeta)} v(t,z)=h_0(\zeta),\ \textit{and}\ \lim_{t\to0^+}v(t,\zeta)=h_0(\zeta).$$

If h_0 is continuous on $\bar\Omega$ then v can be chosen to be continuous on $[0,T[\times\bar\Omega$.

Proof. – By assumption on h we have, for all $(t,z)\in[0,T[\times\partial\Omega$,

$$h(0,z)\leq h(t,z)+\kappa t.$$

Set, for $(t,z)\in\Omega_T$,

$$v(t,z):=h_0(z)+t(\rho(z)-C)+n[t\log(t/T)-t],$$

where $C:=\kappa_h+M_F-\min(n\log T,0)$. Then $v\in\mathcal{S}_{h,g,F}(\Omega_T)$ and v is continuous on $[0,T[\times\bar\Omega$ if h_0 is continuous on $\bar\Omega$. □

3.3. Super-barriers

3.3.1. *Super-barriers at boundary points of Dirichlet type.* – For each $t \in [0, T[$, we let H_t be the unique harmonic function in Ω with boundary value h_t on $\partial\Omega$ and set $H(t, z) := H_t(z)$ (the existence of H_t is a classical fact; see e.g., [12, Theorem 2.14]). Recall that H_t can be defined as the upper envelope of all subharmonic functions in Ω with boundary values $\leq h_t$. Observe that $h_0 \leq H(0, \cdot)$ in Ω, with equality at the boundary.

LEMMA 3.9. – *For all $(t, z) \in [0, T[\times \Omega$ we have $U^*(t, z) \leq H(t, z)$. In particular, for all $(s, \zeta) \in [0, T[\times \partial\Omega$,*

$$\limsup_{(t,z)\to(s,\zeta)} U^*(t, z) \leq h(s, \zeta).$$

Proof. – It follows directly from the maximum principle for subharmonic functions that $U_t \leq H_t$, for all $t \in [0, T[$. Fix $S \in]0, T[$. Since the family $\{h(\cdot, z) ; z \in \partial\Omega\}$ is equicontinuous in $[0, S]$, it follows by definition that the family $\{H(\cdot, z) ; z \in \bar{\Omega}\}$ is equicontinuous in $[0, S]$, hence the function H is continuous in $[0, T[\times \bar{\Omega}$, by Lemma 1.5. Then $U^*(t, z) \leq H(t, z)$ for any $(t, z) \in \Omega_T$. From the continuity of H, it follows that $U^* \leq H$ in $[0, T[\times \bar{\Omega}$. □

3.3.2. *Boundary behavior at Cauchy boundary points.* – So far we have constructed enough barriers to ensure that the envelope of subsolutions either matches the boundary data (at Dirichlet points), or stays below it. The following average argument will allow us to conclude that it also coincides with the boundary data at Cauchy points :

LEMMA 3.10. – *Let $\varphi \in \mathcal{P}(\Omega_T) \cap L^\infty(\Omega_T)$ be a subsolution to* (CMAF) *such that $\int_D (dd^c\varphi_t)^n \leq C$, for every $t \in [0, T[$, for some $C > 0$, where D is an open set in Ω. Then, for each positive continuous test function χ in D, there exists $A > 0$ such that*

$$t \mapsto \int_D \chi\varphi_t g dV - At$$

is decreasing in $]0, T[$.

Proof. – Since φ is a subsolution to (CMAF) we obtain for a.e. $t \in]0, T[$,

$$\int_D \chi e^{\dot{\varphi}_t + m_F} g dV \leq \int_D \chi e^{\dot{\varphi}_t + F} g dV \leq \int_D \chi(dd^c\varphi_t)^n \leq C,$$

where $m_F := \inf_{[0,T[\times\bar{\Omega}\times[-M_U,M_U]} F$. It follows from Jensen's inequality that

$$\int_D \chi\dot{\varphi}_t g dV \leq C_2,$$

for a.e. $t \in]0, T[$, where $C_2 > 0$ is a uniform constant. We then infer that the function $t \mapsto \int_D \chi\varphi_t g dV - C_2 t$ is decreasing in $]0, T[$. □

COROLLARY 3.11. – *Assume $\{u^j\} \subset \mathcal{S}_{h,g,F}(\Omega_T)$ is a bounded sequence which is locally uniformly Lipschitz in $]0, T[$ (with Lipschitz constant independent of j). If $\{u^j\}$ converges in $L^1_{\mathrm{loc}}(\Omega_T)$ to $u \in \mathcal{P}(\Omega_T)$ then*

$$\limsup_{(t,z)\to(s,\zeta)} u(t, z) \leq h(s, \zeta), \ \forall (s, \zeta) \in \partial_0\Omega_T.$$

Recall that $h : \partial_0\Omega_T \to \mathbb{R}$ is a boundary data (see the last section of the introduction) and $h_0 := h(0,\cdot)$ is a bounded psh function in Ω.

Proof. – For $(s,\zeta) \in]0,T[\times \partial\Omega$ the desired inequality holds thanks to Lemma 3.9. Fix $D \Subset \Omega$ and let χ be a positive continuous test function in Ω. It follows from the Chern-Levine-Nirenberg inequality [17, Theorem 3.9] that $\int_D (dd^c u_t^j)^n$ is uniformly bounded. Lemma 3.10 therefore provides us with a uniform constant $A > 0$ such that

$$\int_D \chi u_t^j g dV \leq \int_D \chi h_0 g dV + At, \ \forall t \in]0,T[, \ \forall j.$$

Letting $j \to +\infty$, Lemma 1.11 ensures that

$$\int_D \chi u_t g dV \leq \int_D \chi h_0 g dV + At, \ \forall t \in]0,T[.$$

If v is a cluster point of u_t as $t \to 0$ then the above estimate yields $v \leq h_0$ on D. Since D was chosen arbitrarily, $v \leq h_0$ on Ω. The conclusion thus follows from Lemma 1.6. □

3.4. Boundary behavior of the Perron envelope

THEOREM 3.12. – *Assume h satisfies* (3.3). *Then the upper semi-continuous regularization of the envelope $U = U_{h,g,F,\Omega_T}$ satisfies*

(i) *for any* $(s,\zeta) \in [0,T[\times \partial\Omega$, $\lim_{\Omega_T \ni (t,z) \to (s,\zeta)} U^*(t,z) = h(s,\zeta)$.

(ii) *for any* $z_0 \in \Omega$,

$$\lim_{t\to 0^+} U^*(t,z_0) = h(0,z_0), \quad \text{and} \quad \limsup_{\Omega_T \ni (t,z)\to(0,z_0)} U^*(t,z) = h(0,z_0).$$

Here U^* denotes the u.s.c. regularization of U in the variable (t,z) in Ω_T.

Proof. – Fix $(s,\zeta) \in [0,T[\times \partial\Omega$. Lemma 3.7 and Lemma 3.9 yield (i).

In view of Lemma 3.8 it remains to prove that for all $z_0 \in \Omega$,

$$\limsup_{\Omega_T \ni (t,z)\to(0,z_0)} U^*(t,z) \leq h_0(z_0).$$

The envelope U is locally uniformly Lipschitz in $]0,T[$, as follows from Theorem 4.2. We can thus apply Lemma 1.7 to conclude that $U^*(t,\cdot) = U_t^*$ in Ω for any $t \in]0,T[$, where $U_t^* = (U_t)^*$ is the u.s.c. regularization of the function U_t (t fixed) in Ω. Using Lemma 1.6 it is then enough to show that

$$\limsup_{t\to 0} U_t^*(z_0) \leq h_0(z_0), \forall z_0 \in \Omega.$$

Observe that U can be seen as the upper envelope of all $\varphi \in \mathcal{S}_{h,g,F}(\Omega_T)$ such that $\sup_{\Omega_T} |\varphi| \leq M_U$, where M_U is given in Lemma 3.6.

Fix χ a continuous positive test function in Ω. We claim that there exists a constant $C > 0$ such that for all $t \in]0,T[$,

$$\int_{\Omega_1} \chi U_t^* g dV \leq \int_{\Omega_1} \chi h_0 g dV + Ct. \tag{3.5}$$

Indeed, fix $t_0 \in]0,T[$. Since the set of subsolutions is stable under maximum, by Choquet's lemma, $U_{t_0}^* = (\lim_{j\to+\infty} \varphi_{t_0}^j)^*$ in Ω, where $\{\varphi^j\}$ is an increasing sequence in $\mathcal{S}_{h,g,F}(\Omega_T)$ with $|\varphi^j| \leq M_U$. The sequence $\{\varphi^j\}$ depends on t_0 but, as will be shown later, the constant C does not depend on t_0. Now fix $j \in \mathbb{N}$, $\Omega_1 \Subset \Omega_2 \Subset \Omega$ compact subsets of Ω. It follows from the Chern-Levine-Nirenberg inequality [17, Theorem 3.9] that

$$\int_{\Omega_1} \chi (dd^c \varphi_t^j)^n \leq C_1, \text{ for all } t \in]0, T[,$$

where C_1 depends only on Ω_1, Ω_2, χ and M_U. It thus follows from Lemma 3.10 that

$$\int_{\Omega_1} \chi \varphi_t^j g dV \leq \int_{\Omega_1} \chi h_0 g dV + C_2 t, \text{ for all } t \in]0, T[,$$

for a uniform constant $C_2 > 0$. A classical theorem of Lelong (see [17, Proposition 1.40]) ensures that

$$\left\{ z \in \Omega \; ; \; \lim_{j \to +\infty} \varphi_{t_0}^j(z) < (U_{t_0})^*(z) \right\}$$

has volume zero in Ω. Therefore taking the limit as $j \to +\infty$ in the previous inequality for $t = t_0$, we deduce that

$$\int_{\Omega_1} \chi U_{t_0}^* g dV \leq \int_{\Omega_1} \chi h_0 g dV + C_2 t_0.$$

Since C_2 does not depend on t_0, the claim is proved.

Let $w_0 \in \mathrm{PSH}(\Omega)$ be any cluster point of U_t^* as $t \to 0^+$. We can assume that U_t^* converge to w_0 in $L^q(\Omega)$ for any $q > 1$. Then $U_t^* g$ converge to $w_0 g$ in $L^1(\Omega)$. Thus, by (3.5), $\int_{\Omega_1} \chi w_0 g dV \leq \int_{\Omega_1} \chi h_0 g dV$. Since $\chi \geq 0$ was chosen arbitrarily, we infer that $w_0 \leq h_0$ almost everywhere in Ω_1 with respect to $g dV$. The assumption on g finally yields $w_0 \leq h_0$ on Ω_1. By letting $\Omega_1 \to \Omega$ we can then conclude that $\limsup_{t \to 0} U_t^* \leq h_0$ in Ω. □

LEMMA 3.13. – *If h_0 is continuous on $\bar{\Omega}$ then $U^*(t, \cdot)$ uniformly converges to h_0 as $t \to 0^+$.*

Note that in Lemma 3.13 we merely assume that h is locally uniformly Lipschitz in $t \in]0, T[$.

Proof. – We first assume that h satisfies (3.3). It follows from Lemma 3.8 that there exists a continuous subsolution $u \in \mathcal{S}_{h,g,F}(\Omega_T)$:

$$u(t, z) := h_0(z) + t(\rho(z) - C) + \eta(t),$$

where C is a uniform constant, $\eta(t) \to 0$ as $t \to 0$ and ρ is defined by (0.7).

For each $t \in [0, T[$, let H_t be the unique continuous harmonic function in Ω with boundary value h_t. Then

$$u \leq U^* \leq H.$$

It follows moreover from Theorem 3.12 that $U^*(t, \cdot)$ converges in $L^1(\Omega)$ to h_0 as $t \to 0$. Hartogs' lemma thus yields

$$\limsup_{t \to 0} \max_{z \in K} (U^*(t, z) - h_0(z)) \leq 0,$$

for any compact $K \Subset \Omega$. Since u_t uniformly converges to h_0 as $t \to 0^+$ we infer, for any compact $K \Subset \Omega$,

$$\lim_{t \to 0^+} \sup_{z \in K} |U^*(t, z) - h_0(z)| = 0. \tag{3.6}$$

Fix $\varepsilon > 0$. Since H_0 and h_0 are continuous on $\bar{\Omega}$ with $h_0 = H_0$ on $\partial\Omega$, there exists $\delta > 0$ small enough such that

$$\sup_{z \in \Omega_\delta} |H_0(z) - h_0(z)| \leq \varepsilon,$$

where $\Omega_\delta := \{z \in \bar{\Omega} \ ; \ \mathrm{dist}(z, \partial\Omega) < \delta\}$. We also have, for $(t, z) \in [0, T[\times \bar{\Omega}$,

$$U^*(t, z) - h_0(z) \leq H_t(z) - h_0(z) \leq H_0(z) - h_0(z) + \kappa_h t.$$

Using this and the uniform convergence of u_t to h_0 as $t \to 0$ we obtain

$$\lim_{t \to 0} \sup_{z \in \Omega_\delta} |U^*(t, z) - h_0(z)| \leq \varepsilon.$$

Using (3.6) we infer

$$\lim_{t \to 0} \sup_{z \in \bar{\Omega}} |U^*(t, z) - h_0(z)| \leq \varepsilon.$$

Letting $\varepsilon \to 0^+$ yields the conclusion.

For the general case (i.e., h is locally uniformly Lipschitz in $]0, T[$ with h_0 continuous on $\bar{\Omega}$), we proceed by approximation. Fix $S \in]T/2, T[$, $\varepsilon > 0$ small enough. Proposition 4.1 below ensures that $U_{h,g,F,\Omega_S} = U_{h,g,F,\Omega_T}$ in Ω_S. Set

$$\begin{cases} h^\epsilon(t, \zeta) := h(t + \epsilon, \zeta) & \text{if } (t, \zeta) \in [0, S] \times \partial\Omega \\ h^\epsilon(0, z) = h_0(z) + \psi^\epsilon(z) & \text{if } \ z \in \Omega, \end{cases}$$

where ψ^ϵ is the maximal plurisubharmonic function in Ω such that $\psi^\epsilon(\zeta) = h(\epsilon, \zeta) - h(0, \zeta)$ in $\partial\Omega$. Recall that ψ^ϵ is the upper envelope of all psh functions ψ in Ω whose boundary values satisfy $\psi^* \leq h(\epsilon, \zeta) - h(0, \zeta)$ on $\partial\Omega$.

Since $h^\epsilon(0, \cdot) = h(\varepsilon, \cdot) \to h(0, \cdot)$ uniformly on $\partial\Omega$ as $\epsilon \to 0$, it follows that $\psi^\epsilon \to 0$ uniformly in $\bar{\Omega}$ as $\epsilon \to 0$. Therefore $\{h^\epsilon\}$ uniformly converges on $\partial_0\Omega_S$ to h as $\epsilon \to 0$. Set $U^\varepsilon := U_{h^\varepsilon,g,F,\Omega_S}$. Then $(U^\varepsilon)^*$ uniformly converges to U^* in Ω_S. Since h^ϵ is uniformly Lipschitz in $t \in [0, S]$, the previous step (using Theorem 3.12) guarantees that $(U^\varepsilon)^*(t, \cdot)$ uniformly converges to h_0 as $t \to 0$, hence U_t^* uniformly converges to h_0 as $t \to 0$. □

4. Time regularity of parabolic envelopes

We establish in this section time regularity of the envelope $U := U_{h,g,F,\Omega_T}$ by using and adapting some classical ideas of pluripotential theory.

We work in Ω_S for each $0 < S < T$ and eventually let $S \to T$. We thus assume $T < +\infty$, the family $\{F(\cdot, z, \cdot) \ ; \ z \in \Omega\}$ is uniformly Lipschitz and semi-convex in $[0, T] \times J$ for each $J \Subset \mathbb{R}$, and h satisfies

$$t|\partial_t h(t, z)| \leq \kappa_h, \text{ for all } (t, z) \in]0, T[\times \partial\Omega, \tag{4.1}$$

for some positive constant κ. The condition (4.1) is equivalent to the fact that for all $(t, z) \in \Omega_T$ and $s > 0$ with $st < T$, we have

$$|h(t, z) - h(st, z)| \leq \kappa_h \frac{|s - 1|}{\min(s, 1)}, \quad z \in \partial\Omega. \tag{4.2}$$

If h is uniformly Lipschitz in $t \in [0, T[$ (as in (3.3)) then the above condition is automatically satisfied. On the other hand the condition above implies that $h(\cdot, z)$, $z \in \partial\Omega$ is locally uniformly Lipschitz in $]0, T[$.

4.1. Lipschitz control in the time variable

The following identity principle plays a crucial role in the sequel. For simplicity we will denote the restriction of h on $\partial_0\Omega_S$, for $0 < S < T$, by h.

PROPOSITION 4.1. – *For all $S \in]0, T[$ we have $U_{h,g,F,\Omega_T} = U_{h,g,F,\Omega_S}$ in Ω_S.*

Proof. – Set $V := U_{h,g,F,\Omega_S}$ and $U := U_{h,g,F,\Omega_T}$. Fix $u \in \mathcal{S}_{h,g,F}(\Omega_S)$ and $t_0 \in]0, S[$, such that

$$(dd^c u(t_0,\cdot))^n \geq e^{\partial_t u(t_0,\cdot)+F(t_0,\cdot,u(t_0,\cdot))} g dV.$$

Set $M_1 := \sup_\Omega |\partial_t u(t_0,\cdot)| < +\infty$. If $A \geq M_1$ the function

$$\Omega_T \ni (t,z) \mapsto v(t,z) := \begin{cases} u(t,z), \text{if } t \in [0,t_0], \\ u(t_0,z) - A(t-t_0) \text{ if } t \in [t_0,T[, \end{cases}$$

is again a subsolution to (CMAF) in Ω_T. Applying (3.3) on the interval $J := [t_0, T[$, we obtain that $v^* \leq h$ on $\partial_0\Omega_T$ if $A \geq \kappa_J(h)$.

We therefore choose $A \geq \max\{M_1, \kappa_J(h)\}$. Then $v \in \mathcal{S}_{h,g,F}(\Omega_T)$ hence $v \leq U$ in Ω_T. In particular $u \leq U$ on Ω_{t_0}. Taking supremum over all candidates u we obtain $V \leq U$ in Ω_{t_0}. Using Proposition 3.2 we can let $t_0 \to S$ to obtain $V \leq U$ in Ω_S. The reverse inequality is clear. □

THEOREM 4.2. – *If h satisfies* (4.1)*, then the envelope $U := U_{h,g,F,\Omega_T}$ satisfies*

$$t|\partial_t U(t,z)| \leq \kappa_U, \ \forall (t,z) \in \Omega_T,$$

where $\kappa_U > 0$ is a uniform constant.

We will show that the constant κ_U is actually explicit,

$$\kappa_U = (T+1)(3M_U + 2\kappa_h + 2n + \kappa_F(T + M_U)). \tag{4.3}$$

This quantitative information will be crucial in perturbation arguments, to obtain uniform Lipschitz constants of the approximants.

The proof of this theorem follows and adapt ideas developed by Bedford and Taylor in their study of Dirichlet problems for elliptic complex Monge-Ampère equations (see [1, Theorem 6.7], [5]).

Proof. – By the assumption on F, there exists a constant κ_F such that, for all $z \in \Omega$ and $(t_j, r_j) \in [0, T[\times [-2M_U, 2M_U]$, $j = 1, 2$,

$$|F(t_1,z,r_1) - F(t_2,z,r_2)| \leq \kappa_F(|t_1 - t_2| + |r_1 - r_2|). \tag{4.4}$$

Fix $u \in \mathcal{S}_{h,g,F}(\Omega_T)$ such that $\sup_{\Omega_T} |u| \leq M_U$, where M_U is defined in Lemma 3.6. Fix $0 < S < T$ and $s \geq 1/2$ close enough to 1 such that $sS < T$. Set, for $(t,z) \in \Omega_S$,

$$v^s(t,z) := s^{-1}u(st,z) - C|s-1|(t+1),$$

where

$$C := 2M_U + 2\kappa_h + 2n + \kappa_F(T + M_U). \tag{4.5}$$

We are going to prove that $v^s \in \mathcal{S}_{h,g,F}(\Omega_S)$. Since u is a subsolution to (CMAF), for a.e. $t \in]0,S[$ we have

$$
\begin{aligned}
(dd^c v^s(t,\cdot))^n &= s^{-n}(dd^c u(st,\cdot)^n \\
&\geq e^{-n\log s+\partial_\tau u(st,\cdot)+F(st,\cdot,u(st,\cdot))} g dV \\
&\geq e^{\partial_t v^s(t,\cdot)+C|s-1|+F(t,\cdot,s^{-1}u(st,\cdot))-n\log s-\kappa_F(T|s-1|+|s^{-1}-1|M_U)} g dV \\
&\geq e^{\partial_t v^s(t,\cdot)+F(t,\cdot,v^s(t,\cdot))} g dV,
\end{aligned}
$$

where in the last line we use (4.5) and the fact that F is increasing in r.

We now take care of the boundary values. For $t \in [0,S]$, $z \in \partial\Omega$ we have

$$
\begin{aligned}
v^s(t,z) &\leq -C|s-1| + |s^{-1}-1|M_U + h(st,z) \\
&\leq -C|s-1| + 2|s-1|M_U + h(t,z) + 2\kappa_h|s-1| \\
&\leq h(t,z),
\end{aligned}
$$

where in the second line we use (4.2), and in the last line we use again (4.5). For $z \in \Omega$ we similarly get $(v^s)^*(0,z) \leq h_0(z)$.

The computations above show that $v^s \in \mathcal{S}_{h,g,F}(\Omega_S)$. Proposition 4.1 thus yields $v^s \leq U$ in Ω_S. Taking supremum over u we arrive at

$$s^{-1}U(st,z) - C|s-1|(t+1) \leq U(t,z), \text{ for all } (t,z) \in \Omega_S.$$

Letting $s \to 1$ we infer, for all $(t,z) \in \Omega_S$,

$$t|\partial_t U(t,z)| \leq M_U + C(T+1).$$

Letting $S \to T$ yields the conclusion. □

Definition 4.3. – Given a constant $\kappa > 0$ we let $\mathcal{S}^\kappa := \mathcal{S}^\kappa_{h,g,F}(\Omega_T)$ denote the set of all $u \in \mathcal{S}_{h,g,F}(\Omega_T)$ such that, for all $t \in]0,T[$,

$$\sup_\Omega |\partial_t u(t,z)| \leq \kappa/\min(t,b), \tag{4.6}$$

where $b = \min(1, T/2)$, and we set

$$U^\kappa := U^\kappa_{h,g,F,\Omega_T} := \sup\{u \,;\, u \in \mathcal{S}^\kappa_{h,g,F}(\Omega_T)\}.$$

We will need the following identity principle :

Proposition 4.4. – *For all $S \in]T-\varepsilon, T[$ and $\kappa \geq 2T\kappa_h$ we have*

$$U^\kappa_{h,g,F,\Omega_T} = U^\kappa_{h,g,F,\Omega_S} \ \textit{in } \Omega_S.$$

Proof. – The proof is similar to that of Proposition 4.1. Fix $S \in]T-\varepsilon, T[$ and set $V := U^\kappa_{h,g,F,\Omega_S}$, $W := U^\kappa_{h,g,F,\Omega_T}$. Fix $u \in \mathcal{S}^\kappa_{h,g,F}(\Omega_S)$. Using Proposition 3.2 we fix $t_0 \in]T/2, S[$, such that

$$(dd^c u(t_0,\cdot))^n \geq e^{\partial_t u(t_0,\cdot)+F(t_0,\cdot,u(t_0,\cdot))} g dV.$$

Since $\sup_\Omega |\partial_t u(t_0,\cdot)| \leq \kappa/b$, the function

$$\Omega_T \ni (t,z) \mapsto v(t,z) := \begin{cases} u(t,z), \text{if } t \in [0,t_0], \\ u(t_0,z) - \kappa b^{-1}(t-t_0) \text{ if } t \in [t_0,T[, \end{cases}$$

is still a subsolution to (CMAF) in Ω_T. It follows from (4.2) that

$$|h(t,z)-h(t_0,z)| \le \frac{2\kappa_h}{T}|t-t_0|, \text{ for all } t \in [t_0, T[.$$

Using that $\kappa \ge 2T\kappa_h$ and $b<1$, we thus obtain $v^* \le h$ on $\partial_0\Omega_T$. By construction, v satisfies (4.6). Therefore $v \in \mathcal{S}^{\kappa}_{h,g,F}(\Omega_T)$, hence $v \le W$ in Ω_T. We infer in particular $u \le W$ on Ω_{t_0}. Taking supremum over all candidates u we obtain $V \le W$ in Ω_{t_0}. Using Proposition 3.2 we can let $t_0 \to S$ to obtain $V \le W$ in Ω_S. The reverse inequality is obvious. □

THEOREM 4.5. – *There exists an explicit $\kappa_0 > 0$ such that, for all $\kappa > \kappa_0$,*

$$\sup_{\Omega_T} t|\partial_t U^{\kappa}| \le \kappa_0.$$

Proof. – We use the same notations as in the proof of Theorem 4.2. By approximating F we can assume that $F(t,z,r)$ does not depend on $t \in [T-\varepsilon, T[$ for some $\epsilon > 0$. Define

$$C := \kappa_F T + 2\kappa_F M_U + 2M_F + 2\kappa_h + 2M_h + 2n, \tag{4.7}$$

and

$$\kappa_0 := 2M_U + 3C(T+1) + 2\sup_{\Omega}|\rho|. \tag{4.8}$$

Fix $\kappa > \kappa_0$. By definition of κ_0 we have, for all $t \in]0,T[$, $2\kappa_h \le \kappa_0/t$. Proposition 4.4 thus ensures that, for all $T-\varepsilon < S < T$,

$$U^{\kappa}_{h,g,F,\Omega_T} = U^{\kappa}_{h,g,F,\Omega_S} \text{ in } \Omega_S. \tag{4.9}$$

Fix $u \in \mathcal{S}^{\kappa}$, $T-\varepsilon < S < T$, $s>0$ close enough to 1 and set, for $(t,z) \in \Omega_S$,

$$w(t,z) := as^{-1}u(st,z) + (1-a)\rho - C(1-a)(t+1),$$

where $a = 1-2|s-1| > 0$, ρ is defined in (0.7).

Since u is a subsolution to (CMAF) we have, for almost all $t \in]0,T[$,

$$(dd^c s^{-1}u(st,\cdot))^n \ge \exp\{-n\log s + \partial_\tau u(st,z) + F(st,z,u(st,z))\} g dV.$$

It thus follows from Lemma 2.10 that

$$\begin{aligned}(dd^c w)^n &\ge \exp\{a\partial_t u(st,z) + aF(st,z,u(st,z)) - an\log s\} g dV\\ &= \exp\{\partial_t w(t,z) + C(1-a) - an\log s + aF(st,z,u(st,z))\} g dV.\end{aligned}$$

From (4.4) and the assumption that F is increasing in r we obtain

$$\begin{aligned}aF(st,z,u(st,z)) &= F(st,z,u(st,z)) + (1-a)F(st,z,u(st,z))\\ &\ge F(t,z,as^{-1}u(st,z)) - |s-1|(\kappa_F T + 2\kappa_F M_U + 2M_F)\\ &\ge F(t,z,w(t,z)) - |s-1|(\kappa_F T + 2\kappa_F M_U + 2M_F).\end{aligned}$$

For $(\tau,\zeta) \in \partial_0\Omega_S$ we have

$$\begin{aligned}w(\tau,\zeta) &\le as^{-1}h(st,\zeta) - 2C|s-1|\\ &\le h(st,\zeta) + |as^{-1}-1|M_h - 2C|s-1|\\ &\le h(t,\zeta) + 2|s-1|\kappa_h + 2M_h|s-1| - 2C|s-1|.\end{aligned}$$

The choice of C in (4.7) and the previous computations ensure that $w \in \mathcal{S}_{h,g,F}(\Omega_S)$. Moreover, for $s \in [1/2, 3/2]$, $t \in]0, S[$,

$$\sup_{\Omega} |\partial_t w(t,z)| \leq \frac{(1-2|s-1|)\kappa}{\min(st,b)} + 2C|s-1|.$$

Since $\kappa/t > \kappa_0/t > 3C$, it follows that for $s \in [1, 3/2]$, $t \in]0, S[$,

$$\sup_{\Omega} |\partial_t w(t,z)| \leq \frac{(3-2s)\kappa}{\min(t,b)} + \frac{2(s-1)\kappa}{3t} \leq \frac{\kappa}{\min(t,b)}.$$

Hence $w \in \mathcal{S}^{\kappa}_{h,g,F}(\Omega_S)$. By definition of U^{κ} and (4.9) we have $w \leq U^{\kappa}$ on Ω_S. Taking supremum over $u \in \mathcal{S}^{\kappa}_{h,g,F}(\Omega_T)$ we obtain, for all $(t,z) \in \Omega_S$,

$$as^{-1}U^{\kappa}(st,z) - 2C|s-1|(t+1) + 2|s-1|\rho(z) \leq U^{\kappa}(t,z).$$

Letting $s \to 1$ yields

$$t|\partial_t U^{\kappa}(t,z)| \leq 2M_U + 2C(T+1) + 2\sup_{\Omega}|\rho| \leq \kappa_0,$$

where in the last inequality we use (4.8). This concludes the proof. □

EXAMPLE 4.6. – Consider $\varphi(t,z) := t\psi(z) + n(t\log t - t)$, where ψ is a bounded plurisubharmonic function in Ω with zero boundary values, solution of

$$(dd^c\psi)^n = e^{\psi} g dV.$$

Then φ is a parabolic potential solution of

$$t^n(dd^c\psi)^n \wedge dt = (dd^c\varphi)^n \wedge dt = e^{\partial_t\varphi} g dV \wedge dt.$$

This example shows that one cannot expect the solutions to be Lispchitz at time zero.

4.2. The maximal subsolution

We now prove that $U \in \mathcal{S}_{h,g,F}(\Omega_T)$.

THEOREM 4.7. – *Assume h satisfies* (4.1) *and set $U := U_{h,g,F,\Omega_T}$. Then $U \in \mathcal{S}_{h,g,F}(\Omega_T)$ and satisfies the following properties:*

1. $\lim_{\Omega_T \ni (t,z) \to (s,\zeta)} U(t,z) = h(s,\zeta)$ *for all* $(s,\zeta) \in [0,T[\times \partial\Omega$*;*
2. $\limsup_{\Omega_T \ni (t,z) \to (0,z_0)} U(t,z) = h(0,z_0)$ *for all* $(0,z_0) \in \{0\} \times \Omega$*;*
3. $\lim_{t\to 0} U_t(z) = h_0(z)$ *for all* $z \in \Omega$.

If h_0 is continuous then for all $(s,\zeta) \in \partial_0\Omega_T$,

$$\lim_{\Omega_T \ni (t,z) \to (s,\zeta)} U(t,z) = h(s,\zeta).$$

Proof. – We proceed in several steps.

Step 1. – Assume h satisfies (3.3).

Theorem 3.12 ensures that U^* has the desired boundary values. We are going to prove that U^* is a subsolution to (CMAF).

Step 1.1. – Assume h_0 is continuous on $\bar{\Omega}$.

Fix $\kappa \geq \kappa_0$, where κ_0 is defined in Theorem 4.5.

CLAIM 1. – *We have $U^\kappa = (U^\kappa)^* \in \mathcal{S}^\kappa_{h,g,F}(\Omega_T)$.*

Indeed, since $U^\kappa \leq U$, the boundary condition $(U^\kappa)^*|_{\partial_0\Omega_T} \leq h$ is satisfied. We now prove that $(U^\kappa)^*$ is a subsolution to (CMAF). A classical lemma of Choquet ensures that there exists a sequence $\{u^j\}$ in $\mathcal{S}^\kappa(h,g,F,\Omega_T)$ such that

$$(U^\kappa)^* = \left(\sup_{j\in\mathbb{N}} u^j\right)^* \text{ in } \Omega_T.$$

By Lemma 3.6, we can assume $\sup_{\Omega_T}|u^j| \leq M_U$. Since $\mathcal{S}^\kappa$ is stable under taking maximum we can assume that $\{u^j\}$ is increasing. By definition of $\mathcal{S}^\kappa$, $\lim_j u^j$ is locally uniformly Lipschitz in $t \in]0,T[$. Hence from Lemma 1.7 it follows that u^j increases to $(U^\kappa)^*$ almost everywhere in Ω_T. We infer that $dt \wedge (dd^c u_j)^n \to dt \wedge (dd^c(U^\kappa)^*)^n$ weakly in Ω_T. Moreover, the sequence $\{\psi_j\} := \{\partial_t u^j + F(t,z,u^j)\}$ is bounded and converges in the sense of distributions to $\partial_t(U^\kappa)^* + F(t,z,(U^\kappa)^*)$. Proposition 2.6 thus yields

$$e^{\partial_t(U^\kappa)^*+F(t,z,(U^\kappa)^*)} g\,dt\wedge dV \leq \liminf_j e^{\partial_t u^j+F(t,z,u^j)} g\,dt\wedge dV,$$

weakly in Ω_T. Therefore, $(U^\kappa)^*$ is a subsolution to (CMAF) in Ω_T. Hence $(U^\kappa)^* = U^\kappa$ and Claim 1 is proved.

It now follows from Theorem 4.5 that $U^\kappa = U^{\kappa_0}$, for all $\kappa > \kappa_0$.

CLAIM 2. – *We have $U = U^{\kappa_0}$ in Ω_T.*

Fix $v \in \mathcal{S}_{h,g,F}(\Omega_T)$, $S \in]T/2,T[$, $\varepsilon > 0$ small enough. Define, for $(t,z) \in [0,S]\times\Omega$,

$$u(t,z) := v(t+\varepsilon,z) - C\varepsilon(1+t) - \theta(\varepsilon),$$

where $C > 0$ is a uniform constant and $\theta(\varepsilon) := \sup_{\bar{\Omega}}|U^*(\varepsilon,z) - h_0(z)|$ converges to 0 (by Lemma 3.13). Since $v^* \leq h$ on $\partial_0\Omega_T$, we obtain for all $(\tau,\zeta) \in [0,S]\times\partial\Omega$,

$$u(\tau,\zeta) \leq h(\tau+\varepsilon,\zeta) - C\varepsilon \leq h(\tau,\zeta),$$

if $C \geq \kappa_h$. By definition of $\theta(\varepsilon)$ we also have $u(0,z) \leq h_0(z)$ in Ω.

A direct computation shows that, for $C > 0$ large enough, $u \in \mathcal{S}_{h,g,F}(\Omega_S)$. Since u is uniformly Lipschitz in $[0,S]$, $u \in \mathcal{S}^\kappa_{h,g,F}(\Omega_S)$ for some $\kappa > 0$ large enough. Hence $u \leq U^{\kappa_0}$ in Ω_S. Letting $\varepsilon \to 0$ we obtain $v \leq U^{\kappa_0}$ in Ω_S. Letting $S \to T$ we arrive at $v \leq U^{\kappa_0}$, hence $U \leq U^{\kappa_0}$. Therefore $U = U^{\kappa_0}$ is the maximal subsolution to (CMAF) with boundary value h.

Step 1.2. – We now remove the continuity assumption on h_0.

Using Lemma 2.11 we can find a sequence h_0^j of psh functions in Ω such that h_0^j is continuous on $\bar{\Omega}$, $h_0^j = h_0$ on $\partial\Omega$, and $h_0^j \downarrow h_0$ in Ω. We then define $h^j(t,z) := h(t,z)$ for $(t,z) \in [0,T[\times\partial\Omega$ and $h^j(0,z) = h_0^j(z)$ for $z \in \Omega$. We thus obtain a sequence of continuous Cauchy-Dirichlet boundary data for Ω_T such that $h^j = h$ on $[0,T[\times\partial\Omega$ and h^j decreases pointwise to h. The previous step ensures that $U^j := U_{h^j,g,F,\Omega_T}$ is a subsolution to (CMAF). Theorem 4.2 and Theorem 4.8 provide a uniform Lipschitz constant for U^j. Since h^j decreases to h, $U \leq U^j$ decreases to some $V \in \mathscr{P}(\Omega_T)$. We thus have $V^*|_{\partial_0\Omega_T} \leq h$, and Proposition 2.6 reveals that V is a subsolution to (CMAF). It then follows that $V = U$.

Step 2. – To treat the general case we proceed by approximation as in the proof of Lemma 3.13. Fix $0 < S < T$ and $0 < \varepsilon < (T-S)/2$. Define

$$\begin{cases} h^\epsilon(t,\zeta) := h(t+\epsilon,\zeta) & \text{if } (t,\zeta) \in [0,S]\times\partial\Omega \\ h^\epsilon(0,z) = h_0(z) + \psi^\epsilon(z) & \text{if } z \in \Omega, \end{cases}$$

where ψ^ϵ is the maximal psh function in Ω such that $\psi^\epsilon(\zeta) = h(\epsilon,\zeta) - h(0,\zeta)$ in $\partial\Omega$. Then $\{h^\epsilon\}$ uniformly converges on $\partial_0\Omega_S$ to h as $\epsilon \to 0$. Since h^ϵ is uniformly Lipschitz in $t \in [0,S]$, the previous step and Theorem 3.12 ensure that $U^\varepsilon := U_{h^\varepsilon,g,F,\Omega_S} \in \mathcal{S}_{h^\varepsilon,F,g}(\Omega_S)$ satisfies the boundary conditions (1.), (2.), (3.). Moreover, it follows from Proposition 4.1 that the envelopes U^ε uniformly converge in Ω_S to U as $\varepsilon \to 0$. Hence, Proposition 2.6 and Proposition 2.3 (together with Remark 2.4) yield that U is a subsolution to (CMAF) and U satisfies the boundary conditions (1.), (2.), (3.).

If h_0 is continuous on $\bar{\Omega}$ then Lemma 3.13 and the three boundary conditions (1.), (2.), (3.) give the last statement. □

4.3. Semi-concavity in the time variable

In this section we assume that h satisfies (4.1) and there exists $C_h > 0$ such that, for all $z \in \partial\Omega$,

$$\partial_t^2 h(t,z) \leq C_h t^{-2} \tag{4.10}$$

in the sense of distributions in $]0,T[$. Condition (4.10) is equivalent to the fact that $t \mapsto h(t,z) + C_h \log t$ is concave in $]0,T[$. It implies in particular that h is locally uniformly semi-concave in the t-variable.

THEOREM 4.8. – *Assume h satisfies* (4.1) *and* (4.10). *The envelope $U := U_{h,g,F,\Omega_T}$ is locally uniformly semi-concave in* $]0,T[$ *: for all $z \in \Omega$,*

$$\partial_t^2 U(t,z) \leq C_U t^{-2}$$

in the sense of distributions in $]0,T[$*, for some uniform constant $C_U > 0$.*

We will show that the constant C_U is actually explicit,

$$C_U := C_h + 2M_h + 8\kappa_h + (2\kappa_F + 3)(M_U + 5\kappa_U + 1 + C_F T^2 + 16\kappa_U^2). \tag{4.11}$$

This quantitative information is important in perturbation arguments, to obtain uniform semi-concavity constants of the approximants.

By the assumption on F, there is a constant $C_F > 0$ such that for all $z \in \Omega$, the function

$$(t, r) \mapsto F(t, z, r) + C_F(t^2 + r^2) \text{ is convex in } [0, T] \times [-2M_U, 2M_U]. \tag{4.12}$$

Proof. – It follows from Theorem 4.7 that $U \in \mathcal{S}_{h,g,F}(\Omega_T)$. Fix $0 < S < T$, and $s > 1/2$ close to 1 enough such that $sS < T$. Set, for $(t, z) \in \Omega_S$,

$$v^s(t, z) := \frac{s^{-1}U(st, z) + sU(s^{-1}t, z)}{2} - C(t + 1)(s - 1)^2,$$

where $C > 0$ is defined as

$$C := C_h + 1 + 2M_h + 8\kappa_h + 2\kappa_F(M_U + 4\kappa_U + T + C_F T^2 + 16\kappa_U^2). \tag{4.13}$$

We are going to prove that $v^s \in \mathcal{S}_{h,g,F}(\Omega_S)$.

Boundary values of v^s.– It follows from (4.10) that for all $z \in \partial\Omega$, $t \in]0, S[$,

$$\begin{aligned}
\frac{h(st, z) + h(s^{-1}t, z)}{2} &\leq h\left(\frac{(s + s^{-1})t}{2}, z\right) + C_h \log\left(\frac{s + s^{-1}}{2}\right) \\
&\leq h\left(\frac{(s + s^{-1})t}{2}, z\right) + C_h(s - 1)^2 \\
&\leq h(t, z) + (C_h + 1)(s - 1)^2,
\end{aligned}$$

where in the last line we use (4.2). We claim that for all $(t, z) \in]0, S[\times \partial\Omega$,

$$s^{-1}h(st, z) + sh(s^{-1}t, z) \leq h(st, z) + h(s^{-1}t, z) + (2M_h + 3\kappa_h)(s - 1)^2.$$

Indeed, write $s = 1 - \sigma$ and observe that $s^{-1} = 1 + \sigma + O(\sigma^2)$, where $|O(\sigma^2)| \leq 2\sigma^2$ for $|\sigma| \leq 1/2$. Thus for all $(t, z) \in]0, S[\times \partial\Omega$,

$$\begin{aligned}
s^{-1}h(st, z) + sh(s^{-1}t, z) &\leq (1 + \sigma)h(st, z) + (1 - \sigma)h(s^{-1}t, z) + 2M_h\sigma^2 \\
&\leq h(st, z) + h(s^{-1}t, z) + \sigma(h(st, z) - h(s^{-1}t, z)) + 2M_h\sigma^2.
\end{aligned}$$

Using (4.2), we obtain

$$s^{-1}h(st, z) + sh(s^{-1}t, z) \leq h(st, z) + h(s^{-1}t, z) + (2M_h + 4\kappa_h)(s - 1)^2,$$

which proves the claim.

Since $U^*|_{\partial_0\Omega_T} \leq h$, the above estimate implies that $(v^s)^* \leq h$ on $\partial_0\Omega_S$. Using similarly the estimate in Theorem 4.2, we obtain the following estimate which will be useful later: for all $(t, z) \in]0, S[\times \bar{\Omega}$,

$$|(U(st, z) + U(s^{-1}t, z)) - (s^{-1}U(st, z) + sU(s^{-1}t, z))| \leq (2M_U + 4\kappa_U)(s - 1)^2. \tag{4.14}$$

Estimating the Monge-Ampère measure of v^s. It follows from Proposition 3.2 that for almost all $t \in]0, S[$,

$$(dd^c s^{-1}U(st, \cdot))^n \geq e^{n \log s^{-1} + \partial_\tau U(st, \cdot) + F(st, \cdot, U(st, \cdot))} g dV.$$

Using Lemma 2.10 we infer

$$(dd^c v^s(t, \cdot))^n \geq e^{a(s) + a(s^{-1})} g dV,$$

where

$$a(s) = \frac{1}{2}\left(\partial_\tau U(st, \cdot) + F(st, \cdot, U(st, \cdot))\right).$$

By the semi-convexity assumption (4.12) on F, for $\lambda \in]0,1[$, $t_1, t_2 \in [0,T]$, $r_1, r_2 \in [-2M_U, 2M_U]$ we have

$$F(\lambda(t_1,r_1) + (1-\lambda)(t_2,r_2)) \\ \leq \lambda F(t_1,r_1) + (1-\lambda)F(t_2,r_2) + C_F\lambda(1-\lambda)\left((t_1-t_2)^2 + (r_1-r_2)^2\right).$$

Applying this for $(t,r) \mapsto F(t,z,r)$, $z \in \Omega$, $\lambda = 1/2$, $t_1 = st$, $t_2 = s^{-1}t$, $r_1 = U(st,z)$, $r_2 = U(s^{-1}t,z)$, we obtain

$$\begin{aligned}\frac{1}{2}F(st,z,&U(st,z)) + \frac{1}{2}F(s^{-1}t,z,U(s^{-1}t,z)) \\ &\geq F\left(\frac{(s+s^{-1})t}{2}, z, (U(st,z)+U(s^{-1}t,z))/2\right) \\ &\quad - \frac{C_F}{4}\left(t^2(s-s^{-1})^2 + (U(st,\cdot)-U(s^{-1}t,\cdot))^2\right).\end{aligned}$$

Using (4.4), (4.14), and the fact that F is increasing in r, we thus get

$$\begin{aligned}\frac{1}{2}F(st,z,&U(st,z)) + \frac{1}{2}F(s^{-1}t,z,U(s^{-1}t,z)) \\ &\geq F(t,z,v^s(t,z)) - \kappa_F(M_U+2\kappa_U+t)(s-1)^2 \\ &\quad - \frac{C_F}{4}\left(t^2(s-s^{-1})^2 + (U(st,\cdot)-U(s^{-1}t,\cdot))^2\right) \\ &\geq F(t,z,v^s(t,z)) - (\kappa_F(M_U+2\kappa_U+T) + 2C_F(T^2+2\kappa_U^2))(s-1)^2.\end{aligned}$$

The choice of C (4.13) ensures that

$$a(s) + a(s^{-1}) \geq \partial_t v^s(t,\cdot) + F(t,\cdot,v^s(t,\cdot)).$$

Altogether we conclude that $v^s \in \mathcal{S}_{h,g,F}(\Omega_S)$. Using Proposition 4.1 we infer $v^s \leq U$ in Ω_S. From this and (4.14) we obtain that for all $(t,z) \in \Omega_S$,

$$\frac{U(st,z) + U(s^{-1}t,z)}{2} - U(t,z) \leq (C + 2M_U + 8\kappa_U)(s-1)^2.$$

An elementary computation then yields (letting $s \to 1$) that $\forall (t,z) \in \Omega_S$,

$$t^2\partial_t^2 U(t,z) \leq (9\kappa_U + 2M_U + C).$$

We finally let $S \to T$ to conclude the proof. □

5. Space regularity of parabolic envelopes

We establish the first steps of a balayage process by studying solutions constructed in small balls, and establishing space regularity of $U_{h,g,F,\mathbb{B}_T}$: assuming adequate regularity conditions on the data we prove that $U_{h,g,F,\mathbb{B}_T}$ is locally $\mathcal{C}^{1,1}$ in $z \in \mathbb{B}$.

We assume that $T < +\infty$, and h satisfies (4.1) and (4.10).

5.1. Continuity in the space variable

Let (Y, d_Y) be a metric space. The uniform partial modulus of continuity in the space variable $y \in Y$ of a function $u : [0,T[\times Y \longrightarrow \mathbb{R}$ is

$$\eta(u,\delta) := \sup\{|u(t,y_1) - u(t,y_2)| \; ; \; t \in [0,T[,\; y_1, y_2 \in Y,\; d_Y(y_1,y_2) \le \delta\}.$$

In particular, the uniform partial modulus of continuity of F is defined as above with $Y := \Omega \times \mathbb{R}$.

THEOREM 5.1. – *Assume the following conditions :*

— $G := \log g$ *is continuous in* Ω *;*

— *there exists* $u \in \mathcal{S}_{h,g,F}(\Omega_T) \cap \mathcal{C}([0,T[\times \bar{\Omega})$*, such that* $u = h$ *on* $\partial_0 \Omega_T$.

Then $U := U_{h,g,F,\Omega_T}$ *is continuous on* $[0,T[\times \bar{\Omega}$ *and*

$$\eta(U,\delta) \le \eta(u,\delta) + \eta(H,\delta) + (\eta(F,\delta) + \eta(G,\delta))T. \tag{5.1}$$

Recall that H_t is the unique harmonic function in Ω with $H_t = h_t$ on $\partial\Omega$.

A continuous subsolution which agrees with h on $\partial_0\Omega$ is called a *subbarrier*. Such a subbarrier (for the whole boundary $\partial_0\Omega_T$ as required in the Theorem) exists when h is uniformly Lipschitz in $[0,T[$ and continuous on $\partial_0\Omega_T$ by Lemma 3.7, Lemma 3.8 and Lemma 3.9.

Proof. – It follows from Theorem 3.12 that U continuously extends to the boundary $\partial_0\Omega_T$ so that $U = h$ on $\partial_0\Omega_T$. We use the perturbation method of Walsh [31] to extend this property to the interior and prove that U is continuous on $[0,T[\times \bar{\Omega}$.

Fix $\delta > 0$ small enough. Since $u = h = U$ in $[0,T[\times \partial\Omega$, we infer that for all $t \in [0,T[$, $z \in \Omega, \zeta \in \partial\Omega$ with $|z - \zeta| \le \delta$,

$$U(t,\zeta) = u(t,\zeta) \le u(t,z) + \eta(u,\delta) \le U(t,z) + \eta(u,\delta). \tag{5.2}$$

Fix $\xi \in \mathbb{C}^n$ such that $|\xi| \le \delta$ and set $\Omega_\xi := \Omega - \xi$ and consider

$$W(t,z) := \begin{cases} U(t,z), & \text{if } t \in [0,T[, z \in \Omega \setminus \Omega_\xi, \\ \max\{U(t,z), U(t,z+\xi) - \eta(u,\delta)\}, & \text{if } t \in [0,T[, z \in \Omega \cap \Omega_\xi. \end{cases}$$

By (5.2) the two definitions coincide when $(t,z) \in [0,T[\times \Omega$ and $z + \xi \in \partial\Omega$. Therefore $W \in \mathcal{P}(\Omega_T)$. We are going to prove that $W - O(\delta)(t+1) \in \mathcal{S}_{h,g,F}(\Omega_T)$ for some small error term $O(\delta)$.

The subsolution property. By Lemma 3.4, for a.e. $(t,z) \in [0,T[\times (\Omega \cap \Omega_\xi)$,

$$\partial_t W(t,z) = 1_{\{U(t,z) < \tilde{U}(t,z)\}} \partial_t \tilde{U}(t,z) + 1_{\{U(t,z) \ge \tilde{U}(t,z)\}} \partial_t U(t,z),$$

where

$$\tilde{U}(t,z) := U(t,z+\xi) - \eta(u,\delta) \text{ in } \Omega \cap \Omega_\xi.$$

Moreover

$$\begin{aligned} &e^{\partial_t \tilde{U}(t,z) + F(t,z,\tilde{U}(t,z)) + G(z)} dt \wedge dV(z) \\ &\qquad \le e^{\partial_t U(t,z+\xi) + F(t,z+\xi,U(t,z+\xi)) + \eta(F,\delta) + G(z+\xi) + \eta(G,\delta)} dt \wedge dV(z) \\ &\qquad \le e^{\eta(F,\delta) + \eta(G,\delta)} dt \wedge (dd^c \tilde{U})^n, \end{aligned}$$

in the weak sense on $]0,T[\times(\Omega\cap\Omega_\xi)$. We thus obtain

$$e^{\partial_t W(t,z)+F(t,z,W(t,z))+G(z)}dt\wedge dV(z)\le e^{b(\delta)}dt\wedge(dd^cW)^n,$$

i.e., the function defined on $[0,T[\times\Omega$ by $W_1(t,z):=W(t,z)-b(\delta)t$, is a subsolution to (CMAF) in $]0,T[\times\Omega$. Here $b(\delta):=\eta(F,\delta)+\eta(G,\delta)$.

Estimating boundary values.– It follows from Theorem 3.12 that

$$\lim_{(t,z')\to(0,z)}U(t,z')=h_0(z),\ z\in\Omega.$$

By definition of W and the assumption that $h_0=u$ on $\{0\}\times\Omega$, we obtain

$$\lim_{(t,z')\to(0,z)}W(t,z')\le h_0(z),\ \text{for all } z\in\Omega.$$

Fix $(\tau,\zeta)\in[0,T[\times\partial\Omega$.

Since $U\le H$ in Ω_T and $U=h$ in $[0,T[\times\partial\Omega$, we infer

$$\lim_{]0,T[\times(\Omega\cap\Omega_\xi)\ni(t,z)\to(\tau,\zeta)}W(t,z)\le\max(h(\tau,\zeta),H(\tau,\zeta+\xi))\le h(\tau,\zeta)+\eta(H,\delta),$$

and

$$\lim_{]0,T[\times(\Omega\setminus\Omega_\xi)\ni(t,z)\to(\tau,\zeta)}W(t,z)=\lim_{]0,T[\times(\Omega\setminus\Omega_\xi)\ni(t,z)\to(\tau,\zeta)}U(t,\zeta)=h(\tau,\zeta).$$

From the computations above we conclude that $W_1-\eta(H,\delta)\in\mathcal{S}_{h,g,F}(\Omega_T)$. Thus $W_1-\eta(H,\delta)\le U$ in Ω_T, hence

$$U(t,z+\xi)-\eta(u,\delta)-\eta(H,\delta)-(\eta(F,\delta)+\eta(G,\delta))t\le U(t,z),$$

for $(t,z)\in[0,T[\times(\Omega\cap\Omega_\xi)$ and $\xi\in\mathbb{C}^n$ with $|\xi|\le\delta$. This gives (5.1).

The continuity of U on $[0,T[\times\bar\Omega$ follows from Theorem 4.7, the continuity of h_0, the continuity of each slice $U(t,\cdot)$ on $\bar\Omega$ and the fact that U is locally uniformly Lipschitz in $t\in]0,T[$. □

Corollary 5.2. – *Assume that*

1. $G:=\log g$ *is Lipschitz in* Ω*;*
2. *the family* $\{h(\cdot,z)\,;\,z\in\partial\Omega\}$ *is uniformly Lipschitz in* $[0,T[$*;*
3. h_0 *is Lipschitz on* $\bar\Omega$*;*
4. *the family* $\{h(t,\cdot)\,;\,t\in]0,T[\}$ *is uniformly* $\mathcal{C}^{1,1}$ *on* $\partial\Omega$*;*
5. *the function* F *is Lipschitz on* $[0,T[\times\Omega\times J$*, for each* $J\Subset\mathbb{R}$.

Then the family $\{U(t,\cdot)\,;\,t\in[0,T[\}$ *is uniformly Lipschitz on* $\bar\Omega$.

Proof. – It follows from Lemma 3.7, Lemma 3.8 and assumption (2.) that there exists $u\in\mathcal{S}_{g,h,F}(\Omega_T)\cap\mathcal{C}([0,T[\times\bar\Omega)$ with $u|_{\partial_0\Omega_T}=h$. [17, Theorem 5.12] and (3.), (4.) ensure that the family $\{u(t,\cdot)\,;\,t\in[0,T[\}$ is uniformly Lipschitz on $\bar\Omega$. We now invoke Theorem 5.1 to finish the proof. □

5.2. $\mathcal{C}^{1,1}$-regularity in the space variable

We prove the following regularity result:

THEOREM 5.3. – *Assume $\Omega = \mathbb{B}$ is the unit ball, $T < +\infty$, and*

1. $G := \log g \in C^{1,1}(\bar{\mathbb{B}})$*;*
2. *h satisfies the assumptions of Corollary 5.2;*
3. *F is Lipschitz and semi-convex in $[0, T[\times \bar{\mathbb{B}} \times J$, for each $J \Subset \mathbb{R}$.*

Then the envelope $U_{h,g,F,\mathbb{B}_T}$ is locally uniformly $\mathcal{C}^{1,1}$ in $z \in \mathbb{B}$.

By scaling and translating, the result still holds for any ball $B(z_0, r) \Subset \mathbb{C}^n$. In the proof below we use C to denote various uniform constants which may be different from place to place.

Proof. – The proof is a parabolic analogue of [1, Theorem 6.7]. We follow closely the presentation of [17, Theorem 5.13].

Recall from Corollary 5.2 that the family $\{U(t,\cdot)\ ;\ t \in [0, T[\}$ is uniformly Lipschitz on $\bar{\mathbb{B}}$.

Automorphisms of the ball $\mathbb{B}$. – For $a \in \mathbb{B}$, we set

$$\mathcal{T}_a(z) = \frac{P_a(z) - a + \sqrt{1-|a|^2}(z - P_a(z))}{1 - \langle z, a\rangle} \quad ; \quad P_a(z) = \frac{\langle z, a\rangle}{|a|^2} a,$$

where $\langle \cdot, \cdot \rangle$ denote the Hermitian product in $\mathbb{C}^n$. It is well known (see [22, Lemma 4.3.1]) that $\mathcal{T}_a$ is a holomorphic automorphism of the unit ball such that $\mathcal{T}_a(a) = 0$ and $\mathcal{T}_a(\partial\mathbb{B}) = \partial\mathbb{B}$. Note that $\mathcal{T}_0$ is the identity. We set

$$\xi = \xi(a, z) := a - \langle z, a\rangle z.$$

Observe that $\xi(-a, z) = -\xi(a, z)$. If $|a| \le 1/2$ then

$$\mathcal{T}_a(z) = z - \xi + O(|a|^2),$$

where $O(|a|^2) \le C_0|a|^2$, with C_0 a numerical constant independent of $z \in \mathbb{B}$ when $|a| \le 1/2$. Thus $\mathcal{T}_{\pm a}$ is the translation by $\mp\xi$ up to small second order terms, when $|a|$ is small enough.

We set, for $(t, z) \in \mathbb{B}_T$,

$$V_a(t, z) := \frac{1}{2}(U(t, \mathcal{T}_a(z)) + U(t, \mathcal{T}_{-a}(z)).$$

We are going to prove that, for a uniform constant $C > 0$, the function $V_a - C|a|^2(t+1)$ belongs to $\mathcal{S}_{h,g,F}(\mathbb{B}_T)$. We proceed in two steps.

Step 1: Boundary values of V_a. – If q is $\mathcal{C}^{1,1}(\bar{\mathbb{B}})$ then, as in [17, Page 145],

$$\left|q(\mathcal{T}_a(z)) + q(\mathcal{T}_{-a}(z)) - 2q(z)\right| \le 2C(q)|a|^2, \tag{5.3}$$

where $C(q) > 0$ depends on the uniform $\mathcal{C}^{1,1}$-norm of q on $\bar{\mathbb{B}}$.

Since the family $\{h(t,\cdot)\ ;\ t \in [0, T[\}$ is uniformly Lipschitz in $\partial\mathbb{B}$, applying (5.3) yields

$$h(t, \mathcal{T}_a(z)) + h(t, \mathcal{T}_{-a}(z)) \le 2h(t, z) + 2C(h)|a|^2,$$

for $z \in \partial\mathbb{B}$, $|a|$ small enough, where $C(h) > 0$ depends on the uniform $\mathcal{C}^{1,1}$-bound of $h(t,\cdot)$ in a neighborhood of $\partial\mathbb{B}$. We infer, for all $(t, \zeta) \in \partial_0\mathbb{B}_T$,

$$V_a(t, \zeta) \le h(t, \zeta) + C(h)|a|^2.$$

Step 2: Estimating the Monge-Ampère measure of V_a. – Since U is a subsolution to (CMAF) a direct computation shows that

$$\begin{aligned}(dd^c U \circ \mathcal{T}_a)^n &= |\det \mathcal{T}'_a|^2 (dd^c U)^n \circ \mathcal{T}_a \\ &\geq |\det \mathcal{T}'_a|^2 \exp\left(\partial_t U(t, \mathcal{T}_a(z)) + F(t, \mathcal{T}_a(z), U(t, \mathcal{T}_a(z))) + G(\mathcal{T}_a(z))\right).\end{aligned}$$

Since the function $(a, z) \mapsto \theta(0, z) := \log|\det \mathcal{T}'_a(z)|^2 + \log|\det \mathcal{T}'_{-a}(z)|^2$ is smooth in $\mathbb{B}_{1/2} \times \bar{\mathbb{B}}$ and $\theta(0, z) = 0$, the Taylor expansion yields

$$\theta(a, z) + \theta(-a, z) = O(|a|^2).$$

The assumption (3) provides us with a uniform constant C such that

$$\begin{aligned}\frac{1}{2}\{F(t, \mathcal{T}_a(z), &U(t, \mathcal{T}_a(z))) + F(t, \mathcal{T}_{-a}(z), U(t, \mathcal{T}_{-a}(z)))\} \\ &\geq F\left(t, \frac{\mathcal{T}_a(z) + \mathcal{T}_{-a}(z)}{2}, V_a(t, z)\right) \\ &\quad - C(\|\mathcal{T}_a(z) - \mathcal{T}_{-a}(z)\|^2 + (U(t, \mathcal{T}_a(z) - U(t, \mathcal{T}_{-a}(z))^2) \\ &\geq F(t, z, V_a(t, z)) - C|a|^2,\end{aligned}$$

where in the last inequality we have used $\mathcal{T}_a(z) + \mathcal{T}_{-a}(z) - 2z = O(|a|^2)$, $\mathcal{T}_a(z) - \mathcal{T}_{-a}(z) = O(|a|)$, and the Lipschitz regularity (in $z \in \bar{\mathbb{B}}$) of U. Using this, and applying Lemma 2.10 and the uniform estimate (5.3) to the function G we obtain

$$(dd^c V_a(t, \cdot))^n \geq \exp\left\{\partial_t V_a + F(t, z, V_a(t, z)) + G(z) - C|a|^2\right\}.$$

By the computations above we conclude that the function

$$\mathbb{B}_T \ni (t, z) \mapsto W_a(t, z) = V_a(t, z) - C|a|^2(t + 1)$$

belongs to $\mathcal{S}_{h,g,F}(\mathbb{B}_T)$. Therefore, for all $(t, z) \in \mathbb{B}_T$,

$$V_a(t, z) - (T + 1)C|a|^2 \leq U(t, z).$$

From this estimate, we proceed as in [17, page 146-147] to prove that the second order partial derivatives (in z) of U are locally bounded in $\mathbb{B}$. □

We now show that U admits a Taylor expansion up to order $(1, 2)$:

LEMMA 5.4. – *Assume $(h, g, F, \mathbb{B}_T)$ is as in Theorem 5.3. Then the envelope U admits the following Taylor expansion at almost every point $(t_0, z_0) \in \mathbb{B}_T$,*

$$\begin{aligned}U(t, z) = U(t_0, z_0) &+ (t - t_0)\partial_t U(t_0, z_0) + \Re P(z - z_0) + L(z - z_0) \\ &+ o(|t - t_0| + |z - z_0|^2),\end{aligned}$$

where P is a holomorphic polynomial of degree 2 and L is the Levi form of $U(t_0, z)$ at z_0.

Proof. – It follows from Theorem 4.8 that U is locally uniformly semi-concave in $t \in]0, T[$. Theorem 5.3 ensures that U is locally uniformly Lipschitz in $z \in \mathbb{B}$, hence, for all $t \in]0, T[$, $U(t, \cdot)$ is twice differentiable at a.e. $z \in \mathbb{B}$.

Let A_1 be the set of points $(t_0, z_0) \in \Omega_T$ such that $U(\cdot, z_0)$ is not differentiable at t_0 and A_2 be the set of points $(t_0, z_0) \in \Omega_T$ such that $U(t_0, \cdot)$ is not twice differentiable at z_0. It follows from Fubini's Theorem that the set $A := A_1 \cup A_2$ is of Lebesgue measure zero in Ω_T.

We show that the Taylor expansion holds at any point $(t_0, z_0) \notin A$. Fix $\varepsilon > 0$ and $(t_0, z_0) \notin A$. We first write for $(t, z) \in \Omega_T$,

$$U(t,z) - U(t_0, z_0) = U(t,z) - U(t_0, z) + U(t_0, z) - U(t_0, z_0).$$

Since $(t_0, z_0) \notin A_1$, the function $U(t_0, \cdot)$ is twice differentiable at z_0. Thus there exists $r > 0$ such that for $|z - z_0| < r$,

$$|U(t_0, z) - U(t_0, z_0) - \Re P(z - z_0) - L(z - z_0)| \leq \varepsilon |z - z_0|^2. \tag{5.4}$$

On the other hand since $(t_0, z_0) \notin A_2$, $\partial_t U(t_0, z_0)$ exists and we have

$$\begin{aligned} U(t,z) - U(t_0, z) - (t - t_0)\partial_t U(t_0, z_0) &= \int_{t_0}^{t} (\partial_\tau^+ U(\tau, z) - \partial_t U(t_0, z_0)) d\tau \\ &= \int_{t_0}^{t} (\partial_\tau^- U(\tau, z) - \partial_t U(t_0, z_0)) d\tau. \end{aligned}$$

Since $\partial_\tau^+ U$ is lsc and $\partial_\tau^- U$ is usc, we can choose r so small that for $|t - t_0| + |z - z_0| < r$,

$$|U(t,z) - U(t_0, z) - (t - t_0)\partial_t U(t_0, z_0)| \leq \varepsilon |t - t_0|. \tag{5.5}$$

The Taylor expansion thus follows from (5.4) and (5.5). □

6. Pluripotential solutions

We finally prove in this section that U_{h,g,F,Ω_T} is the unique pluripotential solution to the Cauchy-Dirichlet problem for (CMAF) which is locally uniformly semi-concave.

6.1. The case of Euclidean balls

We first treat the case when Ω is a Euclidean ball in $\mathbb{C}^n$. By scaling and translating, it suffices to treat the case of the unit ball.

THEOREM 6.1. – *Let $\Omega = \mathbb{B}$ be the unit ball in $\mathbb{C}^n$, $T < +\infty$, and assume that*

1. *$G := \log g$ is $\mathcal{C}^{1,1}$ in $\bar{\mathbb{B}}$;*
2. *h is uniformly $\mathcal{C}^{1,1}$ in $z \in \partial\mathbb{B}$, h_0 is $\mathcal{C}^{1,1}$ in $\bar{\mathbb{B}}$;*
3. *h is uniformly Lipschitz in $t \in [0, T[$ and $\partial_t^2 h \leq C t^{-2}$ on $]0, T[\times \partial\mathbb{B}$;*
4. *F is Lipschitz and semi-convex in $(t, z, r) \in [0, T[\times \bar{\mathbb{B}} \times J$ for each $J \Subset \mathbb{R}$.*

Then for almost every $(t, z) \in \mathbb{B}_T$,

$$\det\left(\frac{\partial^2 U}{\partial z_j \partial \bar{z}_k}(t,z)\right) = e^{\dot{U}(t,z) + F(t,z,U(t,z)) + G(z)}.$$

In particular U is a pluripotential solution to the Cauchy-Dirichlet problem for the parabolic Equation (CMAF) *with boundary data h.*

Proof. – Theorem 4.7 and the Lipschitz assumption on h ensure that U is a subsolution to (CMAF) with $U = h$ on $\partial_0 \Omega_T$. It follows from Corollary 5.2 and Theorem 5.3 that U is uniformly Lipschitz in $z \in \bar{\mathbb{B}}$ and locally $\mathcal{C}^{1,1}$ in $\mathbb{B}$. In particular U is twice differentiable in z almost everywhere in Ω_T, hence

$$(dd^c U)^n = \det(U_{j,\bar{k}}(t,z)) dV(z).$$

As U is also almost everywhere differentiable in t and a subsolution to the parabolic Equation (CMAF), we infer by Proposition 3.2,

$$\det(U_{j,\bar{k}}(t,z)) \geq e^{\partial_t U(t,z) + F(t,z,U(t,z)) + G(z)}, \tag{6.1}$$

almost everywhere in $\mathbb{B}_T$.

We want to prove that equality holds in (6.1). We use the notation of the proof of Lemma 5.4 and set $E = \mathbb{B}_T \setminus A$. Arguing by contradiction we assume that

$$\det\left(U_{j,\bar{k}}(t_0,z_0) - \varepsilon I_n\right) > e^{\dot{U}(t_0,z_0) + F(t_0,z_0,U(t_0,z_0)) + G(z_0) + \varepsilon},$$

at some point $(t_0, z_0) \in E$, for a small constant $\varepsilon > 0$.

We use a bump construction to produce a subsolution $v \in \mathcal{S}_{h,g,F}(\mathbb{B}_T)$ which satisfies $v(t_0,z_0) > U(t_0,z_0)$ providing a contradiction. It follows from Lemma 5.4 that

$$\begin{aligned} U(t,z) - U(t_0,z_0) = (t-t_0)\partial_t U(t_0,z_0) + \Re P(z-z_0) + L(z-z_0) \\ + o(|t-t_0| + |z-z_0|^2). \end{aligned} \tag{6.2}$$

Set $D_r := \{(t,z)\,;\ |t-t_0| + |z-z_0|^2 < r\}$ and define

$$\begin{aligned} w(t,z) := U(t_0,z_0) + \partial_t U(t_0,z_0)(t-t_0) + \Re P(z-z_0) \\ + L(z-z_0) + \delta - \gamma(|z-z_0|^2 + |t-t_0|), \end{aligned}$$

where $\delta, \gamma > 0$ are constants to be specified later. Note that if γ is small enough then $w \in \mathcal{P}(D_r)$. For any $(t,z) \in D_r$, the Taylor expansion (6.2) ensures that

$$U(t,z) \geq w(t,z) + \gamma(|t-t_0|) + |z-z_0|^2) - \delta + o(r).$$

Hence for any $(t,z) \in D_r \setminus D_{r/2}$,

$$U(t,z) \geq w(t,z) + \gamma r/2 - \delta + o(r) > w(t,z),$$

if $\delta = \gamma r/4$, and $r > 0$ is small enough. On the other hand for $(t,z) \in D_r$,

$$(dd^c w)^n = dd^c(U - \gamma|z-z_0|^2)^n(t_0,z_0),$$

and for $(t,z) \in D_r$, $t \neq t_0$,

$$\partial_t w(t,z) = \partial_t U(t_0,z_0) - \gamma(t-t_0)/|t-t_0|.$$

Thus if $\gamma < \varepsilon$, we obtain for any $(t,z) \in D_r$,

$$\begin{aligned} (dd^c w(t,z))^n &\geq e^{\partial_t w(t,z) + \gamma(t-t_0)/|t-t_0|)) + F(t_0,z_0,U(t_0,z_0)) + G(z_0) + \varepsilon} dV \\ &\geq e^{\partial_t w(t,z) - \gamma + F(t,z,w(t,z)) + G(z) + R(t,z) + \varepsilon} dV, \end{aligned}$$

where

$$R(t,z) := F(t_0,z_0,U(t_0,z_0)) - F(t,z,w(t,z)) + (G(z_0) - G(z)).$$

Since U and F are locally Lipschitz, there exists $A > 0$ such that for $r > 0$ small enough and $(t, z) \in D_r$,

$$R(t, z) \geq -A\sqrt{r} \geq \gamma - \varepsilon.$$

The function w is therefore a subsolution to (CMAF) in D_r.

The previous estimates ensure that the function

$$v(t, z) := \begin{cases} \max\{U(t, z), w(t, z)\} & \text{if} \quad (t, z) \in D_r \\ U(t, z) & \text{if} \ (t, z) \in \mathbb{B}_T \setminus D_r \end{cases}$$

belongs to $\mathcal{S}_{h,g,F}(\mathbb{B}_T)$, hence $v \leq U$ in $\mathbb{B}_T$. In particular, $w \leq U$ in D_r which is a contradiction since $w(t_0, z_0) = U(t_0, z_0) + \delta > U(t_0, z_0)$. □

We now relax the regularity assumptions in Theorem 6.1.

Proposition 6.2. – *Assume $\Omega = \mathbb{B}$ is the unit ball in $\mathbb{C}^n$, $T < +\infty$, and*

— *$G := \log g$ is continuous in $\bar{\mathbb{B}}$;*

— *h is continuous on $\partial_0 \mathbb{B}_T$ and satisfies* (4.1) *and* (4.10)*;*

— *F extends as a continuous function on $[0, T[\times \bar{\mathbb{B}} \times \mathbb{R}$ which is uniformly Lipschitz and uniformly semi-convex in $(t, r) \in [0, T[\times J$ for each $J \Subset \mathbb{R}$.*

Then $U_{h,g,F,\mathbb{B}_T}$ is a continuous solution to (CMAF) *with boundary values h.*

Proof. – It follows from Theorem 4.7 that $U \in \mathcal{S}_{h,g,F}(\mathbb{B}_T)$ satisfies the boundary conditions (0.2), (0.3). It remains to prove that U is continuous on $[0, T[\times \bar{\Omega}$ and solves (CMAF) in Ω_T. By Proposition 4.1 it suffices to prove these statements in $\mathbb{B}_S$ for each fixed $S < T$. We proceed in several steps.

Step 1. – Assume that $h(\cdot, z)$ is uniformly Lipschitz in $t \in [0, T[$. It follows from Theorem 5.1 that U is continuous on $[0, T[\times \bar{\Omega}$. The goal is to prove that U solves (CMAF) in $\mathbb{B}_S$. We proceed by approximation as follows.

Let $(G_j) = (\log g_j)$ be a sequence of smooth functions uniformly converging to G on $\bar{\mathbb{B}}$. Extending F continuously in an open neighborhood of $[0, S] \times \bar{\mathbb{B}} \times \mathbb{R}$ and taking convolution in (t, z, r) we can find a sequence $F_j : [0, S] \times \bar{B} \times \mathbb{R}$ of functions which are smooth in (t, z, r) and

— Lipschitz and semi-convex in $[0, S] \times \bar{\mathbb{B}} \times J$ for each $J \Subset \mathbb{R}$;

— uniformly converge to F on $[0, S] \times \bar{\mathbb{B}} \times J$, for each $J \Subset \mathbb{R}$.

We extend h as a continuous function in $[0, T[\times \{|z| \geq 1/4\}$ by setting

$$h(t, z) := h\left(t, \frac{z}{|z|}\right), z \in \mathbb{C}^n, |z| \geq 1/4;$$

The extension h satisfies (4.1) and (4.10) for all $|z| \geq 1/4$ (with the same constants κ_h, C_h as the original function h defined on $\partial_0 \mathbb{B}_T$). Taking convolution in the z variable we can find a sequence $(\hat{h}^j)$ of functions in $[0, T[\times \{|z| > 1/3\}$ which are smooth in z and

— are uniformly Lipschitz in t;

— satisfy (4.10) with the same uniform constant C_h;

— uniformly converge to h on $[0, S] \times \partial \mathbb{B}$.

Fix $j \in \mathbb{N}$ and define h^j by

$$\begin{cases} h^j(t,z) := \hat{h}_j(t,z) & \text{if } (t,z) \in]0,T[\times \partial\mathbb{B} \\ h^j(0,z) = h_0 + H^j & \text{if } (t,z) \in \{0\} \times \mathbb{B}, \end{cases}$$

where H^j is the maximal plurisubharmonic function in $\mathbb{B}$ with boundary values $\hat{h}^j(0,\cdot) - h_0$. Observe that h^j is a Cauchy-Dirichlet boundary data on $\mathbb{B}_T$ which satisfies the assumptions of Theorem 6.1. Note also that h^j uniformly converges to h on $\partial_0\mathbb{B}_T$, since H^j uniformly converges to 0.

Set $U^j := U_{h^j,g_j,F_j}(\mathbb{B}_S)$, $j \in \mathbb{N}$. Theorem 6.1 ensures that U^j is a pluripotential solution to the Equation (CMAF) and $U^j = h^j$ on $\partial_0\mathbb{B}_S$. It also follows from Theorem 4.2 and Theorem 4.8 that U^j is locally uniformly semi-concave in $t \in]0,S]$. Moreover, (4.5) and (4.13) ensure that the Lipschitz and semi-concave constants of U^j are uniform. By definition of the envelope, U^j uniformly converges to U as $j \to +\infty$. It thus follows from Proposition 2.9, Proposition 2.3, Remark 2.4 and Lemma 2.8 that U is a pluripotential solution to (CMAF) in $\mathbb{B}_S$.

Step 2. – To treat the general case, we approximate h by a family of functions h^ε which are Lipschitz up to zero, in such a way that they satisfy (4.1) and (4.10) with constants independent of ε. Here, Lipschitz up to zero means that for each $S \in]0,T[$, there exists a constant C_S such that

$$|h^\varepsilon(t,z) - h^\varepsilon(t',z)| \leq C_S|t-t'|, \ t,t' \in [0,S], \ z \in \partial\Omega.$$

We proceed as in the proof of Theorem 4.7. Fix $S > 0$ and $\varepsilon > 0$ such that $S + \varepsilon < T$, and define

$$\begin{cases} h^\varepsilon(t,\zeta) = h(t+\varepsilon,\zeta) & \text{if } (t,\zeta) \in [0,S] \times \partial\mathbb{B} \\ h^\varepsilon(0,z) = h_0(z) + \phi_\varepsilon(z) & \text{if } z \in \mathbb{B}, \end{cases}$$

where ϕ_ε is the maximal plurisubharmonic function in $\mathbb{B}$ such that $\phi_\varepsilon(\zeta) = h(\varepsilon,\zeta) - h_0(\zeta)$ on $\partial\mathbb{B}$. Then h^ε uniformly converges to h on $\partial_0\mathbb{B}_S$.

Observe that h^ε is a Cauchy-Dirichlet boundary data satisfying (4.1) and (4.10) with constants independent of ε. By construction h^ε is uniformly Lipschitz in $t \in [0,S]$. The previous step shows that $U^\varepsilon := U_{h^\varepsilon,g,F,\mathbb{B}_S}$ is a continuous pluripotential solution to (CMAF) with boundary data h^ε. By Proposition 4.1, U^ε uniformly converges to U on $\mathbb{B}_S$. The continuity of U^ε ensures that U is continuous in $\mathbb{B}_S$. Since h_0 is continuous, Theorem 3.12 ensures that U is continuous in $[0,S[\times \bar{\mathbb{B}}$. It follows from Theorem 4.2 and Theorem 4.8 that the family U^ε is locally uniformly semi-concave in $t \in]0,S[$ with constants independent of ε; see (4.3) and (4.11). Arguing as in the last part of Step 1 we conclude that U solves (CMAF) in $\mathbb{B}_S$. □

EXAMPLE 6.3. – It is difficult to provide explicit examples of solutions, as the equation is highly non linear. Here is a simple smooth radial solution for the unit ball: the function

$$(t,z) \mapsto \psi(t,z) = ||z||^2 + at + b$$

is a parabolic potential on $\mathbb{R}^+ \times \mathbb{B}$, solution of (CMAF) in $\mathbb{B}_{+\infty}$ with $F \equiv 0$, $G \equiv$ constant and smooth boundary values.

6.2. The case of bounded strictly pseudoconvex domains

We now consider the case of a smooth bounded strictly pseudoconvex domain.

We first prove the existence result in a particular case.

PROPOSITION 6.4. – *Assume $T < +\infty$, h satisfies* (4.1) *and* (4.10). *Then $U_{h,g,F}$ is a pluripotential solution to the Cauchy-Dirichlet problem for the parabolic Equation* (CMAF) *in Ω_T with boundary conditions* (0.2) *and* (0.3).

Proof. – It follows from Theorem 4.2, Theorem 4.7, Theorem 4.8 that U is locally uniformly semi-concave in $t \in]0,T[$, $U \in \mathcal{S}_{h,g,F}(\Omega_T)$ and it satisfies the boundary conditions (0.2) and (0.3). It remains to verify that U solves (CMAF). We proceed in several steps.

Step 1. – We first assume that h_0 and $G := \log g$ are continuous in $\bar{\Omega}$. Then U is also continuous on $[0,T[\times \bar{\Omega}$ thanks to Theorem 5.1.

Let $B \Subset \Omega$ be a small ball and h_B denote the restriction of U on the parabolic boundary of B_T. The boundary data h_B for the Cauchy-Dirichlet problem for (CMAF) satisfies the assumption of Proposition 6.2. Also, the restriction of U on $[0,T[\times B$ is a continuous subsolution to the Cauchy Dirichlet problem (CMAF) in B_T with boundary data h_B. It follows from Proposition 6.2 that $U_B := U_{h_B,g,F,B_T}$ is a pluripotential solution to (CMAF) with boundary data h_B and $U_B \geq U$ in B_T.

The function V, which is defined as U_B in B_T and U in $\Omega_T \setminus B_T$, belongs to $\mathcal{S}_{h,g,F}(\Omega_T)$. Hence $V = U$ is a pluripotential solution to (CMAF).

Step 2. – We next assume h_0 is continuous, but we merely assume $g \in L^p$.

Let (g_j) be a sequence of strictly positive continuous functions in $\bar{\Omega}$ that converges to g in $L^p(\Omega)$. Set $U^j := U_{h,g_j,F}$ and $U := U_{h,g,F}$. Since the L^p-norm of g_j is uniformly bounded, Theorem 4.2 and Theorem 4.8 ensure that the functions U^j are locally uniformly semi-concave (with constants independent of j). It thus follows from Proposition 1.17 that a subsequence of U^j, still denoted by U^j, converges almost everywhere in Ω_T to a function $V \in \mathcal{P}(\Omega_T)$. Lemma 1.11 ensures that U^j_t converges in $L^1(\Omega)$ to V_t, for all $t \in]0,T[$. By Proposition 2.9, for almost all $t \in]0,T[$, $\partial_t U^j(t,\cdot)$ converges pointwise to $\partial_t V(t,\cdot)$. Thus, for almost all $t \in]0,T[$,

$$e^{\partial_t U^j(t,\cdot)+F(t,\cdot,U^j)} g_j \overset{L^p(\Omega)}{\longrightarrow} e^{\partial_t V(t,\cdot)+F(t,\cdot,V)} g.$$

A result due to Kołodziej (see [24, end of the proof of Theorem 3], see also [8, Theorem 2.8]) ensures that $U^j(t,\cdot)$ uniformly converges to $V(t,\cdot)$ and $(dd^c U^j(t,\cdot))^n$ converges in the sense of positive measures to $(dd^c V(t,\cdot))^n$. Thus $dt \wedge (dd^c U^j)^n$ weakly converges in Ω_T to $dt \wedge (dd^c V)^n$ (see the proof of Proposition 2.3). Hence V solves (CMAF) in Ω_T. Lemma 3.9 and Corollary 3.11 ensure that $V^*|_{\partial_0 \Omega_T} \leq h$. Thus $V \leq U$.

To prove that $U \leq V$ we now use a perturbation argument following an idea of Kołodziej [24] (see also [13]). For each j let θ_j be the unique continuous psh function in $\bar{\Omega}$, vanishing on $\partial\Omega$ such that $(dd^c \theta_j)^n = |g_j - g| dV$. It follows from [25] that

$$\lim_{j \to +\infty} \sup_{\bar{\Omega}} |\theta_j| = 0.$$

Fix $0 < S < T$, $\varepsilon > 0$ small enough and set, for $(t,z) \in \Omega_S$,

$$W^j(t,z) := W^{j,\varepsilon}(t,z) := U(t+\varepsilon,z) - \delta(\varepsilon)t + C(\varepsilon)\theta_j(z),$$

where $\delta(\varepsilon) > 0$, $C(\varepsilon) > 0$ are constants to be chosen in such a way that $\delta(\varepsilon) \to 0$ but $C(\varepsilon)$ may blow up as $\varepsilon \to 0$. The goal is to prove that $W^j \in \mathcal{S}_{h,g_j,F}(\Omega_S)$. It follows from Lemma 3.13 that U_t uniformly converges on $\bar{\Omega}$ to h_0, ensuring that

$$b(\varepsilon) := \sup_{\partial_0\Omega_S} |U(t+\varepsilon,z) - h(t,z)| \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

A direct computation shows that

$$\begin{aligned}(dd^c W^j)^n &\geq (dd^c U(t+\varepsilon,\cdot))^n + C(\varepsilon)^n (dd^c\theta_j)^n \\ &\geq e^{\partial_t U(t+\varepsilon,\cdot)+F(t+\varepsilon,\cdot,U(t+\varepsilon,\cdot))} g(z)dV + C(\varepsilon)^n |g - g_j| dV.\end{aligned}$$

By the Lipschitz condition (4.4) on F we can write

$$|F(t+\varepsilon,\cdot,U(t+\varepsilon,\cdot)) - F(t,\cdot,U(t+\varepsilon,\cdot))| \leq \varepsilon\kappa_F.$$

Since $r \mapsto F(t,z,r)$ is increasing,

$$F(t,\cdot,U(t+\varepsilon,\cdot)) \geq F(t,\cdot,W^j(t,\cdot)) - A\varepsilon,$$

where $A > 0$ depends on κ_F, M_U. We choose $\delta(\varepsilon) := b(\varepsilon) + A\varepsilon$. Then

$$\begin{aligned}(dd^c W^j)^n &\geq e^{\partial_t U(t+\varepsilon,\cdot)+F(t,\cdot,W^j(t,\cdot))-A\varepsilon} g(z)dV + C(\varepsilon)^n|g-g_j|dV \\ &\geq e^{\partial_t W^j(t,\cdot)+F(t,\cdot,W^j(t,\cdot))} g(z)dV + C(\varepsilon)^n|g-g_j|dV\end{aligned}$$

and $W^j\big|_{\partial_0\Omega_S} \leq h$. We now choose

$$C(\varepsilon) := \left(\sup_{\Omega_S} \exp\{\partial_t U(t+\varepsilon,z) + F(t,z,U(t+\varepsilon,z))\}\right)^{1/n} < +\infty.$$

Since $r \mapsto F(t,z,r)$ is increasing we obtain

$$\begin{aligned}(dd^c W^j)^n &\geq e^{\partial_t W^j(t,\cdot)+F(t,\cdot,W^j(t,\cdot))} g dV + e^{\partial_t W^j(t,\cdot)+F(t,\cdot,W^j(t,\cdot))}|g-g_j|dV \\ &\geq e^{\partial_t W^j(t,\cdot)+F(t,\cdot,W^j(t,\cdot))} g_j dV.\end{aligned}$$

Thus $W^j \in \mathcal{S}_{h,g_j,F}(\Omega_S)$. Together with Proposition 4.1 this yields

$$W^{j,\varepsilon} \leq U_{h,g_j,F,\Omega_T}, \text{ for all } (t,z) \in \Omega_S. \tag{6.3}$$

In (6.3) we first let $j \to +\infty$ and then $\varepsilon \to 0$ to arrive at $U \leq V$. Hence $U = V$ is a pluripotential solution to the parabolic Monge-Ampère Equation (CMAF) with boundary data h.

Step 3. – We finally remove the continuity assumption on h_0. Using Lemma 2.11 we find a sequence h^j of continuous Cauchy-Dirichlet boundary data for Ω_T such that $h^j = h$ on $[0, T[\times \partial\Omega$ and h^j decreases pointwise to h.

The previous step ensures that $U^j := U(h^j, g, F)$ solves (CMAF). Theorem 4.2 and Theorem 4.8 provide uniform concavity constants for U^j. Since h^j decreases to h, $U \leq U^j$ decreases to some $V \in \mathcal{P}(\Omega_T)$. We thus have $V^*|_{\partial_0\Omega_T} \leq h$, and Proposition 2.9 and Proposition 2.3 reveal that V solves (CMAF). Thus V is a candidate defining U, hence $U = V$. □

We are now ready to prove a general existence result. Here T may take the value $+\infty$. We assume that, for each $0 < S < T$, there exists a constant $C(S) > 0$ such that for all $(t, z) \in]0, S] \times \partial\Omega$,

$$t|\partial_t h(t, z)| \leq C(S) \ ; \ t^2\partial_t^2 h(t, z) \leq C(S). \tag{6.4}$$

THEOREM 6.5. – *If h satisfies* (6.4) *then $U := U_{h,g,F}$ is a pluripotential solution to the Cauchy-Dirichlet problem for* (CMAF) *in Ω_T with boundary values h. Moreover, U is continuous in* $]0, T[\times \bar{\Omega}$ *and locally uniformly semi-concave in* $t \in]0, T[$.

In particular, if h_0 is continuous on $\bar{\Omega}$ then U is continuous on $[0, T[\times \bar{\Omega}$.

Proof. – For $S \in]0, T[$ we define $U^S := U_{h,g,F,\Omega_S}$. Proposition 6.4 ensures that U^S solves (CMAF) with $U^S = h$ on $\partial_0\Omega_S$. It follows from Prop. 4.1 that, for $0 < S_1 < S_2 < T$, $U^{S_1} = U^{S_2}$ on Ω_{S_1}. Letting $S \to T$ we obtain a function $V \in \mathcal{P}(\Omega_T)$ which solves (CMAF) and satisfies $V = h$ on $\partial_0\Omega_T$. Obviously $U \leq U^S$, for all $S \in]0, T[$, hence $U \leq V$. But V is also a candidate defining U, hence $V \leq U$. Therefore $V = U$ solves (CMAF) in Ω_T. Moreover, by Theorem 4.2 and Theorem 4.8, U^S is locally uniformly Lipschitz and semiconcave in $t \in]0, S[$, hence so is U.

It follows from Proposition 3.2 and Remark 3.3 that

$$(dd^c U_t)^n = e^{\partial_t U_t + F(t,\cdot,U_t)} g dV$$

for almost every $t \in]0, T[$. Since $\partial_t U$ is locally bounded and h_t is continuous on $\partial\Omega$ for all $t \in]0, T[$, [25] ensures that U_t is continuous on $\bar{\Omega}$ for almost all $t \in]0, T[$. Since U is locally uniformly Lipschitz in t we infer that U is continuous in $]0, T[\times \bar{\Omega}$.

If h_0 is continuous on $\bar{\Omega}$ then Theorem 4.7 and the continuity of $U(t, \cdot)$ (for each $t \in]0, T[$ fixed) ensure that U is continuous on $[0, T[\times \bar{\Omega}$. □

6.3. Uniqueness

We have proved in Section 6.2 the existence of a pluripotential solution to (CMAF) which is locally uniformly semi-concave in t. Our next goal is to prove that this is the unique such solution :

THEOREM 6.6. – *Let $\Phi, \Psi \in \mathcal{P}(\Omega_T) \cap L^\infty(\Omega_T)$ with boundary data h_Φ, h_Ψ. Assume that*

1. *Ψ is locally uniformly semi-concave in $t \in]0, T[$;*
2. *Φ is a subsolution while Ψ is a supersolution to* (CMAF) *in Ω_T;*
3. *h_Φ satisfies* (6.4).

Then $h_\Phi \leq h_\Psi \Longrightarrow \Phi \leq \Psi$.

Here h_Φ, h_Ψ are Cauchy Dirichlet boundary data in Ω_T. In particular, $h_\Psi(t,\cdot)$ is continuous on $\partial\Omega$, and the supersolution property of Ψ implies that Ψ is continuous in $]0,T[\times\bar\Omega$ (see Theorem 6.5).

An important consequence of this comparison principle is the following uniqueness result:

COROLLARY 6.7. – *Assume that $\Phi,\Psi \in \mathcal{P}(\Omega_T)\cap L^\infty(\Omega_T)$ are two pluripotential solutions to* (CMAF) *with boundary values h satisfying* (6.4). *If Φ,Ψ are locally uniformly semi-concave in $t\in]0,T[$ then $\Phi=\Psi$ in Ω_T.*

Proof. – Let $U := U_{h,g,F,\Omega_T}$. Then Theorem 6.5 ensures that U solves (CMAF) and U,Φ,Ψ are continuous on $]0,T[\times\bar\Omega$. By definition, $\Phi,\Psi\le U$. It follows from Theorem 6.6 that $U\le\Phi,\Psi$, hence equality. □

We first establish Theorem 6.6 under extra assumptions :

LEMMA 6.8. – *With the same assumptions as in Theorem 6.6, assume moreover that Φ is $\mathcal{C}^1$ in t, continuous on* $]0,T[\times\bar\Omega$, *and Ψ is continuous on* $[0,T[\times\bar\Omega$. *Then $h_\Phi\le h_\Psi \Longrightarrow \Phi\le\Psi$.*

The first assumption (that Φ is $\mathcal{C}^1$ in t) means that $(t,z)\mapsto\partial_t\Phi(t,z)$ exists and it is continuous on $]0,T[\times\Omega$.

Proof. – We fix $S\in]0,T[$, $\varepsilon>0$ small enough, and prove that

$$\Phi\le\Psi+2\varepsilon t \text{ in } \Omega_S.$$

The function

$$[0,S]\times\bar\Omega\ni(t,z)\mapsto W(t,z):=\Phi(t,z)-\Psi(t,z)-2\varepsilon t$$

is upper semi-continuous and bounded. We are done if the maximum is attained on $\partial_0\Omega_S$. We thus assume that $\max W$ is reached at some point $(t_0,z_0)\in]0,S]\times\Omega$. We want to prove that $W(t_0,z_0)\le 0$. Assume, by contradiction that it is not the case. Then the set

$$K:=\{z\in\Omega\ ;\ W(t_0,z)=W(t_0,z_0)\}$$

is compact and the maximum principle ensures that

$$\partial_t\Phi(t_0,z)\ge\partial_t^-\Psi(t_0,z)+2\varepsilon,\ \text{for all } z\in K.$$

Since Ψ is locally uniformly semi-concave in $t\in]0,T[$ and continuous on $[0,T[\times\bar\Omega$, the left derivative $\partial_t^-\Psi(t,z)$ exists and it is upper semi-continuous in $]0,T[\times\Omega$. Hence we can find $r>0$ so small that

$$\partial_t\Phi(t_0,z)\ge\partial_t^-\Psi(t_0,z)+\varepsilon,\ \text{for all } z\in B,$$

where $B=B_r:=\{z\in\Omega\ ;\ \mathrm{dist}(z,K)<r\}$.

Since Φ is a subsolution (which is $\mathcal{C}^1$ in t) while Ψ is a supersolution to (CMAF), Proposition 3.2 and Remark 3.3 ensure that

$$(dd^c\varphi)^n\ge e^{F(t_0,z,\varphi(z))-F(t_0,z,\psi(z))+\varepsilon}(dd^c\psi)^n,$$

setting $\varphi:=\Phi(t_0,\cdot)$, $\psi:=\Psi(t_0,\cdot)$. Since φ and ψ are continuous in Ω, F is increasing in r, and $\varphi(z)\ge\psi(z)+2\varepsilon t_0$ on K, up to shrinking B we can assume that

$$(dd^c\varphi)^n\ge e^\varepsilon(dd^c\psi)^n \text{ in } B.$$

Set now $\varphi_r := \varphi + m_r$, where $m_r := \min_{\partial B}(\psi - \varphi)$. Since $\psi \geq \varphi_r$ on ∂B, the comparison principle [1] yields

$$\int_{\{\psi<\varphi_r\}\cap B} e^{\varepsilon}(dd^c\psi)^n \leq \int_{\{\psi<\varphi_r\}\cap B} (dd^c\varphi_r)^n \leq \int_{\{\psi<\varphi_r\}\cap B} (dd^c\psi)^n.$$

Therefore $(dd^c\psi)^n$ does not charge the set $\{z \in B \,;\, \psi(z) < \varphi_r(z)\}$ and the domination principle (see e.g., [13, Proposition 1.2]) yields $\varphi_r \leq \psi$ in B. In particular

$$\varphi(z_0) - \psi(z_0) + \min_{\partial B}(\psi - \varphi) = \varphi_r(z_0) - \psi(z_0) \leq 0.$$

Since $K \cap \partial B = \emptyset$, we obtain, for all $z \in \partial B$, $W(t_0, z) < W(t_0, z_0)$ hence

$$\varphi(z) - \psi(z) < \varphi(z_0) - \psi(z_0) \leq \max_{\partial B}(\varphi - \psi),$$

a contradiction. Thus $\Phi \leq \Psi + 2\varepsilon t$ and we conclude by letting $\varepsilon \to 0$. □

We next establish an estimate for supersolutions to (CMAF).

LEMMA 6.9. – *Assume* $\Psi \in \mathscr{P}(\Omega_T)$ *has boundary data* h_Ψ. *If* Ψ *is a pluripotential supersolution to* (CMAF) *then for all* $(t,z) \in \Omega_T$,

$$\Psi(t,z) \geq h_\Psi(0,z) - c(t),$$

where $c(t) > 0$ *satisfies* $\lim_{t\to 0^+} c(t) = 0$.

Proof. – Fix $0 < S < T$. For $s > 0$ small enough we set

$$\delta(s) := \sup\{|h_\Psi(\tau,z) - h_\Psi(t,z)| \,;\, z \in \partial\Omega,\ t,\tau \in [0,S],\ |t-\tau| \leq s\}. \tag{6.5}$$

Since h_Ψ is continuous on $[0,T[\times\partial\Omega$, we have $\lim_{s\to 0^+}\delta(s) = 0$.

Fix $s \in]0,(T-S)/2[$. We are going to prove that

$$\Psi(s,z) \geq h_\Psi(0,z) - \delta(s) + s(\rho(z) - C) + n(s\log(s/T) - s),$$

where ρ is defined in (0.7) and C is a uniform constant.

Fix $\varepsilon \in]0,s]$ and let h^ε denote the restriction of $(t,z) \mapsto \Psi(t+\varepsilon,z)$ on $\partial_0\Omega_s$. Then h^ε is a continuous boundary data on Ω_s. Set, for $(t,z) \in \Omega_s$,

$$u^\varepsilon(t,z) := \Psi(\varepsilon,z) - \delta(s) + t(\rho(z) - C_1) + n(t\log(t/T) - t),$$

where C_1 is a positive constant. By definition of $\delta(s)$ we have

$$u^\varepsilon(t,z) \leq \Psi(t+\varepsilon,z) = h^\varepsilon(t,z), \text{ for all } (t,z) \in \partial_0\Omega_s.$$

Arguing as in the proof of Lemma 3.8 we see that for $C_1 > 0$ big enough (depending on M_F), u^ε is a pluripotential subsolution to (CMAF) in Ω_s. Moreover, u_ε is of class $\mathcal{C}^1$ in $t \in [0,s]$. On the other hand, a direct computation shows that, for $C_2 > 0$ large enough and under control (depending on κ_F), the function

$$[0,s]\times\Omega \ni (t,z) \mapsto w^\varepsilon(t,z) := \Psi(t+\varepsilon,z) + C_2\varepsilon t$$

is a pluripotential supersolution to (CMAF) and $w^\varepsilon \geq h^\varepsilon$ on $\partial_0\Omega_s$. By assumption on Ψ, w^ε is continuous on $[0,s]\times\bar{\Omega}$. It thus follows from Lemma 6.8 that $w^\varepsilon \geq u^\varepsilon$ on $[0,s]\times\Omega$. We conclude by letting $\varepsilon \to 0$. □

We next remove the continuity assumption on Ψ in Lemma 6.8.

LEMMA 6.10. – *With the same assumptions as in Theorem 6.6, assume moreover that Φ is $\mathcal{C}^1$ in t and continuous on $]0,T[\times\bar{\Omega}$. Then*

$$h_\Phi \le h_\Psi \Longrightarrow \Phi \le \Psi.$$

Proof. – Since h_Ψ is continuous on $[0,T[\times\partial\Omega$, the proof of Theorem 6.5 shows that Ψ is continuous in $]0,T[\times\bar{\Omega}$, but it may not be continuous on $[0,T[\times\bar{\Omega}$. We use an idea in [6], exploiting the regularity of Ψ at positive times close to zero. We fix $S\in]0,T[$ and prove that $\Phi\le\Psi$ on Ω_S.

Fix $s\in]0,(T-S)/2[$ and set, for $(t,z)\in[0,S]\times\bar{\Omega}$,

$$v(t,z) := \Psi(t+s,z)+c(s)+\delta(s)+Ast,$$

where $\delta(s)$ is defined in (6.5), $A>0$ is a constant, and $c(s)>0$ is as in Lemma 6.9 (which ensures $\Psi(s,z)\ge h_\Psi(0,z)-c(s)$).

From the definition of $\delta(s)$ it follows that $v(t,z)\ge\Psi(t,z)=h_\Psi(t,z)$ on $[0,S]\times\partial\Omega$. For $A>0$ large enough (depending on κ_F), a direct computation shows that v is a supersolution to (CMAF). Since v is continuous on $[0,S]\times\bar{\Omega}$, Lemma 6.8 then applies and yields $\Phi(t,z)\le v(t,z)$ on $[0,S]\times\Omega$. We conclude by letting $s\to 0$. □

We are now ready to prove the comparison principle.

Proof of Theorem 6.6. – We can assume without loss of generality that $\Phi=U_{h_\Phi,g,F}$. From assumption (3.) and Theorem 6.5 we deduce that $U_{h_\Phi,g,F}$ is continuous on $]0,T[\times\bar{\Omega}$. We would like to apply Lemma 6.10 but Φ is a priori not $\mathcal{C}^1$ in t. We are going to regularize Φ by taking convolution in t.

Fix $0<S<T$. For $s>0$ near 1 we set, for $(t,z)\in\Omega_S$,

$$W^s(t,z) := s^{-1}\Phi(st,z)-C|s-1|(t+1).$$

If $C>0$ is large enough, the proof of Theorem 4.2 ensures that $W^s\in\mathcal{S}_{h_\Phi,g,F}(\Omega_S)$. Let $\{\chi_\varepsilon\}_{\varepsilon>0}$ be a family of smoothing kernels in $\mathbb{R}$ approximating the Dirac mass δ_1. For $\varepsilon>0$ small enough we define

$$\Phi^\varepsilon(t,z) := \int_{\mathbb{R}} W^s(t,z)\chi_\varepsilon(s)ds. \tag{6.6}$$

We are going to prove that Φ^ε (or $\Phi^\varepsilon-O(\varepsilon)$) is again a subsolution and use the previous step to conclude.

Let $\mathcal{H}$ denote the space of Hermitian positive definite matrices H that are normalized by $\det H=1$, and let Δ_H denote the Laplace operator

$$\Delta_H\varphi := \frac{1}{n}\sum_{j,k=1}^n h_{jk}\frac{\partial^2\varphi}{\partial z_j\partial\bar{z}_k}.$$

Fix $H\in\mathcal{H}$. Since $W^s\in\mathcal{S}_{h_\Phi,g,F}(\Omega_T)$, Proposition 3.2 and [14, Main Theorem] yield

$$\Delta_H W^s(t,z)\ge\exp\left(\frac{\partial_t W^s(t,z)+F(t,z,W^s(t,z))}{n}\right)g(z)^{1/n}.$$

By definition of Φ^ε we obtain, using the convexity of the exponential,

$$\begin{aligned}\Delta_H \Phi^\varepsilon(t,z) &= \int_{\mathbb{R}} \Delta_H W^s(t,z)\chi_\varepsilon(s)ds \\ &\geq g(z)^{1/n} \int_{\mathbb{R}} \exp\left(\frac{\partial_t W^s(t,z) + F(t,z,W^s(t,z))}{n}\right)\chi_\varepsilon(s)ds \\ &\geq g(z)^{1/n} \exp\left(\frac{1}{n}\left(\int_{\mathbb{R}} \left(\partial_t W^s(t,z) + F(t,z,W^s(t,z))\right)\chi_\varepsilon(s)ds\right)\right).\end{aligned}$$

Step 1. To simplify we first treat the case when F is convex in r. – Thus

$$\begin{aligned}\Delta_H \Phi^\varepsilon(t,z) &= \int_{\mathbb{R}} \Delta_H W^s(t,z)\chi_\varepsilon(s)ds \\ &\geq g(z)^{1/n} \exp\left(\frac{1}{n}\left(\partial_t \Phi^\varepsilon(t,z) + F\left(t,z,\int_{\mathbb{R}} W^s(t,z)\chi_\varepsilon(s)ds\right)\right)\right) \\ &= g(z)^{1/n} \exp\left(\frac{1}{n}\left(\partial_t \Phi^\varepsilon(t,z) + F(t,z,\Phi^\varepsilon(t,z))\right)\right).\end{aligned}$$

Using Proposition 3.2 and [14, Main Theorem] again, we infer that Φ^ε is a subsolution to (CMAF) in Ω_S.

We now check that $(\Phi^\varepsilon)^* - O(\varepsilon) \leq h_\Phi$ on $\partial_0\Omega_S$. Indeed, for $z \in \partial\Omega$ we have $W^s(t,z) \leq h_\Phi(t,z)$, for all s, thus $\Phi^\varepsilon(t,z) \leq h_\Phi(t,z)$ for all $(t,z) \in [0,S] \times \partial\Omega$. It remains to check that $(\Phi^\varepsilon)^*(0,z) \leq h_\Phi(0,z)$, for all $z \in \Omega$. It follows from Theorem 6.5 that U_{h_Φ,g,F,Ω_T} has boundary value h_Φ, hence, for C large enough

$$\lim_{t\to 0} W^s(t,z) \leq h_\Phi(0,z) - C|s-1|, \text{ for all } z \in \bar{\Omega}.$$

From the definition of Φ^ε in (6.6) it follows that

$$\lim_{t\to 0} \Phi^\varepsilon(t,z) \leq h_\Phi(0,z), \ \forall z \in \bar{\Omega}.$$

Hence $\Phi^\varepsilon - O(\varepsilon)t \in \mathcal{S}_{h_\Phi,g,F}(\Omega_S)$. Moreover, Φ^ε is of class $\mathcal{C}^1$ in $t \in]0,S[$ and Φ^ε converges pointwise to Φ as $\varepsilon \to 0$. Using Lemma 6.10 we obtain $\Phi^\varepsilon \leq \Psi$ in Ω_S. The conclusion follows by letting $\varepsilon \to 0$.

Step 2. We now treat the case when F is merely uniformly semi-convex in r. – It follows from (4.2) and Theorem 4.2 that the functions $s \mapsto W^s(t,z)$, $(t,z) \in \Omega_S$, are uniformly Lipschitz in $[1/2,3/2]$. Note also that W^s and Φ are uniformly bounded in Ω_S. Thus for all $(t,z) \in \Omega_S$, $s \in [1/2,3/2]$,

$$|W^s(t,z) - \Phi(t,z)| \leq C|s-1|, \text{ and } |(W^s(t,z))^2 - \Phi^2(t,z)| \leq C|s-1|,$$

for some uniform constant C, hence

$$\int_{\mathbb{R}} W^s(t,z)\chi_\varepsilon(s)ds = \Phi(t,z) + O(\varepsilon), \text{ and } \int_{\mathbb{R}} (W^s)^2(t,z)\chi_\varepsilon(s)ds = \Phi^2(t,z) + O(\varepsilon).$$

We thus have

$$\int_{\mathbb{R}} (W^s)^2(t,z)\chi_\varepsilon(s)ds - \left(\int_{\mathbb{R}} W^s(t,z)\chi_\varepsilon(s)ds\right)^2 = O(\varepsilon). \tag{6.7}$$

Recall (assumption (0.5)) that the function $r \mapsto F(t,z,r) + C_F r^2$ is convex in a large interval $J \Subset \mathbb{R}$, for fixed $(t,z) \in \Omega_S$. Jensen's inequality yields

$$\int_{\mathbb{R}} \big(F(t,z,W^s(t,z)) + C_F (W^s(t,z))^2\big)\chi_\varepsilon(s)ds \geq F\left(t,z,\int_{\mathbb{R}} W^s(t,z)\chi_\varepsilon(s)ds\right) + C_F\left(\int_{\mathbb{R}} W^s(t,z)\chi_\varepsilon(s)ds\right)^2.$$

Using this and (6.7) we obtain

$$\int_{\mathbb{R}} F(t,z,W^s(t,z))\chi_\varepsilon(s)ds - F\left(t,z,\int_{\mathbb{R}} W^s(t,z)\chi_\varepsilon(s)ds\right) \geq O(\varepsilon).$$

We repeat the previous step to conclude that $\Phi^\varepsilon - O(\varepsilon)t \in \mathcal{S}_{h_\Phi,g,F}(\Omega_S)$. □

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APPLICATIONS OF THE MORAVA K-THEORY TO ALGEBRAIC GROUPS

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Abstract. – In this article we discuss an approach to cohomological invariants of algebraic groups based on the Morava K-theories.

We show that the second Morava K-theory detects the triviality of the Rost invariant and, more generally, relate the triviality of cohomological invariants and the splitting of Morava motives.

We compute the Morava K-theory of generalized Rost motives and of some affine varieties and characterize the powers of the fundamental ideal of the Witt ring with the help of the Morava K-theory. Besides, we obtain new estimates on torsion in Chow groups of quadrics and investigate torsion in Chow groups of $K(n)$-split varieties. An important role in the proofs is played by the gamma filtration on Morava K-theories, which gives a conceptual explanation of the nature of the torsion.

Furthermore, we show that under some conditions if the $K(n)$-motive of a smooth projective variety splits, then its $K(m)$-motive splits for all $m \leq n$.

Résumé. – Dans cet article nous présentons une approche des invariants cohomologiques des groupes algébriques basée sur les K-théories de Morava.

Nous montrons que la deuxième K-théorie de Morava détecte la trivialité de l'invariant de Rost et, plus généralement, établissons un rapport entre la trivialité des invariants cohomologiques et le déploiement des motifs de Morava.

Nous calculons la K-théorie de Morava des motifs généralisés de Rost et de quelques variétés affines et caractérisons les puissances de l'idéal fondamental de l'anneau de Witt à l'aide de la K-théorie de Morava. Par ailleurs, nous obtenons de nouvelles estimations de la torsion dans les groupes de Chow des quadriques et étudions la torsion dans les groupes de Chow des variétés $K(n)$-déployées. La gamma-filtration de K-théorie de Morava joue un rôle important dans les preuves, et fournit une explication conceptuelle de la nature de la torsion.

De plus, nous montrons que sous certaines conditions, si le $K(n)$-motif d'une variété projective lisse est déployé, alors son $K(m)$-motif est déployé pour tout $m \leq n$.

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1. Introduction

The present article is devoted to applications of the Morava K-theory to cohomological invariants of algebraic groups and to computations of the Chow groups of quadrics.

1.1. Cohomological invariants

In his celebrated article on irreducible representations [55] Jacques Tits introduced the notion of a Tits algebra, which is an example of cohomological invariants of algebraic groups of degree 2. This invariant of a linear algebraic group G plays a crucial role in the computation of the K-theory of twisted flag varieties by Panin [39] and in the index reduction formulas by Merkurjev, Panin and Wadsworth [34]. It has important applications to the classification of linear algebraic groups and to the study of associated homogeneous varieties.

The idea to use cohomological invariants in the classification of algebraic groups goes back to Jean-Pierre Serre. In particular, Serre conjectured the existence of an invariant of degree 3 for groups of type F_4 and E_8. This invariant was later constructed by Markus Rost for all G-torsors, where G is a simple simply-connected algebraic group, and is now called the Rost invariant (see [9]).

Moreover, the Serre-Rost conjecture for groups of type F_4 says that the map

$$H^1_{\mathrm{et}}(F, \mathrm{F}_4) \hookrightarrow H^3_{\mathrm{et}}(F, \mathbb{Z}/2) \oplus H^3_{\mathrm{et}}(F, \mathbb{Z}/3) \oplus H^5_{\mathrm{et}}(F, \mathbb{Z}/2)$$

induced by the invariants f_3, g_3 and f_5 described in [23, §40] (f_3 and g_3 are the modulo 2 and modulo 3 components of the Rost invariant), is injective. The validity of the Serre-Rost conjecture would imply that one can exchange the study of the *set* $H^1_{\mathrm{et}}(F, \mathrm{F}_4)$ of isomorphism classes of groups of type F_4 over F (equivalently of isomorphism classes of F_4-torsors or of isomorphism classes of Albert algebras) by the study of the *abelian group* $H^3_{\mathrm{et}}(F, \mathbb{Z}/2) \oplus H^3_{\mathrm{et}}(F, \mathbb{Z}/3) \oplus H^5_{\mathrm{et}}(F, \mathbb{Z}/2)$.

In the same spirit one can formulate the Serre Conjecture II, saying in particular that $H^1_{\mathrm{et}}(F, \mathrm{E}_8) = 1$ if the field F has cohomological dimension 2. Namely, for such fields $H^n_{\mathrm{et}}(F, M) = 0$ for all $n \geq 3$ and all torsion modules M. In particular, for groups over F there are no invariants of degree ≥ 3, and the Serre Conjecture II predicts that the groups of type E_8 over F themselves are split.

Furthermore, the Milnor conjecture on quadratic forms (proven by Orlov, Vishik and Voevodsky) together with the Milnor conjecture on the étale cohomology (proven by Voevodsky) provides a classification of quadratic forms over fields in terms of the Galois cohomology, i.e., in terms of cohomological invariants.

In the present article we will relate the Morava K-theory with some cohomological invariants of algebraic groups.

1.2. Morava K-theory and Morava motives

Let n be a positive integer and let p be a prime. The Morava K-theory $K(n)^*$ is a free oriented cohomology theory in the sense of Levine-Morel [28] whose coefficient ring is $\mathbb{Z}_{(p)}$, whose formal group law modulo p has height n, and the logarithm is of the type

$$\log_{K(n)}(x) = x + \frac{a_1}{p} x^{p^n} + \frac{a_2}{p^2} x^{p^{2n}} + \cdots$$

with $a_i \in \mathbb{Z}_{(p)}^{\times}$. If $n = 1$ and all a_i are equal to 1, then the theory $K(1)^*$ is isomorphic to Grothendieck's $K^0 \otimes \mathbb{Z}_{(p)}$ as a presheaf of rings. Moreover, there is some kind of analogy between Morava K-theory in general and K^0.

More conceptually, algebraic cobordism of Levine-Morel can be considered as a functor to the category of graded comodules over the Hopf algebroid $(\mathbb{L}, \mathbb{L}B)$, where $\mathbb{L}$ is the Lazard ring and $\mathbb{L}B = \mathbb{L}[b_1, b_2, \ldots]$. This Hopf algebroid parametrizes the groupoid of formal group laws with strict isomorphisms between them, and the category of comodules over it can be identified with the category of quasi-coherent sheaves over the stack of formal groups $\mathcal{M}_{\mathrm{fg}}$. This stack modulo p has a descending filtration by closed substacks $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ which classify the formal group laws of height $\geq n$. Moreover, $\mathcal{M}_{\mathrm{fg}}^{\geq n} \setminus \mathcal{M}_{\mathrm{fg}}^{\geq n+1}$ has an essentially unique geometric point which corresponds to the Morava K-theory $K(n)^* \otimes \overline{\mathbb{F}}_p$. This chromatic picture puts $K(n)^*$ into an intermediate position between K^0 and CH^*.

We remark also that Levine and Tripathi construct in [29] a higher Morava K-theory in algebraic geometry.

1.3. Morava K-theories, split motives and vanishing of cohomological invariants

There are three different types of results in this article which fit into the following guiding principle. The leading idea of this principle has been probably well understood already by Voevodsky, since he considered the Morava K-theory in his program on the proof of the Bloch-Kato conjecture in [63].

GUIDING PRINCIPLE. – *Let X be a projective homogeneous variety, let p be a prime number and let $K(n)^*$ denote the corresponding Morava K-theory.*

Then vanishing of cohomological invariants of X with p-torsion coefficients in degrees no greater than $n + 1$ should correspond to the splitting of the $K(n)^*$-motive of X.

First of all, due to the Milnor conjecture the associated graded ring of the Witt ring $W(F)$ of a field F of characteristic not 2 is canonically isomorphic to the étale cohomology of the base field with $\mathbb{Z}/2$-coefficients: $H^n_{\mathrm{et}}(F, \mathbb{Z}/2) \simeq I^n/I^{n+1}$, where I denotes the fundamental ideal of $W(F)$. Therefore, the projective quadric which corresponds to a quadratic form $q \in I^n$ has a canonical cohomological invariant of degree n. The guiding principle suggests that the $K(n)^*$-motive of an even-dimensional projective quadric is split if and only if the class of the corresponding quadratic form in the Witt ring lies in the ideal I^{n+2}. Indeed, we prove this statement in Proposition 6.18.

Secondly, we relate cohomological invariants of simple algebraic groups to Morava K-theories. We show in Section 9 that for a simple simply-connected group G with trivial Tits algebras the Morava K-theory $K(2)^*$ detects the triviality of the Rost invariant of G. Note that in a similar spirit Panin showed in [39] that the Grothendieck's K^0 detects the triviality of Tits algebras. Moreover, for a group G of type E_8 the Morava K-theory $K(4)^*$ for $p = 2$ detects the splitting of the variety of Borel subgroups of G over a field extension of odd degree (Theorem 9.1). All these results agree with the guiding principle.

Thirdly, we relate the property of being split with respect to Morava K-theories $K(n)^*$ for different n. Namely, we prove in Proposition 7.10 that if a smooth projective geometrically cellular variety X over a field F of characteristic 0 satisfies the Rost nilpotence principle for

Morava K-theories and has a split $K(n)^*$-motive, then it has a split $K(m)^*$-motive for all $m \leq n$. In particular, Morava motives provide a linearly ordered series of obstructions for a projective homogeneous variety to be isotropic over a base field extension of a prime-to-p degree.

1.4. Operations in Morava K-theories and applications to Chow groups

The study of cohomological invariants of algebraic groups is partially motivated by the interest in Chow groups of torsors. Whenever some cohomological invariants vanish, one may ask whether this yields any restrictions on the structure of Chow groups, e.g., existence, order or cardinality of torsion in certain codimensions. We approach this question by studying projective homogeneous (or, more generally, geometrically cellular) varieties X for which the $K(n)^*$-motive is split. In order to obtain information about Chow groups from Morava K-theories we use operations.

The first author constructed in [49] and [51] generators of all (not necessarily additive) operations from the Morava K-theory to $\mathrm{CH}^* \otimes \mathbb{Z}_{(p)}$ and from the Morava K-theory to itself. The latter allows one to define the gamma filtration on the Morava K-theory, and it turns out that its i-th graded factor maps surjectively onto $\mathrm{CH}^i \otimes \mathbb{Z}_{(p)}$ for all $i \leq p^n$. These operations and their various properties are constructed using the classification of operations given in a series of articles by Vishik (see [60], [61]).

Let X be a smooth geometrically cellular variety such that the pullback map from $K(n)^*(X)$ to $K(n)^*(X_E)$ is an isomorphism, where $X_E = X \times_F E$ is the base change to a field E for which X_E becomes cellular. The operations above as well as symmetric operations of Vishik allow us to show that there is no p-torsion in Chow groups of X in codimensions up to $\frac{p^n-1}{p-1}$ (Theorem 7.19). Moreover, we prove that p-torsion is finitely generated in Chow groups of codimension up to p^n, and we provide a combinatorial method to estimate this torsion (Theorem 7.23).

For quadratic forms from the ideal I^{m+2} of the Witt ring of a field F of characteristic zero the $K(m)^*$-motive of the corresponding quadric is split as mentioned above. Thus, we obtain that there is no torsion in Chow groups of codimensions less than 2^m and we also calculate uniform finite upper bounds on the torsion in CH^{2^m} which do not depend on the quadric (see Theorem 8.14). In this way Morava K-theory provides a conceptual explanation of the nature of this torsion.

These results fit well in the quite established history of estimates on torsion of quadrics obtained among others by Karpenko, Merkurjev and Vishik. In particular, Karpenko conjectured in [16, Conjecture 0.1] that for every integer l the Chow group CH^l of an n-dimensional quadric over F is torsion-free whenever n is bigger than some constant which depends only on l. This was confirmed only for $l \leq 4$. Recall that by the Arason-Pfister Hauptsatz every anisotropic non-zero quadratic form from I^m has dimension at least 2^m and therefore, the absence of torsion in Chow groups of small codimensions of corresponding quadrics can be considered as an instance of the Karpenko conjecture. Note also that there are examples of quadrics from I^{m+2} having non-trivial torsion in CH^{2^m}.

Finally, we discuss an approach to cohomological invariants which uses an exact sequence (3.6) of Voevodsky (see below). This exact sequence involves motivic cohomology of some simplicial varieties. For example, this sequence was used in [52] to construct an invariant of degree 5 modulo 2 for groups of type E_8 with trivial Rost invariant and to solve a problem posed by Serre.

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2. Definitions and notation

In the present article we assume that F is a field of characteristic 0. By F_{sep} we denote a separable closure of F.

Let G be a semisimple linear algebraic group over a field F (see [54], [23]). A G-torsor over F is an algebraic variety P equipped with an action of G such that $P(F_{\mathrm{sep}}) \neq \emptyset$ and the action of $G(F_{\mathrm{sep}})$ on $P(F_{\mathrm{sep}})$ is simply transitive.

The set of isomorphism classes of G-torsors over F is a pointed set (with the base point given by the trivial G-torsor G) which is in natural one-to-one correspondence with the (non-abelian) Galois cohomology set $H^1_{\mathrm{et}}(F, G)$.

Let A be some algebraic structure over F (e.g., an algebra or quadratic space) such that $\mathrm{Aut}(A)$ is an algebraic group over F. Then an algebraic structure B is called a *twisted form* of A, if over a separable closure of F the structures A and B are isomorphic. There is a natural bijection between $H^1_{\mathrm{et}}(F, \mathrm{Aut}(A))$ and the set of isomorphism classes of the twisted forms of A.

For example, if A is an octonion algebra over F, then $\mathrm{Aut}(A)$ is a group of type G_2 and $H^1_{\mathrm{et}}(F, \mathrm{Aut}(A))$ is in one-to-one correspondence with the twisted forms of A, i.e., with the octonion algebras over F (since any two octonion algebras over F are isomorphic over a separable closure of F and since any algebra, which is isomorphic to an octonion algebra over a separable closure of F, is an octonion algebra).

By $\mathbb{Q}/\mathbb{Z}(n)$ we denote the Galois-module $\operatorname{colim} \mu_l^{\otimes n}$ taken over all l (see [23, p. 431]).

In the article we use notions from the theory of quadratic forms over fields (e.g., Pfister-forms, Witt-ring). We follow [23], [24], and [6]. Further, we use the notion of motives; see [31], [6].

3. Geometric constructions of cohomological invariants

First, we describe several geometric constructions of cohomological invariants of torsors of degree 2 and 3.

Let G be a semisimple algebraic group over a field F. In general, a cohomological invariant of G-torsors of degree n with values in a Galois-module M is a transformation of functors $H^1_{\mathrm{et}}(-,G) \to H^n_{\mathrm{et}}(-,M)$ from the category of field extensions of F to the category of pointed sets (see [23, 31.B]).

3.1. Tits algebras and the Picard group

In his celebrated article [55] Jacques Tits introduced invariants of degree 2, called nowadays the *Tits algebras*.

There exists a construction of Tits algebras based on the Hochschild-Serre spectral sequence. For a smooth variety X over F one has

$$H^p(\Gamma, H^q_{\mathrm{et}}(X_{\mathrm{sep}}, \mathcal{G})) \Rightarrow H^{p+q}_{\mathrm{et}}(X, \mathcal{G}),$$

where Γ is the absolute Galois group, $X_{\mathrm{sep}} = X \times F_{\mathrm{sep}}$ and $\mathcal{G}$ is an étale sheaf. The first terms of the induced exact sequence are

$$0 \to H^1(\Gamma, H^0_{\mathrm{et}}(X_{\mathrm{sep}}, \mathcal{G})) \to H^1_{\mathrm{et}}(X, \mathcal{G}) \to H^0(\Gamma, H^1_{\mathrm{et}}(X_{\mathrm{sep}}, \mathcal{G})) \to H^2(\Gamma, H^0_{\mathrm{et}}(X_{\mathrm{sep}}, \mathcal{G})).$$

Let $\mathcal{G} = \mathbb{G}_m$ and let X be a smooth projective geometrically irreducible variety. Then

$$H^1(\Gamma, H^0_{\mathrm{et}}(X_{\mathrm{sep}}, \mathbb{G}_m)) = H^1(\Gamma, F^\times_{\mathrm{sep}}) = 0,$$

by Hilbert's Theorem 90, $H^1_{\mathrm{et}}(X, \mathbb{G}_m) = \mathrm{Pic}(X)$, $H^0(\Gamma, H^1_{\mathrm{et}}(X_{\mathrm{sep}}, \mathbb{G}_m)) = (\mathrm{Pic}\, X_{\mathrm{sep}})^\Gamma$, and $H^2(\Gamma, H^0_{\mathrm{et}}(X_{\mathrm{sep}}, \mathbb{G}_m)) = H^2(\Gamma, F^\times_{\mathrm{sep}}) = \mathrm{Br}(F)$. Thus, we obtain an exact sequence

$$0 \to \mathrm{Pic}\, X \to (\mathrm{Pic}\, X_{\mathrm{sep}})^\Gamma \xrightarrow{f} \mathrm{Br}(F). \tag{3.2}$$

The map $\mathrm{Pic}\, X \to (\mathrm{Pic}\, X_{\mathrm{sep}})^\Gamma$ is the restriction map and the homomorphism

$$(\mathrm{Pic}\, X_{\mathrm{sep}})^\Gamma \xrightarrow{f} \mathrm{Br}(F)$$

was described by Merkurjev and Tignol in [35, Section 2]. If X is the variety of Borel subgroups of a semisimple algebraic group G, then the Picard group of X_{sep} can be identified with the free abelian group with basis $\omega_1, \dots, \omega_n$ consisting of the fundamental weights, i.e., $\mathrm{Pic}\, X_{\mathrm{sep}} = \Lambda$, where Λ denotes the weight lattice. If ω_i is Γ-invariant (e.g., if G is of inner type), then $f(\omega_i) = [A_i]$ is the Brauer class of the Tits algebra of G corresponding to the (fundamental) representation with the highest weight ω_i (see [35] for a general description of the homomorphism f).

Moreover, one can continue the exact sequence (3.2), namely, the sequence

$$0 \to \mathrm{Pic}\, X \to (\mathrm{Pic}\, X_{\mathrm{sep}})^\Gamma \xrightarrow{f} \mathrm{Br}(F) \to \mathrm{Br}(F(X)) \tag{3.3}$$

is exact, where the last map is the restriction homomorphism (see [35]).

3.4. Tits algebras and K^0

There is another interpretation of the Tits algebras related to Grothendieck's K^0 functor. Let G be a semisimple algebraic group over F of inner type and let X be the variety of Borel subgroups of G. By Panin [39] the K^0-motive of X is isomorphic to a direct sum of $|W|$ motives, where W denotes the Weyl group of G. Denote these motives by L_w, $w \in W$.

For $w \in W$ consider

$$\rho_w = \sum_{\{\alpha_k \in \Pi | w^{-1}(\alpha_k) \in \Phi^-\}} w^{-1}(\omega_k) \in \Lambda,$$

where Π is the set of simple roots, Φ^- is the set of negative roots, and Λ is the weight lattice.

Let Λ_r be the root lattice and

$$\beta \colon \Lambda/\Lambda_r \to \mathrm{Br}(F)$$

be the Tits homomorphism, which sends a fundamental weight ω_i to $[A_i]$ (see [55]). In particular, the homomorphism β is essentially the homomorphism f from Section 3.1. Then over a splitting field K of G, the motive $(L_w)_K$ is isomorphic to a Tate motive and the restriction homomorphism

$$K^0(L_w) \to K^0((L_w)_K) = \mathbb{Z}$$

is an injection $\mathbb{Z} \to \mathbb{Z}$ given by the multiplication by $\operatorname{ind} A_w$, where $[A_w] = \beta(\rho_w)$. In particular, different motives L_w can be parametrized by the Tits algebras.

Moreover, if all Tits algebras of G are split, then the K^0-motive of X is a direct sum of Tate motives over F.

3.5. Tits algebras and simplicial varieties

Let Y be a smooth irreducible variety over F. Consider the Čech simplicial scheme $\mathcal{X}_Y$ associated with Y, i.e., the simplicial scheme

$$Y \leftleftarrows Y \times Y \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}}\; Y \times Y \times Y \cdots$$

Then for all $n \geq 2$ there is a long exact sequence of cohomology groups (see [48, Corollary 2.2] and [65, Proof of Lemma 6.5]):

$$0 \to H^{n,n-1}_{\mathcal{M}}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z}) \xrightarrow{g} H^n_{\text{et}}(F, \mathbb{Q}/\mathbb{Z}(n-1)) \to H^n_{\text{et}}(F(Y), \mathbb{Q}/\mathbb{Z}(n-1)), \tag{3.6}$$

where $H^{n,n-1}_{\mathcal{M}}$ is the motivic cohomology and the homomorphism g is induced by the change of topology from Nisnevich to étale (note that by [64, Lemma 7.3] $\mathcal{X}_Y$ is contractible in the étale topology).

Let $n = 2$ and let Y be the variety of Borel subgroups of a semisimple algebraic group G of inner type. Then $H^2_{\text{et}}(F, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(F)$ and we have a long exact sequence

$$0 \to H^{2,1}_{\mathcal{M}}(\mathcal{X}_Y) \xrightarrow{g} \mathrm{Br}(F) \to \mathrm{Br}(F(Y)).$$

Thus, combining this exact sequence with exact sequence (3.3) and using explicit description of the homomorphism f from Section 3.1, we obtain that $H^{2,1}_{\mathcal{M}}(\mathcal{X}_Y) = \Lambda/\Lambda'$, where Λ' denotes the kernel of f. Note also that $\Lambda_r \subset \Lambda'$. Thus, the Tits homomorphism β factors through $H^{2,1}_{\mathcal{M}}(\mathcal{X}_Y)$ by means of the homomorphism g. This gives one more interpretation of the Tits algebras via a change of topology.

3.7. Rost invariant

If G is a simple simply connected algebraic group, then there exists an invariant

$$H^1_{\mathrm{et}}(-,G) \to H^3_{\mathrm{et}}(-,\mathbb{Q}/\mathbb{Z}(2))$$

of degree 3 of G-torsors which is called the *Rost invariant* (see [9]). In a particular case when G is the spinor group, this invariant is called the *Arason invariant*.

If G is of inner type, the Rost invariant can be constructed as follows. Let Y be a G-torsor. Then there is a long exact sequence (see [9, Section 9])

$$(3.8)\qquad \begin{aligned} &0 \to A^1(Y,K_2) \to A^1(Y_{\mathrm{sep}},K_2)^{\Gamma} \\ &\qquad \xrightarrow{h} \operatorname{Ker}\big(H^3_{\mathrm{et}}(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{\mathrm{et}}(F(Y),\mathbb{Q}/\mathbb{Z}(2))\big) \to \mathrm{CH}^2(Y),\end{aligned}$$

where $A^1(-,K_2)$ is the K-cohomology group (see [46], [9, Section 4]), Γ is the absolute Galois group, and $Y_{\mathrm{sep}} = Y \times_F F_{\mathrm{sep}}$. Moreover, $A^1(Y_{\mathrm{sep}},K_2)^{\Gamma} = \mathbb{Z}$ and $\mathrm{CH}^2(Y) = 0$. The Rost invariant of Y is the image of $1 \in A^1(Y_{\mathrm{sep}},K_2)^{\Gamma}$ under the homomorphism h. We remark that sequence (3.8) for the Rost invariant is analogous to the sequence (3.3) for the Tits algebras arising from the Hochschild-Serre spectral sequence.

We remark also that if G is a group of inner type with trivial Tits algebras (simply-connected or not), then there is a well-defined Rost invariant of G itself (not of G-torsors); see [12, Section 2].

4. Oriented cohomology theories and the Morava K-theory

In this section we will introduce a cohomology theory—the Morava K-theory. We will prove later that it detects the triviality of some cohomological invariants (in particular, of the Rost invariant) of algebraic groups.

4.1. Characteristic numbers

Let X be a smooth projective irreducible variety over a field F. Given a partition $J = (l_1,\dots,l_r)$ of length $r \geq 0$ with $l_1 \geq l_2 \geq \cdots \geq l_r > 0$ one can associate with it a *characteristic class*

$$c_J(X) \in \mathrm{CH}^{|J|}(X) \qquad (|J| = \sum_{i\geq 1} l_i)$$

of X as follows. Let $P_J(x_1,\dots,x_r)$ be the smallest symmetric polynomial (i.e., with a minimal number of non-zero coefficients) containing the monomial $x_1^{l_1}\cdots x_r^{l_r}$. We can express P_J as a polynomial on the standard symmetric functions $\sigma_1,\dots,\sigma_r$ as

$$P_J(x_1,\dots,x_r) = Q_J(\sigma_1,\dots,\sigma_r)$$

for some polynomial Q_J. Let $c_i = c_i(-T_X)$ denote the i-th Chern class of the virtual normal bundle of X. Then

$$c_J(X) = Q_J(c_1,\dots,c_r).$$

For $|J| = \dim(X)$, the degrees of the characteristic classes are called the *characteristic numbers*.

If $J = (1, \dots, 1)$ (i times), then $c_J(X) = c_i(-T_X)$ is the usual Chern class. If $\dim X = p^n - 1$ and $J = (p^n - 1)$ for some prime number p, we write $c_J(X) = S_{p^n-1}(X)$. The degree of the class $S_{\dim X}(X)$ is always divisible by p and we set

$$s_{\dim X}(X) = \frac{\deg S_{\dim X}(X)}{p}$$

and call it the *Milnor number* of X (see [28, Section 4.4.4], [52, Section 2]).

Definition 4.2. – Let p be a prime. A smooth projective variety X is called a v_n-variety if $\dim X = p^n - 1$, all characteristic numbers of X are divisible by p and $s_{\dim X}(X) \neq 0 \bmod p$.

4.3. Oriented cohomology theories and Borel-Moore homology theories

In this article we consider oriented cohomology theories A^* in the sense of Levine-Morel (see [28, Definition 1.1.2]). By a variety we always mean a quasi-projective variety.

For a smooth variety X over F with the irreducible components $X_1, \dots, X_l$ we set $A_*(X) := \bigoplus_{i=1}^{l} A^{\dim X_i - *}(X_i)$. Then the assignment $X \mapsto A_*(X)$ defines an oriented Borel-Moore homology theory in the sense of Levine-Morel (see [28, Definition 5.1.3]). Moreover, by [28, Proposition 5.2.1] this gives a one-to-one correspondence between oriented cohomology theories and oriented Borel-Moore homology theories on the category of smooth varieties over F.

Given an oriented Borel-Moore homology theory on the category of smooth varieties over F we extend it to all separated schemes of finite type over F via

$$A_*(Y) := \operatorname{colim}_{V \to Y} A_*(V),$$

where the colimit runs over all projective morphisms $V \to Y$, where V are smooth varieties over F, and with push-forward maps as transition maps.

For an oriented Borel-Moore homology theory A_* we say that it satisfies the *localization axiom*, if for every quasi-projective F-scheme X and a closed F-embedding $j\colon Z \to X$ with the open complement $i\colon U \to X$ the sequence

$$A_*(Z) \xrightarrow{j_*} A_*(X) \xrightarrow{i^*} A_*(U) \to 0$$

is exact.

4.4. Free theories

Consider the algebraic cobordism Ω^* of Levine-Morel (see [28]). By [28, Theorem 1.2.6] the algebraic cobordism is a universal oriented cohomology theory, i.e., there is a (unique) morphism of theories $\Omega^* \to A^*$ for every oriented cohomology theory A^* in the sense of Levine-Morel.

Each oriented cohomology theory A^* is equipped with a 1-dimensional commutative formal group law $\mathcal{F}_A$. For Ω^* the respective formal group law $\mathcal{F}_\Omega$ is the universal one, and the canonical morphism $\mathbb{L} \to \Omega^*(\operatorname{Spec} F)$ from the Lazard ring is an isomorphism (see [28, Theorem 1.2.7]).

In this article, when necessary, we consider our oriented cohomology theories as Borel-Moore homology theories and extend them to all separated schemes of finite type over F

as in Section 4.3. Conversely, every oriented Borel-Moore homology theory restricted to the category of smooth varieties gives an oriented cohomology theory.

DEFINITION 4.5 (Levine-Morel, [28, Remark 2.4.14(2)]). – Let R be a commutative ring, let $\mathcal{F}_R$ be a formal group law over R, and let $\mathbb{L} \to R$ be the respective ring morphism. Then $\Omega_* \otimes_{\mathbb{L}} R$ is an oriented Borel-Moore homology theory which is called a *free theory*. Its ring of coefficients is R, and its associated formal group law is $\mathcal{F}_R$.

For example, the Chow theory is a free theory with the additive formal group law and with the coefficient ring $\mathbb{Z}$ (see [28, Theorem 1.2.19]). In this article K^0 stands for a free theory with the multiplicative formal group law and with the coefficient ring $\mathbb{Z}$. If X is a smooth variety over F, then $K^0(X)$ is Grothendieck's K^0-theory of locally free coherent sheaves on X (see [28, Theorem 1.2.18]).

By [28, Corollary 4.4.3] every free theory A_* is *generically constant*, i.e., for every integral scheme X over F the canonical map

$$A_*(\operatorname{Spec} F) \to A_*(\operatorname{Spec} F(X)) := \operatorname{colim}_{U\subset X} A_{*+\dim X}(U)$$

is an isomorphism, where the colimit is taken over all non-empty open subschemes of X.

By [28, Theorem 3.2.7] the algebraic cobordism theory satisfies the localization axiom. Hence, every free theory satisfies the localization axiom as well.

In [60, Definition 4.1] Vishik defines *theories of rational type* in geometric terms and proves in [60, Proposition 4.7] that the generically constant theories of rational type are precisely the free theories. Vishik's definition allows to describe efficiently the sets of operations between such theories and Riemann-Roch type results for them.

4.6. Brown-Peterson cohomology and Morava K-theories

For a prime number p and a positive integer n we consider the n-th Morava K-theory $K(n)^*$ with respect to p. Note that we do not include p in the notation. We define this theory as a free theory with the coefficient ring $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ where $\deg v_n = -(p^n - 1)$ and with a formal group law which we will describe below.

The variable v_n, as it is invertible, does not play an important role in computations with Morava K-theories, and sometimes we will prefer to set it to be equal to 1. It will be always clear from the context which n-th Morava K-theory we use, i.e., with $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$- or $\mathbb{Z}_{(p)}$-coefficients.

We follow [14] and [43]. There exists a universal p-typical formal group law $\mathcal{F}_{BP}$ over a ring BP. The latter ring is non-canonically isomorphic to the ring $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, and from now on we choose the isomorphism defined by Hazewinkel (see [43, Appendix 2]). The canonical morphism from $\mathcal{F}_\Omega$ over $\mathbb{L}_{(p)} = \mathbb{L} \otimes \mathbb{Z}_{(p)}$ to $\mathcal{F}_{BP}$ over the ring $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ defines a multiplicative projector on $\Omega^*_{(p)} := \Omega^* \otimes \mathbb{Z}_{(p)}$ whose image is the *Brown-Peterson cohomology BP^**.

The logarithm of the formal group law of the Brown-Peterson theory equals

$$l(t) = \sum_{i\geq 0} m_i t^{p^i},$$

where $m_0 = 1$ and the remaining variables m_i are related to v_j following Hazewinkel as follows:

$$m_j = \frac{1}{p} \cdot \big(v_j + \sum_{i=1}^{j-1} m_i v_{j-i}^{p^i}\big),$$

see, e.g., [43, Appendix 2.2.1]. Let $e(t)$ be the compositional inverse of $l(t)$. Then the Brown-Peterson formal group law is given by $e(l(x) + l(y))$. We remark that the coefficients of the logarithm $l(t)$ lie in $\mathbb{Q}[v_1, v_2, \ldots]$, but the coefficient ring of BP^* is $BP = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$. Note also that $\deg v_i = -(p^i - 1)$.

We define an n-th Morava K-theory $K(n)^*$ as a free theory with a p^n-typical formal group law $\mathcal{F}$ over $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$ (or over $\mathbb{Z}_{(p)}$) such that the height of $\mathcal{F}$ modulo p is n (see [51, Definition 3.9]). Thus, even for a fixed prime p and a fixed height n there exist non-isomorphic n-th Morava K-theories (which are though isomorphic as presheaves of abelian groups, see [51, Theorem 5.3]).

As in topology we denote by $K(0)^*$ the theory $\mathrm{CH}^* \otimes \mathbb{Q}$ (independently of a prime p).

In the classical construction of the n-th Morava formal group law one takes the BP formal group law and sends all v_j with $j \neq n$ to zero. Modulo the ideal J generated by p, x^{p^n}, y^{p^n} the formal group law for the n-th Morava K-theory equals then

$$\mathcal{F}_{K(n)}(x, y) = x + y - v_n \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^{ip^{n-1}} y^{(p-i)p^{n-1}} \mod J$$

and the logarithm of the corresponding particular n-th Morava K-theory equals

$$\log_{K(n)}(t) = \sum_{i=0}^{\infty} \frac{1}{p^i} v_n^{\frac{p^{in}-1}{p^n-1}} t^{p^{in}}.$$

More generally, every n-th Morava K-theory is obtained from BP^* by sending all v_j with $n \nmid j$ to zero, but v_j with $n|j$ are sent to some multiples of the corresponding powers of v_n (and the set of all thus obtained theories is independent of the choice of variables v_j).

For a variety X over F one has

$$K(n)^*(X) = \Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z}_{(p)}[v_n, v_n^{-1}],$$

and v_n is a ν_n-element in the Lazard ring $\mathbb{L}$.

We remark that classically in topology one considers the Morava K-theory with the coefficient ring $\mathbb{F}_p[v_n, v_n^{-1}]$, but in the present article it is crucial that we consider an integral version. Note also that as was mentioned above two n-th Morava K-theories are additively isomorphic, but are in general not multiplicatively isomorphic.

If $n = 1$, for a particular choice of $K(1)^*$ there exists a functorial (with respect to pullbacks) isomorphism of algebras $K(1)^*(X)/(v_1 - 1) \simeq K^0(X) \otimes \mathbb{Z}_{(p)}$, which can be obtained with the help of the Artin-Hasse exponent (for the latter see [44, Chapter 7, Section 2]).

4.7. Euler characteristic

The Euler characteristic of a smooth projective irreducible variety X with respect to an oriented cohomology theory A^* (A^*-*Euler characteristic*) is defined as the push-forward

$$\pi_*^A(1_X) \in A^*(\operatorname{Spec} F)$$

of the structural morphism $\pi\colon X \to \operatorname{Spec} F$. E.g., for $A^* = K^0 \otimes \mathbb{Z}[v_1, v_1^{-1}]$ with push-forwards defined as in [28, Example 1.1.5] the Euler characteristic of X equals

$$v_1^{\dim X} \cdot \sum (-1)^i \dim H^i(X, \mathcal{O}_X),$$

see [7, Ch. 15]. If X is geometrically irreducible and geometrically cellular, then this element equals $v_1^{\dim X}$ (see [69, Example 3.6]).

For the Morava K-theory $K(n)^*$ and a smooth projective irreducible variety X of dimension $d = p^n - 1$ the Euler characteristic modulo p equals the element $v_n \cdot u \cdot s_d$ for some $u \in \mathbb{F}_p^\times$, where s_d is the Milnor number of X (see [28, Proposition 4.4.22(3)]). In particular, it is invertible, if X is a v_n-variety. If $\dim X$ is not divisible by $p^n - 1$, then the Euler characteristic of X equals zero, since the target graded ring has non-trivial components only in degrees divisible by $p^n - 1$.

4.8. Motives

For a theory A^* we consider the category of A^*-motives over F, which is defined in the same way as the category of Grothendieck's Chow motives with CH^* replaced by A^* (see [31], [6]). Namely, the morphisms between two smooth projective irreducible varieties X and Y over F are given by $A^{\dim Y}(X \times Y)$.

By $\mathbb{T}(l)$, $l \geq 0$, we denote the Tate motives in the category of A^*-motives. They are defined in the same way as the Tate motives in the category of Chow motives. Namely, the A^*-motive of the projective line splits as a direct sum of the A^*-motive of $\operatorname{Spec} F$, which we denote by $\mathbb{T}$, and another motive, which we denote by $\mathbb{T}(1)$. Then $\mathbb{T}(l)$ is defined as $\mathbb{T}(1)^{\otimes l}$ for $l \geq 0$.

Definition 4.9. – For an oriented cohomology theory A^* and a motive M in the category of A^*-motives over F we say that M is *split*, if it is isomorphic to a finite direct sum of Tate motives over F.

Note that this property depends on the theory A^*, i.e., there exist smooth projective varieties whose motives are split for some oriented cohomology theories, but not for all oriented cohomology theories. For example, it follows from Proposition 6.2 below that the n-th Morava K-theory $K(n)^*$ for $p = 2$ of an anisotropic m-fold Pfister quadric over F is split, if $n < m - 1$. On the other hand, the Chow motive of an anisotropic Pfister quadric is never split.

4.10. Rost nilpotence for oriented cohomology theories

Let A^* be an oriented cohomology theory and consider the category of A^*-motives over F. Let M be an A^*-motive over F. We say that the Rost nilpotence principle holds for M, if the kernel of the restriction homomorphism

$$\mathrm{End}(M) \to \mathrm{End}(M_E)$$

consists of nilpotent elements for all field extensions E/F.

By [5, Section 8] Rost nilpotence holds for Chow motives of all twisted flag varieties.

Rost nilpotence is a tool which allows to descend motivic decompositions over E to motivic decompositions over the base field F. E.g., assume that Rost nilpotence holds for M and that we are given a decomposition $M_E \simeq \bigoplus M_i$ over E into a finite direct sum. The motives M and M_i are defined as pairs (X,ρ) and (X_E,ρ_i), where X is a smooth projective variety over F, $\rho \in A^*(X\times X)$ and $\rho_i \in A^*(X_E\times X_E)$ are some projectors. Assume further that all ρ_i are defined over F, i.e., there exists $\eta_i \in A^*(X\times X)$ such that $(\eta_i)_E = \rho_i$. We would like to modify η_i to make it a projector, while at the moment we only know that the difference $\eta_i^{\circ 2}-\eta_i$ is in the kernel of the map to $A^*(X_E\times X_E)$ and, thus, is nilpotent. In fact, considering a *commutative* subring of $A^*(X\times X)$ generated by η_i for a particular index i, one can show that some power of the element η_i is a projector. It follows then that $M \simeq \bigoplus N_i$ for some motives N_i over F, and the scalar extension $(N_i)_E$ is isomorphic to M_i for every i (for more details see [5, Section 8], [42, Section 2]).

Let M be a Chow motive. By [62, Section 2] there is a unique (up to isomorphism) lift of the motive M to the category of Ω^*-motives and, since Ω^* is the universal oriented cohomology theory, there is a respective motive in the category of A^*-motives for every oriented cohomology theory A^*. We denote this A^*-motive by M^A.

By [13, Corollary 4.5] if $M = (X,\pi)$ is a direct summand of the Chow motive of a twisted flag variety, then Rost nilpotence holds for M^A for every oriented cohomology theory obtained from Ω^* by a change of coefficients.

4.11. Generalized Riemann-Roch theorem

We follow [61]. Let A^* be a theory of rational type, let B^* be an oriented cohomology theory and let $\phi\colon A^* \to B^*$ be an operation (which does not necessarily preserve the grading and does not have to be additive).

For a smooth variety Z over a field F and any $c \geq 0$ denote by G_Z^c the composition

$$\begin{aligned} A^*(Z) \to A^*(Z)[[z_1^A,\dots,z_c^A]] &\xrightarrow{\phi_{Z\times(\mathbb{P}^\infty)^{\times c}}} B^*(Z)[[z_1^B,\dots,z_c^B]] \\ \alpha \longmapsto \alpha\cdot z_1^A\cdots z_c^A&, \end{aligned}$$

where we have identified $A^*(Z)[[z_1^A,\dots,z_c^A]]$ with $A^*(Z\times(\mathbb{P}^\infty)^{\times c})$ and similarly for B^*, i.e., z_i is the first Chern class of the pullback along the projection of the canonical line bundle $\mathcal{O}(1)$ over the i-th product component of $(\mathbb{P}^\infty)^{\times c}$.

Note that by the so-called "continuity of operations" ([61, Proposition 5.3]) for every c and Z the series $G_Z^c(1_Z)$ is divisible by $z_1^B\cdots z_c^B$. We denote the quotient by F_Z^c and set $F^c = F_{\mathrm{pt}}^c(1) \in B[[z_1^B,\dots,z_c^B]]$ (we denote $B = B^*(\mathrm{pt})$). We write $G_Z^c(\alpha)|_{z_i^B=y_i}$ when we plug in nilpotent elements $y_i \in B^*(Z)$ in this series (similarly, for F_Z^c and F^c).

Finally, denote by $\omega_t^B \in B[[t]]dt$ the canonical invariant 1-form of the formal group law F_B such that $\omega^B(0) = dt$ (see [60, Section 7.1]).

The following proposition is a particular case of a general form of Riemann-Roch type theorems [61, Theorem 5.19].

PROPOSITION 4.12 (Vishik). – *Let X be a smooth variety over a field F. Let $i\colon Z \hookrightarrow X$ be a closed embedding of a smooth subvariety of codimension c. Let $\alpha \in A^*(Z)$ and denote by $\mu_1, \dots, \mu_c$ the B-roots of the normal bundle $N_{Z/X}$.*

Let $k \geq 0$ and let L_i be line bundles over Z for $1 \leq i \leq k$. Denote by $x_i = c_1^A(L_i)$, $y_i = c_1^B(L_i)$ their first Chern classes.

Then

$$\phi\left(i_*(\alpha \prod_{i=1}^k x_i)\right) = i_* \mathrm{Res}_{t=0} \frac{G_Z^{c+k}(\alpha)|_{z_i^B = t +_B \mu_i, 1 \leq i \leq c,\ z_{c+j}^B = y_j, 1 \leq j \leq k}}{t \cdot \prod_{i=1}^c (t +_B \mu_i)} \omega_t^B.$$

We will need only the following instance of this proposition.

COROLLARY 4.13. – *We have $\phi(i_* 1_Z) = i_*(\pi^*(F^c)|_{z_i^B = \mu_i})$, where*

$$\pi^*\colon B[[z_1^B, \dots, z_c^B]] \to B^*(Z)[[z_1^B, \dots, z_c^B]]$$

is induced by the pullback of the structure map $\pi\colon Z \to \operatorname{Spec} F$.

In particular, the right-hand side depends only on the action of operation ϕ on products of projective spaces and the B-Chern classes of the normal bundle of Z.

Proof. – Indeed, $1_Z = \pi^*(1)$ and $G_Z^c(\pi^*(1)) = \phi(\pi^*(1) z_1^A \cdots z_c^A) = \pi^*(\phi(z_1^A \cdots z_c^A)) = \pi^* G_{\mathrm{pt}}^c(1)$. It follows that $F_Z^c(1_Z) = \pi^* F_{\mathrm{pt}}^c(1) = \pi^* F^c$.

We can rewrite the formula in Proposition 4.12 as

$$\phi(i_* 1_Z) = i_* \mathrm{Res}_{t=0} F_Z^c(1_Z)|_{z_i = t +_B \mu_i} \frac{\omega_t^B}{t} \prod_i \frac{t +_B \mu_i}{t +_B \mu_i}.$$

Since $\omega_t^B(0) = dt$, we get the required formula. □

4.14. Topological filtration on free theories

For a free theory A^* and a smooth variety X we define the *topological filtration* (sometimes referred to as a filtration by codimension of support) as the kernel of the restriction maps to open subvarieties which have a complement of codimension bounded below:

$$\tau^i A^*(X) := \bigcup_{U \subset X\colon \operatorname{codim}_X(X \setminus U) \geq i} \operatorname{Ker}(A^*(X) \to A^*(U)). \tag{4.15}$$

Since the restriction maps commute with pullbacks, it is clear that $\tau^i A^*$ is a subpresheaf of A^*.

We denote by $\tilde{A}^* := \tau^1 A^*$ the subpresheaf consisting of all elements which vanish in the generic points of varieties. Note that since A^* is generically constant, for every irreducible variety X we have a canonical splitting of abelian groups: $A^*(X) = A \oplus \tilde{A}^*(X)$, where A stands for $A^*(\operatorname{Spec} F)$.

One shows using the localization axiom for free theories ([28, Theorem 3.2.7]) that this definition is equivalent to the one given using images of push-forwards as in [28, Section 4.5.2] (cf. [51, Proposition 1.17(1)] for a relation between the topological filtration on Ω^* and on free theories).

There exists a canonical surjective map of $\mathbb{L}$-modules $\rho_\Omega\colon \mathrm{CH}^i\otimes\mathbb{L} \to \tau^i\Omega^*/\tau^{i+1}\Omega^*$ (see [28, Corollary 4.5.8]), and we denote by $\rho_A\colon \mathrm{CH}^i\otimes_{\mathbb{L}}A \to \tau^iA^*/\tau^{i+1}A^*$ the map of A-modules obtained by the change of coefficients of ρ_Ω from $\mathbb{L}$ to A.

Besides, we denote $\mathrm{gr}^i_\tau\tilde{A}^j := \tau^i\tilde{A}^j/\tau^{i+1}\tilde{A}^j$, where $\tau^i\tilde{A}^j$ is defined as in Formula (4.15) with A^* replaced by $\tilde{A}^j$.

5. Gamma filtration on Morava K-theories

5.1. Operations from Morava K-theories

In the article [51] the first author classified all operations from the n-th Morava K-theory to the so called p^n-typical oriented theories whose coefficient ring is a free $\mathbb{Z}_{(p)}$-module.

We will exploit these operations only when the target theory is either the n-th Morava K-theory itself or the Chow theory with p-local coefficients. There exist certain generators of the algebra of all operations constructed in [51] which in these cases are denoted by $c_i^{K(n)}$ and c_i^{CH} respectively, and we summarize their properties in this section.

In this section we consider Morava K-theories with $\mathbb{Z}_{(p)}$-coefficients, i.e., we set $v_n = 1$. This agrees with [49, 51]. This reduction to $\mathbb{Z}_{(p)}$-coefficients does not break the grading completely. Namely, one can show the following proposition.

Proposition 5.2 ([49, Proposition 4.1.5] & [51, Proposition 3.15]).

1. *Morava K-theories $K(n)^*$ are $\mathbb{Z}/(p^n-1)$-graded as presheaves of rings.*
2. *The grading is compatible with push-forwards, i.e., for a projective morphism $f\colon X\to Y$ of codimension c the push-forward map increases the grading in Morava K-theories by c : $f_*\colon K(n)^i(X)\to K(n)^{i+c}(Y)$.*

   In particular, the first Chern class of any line bundle L over a smooth variety X lies in $K(n)^1(X)$.
3. *The topological filtration on the graded component of the n-th Morava K-theory changes only every p^n-1 steps, i.e., we have*

   $$\tau^{j+s(p^n-1)+1}\tilde{K}(n)^j = \tau^{j+s(p^n-1)+2}\tilde{K}(n)^j = \cdots = \tau^{j+(s+1)(p^n-1)}\tilde{K}(n)^j,$$

   where $j\in[1,p^n-1]$, $s\geq 0$.

   In particular, $\mathrm{gr}^j_\tau\tilde{K}(n)^* = \tilde{K}(n)^j/\tau^{j+p^n-1}\tilde{K}(n)^j$ *for* $j\colon 1\leq j\leq p^n-1$.

We denote the graded components of $K(n)^*$ as $K(n)^1, K(n)^2,\cdots,K(n)^{p^n-1}$ and freely use the notation $K(n)^i$, $K(n)^{i \bmod p^n-1}$, $K(n)^{i+r(p^n-1)}$ to denote the component $K(n)^j$ where $j\equiv i \mod p^n-1$, $1\leq j\leq p^n-1$.

Theorem 5.3 ([49, Theorem 4.2.1], [51, Theorem 3.16]). – *There exist operations $c_i^{K(n)}\colon K(n)^{i \bmod p^n-1}\to K(n)^{i \bmod p^n-1}$ and $c_i^{\mathrm{CH}}\colon K(n)^i\to \mathrm{CH}^i\otimes\mathbb{Z}_{(p)}$ for $i\in\mathbb{Z}^{>0}$ satisfying the following properties (we omit the index $K(n)$ resp.* CH *in the notation of c_i, since the index is always clear from the context):*

1. *For any smooth variety X and for every pair of elements $x, y \in K(n)^*(X)$ the modified Cartan's formula holds:*

$$c_{\mathrm{tot}}(x+y) = \mathcal{F}_{K(n)}(c_{\mathrm{tot}}(x), c_{\mathrm{tot}}(y)),$$

where $\mathcal{F}_{K(n)}$ is the formal group law for the Morava K-theory, $c_{\mathrm{tot}} = \sum_{i\geq 1} c_i t^i$, t is a formal variable and we naturally consider each operation c_j to be defined on the whole group $K(n)^*$ via the composition with the natural projection to $K(n)^j$.

The equality takes place in $K(n)^*(X) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[[t]]$ or $\mathrm{CH}^*(X) \otimes \mathbb{Z}_{(p)}[[t]]$.

2. *Every operation from the presheaf $\tilde{K}(n)^*$ to the corresponding target theory can be uniquely expressed as a formal power series in c_i's with $\mathbb{Z}_{(p)}$-coefficients.*

5.4. The gamma filtration

The above operations from the n-th Morava K-theory to itself allow to define the gamma filtration verbatim as for the K-theory. We recall first the classical picture, since the situation with Morava K-theories is very similar.

Recall that for K^0 the Chern classes $c_i\colon K^0 \to \mathrm{CH}^i$ can be restricted to additive maps

$$c_i\colon \mathrm{gr}^i_\tau K^0 \to \mathrm{CH}^i,$$

where $\mathrm{gr}^i_\tau K^0$ stand for the graded components of the topological filtration $\tau^\bullet$ on K^0.

There is also a canonical map $(\rho_{K_0})_i\colon \mathrm{CH}^i \to \mathrm{gr}^i_\tau K^0$ which sends a cycle Z to the class of the coherent sheaf $[\mathcal{O}_Z]$. The compositions $(\rho_{K_0})_i \circ c_i$, $c_i \circ (\rho_{K_0})_i$ are multiplications by $(-1)^{i-1}(i-1)!$. In particular, c_i is surjective if one inverts $(i-1)!$.

The gamma filtration $\gamma^\bullet$ for K^0 is an approximation of the topological filtration. One has $\gamma^i \subset \tau^i$ for all i, and $\gamma^i = \tau^i$ for $i \leq 2$. Moreover, the induced map

$$\mathrm{gr}^i_\gamma K^0 \to \mathrm{gr}^i_\tau K^0 \xrightarrow{c_i} \mathrm{CH}^i$$

is surjective for $i \leq 2$.

A similar picture holds for the Morava K-theories. The canonical additive map

$$(\rho_{K(n)})_i\colon \mathrm{CH}^i \otimes \mathbb{Z}_{(p)} \to \tau^i K(n)^*/\tau^{i+1} K(n)^*$$

is defined using [28, Corollary 4.5.8]. It is possible to calculate the compositions $(\rho_{K(n)})_i \circ c_i^{\mathrm{CH}}$, $c_i^{\mathrm{CH}} \circ (\rho_{K(n)})_i$, and they turn out to be isomorphisms in a bigger range compared to K^0.

PROPOSITION 5.5 ([51, Proposition 6.2]). – *The canonical map*

$$(\rho_{K(n)})_i\colon \mathrm{CH}^i \otimes \mathbb{Z}_{(p)} \to \tau^i K(n)^*/\tau^{i+1} K(n)^*$$

is an isomorphism for $0 \leq i \leq p^n$, and the map c_i^{CH} is its inverse for $1 \leq i \leq p^n$.

In general, it is hard to calculate the topological filtration for $K(n)^*(X)$ even if X is a geometrically cellular variety and $K(n)^*$-motive of X is split. The problem is that the topological filtration is not strictly respected by the base change restrictions like $K(n)^*(X) \to K(n)^*(\overline{X})$. The gamma filtration which we will now describe is a computable approximation to the topological filtration which lacks such "handicap".

DEFINITION 5.6 ([51, Definition 6.1]). – Define the gamma filtration on $K(n)^*$ of a smooth variety X by the following formulas:

$$\gamma^0 K(n)^*(X) = K(n)^*(X),$$

$$\gamma^m K(n)^*(X) := \left\langle c_{i_1}^{K(n)}(\alpha_1)\cdots c_{i_k}^{K(n)}(\alpha_k) \;\middle|\; \sum_j i_j \geq m, i_j \geq 1, k \geq 1, \alpha_j \in K(n)^*(X) \right\rangle,$$

where the $\langle\, ,\rangle$-brackets denote the generation as $\mathbb{Z}_{(p)}$-modules and $m \geq 1$.

It is clear from the definition that $\gamma^m K(n)^*$ is an ideal subpresheaf of $K(n)^*$.

THEOREM 5.7 ([51, Proposition 6.2]). – *The gamma filtration and the topological filtration satisfy the following properties:*

(i) $\gamma^i \subset \tau^i$ *for all* i*;*

(ii) $c_i^{\mathrm{CH}}|_{\tau^{i+1}K(n)^*} = 0$, $c_i^{\mathrm{CH}}|_{\gamma^{i+1}K(n)^*} = 0$*;*

(iii) *the operation* c_i^{CH} *is additive when restricted to* $\tau^i K(n)^*$ *or* $\gamma^i K(n)^*$ *and the map*

$$c_i^{\mathrm{CH}} \otimes \mathrm{id}_{\mathbb{Q}}\colon \mathrm{gr}^i_\gamma K(n)^* \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \to \mathrm{CH}^i \otimes \mathbb{Q}$$

is an isomorphism;

(iv) c_i^{CH} *induces an additive isomorphism between* $\mathrm{gr}^i_\tau K(n)^*$ *and* $\mathrm{CH}^i \otimes \mathbb{Z}_{(p)}$ *for* $1 \leq i \leq p^n$*;*

(v) c_i^{CH} *restricted to* $\gamma^i K(n)^*$ *is surjective for* $1 \leq i \leq p^n$*;*

(vi) $\mathrm{gr}^i_\gamma K(n)^* = \mathrm{gr}^i_\gamma K(n)^{i \bmod p^n-1}$.

In Section 8 we will use the Riemann-Roch formula (Proposition 4.12, Corollary 4.13) to perform computations with the gamma filtration. Let us sketch how it applies.

We follow the notation of Section 4.11. Let $\phi\colon A^* \to B^*$ be an operation, let X be a smooth variety, and let $i\colon Z \hookrightarrow X$ be its smooth closed subvariety of codimension c.

It follows from the Riemann-Roch formula that the value $\phi(i_* 1_Z)$ is equal to $b \cdot 1_Z$ modulo $(c+1)$-st part of the topological filtration, where $b \in B$ is the coefficient of $z_1^B \cdots z_c^B$ in the series $\phi(z_1^A \cdots z_c^A)$. The following technical statements describe this coefficient for some operations for the Morava K-theory.

PROPOSITION 5.8 ([51, Proposition 6.11]). – *Let* $c_{p^n}^{K(n)}$ *be the respective operation from* $K(n)^1$ *to* $\gamma^{p^n} K(n)^1$*.*

Denote by e_j, $j \geq 0$*, the coefficient of the monomial* $z_1 \cdots z_{1+j(p^n-1)}$ *in the series* $c_{p^n}^{K(n)}(z_1 \cdots z_{1+j(p^n-1)}) \in K(n)^1((\mathbb{P}^\infty)^{\times 1+j(p^n-1)})$.

Then for all primes p *and for all* $j \geq 1$ *we have* $e_j \in \mathbb{Z}_{(p)}^\times$*.*

PROPOSITION 5.9 ([51, Proposition 6.13] for $p = 2$). – *Let* $j \geq 0$*.*

There exist operations $\chi, \psi : K(n)^1 \to \gamma^{2^{n+1}-1} K(n)^1$ *which satisfy the following.*

Denote by $g_j, f_j \in \mathbb{Z}_{(2)}$*, the coefficients of the monomial* $z_1 \cdots z_{1+j(2^n-1)}$ *in the series* $\chi(z_1 \cdots z_{1+j(2^n-1)})$, $\psi(z_1 \cdots z_{1+j(2^n-1)}) \in K(n)^1((\mathbb{P}^\infty)^{\times 1+j(2^n-1)})$*, respectively. Then*

1. *we have* $g_j = f_j = 0$ *for* $j = 0, 1$*.*

Let $j \geq 2$*.*

2. *We have* $g_j \in 2^{t_j}\mathbb{Z}_{(2)}^{\times}$ *where* $t_j = \nu_2(j-1)+2$ *if* j *is odd, and* $t_j = 1$ *if* j *is even. Here* ν_2 *denotes the* 2*-adic valuation on integers.*
3. *We have* $f_j \in 2^{2^n}\mathbb{Z}_{(2)}^{\times}$.

6. Some computations of the Morava K-theory

DEFINITION 6.1. – Let $m \geq 2$ and let $\alpha \in H^m_{\mathrm{et}}(F, \mu_p^{\otimes m})$ be a non-zero pure symbol. A motive $R_m = (X, \pi)$ in the category of Chow motives with $\mathbb{Z}_{(p)}$-coefficients is called the *(generalized) Rost motive* for α, if it is indecomposable, splits as a sum of Tate motives over $F(X)$ and for every field extension K/F the following conditions are equivalent:

1. $(R_m)_K$ is decomposable;
2. $(R_m)_K \simeq \bigoplus_{i=0}^{p-1}\mathbb{Z}_{(p)}(b\cdot i)$ with $b = \frac{p^{m-1}-1}{p-1}$;
3. $\alpha_K = 0 \in H^m_{\mathrm{et}}(K, \mu_p^{\otimes m})$.

The fields K from this definition are called splitting fields of R_m.

The Rost motives were constructed by Rost and Voevodsky (see [48], [65]). Namely, for all pure symbols α there exists a smooth geometrically irreducible projective ν_{m-1}-variety X (depending on α) over F such that the Chow motive of X has a direct summand isomorphic to R_m and for every field extension K/F the motive $(R_m)_K$ is decomposable iff X_K has a 0-cycle of degree coprime to p. The variety X is called a *norm variety* of α. Moreover, it follows from [67, Lemma 9.2] that for a given α the respective Rost motive is unique.

E.g., if $p = 2$ and $\alpha = (a_1)\cup\cdots\cup(a_m)$ with $a_i \in F^{\times}$, then one can take for X the projective quadric given by the equation $\langle\langle a_1, \ldots, a_{m-1}\rangle\rangle \perp \langle -a_m\rangle = 0$, where $\langle\langle a_1, \ldots, a_{m-1}\rangle\rangle$ denotes the Pfister form. (We use the standard notation from the quadratic form theory as in [23] and [6].)

As in Section 4.10 using [62, Section 2] one finds a unique lift of the Rost motive R_m to the category of A^*-motives for every oriented cohomology theory A^*. We will denote this A^*-motive by the same letter R_m, since A^* will be always clear from the context. Recall that by $\mathbb{T}(l)$, $l \geq 0$, we denote the Tate motives in the category of A^*-motives. If $A^* = \mathrm{CH}^*\otimes\mathbb{Z}_{(p)}$, we keep the usual notation $\mathbb{T}(l) = \mathbb{Z}_{(p)}(l)$.

Moreover, it follows from [13, Lemma 4.2] that Rost nilpotence holds for R_m with respect to every free theory A^*, since R_m splits over the residue fields of all points of X.

PROPOSITION 6.2. – *Let* p *be a prime number, let* $n \geq 0$ *and* $m \geq 2$ *be integers and* $b = \frac{p^{m-1}-1}{p-1}$. *For a non-zero pure symbol* $\alpha \in H^m_{\mathrm{et}}(F, \mu_p^{\otimes m})$ *consider the respective Rost motive* R_m. *Then*

1. *If* $n < m-1$, *then the* $K(n)^*$*-motive* R_m *is a sum of* p *Tate motives* $\bigoplus_{i=0}^{p-1}\mathbb{T}(b\cdot i)$.
2. *If* $n = m-1$, *then the* $K(n)^*$*-motive* R_m *is a sum of the Tate motive* $\mathbb{T}$ *and an indecomposable motive* L *such that*

$$K(n)^*(L) \simeq (\mathbb{Z}_{(p)}^{\oplus(p-1)} \oplus (\mathbb{Z}/p)^{\oplus(m-2)(p-1)}) \otimes \mathbb{Z}_{(p)}[v_n, v_n^{-1}]. \tag{6.3}$$

For a field extension K/F *the motive* L_K *is isomorphic to a direct sum of Tate motives iff the symbol* $\alpha_K = 0$. *If* $p > 2$, *then this is additionally equivalent to the condition that the motive* L_K *is decomposable.*

3. *If $n > m-1$, then the $K(n)^*$-motive R_m is indecomposable and $K(n)^*(R_m)$ is isomorphic to the group*

$$\mathrm{CH}^*(R_m) \otimes \mathbb{Z}_{(p)}[v_n, v_n^{-1}] \simeq (\mathbb{Z}_{(p)}^{\oplus p} \oplus (\mathbb{Z}/p)^{\oplus(m-2)(p-1)}) \otimes \mathbb{Z}_{(p)}[v_n, v_n^{-1}]. \tag{6.4}$$

For a field extension K/F the motive $(R_m)_K$ is decomposable iff $\alpha_K = 0$. In this case $(R_m)_K$ is a sum of p Tate motives.

Proof. – Let X be a norm variety of dimension $p^{m-1} - 1$ for α. Denote by $\overline{R}_m$ the scalar extension of R_m to its splitting field. By [67, Theorem 10.6] (cf. [62, Theorem 3.5, Proposition 4.4]) the restriction map for the Brown-Peterson theory BP^*

$$\text{res}\colon BP^*(R_m) \to BP^*(\overline{R}_m) = BP^{\oplus p} \tag{6.5}$$

is injective, and the image equals

$$BP^*(R_m) \simeq BP \oplus I(m-1)^{\oplus(p-1)}, \tag{6.6}$$

where $I(m-1)$ is the ideal in the ring $BP = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ generated by the elements $\{v_0, v_1, \ldots, v_{m-2}\}$ where $v_0 = p$. We remark that the article [67, Theorem 10.6] deals with the bigraded version of the Brown-Peterson cohomology theory $ABP^{*,*'}$. Nevertheless, due to Levine's comparison result [27] Yagita identifies $ABP^{2*,*}$ with BP^*.

The projectors for the motive R_m lie in the group $K(n)^{p^{m-1}-1}(R_m \otimes R_m)$, and by [28, Theorem 4.4.7] the elements of $K(n)^{p^{m-1}-1}(R_m \otimes R_m)$ are $\mathbb{Z}_{(p)}$-linear combinations of elements of the form

$$v_n^s \cdot [Y \to X \times X], \quad s \in \mathbb{Z}, \tag{6.7}$$

where Y is a resolution of singularities of a closed irreducible subvariety of $X \times X$, and $-s(p^n - 1) + \operatorname{codim} Y = p^{m-1} - 1$.

(1) Assume first that $n < m-1$. Since the ideal $I(m-1)$ contains v_n for $n < m-1$ and v_n is invertible in $K(n)$, we immediately get that all elements in $K(n)^*(\overline{R}_m)$ are rational, i.e., are defined over the base field.

Therefore, since the motive R_m is geometrically split, all elements in $K(n)^*(\overline{R}_m \otimes \overline{R}_m)$ are rational, and hence by Rost nilpotence for R_m this gives the first statement of the proposition.

(3) Let $n > m-1$. First of all, taking the tensor product $-\bigotimes_{BP(\operatorname{Spec} F)} K(n)$ with Formula (6.5) and using (6.6) one immediately gets Formula (6.4) for $K(n)^*(R_m)$ and $\mathrm{CH}^*(R_m)$.

We have $\dim X = p^{m-1} - 1 < p^n - 1 = -\deg v_n$.

Since every projector in $K(n)^{p^{m-1}-1}(R_m \otimes R_m)$ is a linear combination of elements of the form (6.7) and $\dim(X \times X) = 2(p^{m-1} - 1)$, we must have $s = 0$ in all summands. Therefore, every projector ρ in $K(n)^{p^{m-1}-1}(R_m \otimes R_m)$ comes from the connective Morava K-theory $CK(n)^*$ (the connective Morava K-theory is a free oriented cohomology theory with the same formal group law as the Morava K-theory, but with the coefficient ring $\mathbb{Z}_{(p)}[v_n]$). Thus, we have the following commutative diagram

$$\begin{tikzcd}
CK(n)^*(R_m \otimes R_m) \arrow[r] \arrow[d] & CK(n)^*(\overline{R}_m \otimes \overline{R}_m) \arrow[d, hook] \\
K(n)^*(R_m \otimes R_m) \arrow[r] & K(n)^*(\overline{R}_m \otimes \overline{R}_m)
\end{tikzcd}$$

and the rational projector $\bar{\rho}$ comes from some rational projector $\bar{\tau} \in CK(n)^*(\overline{R}_m \otimes \overline{R}_m)$. Note that the right vertical arrow is injective, since $\overline{R}_m$ is a direct sum of Tate motives.

By [62, Section 2] the $CK(n)^*$-motive R_m is indecomposable, since so is the respective Chow motive. Therefore $\bar{\tau}$ is either zero or the identity projector. Therefore, so is $\bar{\rho}$ and, hence, by Rost nilpotence for R_m so is the projector ρ. Therefore, the $K(n)^*$-motive R_m is indecomposable.

(2) Assume now that $n = m - 1$. Since the $K(n)^*$-Euler characteristic of X equals $u \cdot v_n$ for some $u \in \mathbb{Z}_{(p)}^\times$ (see Section 4.7), the element $v_n^{-1} \cdot u^{-1}(1 \times 1) \in K(n)^*(R_m \otimes R_m)$ is a projector defining the Tate motive $\mathbb{T}$ where $1 \times 1 \in K(n)^0(X \times X)$. Note that this projector lies in $K(n)^*(R_m \otimes R_m)$, since this is true over a splitting field of R_m and since 1×1 is a rational element. Thus, we get the decomposition $R_m \simeq \mathbb{T} \oplus L$ for some motive L.

Taking the tensor product $- \bigotimes_{BP(\operatorname{Spec} F)} K(n)$ with Formula (6.5) and using (6.6) one immediately gets Formula (6.3) for $K(n)^*(L)$.

We claim now that L is indecomposable. If $p = 2$, then this is clear, since in this case L over a splitting field of R_m is a Tate motive. So, we assume that $p > 2$.

We have $\dim X = p^{m-1} - 1 = p^n - 1$. Since every projector in $K(n)^{p^n-1}(R_m \otimes R_m)$ is a linear combination of elements of the form (6.7) and $\dim(X \times X) = 2(p^n - 1)$, we must have $s = 0$, 1 or -1 in all summands.

If a projector contains a summand with $s = -1$, then by dimensional reasons this summand is up to a scalar of the form $v_n^{-1}(1 \times 1)$. Subtracting this summand we obtain a rational element, say $\bar{\sigma}$, in $K(n)^{p^n-1}(\overline{R}_m \otimes \overline{R}_m)$ which comes from a rational element in $CK(n)^{p^n-1}(\overline{R}_m \otimes \overline{R}_m)$. To prove the indecomposability of L it is sufficient to prove its indecomposability modulo p.

We denote by Ch^* the Chow theory modulo p. The Chow motive $\overline{R}_m$ is a direct sum of Tate motives with pairwise distinct twists, the Chow motive R_m is indecomposable over F and some power of any rational cycle in $\operatorname{End}(\overline{R}_m)$ is a rational projector. Therefore, one can see that the only rational cycles in $\mathrm{Ch}^{p^n-1}(\overline{R}_m \otimes \overline{R}_m)$ are scalar multiples of the diagonal.

Thus, by dimensional reasons $\bar{\sigma}$ is of the form $a\Delta_{\overline{R}_m} + bv_n(\mathrm{pt} \times \mathrm{pt})$, where $a, b \in \mathbb{Z}/p$, $\Delta_{\overline{R}_m}$ is the diagonal of $\overline{R}_m$ and $\mathrm{pt} \times \mathrm{pt}$ is the class of a rational point on $\overline{X} \times \overline{X}$.

Therefore, the original rational projector in $K(n)^*(\overline{R}_m \otimes \overline{R}_m)$ is modulo p of the form

$$a\Delta_{\overline{R}_m} + bv_n(\mathrm{pt} \times \mathrm{pt}) + cv_n^{-1}(1 \times 1)$$

for some $c \in \mathbb{Z}/p$. Composing this element with itself and using that

$$\begin{aligned}
(\mathrm{pt} \times \mathrm{pt}) \circ (1 \times 1) &= 1 \times \mathrm{pt}, \\
(1 \times 1) \circ (\mathrm{pt} \times \mathrm{pt}) &= \mathrm{pt} \times 1, \\
(\mathrm{pt} \times \mathrm{pt}) \circ (\mathrm{pt} \times \mathrm{pt}) &= 0, \\
(1 \times 1) \circ (1 \times 1) &= u \cdot v_n(1 \times 1),
\end{aligned}$$

we obtain that this element is a projector only if $(a, b, c) = (0, 0, 0)$ (the trivial projector), $(a, b, c) = (1, 0, 0)$ (the diagonal), $(a, b, c) = (0, 0, u^{-1})$ (the projector $v_n^{-1} \cdot u^{-1}(1 \times 1)$), or $(a, b, c) = (1, 0, -u^{-1})$ (the complementary projector $\Delta_{\overline{R}_m} - v_n^{-1} \cdot u^{-1}(1 \times 1)$). Thus, the motive L is indecomposable. □

REMARK 6.8. – Recall that some generalized Rost motives appear as direct summands of motives of some twisted flag varieties (e.g., Pfister quadrics for $p = 2$ or varieties of type F_4 for $(m, p) = (3, 3)$ or $(5, 2)$ or of type E_8 for $(m, p) = (3, 5)$; see [48], [37], [30], [42, Section 7]). The above proposition demonstrates a difference between K^0 and the Morava $K(n)$-theory, when $n > 1$. By [39] K^0 of all twisted flag varieties is $\mathbb{Z}$-torsion-free. This is not the case for $K(n)^*$, $n > 1$.

Moreover, the same arguments as in the proof of the proposition show that the connective K-theory $CK(1)^*$ (see [2]) of Rost motives R_m for $m > 2$ contains non-trivial $\mathbb{Z}$-torsion.

REMARK 6.9. – The same proof shows that the Johnson-Wilson theory $E(n)^*$ of the Rost motive R_m is split, if $n < m-1$. By definition, the coefficient ring of the Johnson-Wilson theory $E(n)^*$ equals $\mathbb{Z}_{(p)}[v_1, \dots, v_n][v_n^{-1}]$.

REMARK 6.10. – The Chow groups of the Rost motives are known; see [45, Theorem 5], [21, Theorem 8.1], [22, Theorem RM.10], [67, Corollary 10.8], [57, Section 4.1].

The proof of the following proposition is close to [1, Section 8].

PROPOSITION 6.11. – *Let A^* be an oriented generically constant cohomology theory in the sense of Levine-Morel satisfying the localization axiom. Let Z be a smooth variety over a field F. Assume that there exists a smooth projective variety Y with invertible Euler characteristic with respect to A^* and such that for every point $y \in Y$ (not necessarily closed) the natural pullback*

$$A^*(F(y)) \to A^*(Z_{F(y)})$$

is an isomorphism.

Then the pullback of the structural morphism $Z \xrightarrow{\pi} \operatorname{Spec} F$ induces an isomorphism

$$A^*(F) \xrightarrow{\sim} A^*(Z).$$

Before proving this proposition we prove the following lemma.

LEMMA 6.12. – *Let X be a variety over F, let Z be a smooth variety over F and let A^* be as in Proposition 6.11. Assume that the natural pullback $A^*(F(x)) \to A^*(Z_{F(x)})$ is an isomorphism for every point $x \in X$. Then the pullback $A^*(X) \to A^*(Z \times X)$ of the projection is surjective.*

Proof. – We use the Borel-Moore homology theory associated with A^* as explained in Section 4.3.

Let $X_1, \dots, X_l$ be the irreducible components of X with generic points $x_1, \dots, x_l$. We have the following commutative diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^{l} \operatorname*{colim}_{X' \subset X_i} A_*(X') \arrow[r] \arrow[d] & A_*(X) \arrow[r] \arrow[d] & \bigoplus_{i=1}^{l} A_*(F(x_i)) \arrow[r] \arrow[d] & 0 \\
\bigoplus_{i=1}^{l} \operatorname*{colim}_{X' \subset X_i} A_*(Z \times X') \arrow[r] & A_*(Z \times X) \arrow[r] & \bigoplus_{i=1}^{l} A_*(Z_{F(x_i)}) \arrow[r] & 0,
\end{tikzcd}$$

where the vertical arrows are pullbacks of the respective projections, the colimits are taken over all closed codimension ≥ 1 subvarieties of irreducible components of X and the rows are exact by the localization property.

By the assumptions the right vertical arrow is an isomorphism. Note that every closed subvariety X' of X satisfies the assumption of the lemma. Therefore, we can argue by induction on the dimension of varieties X' that the left vertical arrow is surjective. It follows by a diagram chase that the middle vertical arrow is surjective as well. □

Proof of Proposition 6.11. – We omit gradings in the proof.

Let $a: Y \to \operatorname{Spec} F$ be the structural morphism, let $b: Z \times Y \to Y$ and $c: Z \times Y \to Z$ be the projections. Consider now the following commutative diagram:

$$\begin{tikzcd}
A(F) \arrow[rrr, "\pi^*"] \arrow[dr, "a^*"] \arrow[dd, "\simeq"] & & & A(Z) \arrow[dl, "c^*"'] \arrow[dd, "\simeq"] \\
& A(Y) \arrow[r, "b^*"] \arrow[dl, "a_*"] & A(Z \times Y) \arrow[dr, "c_*"] & \\
A(F) \arrow[rrr, "\pi^*"] & & & A(Z).
\end{tikzcd}$$

By Lemma 6.12 applied to the variety Y the homomorphism b^* is surjective. The left and the right vertical arrows are isomorphisms, since they are multiplications by the A^*-Euler characteristic of Y which is invertible.

Therefore, by a diagram chase the bottom horizontal arrow is surjective. But $A(F)$ is a direct summand of $A(Z)$ because the theory A^* is generically constant. Therefore, the bottom arrow is an isomorphism. □

Let $(a_1) \cup \cdots \cup (a_m) \in H^m_{\mathrm{et}}(F, \mathbb{Z}/2)$ be a pure symbol, $a_i \in F^\times$. The quadratic form $q = \langle\langle a_1, \ldots, a_{m-1}\rangle\rangle \perp \langle -a_m\rangle$ is called a *norm form* and the respective projective quadric given by $q = 0$ is called a (projective) *norm quadric*. The respective *affine norm quadric* is an open subvariety of the projective norm quadric given by the equation

$$\langle\langle a_1, \ldots, a_{m-1}\rangle\rangle = a_m,$$

i.e., setting the last coordinate to 1.

Corollary 6.13. – *Let* $0 \leq n < m - 1$ *and set* $p = 2$. *Consider the affine norm quadric* X^{aff} *of dimension* $2^{m-1} - 1$ *corresponding to a pure symbol in* $H^m_{\mathrm{et}}(F, \mathbb{Z}/2)$. *Then the pullback of the structural morphism* $X^{\mathrm{aff}} \xrightarrow{\pi} \operatorname{Spec} F$ *induces an isomorphism*

$$K(n)^*(X^{\mathrm{aff}}) = K(n)^*(F).$$

Proof. – Let $\alpha := (a_1) \cup \cdots \cup (a_m) \in H^m_{\mathrm{et}}(F, \mathbb{Z}/2)$ be our pure symbol, $a_i \in F^\times$, q the norm form for α, and Q the respective projective norm quadric given by $q = 0$. Let Y be the projective norm quadric of dimension $2^n - 1$ corresponding to the subsymbol

$$(a_1) \cup \cdots \cup (a_{n+1}) \in H^{n+1}_{\mathrm{et}}(F, \mathbb{Z}/2).$$

We need to check the conditions of Proposition 6.11. By the choice of Y it is a v_n-variety (see, e.g., [52, Section 2]). Therefore, by Section 4.7 its $K(n)^*$-Euler characteristic is invertible.

Moreover, the quadratic form q is split completely over $F(y)$ for any point y of Y. In particular, $X^{\mathrm{aff}}_{F(y)}$ is a split odd-dimensional affine quadric. The complement $Q' := Q \setminus X^{\mathrm{aff}}$ is the projective Pfister quadric $\langle\langle a_1, \dots, a_{m-1}\rangle\rangle = 0$ of dimension $2^{m-1}-2$, and both Q and Q' are split over $F(y)$.

Let W be a split affine quadric of odd dimension $2k-1 \geq 1$. Then it is well known that W is given by the equation $\sum_{i=1}^{k} x_i y_i = 1$ in the affine space $\mathbb{A}^k \times \mathbb{A}^k$, and the projection $W \to \mathbb{A}^k \setminus \{0\}$, $(x, y) \mapsto y$, is a rank $(k-1)$ affine bundle over $\mathbb{A}^k \setminus \{0\}$. Therefore, by homotopy invariance $K(n)^*(X^{\mathrm{aff}}_{F(y)}) = K(n)^*(W) = K(n)^*(\mathbb{A}^k \setminus \{0\}) = K(n)^*(F(y))$ with $k = 2^{m-2}$. We are done. $\square$

REMARK 6.14. – In the proof of Corollary 6.13 the motive of W in the category DM of Voevodsky (see [32, Lecture 14]) is isomorphic by homotopy invariance to the motive of $\mathbb{A}^k \setminus \{0\}$. The Gysin exact triangle [32, 14.5.5] immediately implies that the motive of $\mathbb{A}^k \setminus \{0\}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}(k)[2k-1]$. In particular, the motive of X^{aff} in the category DM of motives of Voevodsky is not a Tate motive even if the base field is algebraically closed.

Let now B be a central simple F-algebra of a prime degree p and $c \in F^{\times}$. Consider the Merkurjev-Suslin variety

$$\mathrm{MS}(B, c) = \{\alpha \in B \mid \mathrm{Nrd}(\alpha) = c\},$$

where Nrd stands for the reduced norm on B.

COROLLARY 6.15. – *In the above notation the structural morphism induces an isomorphism*

$$A^*(\mathrm{MS}(B, c)) \simeq A^*(F),$$

when A is Grothendieck's K^0 or the first Morava K-theory with respect to the prime p.

Proof. – Let $Y = \mathrm{SB}(B)$ denote the Severi-Brauer variety of B. We need to check the conditions of Proposition 6.11. The variety Y is a geometrically cellular v_1-variety (see [33, Section 7.2]). Thus, by Section 4.7 its A^*-Euler characteristic is invertible.

Over a point $y \in Y$ the variety $\mathrm{MS}(B, c)$ is isomorphic to SL_p, since $\mathrm{MS}(B, c)$ over $F(y)$ is an SL_p-torsor over $F(y)$ and $H^1_{\mathrm{et}}(F(y), \mathrm{SL}_p)$ is trivial. Since GL_p is an open subvariety in $\mathbb{A}^{p^2}$, by the localization sequence $\Omega^*(\mathrm{GL}_p) = \mathbb{L}$. Moreover, GL_p is isomorphic as a variety (not as a group scheme) to $\mathrm{SL}_p \times \mathbb{G}_m$ with the isomorphism sending a matrix α to the pair $(\beta, \det \alpha)$ where β is obtained from α by dividing its first row by $\det \alpha$. The composite morphism

$$\mathrm{SL}_p \hookrightarrow \mathrm{GL}_p \xrightarrow{\simeq} \mathrm{SL}_p \times \mathbb{G}_m \to \mathrm{SL}_p,$$

where the first morphism is the natural embedding and the last morphism is the projection, is the identity. Taking pullbacks in this sequence, one gets that $\Omega^*(\mathrm{SL}_p) = \mathbb{L}$ and, hence, $A^*(\mathrm{SL}_p) = A^*(F(y))$ for A^* as in the statement of the present corollary. We are done. $\square$

Let J be an Albert algebra over F (see [23, Chapter IX]) and N_J denote the cubic norm form on J. For $d \in F^{\times}$ consider the variety

$$Z = \{\alpha \in J \mid N_J(\alpha) = d\}.$$

The group G of isometries of N_J is a group of type ${}^1\mathrm{E}_6$ and it acts on Z geometrically transitively. Note also that Z is in general anisotropic.

Corollary 6.16. – *In the above notation the natural map $K^0(F) \to K^0(Z)$ is an isomorphism.*

Proof. – Let Y be the variety of Borel subgroups of the group G. We need to check the conditions of Proposition 6.11. The K^0-Euler characteristic of Y is invertible, since Y is geometrically cellular.

Let $y \in Y$ be a point. Then G splits over $F(y)$, the variety Z has a rational point over $F(y)$ and its stabilizer is the split group of type F_4, i.e., Z is isomorphic to $\mathrm{E}_6/\mathrm{F}_4$ over $F(y)$, where E_6 and F_4 stand for the split groups of the respective Dynkin types. Finally, by [68, Theorem 2] $K^0(\mathrm{E}_6/\mathrm{F}_4) \simeq K^0(F(y))$. We are done. □

Remark 6.17. – In [68] Yakerson computes the whole higher K-theory of twisted forms of $\mathrm{E}_6/\mathrm{F}_4$ by means of cocycles from $Z^1(F, \mathrm{F}_4)$. Note that such twisted forms are isotropic.

Consider the Witt-ring of the field F and denote by I its fundamental ideal.

Proposition 6.18. – *Let $m \geq 2$ and set $p = 2$. A non-degenerate even-dimensional quadratic form q belongs to I^m iff the $K(n)^*$-motive of the respective projective quadric is split for all $0 \leq n < m-1$.*

Remark 6.19. – Note that by Proposition 7.10 below the $K(m-2)^*$-motive of the respective quadric splits iff its $K(n)^*$-motive is split for all $0 \leq n < m-1$. We do not use this in the proof of Proposition 6.18.

Proof of Proposition 6.18. – Note that the statement of the proposition is clear for $m = 2$, since $K(0)^*$ is defined as $\mathrm{CH} \otimes \mathbb{Q}$ and I^2 consists of all non-degenerate even-dimensional quadratic forms with trivial discriminant. Therefore, q belongs to I^2 iff the $K(0)^*$-motive of the respective projective quadric is split. So, we assume that $m > 2$. It is also clear that it suffices to prove the proposition for $0 < n < m-1$.

Assume that q does not belong to I^m. Let $1 \leq s < m$ be the maximal integer with $q \in I^s$, and assume that the $K(s-1)^*$-motive of the respective quadric is split. If $s = 1$, then as was mentioned above its $K(0)^*$-motive is not split. So, we can assume that $s > 1$.

By [38, Theorem 2.10] there exists a field extension K of F such that the anisotropic part of q_K is similar to an anisotropic s-fold Pfister form, say q'. Thus, q_K is isomorphic to an orthogonal sum of q' (up to a scalar multiple) and a hyperbolic form. Let Q and Q' be the projective quadrics over K associated with q_K and q' resp. By [47, Proposition 2] the Chow motive of Q is isomorphic to a sum of Tate motives and a Tate twist of the motive of Q'. Therefore, by Vishik-Yagita [62, Section 2] the same decomposition holds for the cobordism motives and, hence, for the Morava motives. But by Proposition 6.2 the $K(s-1)^*$-motive of Q' and, hence, of Q is not split. Contradiction.

Conversely, assume that q belongs to I^m and let Q be the respective projective quadric. Let $1 \leq n < m-1$. Then we can present q as a finite sum of (up to proportionality) m-fold Pfister forms. We prove our statement using induction on the length of such a presentation in the Witt-ring. If the length is zero, i.e., if q is a split form, then the $K(n)^*$-motive of Q is split for all n.

Let α be an m-fold Pfister form in the decomposition of q. Let X^{aff} be the affine norm quadric of dimension $2^{n+1}-1$ corresponding to a subsymbol of α from $H^{n+2}_{\mathrm{et}}(F, \mathbb{Z}/2)$ (note that $n+2 \leq m$). Then the length of q over $F(X^{\mathrm{aff}})$ is strictly smaller than the length of q over F.

Applying Lemma 6.12 to the varieties $X = Q \times Q$ and $Z = X^{\mathrm{aff}}$ and using Corollary 6.13 we obtain that the pullback of the natural projection

$$K(n)^*(Q \times Q) \to K(n)^*(X^{\mathrm{aff}} \times Q \times Q)$$

is surjective.

But by the localization sequence

$$K(n)^*(X^{\mathrm{aff}} \times Q \times Q) \to K(n)^*((Q \times Q)_{F(X^{\mathrm{aff}})})$$

is surjective. By the induction hypothesis on the length of q, the restriction homomorphism

$$K(n)^*((Q \times Q)_{F(X^{\mathrm{aff}})}) \to K(n)^*((Q \times Q)_{\widetilde{F}})$$

to a splitting field $\widetilde{F}$ of $Q_{F(X^{\mathrm{aff}})}$ is surjective. Therefore, the restriction homomorphism

$$K(n)^*(Q \times Q) \to K(n)^*((Q \times Q)_{\widetilde{F}})$$

is surjective.

In particular, since the projectors for the Morava motive of Q lie in $K(n)^*(Q \times Q)$, it follows from Rost nilpotence that the $K(n)^*$-motive of Q over F is split. □

REMARK 6.20. – The same statement with a similar proof holds for the variety of totally isotropic subspaces of dimension k for all $1 \leq k \leq (\dim q)/2$. Note that in the case of a Pfister form, the motive of such a variety is still a direct sum of Rost motives.

The same proof also shows the following proposition.

PROPOSITION 6.21. – *If $q \in I^m$ for some $m \geq 2$ and $q' = q \perp \langle c \rangle$ for some $c \in F^\times$, then the $K(n)^*$-motive of the respective projective quadric $q' = 0$ is split for all $0 \leq n < m-1$.*

Conversely, if q' is an odd-dimensional quadratic form such that the $K(n)^*$-motive of the respective projective quadric $q' = 0$ is split for some $n \geq 0$, then in the Witt ring $q' = q \perp \langle c \rangle$ for some $q \in I^{n+2}$ and $c \in F^\times$.

Proof. – The proof is almost verbatim as of Proposition 6.18. For the reader's convenience we sketch the proof of the second part of the proposition.

Set $q = q' \perp \langle -\mathrm{disc}(q') \rangle$ in the Witt ring. Then $q \in I^{n+2}$. Indeed, otherwise as in the proof of Proposition 6.18 over some field extension K of F the anisotropic part of the form q_K will be similar to an anisotropic s-fold Pfister form with $2 \leq s < n+2$. Then the motive of the respective Pfister quadric is a direct sum of Tate twists of Rost motives R_s. But by Proposition 6.2 the $K(n)^*$-motive R_s is not split. Besides, it follows from [47, Theorem 17] that the motive of the quadric $q' = 0$ contains over K a Tate twist of R_s. Hence, the $K(n)^*$-motive of the projective quadric $q' = 0$ is not split over K and, hence, is not split over F. □

7. $K(n)$-split varieties and p-torsion in Chow groups

In this section we obtain several general results concerning Morava K-theories. First, with the help of the Landweber-Novikov operations we prove that if a projective homogeneous variety is split with respect to $K(n)^*$, then it is split with respect to $K(m)^*$ for $m\colon 1 \leq m \leq n$ (Corollary 7.11). Recall that by results of [62] if the motive of a smooth projective variety X is split with respect to the Chow theory, then it is split for every oriented theory.

Thus, to prove the non-splitting of the p-local Chow motive of a projective homogeneous variety one could consecutively check the splitting of the Morava motives $M_{K(1)}$, $M_{K(2)}$, etc. If one of these motives is non-split, then the p-local Chow motive is non-split as well. Conversely, if all Morava motives are split, then the p-local Chow motive is split as well. In fact, by Corollary 7.12 below it suffices to consider the n-th Morava K-theory such that p^n is greater than or equal to the dimension of the variety. In this sense one could interpret $\mathrm{CH}^* \otimes \mathbb{Z}_{(p)}$ as a Morava K-theory of an infinite height.

Secondly, we investigate properties of smooth projective geometrically cellular varieties X for which the pullback restriction map $K(n)^*(X) \to K(n)^*(\overline{X})$ is an isomorphism. Using symmetric operations we show in Theorem 7.19 that $\mathrm{CH}^i(X)$ has no p-torsion for such varieties where $i \leq \frac{p^n-1}{p-1}$. Finally, in Theorem 7.23 we use the gamma filtration on $K(n)^*$ to prove finiteness of p-torsion in Chow groups of such varieties in codimensions up to p^n.

7.1. Landweber-Novikov operations and split $K(n)^*$-motives

As was mentioned in Section 4.4 every formal group law $(R, \mathcal{F}_R)$ yields a free theory $\Omega^* \otimes_{\mathbb{L}} R$. It is natural to ask about relationships between free theories corresponding to isomorphic formal group laws. For simplicity, taking an isomorphism between formal group laws $(R, \mathcal{F}_R)$ and $(R, \mathcal{F}'_R)$ which is the identity on R (i.e., it is a change of the "parameter" of the formal group law), one obtains an isomorphism of the presheaves of rings of corresponding free theories. Moreover, a considerable part of such isomorphisms can be obtained via the specialization of the total Landweber-Novikov operations on the level of algebraic cobordism. These operations put severe constraints on the structure of the algebraic cobordism as an $\mathbb{L}$-module, and we will use this in the study of the Morava K-theories of $K(n)^*$-split varieties.

Recall that there exists a graded Hopf algebroid $(\mathbb{L}, \mathbb{L}B)$ which represents the formal group laws and strict isomorphisms between them (see, e.g., [43, App. 1.1.1, App. 2.1.16]), where $\mathbb{L}B = \mathbb{L}[b_1, b_2, \ldots]$. The total Landweber-Novikov operation

$$S^{\mathrm{tot}}_{L-N}\colon \Omega^* \to \Omega^* \otimes_{\mathbb{L}} \mathbb{L}B$$

is a multiplicative operation which in some sense corresponds to the universal strict isomorphism of formal group laws, see [60, Example 3.9].

Proposition 7.2 ([50, Proposition 2.10]). – *The action of the Landweber-Novikov operations makes Ω^* into a functor to the graded comodules over the Hopf algebroid* $(\mathbb{L}, \mathbb{L}B)$.

In particular, $\mathbb{L} = \Omega^*(\operatorname{Spec} F)$ is canonically a comodule over $(\mathbb{L}, \mathbb{L}B)$, and its subcomodules are the same as the ideals which are invariant with respect to the Landweber-Novikov operations. The only non-zero prime ideals among them are $I(p,m) = (p, v_1, \ldots, v_{m-1})$ and $I(p) = \bigcup_m I(p,m)$, where p is a prime number and v_i's are v_i-elements ([26, Theorem 2.2]).

The situation with BP^* is very similar with Ω^*. For every smooth variety X the BP-module $BP^*(X)$ is a direct summand of $\Omega^*(X) \otimes \mathbb{Z}_{(p)}$ (see, e.g., [50, Proposition 2.4]), and one can restrict the action of the Landweber-Novikov operations to $BP^*(X)$ which makes it a graded comodule over the Hopf algebroid (BP, BP_*BP) (for the latter see [43, Appendix 2.1.27]). In particular, there is an action of the Landweber-Novikov operations on BP and the only non-zero invariant prime ideals are of the form $I(k) = (p, v_1, \ldots, v_{k-1})$ ([26, Theorem 2.2_{BP}]).

The abelian category of comodules over $(\mathbb{L}, \mathbb{L}B)$ (or over (BP, BP_*BP)) was extensively studied by topologists. Note also that $(\mathbb{L}, \mathbb{L}B)$ is canonically isomorphic to $(MU_*, MU_*(MU))$, and the latter notation is often used in the literature.

Proposition 7.3 ([25, Theorem 3.3], [26, Theorem 2.2, 2.3, 2.2_{BP}, 2.3_{BP}]).
Let M be a graded comodule over $(\mathbb{L}, \mathbb{L}B)$ (over (BP, BP_*BP), respectively) which is finitely presented as an $\mathbb{L}$-module (as a BP-module, respectively). Then M has a filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_d = 0$$

such that for every i the module M_i/M_{i+1} is isomorphic to $\mathbb{L}/I(p_i, n_i)$ or $\mathbb{L}$ ($BP/I(m_i)$ or BP, respectively) after a shift of grading, where p_i is a prime number and n_i is a positive integer (m_i is a positive integer, respectively).

Corollary 7.4. – *Fix a prime p and for $s \geq 1$ denote by $K(s) \cong \mathbb{Z}_{(p)}[v_s, v_s^{-1}]$ the $\mathbb{L}$-algebra corresponding to a choice of a formal group law for a Morava K-theory $K(s)^*$. If for M as in Proposition 7.3 we have $M \otimes_{\mathbb{L}} K(n) = 0$ for some $n \geq 1$, then $M \otimes_{\mathbb{L}} K(m) = 0$ for all m: $1 \leq m < n$.*

Proof. – We call by a filtration of an $(\mathbb{L}, \mathbb{L}B)$-comodule M just any filtration from Proposition 7.3. We will prove a stronger statement by induction on the minimal length d of a filtration of M. Namely, if $M \otimes_{\mathbb{L}} K(n) = 0$, then $\mathrm{Tor}_i^{\mathbb{L}}(M, K(m)) = 0$ for all $i \geq 0$ and $m \leq n$, and the graded factors of the filtration on M can be only of the form $\mathbb{L}/I(q,k)$ with $q \neq p$ or with $q = p$ and $n \leq k-1$.

For the base of induction $d = 1$ we just need to check the statement for modules $\mathbb{L}/I(q,k)$. In both cases (if $q \neq p$ or if $q = p$ and $n \leq k-1$) $\mathrm{Tor}_i^{\mathbb{L}}(\mathbb{L}/I(q,k), K(m)) = 0$ because it is naturally both a $K(m)$-module and an $\mathbb{L}/I(q,k)$-module (compatible with the structure of an $\mathbb{L}$-module). If $q \neq p$, then q is invertible in $K(m) = \mathbb{Z}_{(p)}[v_m, v_m^{-1}]$. If $q = p$ and $m \leq k-1$, then v_m is invertible in $K(m)$ and is zero in $\mathbb{L}/I(q,k)$. Therefore, $\mathrm{Tor}_i^{\mathbb{L}}(\mathbb{L}/I(q,k), K(m)) = 0$ for all $i \geq 0$. Clearly, if $q = p$ and $n > k-1$, then $\mathbb{L}/I(q,k) \otimes_{\mathbb{L}} K(n) \neq 0$.

For the induction step suppose that M has a filtration of length $d+1$, which means that there exists a short exact sequence of $(\mathbb{L}, \mathbb{L}B)$-comodules:

$$0 \to N \to M \to \mathbb{L}/I(q,k) \to 0,$$

where N has a filtration of length d. Tensoring this sequence with $K(n)$, we see from the above that either $q \neq p$ or $q = p$ and $n \leq k-1$. Tensoring with $K(m)$, $1 \leq m \leq n$, we obtain that $N \otimes K(n) = 0$ and $\mathrm{Tor}_i^{\mathbb{L}}(N, K(m)) \simeq \mathrm{Tor}_i^{\mathbb{L}}(M, K(m))$ for all $i \geq 0$ and $1 \leq m \leq n$. We then apply the induction hypothesis to N to conclude that $\mathrm{Tor}_i^{\mathbb{L}}(M, K(m)) = 0$ for all $i \geq 0$ and all m, $1 \leq m \leq n$. $\square$

Remark 7.5. – Let $\mathrm{CH}_{(p)}$ denote the coefficient ring of $\mathrm{CH}^* \otimes \mathbb{Z}_{(p)}$ and let M be as in Corollary 7.4. Analogously one can show that if $M \otimes_{\mathbb{L}} \mathrm{CH}_{(p)} = 0$, then $M \otimes_{\mathbb{L}} K(m) = 0$ for all $m \geq 1$.

Remark 7.6. – The language of stacks might provide a more geometric view on the statement above. Indeed, the category of graded comodules over the Hopf algebroid $(\mathbb{L}, \mathbb{L}B)$ can be identified with the category of quasi-coherent sheaves over the stack of formal groups $\mathcal{M}_{\mathrm{fg}}$ (see, e.g., [36]). Working modulo p this stack has an exhaustive descending filtration by closed substacks where the n-th piece of it $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ classifies formal groups of height bigger than or equal to n. Moreover, these substacks are the only irreducible closed (reduced) substacks, and $\mathcal{M}_{\mathrm{fg}}^{\geq n+1}$ is in some sense a divisor in $\mathcal{M}_{\mathrm{fg}}^{\geq n}$ whose complement has a unique geometric point which corresponds to the n-th Morava K-theory.

The support of a coherent sheaf $\mathcal{G}$ over $\mathcal{M}_{\mathrm{fg}}$ is closed, and therefore the reduced support is the closed substack $\mathcal{M}_{\mathrm{fg}}^{\geq m}$ for some m. In particular, the fiber of $\mathcal{G}$ over the points corresponding to the n-th Morava K-theory is zero if $n < m$ and non-zero if $m \geq n$. This gives a vague explanation of Corollary 7.4.

Corollary 7.7. – *Let C be a finitely presented BP-module endowed with the structure of a (BP, BP_*BP)-comodule.*

If $C \otimes_{BP} K(n) = 0$, then $C \otimes_{BP} BP[v_n^{-1}] = 0$.

Proof. – By Proposition 7.3 the BP-module C has a filtration with the graded factors $BP/I(k_i)$. The same proof as of Corollary 7.4 in which one replaces $\mathbb{L}$ with BP and $I(p,k)$ with $I(k)$ shows that if $C \otimes_{BP} K(n) = 0$, then for the graded factors of the filtration above for all i we have $n \leq k_i - 1$, i.e., $v_n \in I(k_i)$. The claim follows. □

The following lemma is straightforward.

Lemma 7.8. – *Let X be a geometrically cellular smooth projective variety over a field F and let A^* be a free oriented cohomology theory. Assume that the A^*-motive $M_A(X)$ satisfies the Rost nilpotence property. Denote $\overline{X} = X \times_F \overline{F}$. Then the following statements are equivalent:*

1. *$M_A(X)$ is split;*
2. *the restriction map $A^*(X \times X) \to A^*(\overline{X} \times_{\overline{F}} \overline{X})$ is an isomorphism;*
3. *the restriction map $A^*(X \times X) \to A^*(\overline{X} \times_{\overline{F}} \overline{X})$ is a surjection;*
4. *the restriction map $A^*(X) \to A^*(\overline{X})$ is an isomorphism;*
5. *the restriction map $A^*(X) \to A^*(\overline{X})$ is a surjection.*

Proof. – To prove the implication (5) $\Rightarrow$ (3) note that $\overline{X}$ is cellular, its motive is split and all elements in $A^*(\overline{X})$ and, therefore, in $A^*(\overline{X} \times_{\overline{F}} \overline{X})$ are rational. The implication (3) $\Rightarrow$ (1) follows from Rost nilpotence. □

Corollary 7.9. – *Assume that two free theories A^* and B^* are isomorphic as presheaves of sets, and the motives $M_A(X)$ and $M_B(X)$ satisfy the Rost nilpotence property. Then $M_A(X)$ is split iff $M_B(X)$ is split.*

Proof. – Indeed, an isomorphism between A^* and B^* commutes with the change of the base field. Thus, whenever one of the maps $A^*(X) \to A^*(\overline{X})$, $B^*(X) \to B^*(\overline{X})$ is surjective, so is the other one. □

In particular, it follows from the above corollary and [13, Corollary 4.5] that for a fixed prime p, a fixed integer n and a projective homogeneous variety X there is a well-defined property for $M_{K(n)}(X)$ to be split which does not depend on the choice of an n-th Morava K-theory.

PROPOSITION 7.10. – *Let $1 \le m \le n$, and let X be a smooth projective geometrically cellular variety such that $M_{K(m)}(X)$ satisfies the Rost nilpotence property.*

If $M_{K(n)}(X)$ is split, then $M_{K(m)}(X)$ is split.

Proof. – By Lemma 7.8 it is sufficient to prove that the map $K(m)^*(X) \to K(m)^*(\overline{X})$ is surjective, whenever $K(n)^*(X) \to K(n)^*(\overline{X})$ is so.

Consider the following short exact sequence of $(\mathbb{L}, \mathbb{L}B)$-comodules:

$$\Omega^*(X) \xrightarrow{\rho} \Omega^*(\overline{X}) \to C \to 0.$$

Clearly, the map $\rho \otimes_{\mathbb{L}} K(m)$ is surjective iff $C \otimes_{\mathbb{L}} K(m) = 0$. However, $C \otimes_{\mathbb{L}} K(n) = 0$ by the assumption, and C is a coherent $\mathbb{L}$-module by [50, Proposition 2.21, Remark 2.24]. Therefore, Corollary 7.4 applies. □

COROLLARY 7.11. – *If X is a projective homogeneous variety such that $M_{K(n)}(X)$ is split, then $M_{K(m)}(X)$ is split for all $1 \le m \le n$.*

Proof. – By [13, Corollary 4.5] for every free theory A^* the motive $M_A(X)$ satisfies the Rost nilpotence property. □

COROLLARY 7.12. – *Let X be a projective homogeneous variety with* $\dim X \le p^n$. *Then the Chow motive of X with $\mathbb{Z}_{(p)}$-coefficients is split if and only if the $K(n)^*$-motive of X is split.*

Proof. – If the Chow motive of X with $\mathbb{Z}_{(p)}$-coefficients is split, then obviously the $K(n)^*$-motive of X is split as well.

Assume now that the $K(n)^*$-motive of X is split. The operations

$$c_i^{\mathrm{CH}}\colon K(n)^*(X) \to \mathrm{CH}^i(X) \otimes \mathbb{Z}_{(p)}$$

are surjective for $i \le p^n$ and commute with extensions of scalars (see Theorem 5.7).

Therefore, condition (5) of Lemma 7.8 is satisfied for $A^* = \mathrm{CH}^* \otimes \mathbb{Z}_{(p)}$. This implies the corollary. □

7.13. Symmetric operations of Vishik and $K(n)^*$-split varieties

We have used above the Landweber-Novikov operations which are stable ([60, Definition 3.4]) and provide constraints on the structure of cobordism which do not “see” the grading. Being interested in the Chow groups and in the topological filtration of small codimension we employ more subtle unstable operations, among which the most powerful are symmetric operations.

Recall that Vishik has defined symmetric operations in algebraic cobordism first for $p = 2$ in [57] using elaborate and elegant constructions and then for all primes in [59] using [60, 61, Theorems 5.1] classifying all operations. We follow the latter approach and explain several properties of these operations.

Fix a set of integers $\bar{i} = \{i_j \mid 0 < j < p\}$ of all representatives of non-zero integers modulo p, and denote $\mathbf{i} = \prod_{j=1}^{p-1} i_j$. There exists a Quillen-type Steenrod operation in algebraic cobordism

$$\mathrm{St}(\bar{i})\colon \Omega^* \to \Omega^*[\mathbf{i}^{-1}][[t]][t^{-1}],$$

which induces a morphism of formal group laws uniquely defined by the power series $\gamma(x) = x \prod_{j=1}^{p-1}(x +_\Omega i_j \cdot_\Omega t)$. We will sometimes drop $\bar{i}$ from the notation of St.

THEOREM 7.14 (Vishik, [59, Theorem 7.1]). – *There exists a unique operation $\Phi(\bar{i})\colon \Omega^* \to \Omega^*[\mathbf{i}^{-1}][t^{-1}]$, called the symmetric operation, such that*

$$(7.15) \qquad (\Box^p - \mathrm{St}(\bar{i}) - \frac{p \cdot_\Omega t}{t}\Phi(\bar{i}))\colon \Omega^* \to \Omega^*[\mathbf{i}^{-1}][[t]]t,$$

where $\Box^p$ is the p-power operation.

It is convenient to use “slices” of the symmetric operation $\Phi(\bar{i})$, defined as the coefficients of the monomials t^l for $l \leq 0$. We will denote these operations as $\Phi_l(\bar{i}) = \Phi_l$.

Fix a prime p and for simplicity we will work p-locally, in particular, using BP^* instead of Ω^*. Recall that there exists a multiplicative projector on $\Omega^* \otimes \mathbb{Z}_{(p)}$ making BP^* a direct summand of $\Omega^* \otimes \mathbb{Z}_{(p)}$. This allows one to restrict the symmetric operation to BP^* even though it is non-additive (see [58, Section 3]).

Recall that following Hazewinkel we have chosen the generators v_n of the ring BP (see Section 4.6). Symmetric operations Φ_l allow to “divide” certain elements of BP^* by elements v_n as was observed, e.g., in [50, Section 3.2]. The following is an instance of this property.

PROPOSITION 7.16. – *Let $k > 0$ and let $\alpha \in BP^{-k(p^n-1)}$ such that $\alpha \equiv v_n^k \mod I(n)$. Then $\Phi_{-k(p-1)(p^n-1)-(p^n-1)}(\alpha) \equiv -v_n^{k-1} \mod I(n)$.*

Proof. – By the definition of the symmetric operation we have the following identity in the ring $BP[[t]][t^{-1}]$ in the coefficients of $t^{\leq 0}$:

$$(7.17) \qquad \alpha^p - \mathrm{St}(\alpha) =^{t^{\leq 0}} [p] \cdot \Phi(\alpha), \qquad [p] := \frac{p \cdot_{BP} t}{t}.$$

Recall that St is a generalized specialization of the total Landweber-Novikov operation ([58, p. 977]), i.e., it can be obtained from the unstable total Landweber-Novikov operation $\Omega^* \to \Omega^* \otimes_{\mathbb{L}} \mathbb{L}[b_0^{\pm 1}, b_1, b_2, \ldots]$, defined by the inverse Todd genus $\sum_{i=0}^\infty b_i t^i$. Therefore, it is an (infinite) BP-linear combination of the Landweber-Novikov operations (see [58, Section 3] for more details). In particular, St preserves the ideals $I(n)$, hence $\mathrm{St}(\alpha) \equiv \mathrm{St}(v_n^k)$ mod $I(n)$.

It follows from the Riemann-Roch theorem for multiplicative operations (see, e.g., [50, Lemma 2.16]) that

$$\mathrm{St}(v_n) \equiv v_n t^{-(p-1)(p^n-1)} \mod I(n).$$

The series $[p]$ appearing above is graded of degree 0 where we take $\deg t = 1$. Moreover, it starts with p, and therefore modulo $I(n)$ the smallest power of t appearing in it is equal to $p^n - 1$ and its coefficient is proportional to v_n. The choice of Hazewinkel implies that it is exactly v_n ([43, A2.2.4]), i.e., $[p] = v_n t^{p^n-1}$ + higher degree terms.

Combining all this together, Equation (7.17) modulo $I(n)$ looks as:

$$v_n^{kp} - v_n^k t^{-k(p-1)(p^n-1)} \equiv^{t^{\leq 0}} (v_n t^{p^n-1} + \text{higher degree terms})\Phi(\alpha) \mod I(n),$$

from which the statement follows using [50, Lemma 3.3] and the fact that $BP/I(n)$ is an integral domain. □

The previous proposition can be used to study rational elements in the BP^*-theory as the following lemma shows. It will be a crucial step in the proof of Theorem 7.19 below.

For an element $z \in BP^r(X)$ we write $\deg z = r$.

LEMMA 7.18. – *Let $f\colon X \to Y$ be a morphism of smooth quasi-projective varieties. Let $z \in BP^r(X)$. Assume that $r > \frac{p^n-1}{p-1}$ and for some $k \geq 0$ the element $v_n^k z$ belongs to the image of $BP^*(Y)$ under the map f^*.*

Then there exists a homogeneous element $\beta \in BP$ such that the element βz belongs to the image of f^* modulo $\tau^{r+1}BP^*(X)$ and

— $\beta \equiv v_n^b \mod I(n)$ *for some* $b \geq 0$*;*

— $\deg(\beta z) = r - b(p^n - 1) > \frac{p^n-1}{p-1}$.

Proof. – Let $x \in BP^*(Y)$ be such that $f^*(x) = v_n^k z$. We will apply the symmetric operation Φ to x several times producing the needed element $y \in BP^*(Y)$ such that $f^*(y) \equiv \beta z \mod \tau^{r+1}BP^*(X)$ for β as in the statement of the proposition. Since all operations commute with pullbacks, we just have to calculate how the operation Φ acts on $v_n^k z$.

Moreover, all operations preserve the topological filtration, and by [59, Proposition 7.14] there is a simple description of the action of the symmetric operation on $\mathrm{gr}_\tau^\bullet BP^*$. Namely, for any $\lambda \in BP$ and z as above we have

$$\Phi(\lambda z) \equiv \mathbf{i}^r \cdot t^{r(p-1)} \cdot \Phi_{\leq -r(p-1)}(\lambda) z \mod \tau^{r+1}BP^*(X),$$

where $\Phi_{\leq -r(p-1)}(\lambda)$ is the part of the polynomial $\Phi(\lambda) \in BP[t^{-1}]$ with the degree of t no greater than $-r(p-1)$.

Thus, to be able to use Proposition 7.16 and "divide" $v_n^k z$ by v_n we need that $k > 0$ (if $k = 0$ we do not have to do anything) and

$$-k(p-1)(p^n-1) - (p^n-1) \leq -r(p-1).$$

Equivalently, $(r - k(p^n-1))(p-1) \leq p^n - 1$ or $\deg(v_n^k z) \leq \frac{p^n-1}{p-1}$.

We can continue this process until we get the desired element βz modulo $\tau^{r+1}BP^*(X)$, where $\deg(\beta z) = r - b(p^n - 1) > \frac{p^n-1}{p-1}$. □

THEOREM 7.19. – *Let X be a smooth projective geometrically cellular variety such that the pullback map $f^*\colon K(n)^*(X) \to K(n)^*(\overline{X})$ is an isomorphism, where $\overline{X} = X \times_F \overline{F}$.*

Then the pullback maps

$$\mathrm{gr}^r_\tau K(n)^*(X) \to \mathrm{gr}^r_\tau K(n)^*(\overline{X}), \quad \mathrm{CH}^r(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}^r(\overline{X}) \otimes \mathbb{Z}_{(p)} \tag{7.20}$$

are isomorphisms for $r \le \frac{p^n-1}{p-1}$.

In particular, $\mathrm{CH}^r(X)$ *has no p-torsion for all $r \le \frac{p^n-1}{p-1}$.*

Proof. – For a smooth projective cellular variety Y and a free theory A^*, the A-module $A^*(Y)$ is free, generated by chosen classes of desingularizations of (closed) cells. We will call these elements classes of cells, and the codimension of the class of a cell is the codimension of the corresponding cell. Moreover, the r-th part of the topological filtration on $A^*(Y)$ is generated by the cells of codimension no less than r.

It follows that $\mathrm{CH}^r(\overline{X}) \otimes \mathbb{Z}_{(p)}$ is torsion-free, and the last claim of the theorem follows from claim (7.20).

For simplicity of notation we switch now from Morava K-theories with $\mathbb{Z}_{(p)}[v_n, v_n^{-1}]$-coefficients to Morava K-theories with $\mathbb{Z}_{(p)}$-coefficients by sending v_n to 1. Clearly, this does not affect neither assumptions, nor conclusions of the theorem.

Under the assumptions of the theorem the pullback map from $\mathrm{gr}^r_\tau K(n)^*(X)$ to $\mathrm{gr}^r_\tau K(n)^*(\overline{X})$ is surjective for $r\colon 0 \le r \le p^n - 1$ since

$$\mathrm{gr}^r_\tau \tilde{K}(n)^* = \tilde{K}(n)^r / \tau^{r+p^n-1}\tilde{K}(n)^r \tag{7.21}$$

in this range of r by Proposition 5.2(3). On the other hand, $\mathrm{gr}^r_\tau K(n)^*(\overline{X})$ is a free $\mathbb{Z}_{(p)}$-module generated by the classes of cells of codimension r. Thus, to prove the theorem it is sufficient to show that preimages under f^* of classes of all cells of codimension greater than $\frac{p^n-1}{p-1}$ lie in $\tau^{>\frac{p^n-1}{p-1}}K(n)^*(X)$. Indeed, this would imply that f^* is an isomorphism between $\tau^{r+p^n-1}K(n)^r(X)$ and $\tau^{r+p^n-1}K(n)^r(\overline{X})$ for $r \le \frac{p^n-1}{p-1}$, and therefore f^* is also an isomorphism on $\mathrm{gr}^r_\tau \tilde{K}(n)^r$ by Formula (7.21).

For the class z of a cell in $BP^r(\overline{X})$ denote by $z_{K(n)}$ its image in $K(n)^*(\overline{X})$. Also abusing notation we denote the preimage of this element in $K(n)^*(X)$ under f^* by the same letter.

We now argue by decreasing induction on r from $\dim X + 1$ to $\frac{p^n-1}{p-1} + 1$ that

$$z_{K(n)} \in \tau^{\frac{p^n-1}{p-1}+1}K(n)^*(X).$$

Base of induction. – The assertion is trivial for $r = \dim X + 1$, since $BP^r(X) = 0$.

Induction step. – Assume that for all classes z_s of cells in $BP^{>r}(\overline{X})$ the classes $z_{s,K(n)}$ lie in $\tau^{\frac{p^n-1}{p-1}+1}K(n)^*(X)$.

Denote by C the cokernel of the map $BP^*(X) \to BP^*(\overline{X})$. It is a finitely presented BP-module with the structure of a comodule over the Hopf algebroid (BP, BP_*BP) ([50, Proposition 2.21, Remark 2.24]). Moreover, $C \otimes K(n) = 0$ by the assumptions of the theorem, and, therefore, by Corollary 7.7 the pullback map

$$BP^*(X)[v_n^{-1}] \to BP^*(\overline{X})[v_n^{-1}]$$

is surjective. In particular, for every class of a cell $z \in BP^r(\overline{X})$ of codimension r there exists $k \geq 0$ such that $v_n^k z$ is a rational element.

If $z \in BP^r(\overline{X})$ is the class of a cell of codimension $r > \frac{p^n-1}{p-1}$, then by Lemma 7.18 applied to $f : \overline{X} \to X$ we obtain that the element $\beta z + \sum_s \alpha_s z_s \in BP^j(\overline{X})$ is rational for some $j > \frac{p^n-1}{p-1}$, $\alpha_s, \beta \in BP$ such that β maps to 1 in $K(n)$ and z_s are classes of cells of bigger codimension (recall that $\tau^{r+1}BP^*(\overline{X})$ is generated by cells of codimension at least $r+1$).

Let y be an element of $BP^j(X)$ which maps to $\beta z + \sum_s \alpha_s z_s \in BP^j(\overline{X})$ under the pullback map. Then $y \in \tau^j BP^j(X)$, since $BP^j = \tau^j BP^j$ (the last formula holds by the definition of the topological filtration and by the fact that BP contains no elements of strictly positive degree). Therefore, the image of y in $K(n)^*(X)$ also lies in $\tau^j K(n)^*(X)$, and at the same time its image in $K(n)^*(\overline{X})$ has the form $z_{K(n)} + \sum [\alpha_s]_{K(n)} z_{s,K(n)}$ where $[\alpha_s]_{K(n)}$ is the image of α_s under the canonical morphism $BP \to K(n) = \mathbb{Z}_{(p)}$. However, by the induction assumption the preimages under the isomorphism f^* of the elements $z_{s,K(n)}$ already lie in $\tau^{\frac{p^n-1}{p-1}+1}K(n)^*(X)$, hence the claim.

As explained above it follows that the pullback map $\mathrm{gr}^i_\tau K(n)^*(X) \to \mathrm{gr}^i_\tau K(n)^*(\overline{X})$ is an isomorphism for $i \leq \frac{p^n-1}{p-1}$. The operation $c_i^{\mathrm{CH}}\colon \mathrm{gr}^i_\tau K(n)^* \to \mathrm{CH}^i \otimes \mathbb{Z}_{(p)}$ commutes with pullbacks by definition and induces an isomorphism for $i \leq p^n$ by Theorem 5.7, (iv)). It follows that the map $\mathrm{CH}^i(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}^i(\overline{X}) \otimes \mathbb{Z}_{(p)}$ is also an isomorphism for $i \leq \frac{p^n-1}{p-1}$. □

7.22. Finiteness of torsion in Chow groups via the gamma filtration

We consider the Morava K-theory $K(n)^*$ with v_n set to be 1.

Above we have used the topological filtration on Morava K-theories to show that there is no p-torsion in Chow groups of certain varieties up to codimension $\frac{p^n-1}{p-1}$. However, calculating graded factors of the topological filtration $\mathrm{gr}^i_\tau K(n)^*$ in the range between $\frac{p^n-1}{p-1}+1$ and p^n seems to be out of reach at the present stage, even though it would still yield $\mathrm{CH}^i \otimes \mathbb{Z}_{(p)}$ by Theorem 5.7. Yet another approach to estimate p-torsion in Chow groups is to use the gamma filtration instead of the topological filtration.

THEOREM 7.23. – *Fix a prime p and let $K(n)^*$ be the corresponding n-th Morava K-theory. Assume that X is a geometrically cellular smooth projective variety such that the restriction map $K(n)^*(X) \to K(n)^*(\overline{X})$ is an isomorphism, where $\overline{X} = X \times_F \overline{F}$.*

Then the p-torsion in $\mathrm{CH}^j(X)$ is a quotient of the p-torsion in $\mathrm{gr}^j_\gamma K(n)^*(\overline{X})$ for $j \leq p^n$.

In particular, the p-torsion in the Chow groups of X is finite in codimensions up to p^n and it can be bounded based on the variety $\overline{X}$ only.

Proof. – As the gamma filtration is defined using the operations which commute with pullbacks by definition, the gamma filtrations on $K(n)^*(X)$ and $K(n)^*(\overline{X})$ coincide via the change of the base field. Therefore, the graded pieces of the gamma filtration of X depend only on $\overline{X}$.

Note that as $\overline{X}$ is cellular, its Chow motive is of Tate type, i.e., is split, and, therefore, its algebraic cobordism motive is of Tate type as well. Therefore, $K(n)^*(\overline{X})$ is a finitely generated free $\mathbb{Z}_{(p)}$-module generated by the classes of desingularizations of the closed cells. This proves that the graded pieces of the gamma filtration (on both $K(n)^*(X)$ and $K(n)^*(\overline{X})$) are finitely generated, and thus have finite torsion.

By Theorem 5.7, (v)) and (iii)) we have surjective additive maps

$$c_j^{\mathrm{CH}}\colon \mathrm{gr}_\gamma^j K(n)^*(X) \to \mathrm{CH}^j(X) \otimes \mathbb{Z}_{(p)}$$

for $j \leq p^n$, which are isomorphisms rationally. Therefore, $\mathrm{CH}^j(X)$ has finite torsion for every $j \leq p^n$ which is bounded above by the torsion of $\mathrm{gr}_\gamma^j K(n)^*(X)$. □

An advantage of this approach is that the calculations are of a purely combinatorial nature and are often amenable as we will show in the case of quadrics in the next section. However, the bounds obtained by the gamma filtration are not exact in general (see Remark 8.15).

8. Bounds on torsion in Chow groups via Morava K-theory

In this section we will provide some bounds on torsion in Chow groups of quadrics. Before doing this we would like to summarize known results in this direction. We apologize in advance in case we forgot to mention some contributions.

8.1. Karpenko's bounds in small codimensions

PROPOSITION 8.2 (Karpenko). – *Let Q be a smooth projective anisotropic quadric of dimension D defined over a field of characteristic not 2.*

[15, Theorem 6.1]*:* $\mathrm{Tors}\,\mathrm{CH}^2(Q) = 0$ *for* $D > 6$*;*

[17, Theorem 6.1]*:* $\mathrm{Tors}\,\mathrm{CH}^3(Q) = 0$ *for* $D > 10$*;*

[17, Theorem 8.5]*:* $\mathrm{Tors}\,\mathrm{CH}^4(Q) = 0$ *for* $D > 22$.

When the dimension of a quadric is smaller than in the above proposition, Karpenko gives some bounds for the torsion in CH^3 and CH^4 and explicitly computes CH^2 (see [15, Theorem 6.1], [18]).

We remark at this point that there are examples of quadrics having infinite torsion in CH^4 (see [20]).

8.3. Rost motives and excellent quadrics

The Chow groups of Pfister quadrics and more generally of excellent quadrics are explicitly known. This was computed by Rost in [45, Theorem 5], see also [21, Theorem 7.1, Theorem 8.1]. More generally, Yagita computed the multiplicative structure of the Chow rings of excellent quadrics (see [66]).

In particular, the following result holds.

PROPOSITION 8.4 (Rost). – *Let Q_α be the Pfister quadric corresponding to a pure non-zero symbol $\alpha \in H^n_{\mathrm{et}}(F, \mathbb{Z}/2)$, $n \geq 3$.*

Then $\operatorname{Tors}\mathrm{CH}^i(Q_\alpha) = 0$ *for* $i < 2^{n-2}$ *and* $\operatorname{Tors}\mathrm{CH}^{2^{n-2}}(Q_\alpha) = \mathbb{Z}/2$.

8.5. Vishik's calculation for generalized Albert's forms

Consider a generalized Albert form of dimension $6{\cdot}2^r$ over F, i.e., a form of the type $\rho\otimes\varphi$, where ρ is an Albert form, i.e., $\rho = \langle a, b, -ab, -c, -d, cd\rangle$ for some $a, b, c, d \in F^\times$, and φ is an r-fold Pfister form. Note that by [56, Lemma 1.4] there exist anisotropic generalized Albert forms of dimension $6 \cdot 2^r$ over suitable fields.

For a quadratic form q denote by Q the respective projective quadric.

PROPOSITION 8.6 (Vishik, [56, Main Theorem]). – *If a generalized Albert form q of dimension $6 \cdot 2^r$ with $r \geq 1$ is anisotropic, then* $\operatorname{Tors}\mathrm{CH}^{2^r+1}(Q) \neq 0$.

Below we will show that there is no torsion in $\mathrm{CH}^j(Q)$ for $j < 2^r + 1$ (Corollary 8.16).

Finally, there are numerous results with computations of the Chow groups of generic quadrics and generic orthogonal Grassmannians (see [19], [40], [53]).

8.7. The gamma and the topological filtration on Morava K-theories of quadrics

In this section $p = 2$. Denote by $\overline{Q}$ a split quadric of dimension D and assume that $D \geq 2^{n+2} - 3$.

Denote by $d := [D/2]$ the dimension of the maximal isotropic projective space inside $\overline{Q}$ and by $\iota\colon \mathbb{P}^d \to \overline{Q}$ the corresponding inclusion map. Denote by $h \in K(n)^*(\overline{Q})$ the first Chern class of the canonical line bundle $\mathcal{O}(1)$. Abusing notation we will denote by the same letter the pullbacks of this class along restrictions to open subsets of $\overline{Q}$.

The following proposition is well-known. We consider the Morava K-theory $K(n)^*$ with v_n set to be 1.

PROPOSITION 8.8. – *The natural linear projection map $b\colon \overline{Q} \setminus \mathbb{P}^d \to \mathbb{P}^d$ induces an isomorphism $b^*\colon K(n)^*(\mathbb{P}^d) \to K(n)^*(\overline{Q} \setminus \mathbb{P}^d)$.*

Moreover, there is a short (split) exact sequence of abelian groups:

$$0 \to \bigoplus_{s=0}^{d} \mathbb{Z}_{(2)} l_s \xrightarrow{\iota_*} K(n)^*(\overline{Q}) \xrightarrow{\pi^*} K(n)^*(\mathbb{P}^d) \to 0,$$

where the map π^* is a morphism of rings compatible with the gamma filtration and l_s is the class of a linear projective space inside $\overline{Q}$ of dimension s.

Proof. – Since the statement of the proposition is well-known, we only sketch the proof.

Let (V, q) be the quadratic space of dimension $D + 2$ with a split quadratic form q. Let $W \subset V$ be the maximal totally isotropic subspace of V. Then $\dim W = d + 1$. The map ι_* is the push-forward of the embedding $\iota: \mathbb{P}^d = \mathbb{P}(W) \hookrightarrow \overline{Q}$.

The quadratic form q induces a natural linear map $V \to W^*$. This map induces a morphism $b: \overline{Q} \setminus \mathbb{P}(W) \hookrightarrow \mathbb{P}(V) \setminus \mathbb{P}(W) \to \mathbb{P}(W^*) = \mathbb{P}^d$ which is an affine bundle of rank $D - d$. Therefore, by homotopy invariance the homomorphism b^* is an isomorphism.

Let $\theta: \overline{Q} \setminus \mathbb{P}(W) \hookrightarrow \overline{Q}$ be the open embedding. Then by the localization axiom the sequence

$$K(n)^*(\mathbb{P}^d) \xrightarrow{\iota_*} K(n)^*(\overline{Q}) \xrightarrow{\theta^*} K(n)^*(\overline{Q} \setminus \mathbb{P}^d) \to 0$$

is exact. Now the homomorphism π^* is defined as $(b^*)^{-1} \circ \theta^*$. Using the fact that all objects here are free $\mathbb{Z}_{(2)}$-modules of suitable ranks one can check that the resulting exact sequence is exact on the left and is split. □

Note that π^* in Proposition 8.8 induces surjective maps of abelian groups

$$\mathrm{gr}^r_\gamma K(n)^*(\overline{Q}) \to \mathrm{gr}^r_\gamma K(n)^*(\mathbb{P}^d)$$

for $r \geq 0$. A direct calculation shows that $\mathrm{gr}^r_\gamma K(n)^*(\mathbb{P}^d)$ has no torsion for all r, i.e., it equals $\mathbb{Z}_{(2)}$ for $0 \leq r \leq d$ and 0 for $r > d$. Thus, we have

$$h^r \in \gamma^r K(n)^*(\overline{Q}) \setminus \gamma^{r+1} K(n)^*(\overline{Q})$$

for $0 \leq r \leq d$ (one could also see this using rational comparisons of Theorem 5.7).

We claim that the torsion in $\mathrm{gr}^r_\gamma K(n)^*(\overline{Q})$ is generated by elements of $\mathrm{Im}\, \iota_*$. Indeed, take an element α from $\gamma^r K(n)^*(\overline{Q})$. By Proposition 8.8 one can express α as a linear combination of elements from $\mathrm{Im}\, \iota_*$ and elements h^k with $k \geq r$. Taking α modulo $\gamma^{r+1} K(n)^*(\overline{Q})$ we may assume that it is a linear combination of elements from $\mathrm{Im}\, \iota_*$ and the element h^r, say, with a coefficient a.

If α gives a torsion element in $\mathrm{gr}^r_\gamma K(n)^*(\overline{Q})$, then it maps to a torsion element in $\mathrm{gr}^r_\gamma K(n)^*(\mathbb{P}^d)$, hence to 0. But it maps to $a\hbar^r$ where $\hbar$ is the first Chern class of $\mathcal{O}(1)$ on $\mathbb{P}^d$. Therefore, $a = 0$.

We recall the multiplication structure in $K(n)^*(\overline{Q})$.

PROPOSITION 8.9. – 1. *We have $h \cdot l_i = l_{i-1}$ where we denote $l_{-1} = 0$.*

2. *If the dimension of the quadric is odd, then $h^{d+1} \equiv 2l_d \mod (l_j \mid 0 \leq j < d)$. Moreover, h^{d+1} is expressible in terms of l_j with $j \equiv d \mod 2^n - 1$.*

Proof. – To prove (1.) note that h can be represented by a general hyperplane section of $\overline{Q}$, so that it intersects transversally the linear subspace representing the class l_i. The product $h \cdot l_i$ is represented by their intersection, which is then a linear subspace of dimension one less.

Part (2.) follows from the well-known multiplication in the Chow ring of $\overline{Q}$. □

For simplicity of notation set $l_r = 0$ for $r < 0$.

Let $D - d \equiv 1 + j \mod 2^n - 1$ where $j \in [0, 2^n - 2]$.

From now on we consider a non-split smooth projective quadric Q of positive dimension such that the restriction map

$$K(n)^*(Q) \to K(n)^*(\overline{Q}) \tag{8.10}$$

to a splitting field of Q is an isomorphism. Note that in this case $\dim Q \geq 2^{n+2} - 3$. Indeed, in view of Lemma 7.8, if $\dim Q$ is even, this follows from Proposition 6.18 and the Arason-Pfister Hauptsatz. If $\dim Q$ is odd, then by Proposition 6.21 the respective quadratic form is of the type $f \perp \langle c \rangle$ for some anisotropic form $f \in I^{n+2}$ and $c \in F^\times$. Therefore, since the dimension of the anisotropic part of $f \perp \langle c \rangle$ is at least $\dim f - 1$, it follows from the Arason-Pfister Hauptsatz that $\dim Q \geq 2^{n+2} - 3$.

Abusing notation we will consider the elements h, l_i of $K(n)^*(\overline{Q})$ defined above also as the corresponding elements of $K(n)^*(Q)$ with respect to the isomorphism (8.10).

Lemma 8.11. – *Let $k \in [0, d]$. Assume that the element l_k lies in*

$$\gamma^r K(n)^*(Q) \mod \bigoplus_{s<k} \mathbb{Z}_{(2)} l_s$$

(resp. in $\tau^r K(n)^*(Q) \mod \bigoplus_{s<k} \mathbb{Z}_{(2)} l_s$) for some $r \geq 1$.

Then for every $u \geq 0$ the element l_{k-u} lies in $\gamma^{r+u} K(n)^*(Q)$ (resp. in $\tau^{r+u} K(n)^*(Q)$).

Proof. – The proof is the same for the gamma and for the topological filtration and exploits only its multiplicativity. We confine ourselves to the case of the gamma filtration. By our assumptions we have $l_k + \sum_{s<k} a_s l_s \in \gamma^r K(n)^*(Q)$ for some $a_s \in \mathbb{Z}_{(2)}$.

We prove the statement by decreasing induction starting with the highest $u = k + 1$. In this case $l_{-1} = 0$ and the claim is trivial.

By Proposition 8.9 we have

$$h^u \cdot (l_k + \sum_{s<k} a_s l_s) = l_{k-u} + \sum_{u \leq s < k} a_s l_{s-u}.$$

The left-hand side lies in $\gamma^{r+u} K(n)^*(Q)$ by the multiplicativity of the gamma filtration and the fact that $h \in \gamma^1 K(n)^*(Q)$, while the "tail" of the right-hand side lies in $\gamma^{r+u+1} K(n)^*(Q)$ by the induction assumption. Therefore, we have $l_{k-u} \in \gamma^{r+u} K(n)^*(Q)$. □

Denote by H the first Chern class of the canonical line bundle $\mathcal{O}(1)$ on Q in the Brown-Peterson cohomology. Again abusing notation, denote by the same letter the corresponding class in $BP^*(\overline{Q})$. Denote by $L_r \in BP^*(\overline{Q})$ the class of a linear subspace inside Q of dimension r. Note that the canonical map of theories

$$\pi_{K(n)}: BP^*(\overline{Q}) \to K(n)^*(\overline{Q})$$

sends H to h and L_r to l_r.

Lemma 8.12. – *We have $l_d \in \tau^{j+2^n} K(n)^*(Q)$.*

Proof. – One could argue as in the proof of Theorem 7.19 to show that $l_d \in \tau^{2^n} K(n)^*(Q)$.

A more direct approach of the use of Theorem 7.19 is the following. Let i be the maximal positive integer such that $l_d \in \tau^i K(n)^*(Q)$. If $i < 2^n$, then l_d defines a non-trivial element of the group $\mathrm{gr}^i_\tau K(n)^*(Q)$. However, this group maps isomorphically to $\mathrm{gr}^i_\tau K(n)^*(\overline{Q})$ where the class of l_d is zero. Contradiction and, therefore, $i \geq 2^n$, i.e., $l_d \in \tau^{2^n} K(n)^*(Q)$.

However, $l_d \in K(n)^{1+j}(Q)$ and $\tau^{2^n} K(n)^{1+j}(Q) = \tau^{j+2^n} K(n)^{1+j}(Q)$ by Proposition 5.2(3). This implies the claim. □

PROPOSITION 8.13. – *In the notation of this section we have*

1. $\mathrm{gr}^s_\tau K(n)^*(Q) = \mathbb{Z}_{(2)}$ *for* $1 \leq s \leq 2^n - 1$*;*
2. *if* $j \neq 0$*, then* $\mathrm{gr}^{2^n}_\tau K(n)^*(Q) = \mathbb{Z}_{(2)}$*;*
3. *if* $j = 0$ *and the dimension of the quadric is odd, then the torsion subgroup of* $\mathrm{gr}^{2^n}_\gamma K(n)^*(Q)$ *is at most* $\mathbb{Z}/2$*;*
4. *if* $j = 0$ *and the dimension of the quadric is even, let* $d = 1 + r(2^n - 1)$ *for some*[(1)] $r \geq 2$*. If* r *is even, then the torsion in* $\mathrm{gr}^{2^n}_\gamma K(n)^*(Q)$ *is at most* $\mathbb{Z}/2$*. If* r *is odd, then the torsion in* $\mathrm{gr}^{2^n}_\gamma K(n)^*(Q)$ *is at most* $\mathbb{Z}/2^s$*, where* $s = \min(\nu_2(r-1) + 2, 2^n)$*. Here we denote by* ν_2 *the* 2*-adic valuation.*

Proof. – (1): This follows from Theorem 7.19.

(2): If $j \neq 0$, then by Lemma 8.12 the element l_d lies in $\tau^{2^n+1} K(n)^*(Q)$ and therefore, by Lemma 8.11 the same holds for l_s, $s < d$. Thus, the graded factors $\mathrm{gr}^s_\tau K(n)^*(Q)$ for $s \leq 2^n$ have to be generated by some power of h and have no torsion.

(3, 4): If $j = 0$, then we will show now that $l_d \in \gamma^{2^n} K(n)^*(Q) \subset \tau^{2^n} K(n)^*(Q)$. Let $\iota\colon \mathbb{P}^d \hookrightarrow \overline{Q}$ be the inclusion of the maximal isotropic linear subspace. In order to calculate $c^{K(n)}_{2^n}(l_d) = c^{K(n)}_{2^n}(\iota_* 1_{\mathbb{P}^d})$ we apply the generalized Riemann-Roch formula (Corollary 4.13). Using Proposition 5.8 we have $c^{K(n)}_{2^n}(\iota_* 1_{\mathbb{P}^d})$ is equal to

$$e_r l_d + \sum_{s>0} b_s l_{d-s(2^n-1)}$$

with $b_s \in \mathbb{Z}_{(2)}$, where r is such that $d = 1 + r(2^n - 1)$ and $e_r \in \mathbb{Z}^\times_{(2)}$. This element lies in $\gamma^{2^n} K(n)^*(Q)$ by the definition of the gamma filtration.

By Lemma 8.11 we obtain that all other elements l_s, $s < d$, lie in the higher parts of the gamma filtration. It follows that the torsion in the group $\mathrm{gr}^{2^n}_\gamma K(n)^*(Q)$ is generated by l_d.

(Only 3): If the dimension of the quadric is odd, then by Proposition 8.9 we have $h^{d+1} = 2l_d + \sum_{s>0} \beta_s l_{d-s(2^n-1)}$ for some $\beta_s \in \mathbb{Z}_{(2)}$. By the multiplicativity of the gamma filtration this element lies in $\gamma^{d+1} K(n)^*(Q)$. Recall that $d \geq 2^{n+1} - 2$ by our assumptions, and therefore, $d + 1 > 2^n$. Thus, by the results above $l_{d-s(2^n-1)} \in \gamma^{2^n+1} K(n)^*(Q)$ for $s > 0$, and we obtain that $2l_d \in \gamma^{2^n+1} K(n)^*(Q)$. This proves the claim.

(Only 4) Let us consider the element $\chi(l_d) \in \gamma^{2^{n+1}-1} K(n)^1(Q)$ for the operation χ from Proposition 5.9. Using the Riemann-Roch formula and Proposition 5.9, 2.) we obtain that this element is equal to $g_r l_d + \sum_{s>0} b_s l_{d-s(2^n-1)}$ for some $b_s \in \mathbb{Z}_{(2)}$ (g_r was defined

(1) The case $r = 2$ is the well-known case of a Pfister quadric.

in Proposition 5.9). Since the elements $l_{d-s(2^n-1)}$ lie in $\gamma^{2^{n+1}-1}K(n)^*(Q)$, we obtain that $g_r l_d \in \gamma^{2^{n+1}-1}K(n)^*(Q)$. If r is even, then $g_r \in 2\mathbb{Z}_{(2)}^{\times}$. If r is odd, then $\nu_2(g_r) = \nu_2(r-1)+2$.

If we use the operation ψ from Proposition 5.9 instead of χ, we obtain that

$$2^{2^n} l_d \in \gamma^{2^{n+1}-1}K(n)^*(Q).$$

The result now follows. □

Combining this together with Theorem 5.7 and Propositions 6.18 and 6.21 we obtain the following theorem.

THEOREM 8.14. – *Let Q be a smooth quadric of positive dimension over a field F such that the corresponding quadratic form q lies either in the ideal I^{n+2} or in the set $\langle c\rangle + I^{n+2}$ inside the Witt ring for some $c \in F^{\times}$. Let D be the dimension of Q, $d := [D/2]$, and let $j \in [0, 2^n-2]$ be such that $D - d \equiv 1 + j \mod 2^n - 1$.*

Then $\mathrm{CH}^{0\leq * \leq 2^n-1}(Q) = \mathbb{Z}$ *and*

1. *if $j \neq 0$, then* $\mathrm{CH}^{2^n}(Q) = \mathbb{Z}$.
2. *if $j = 0$, and the dimension of the quadric is odd, then the torsion in* $\mathrm{CH}^{2^n}(Q)$ *is at most* $\mathbb{Z}/2$;
3. *if $j = 0$ and the dimension of the quadric is even, $d = 1 + r(2^n - 1)$, then the torsion in* $\mathrm{CH}^{2^n}(Q)$ *is at most $\mathbb{Z}/2^s$ where $s = 1$, if r is even, and*

$$s = \min(\nu_2(r-1)+2, 2^n)$$

otherwise. Here we denote by ν_2 the 2-adic valuation.

REMARK 8.15. – One can show that the estimates one gets using just the gamma filtration are not so strong if $j \neq 0$. Namely, if $j \neq 0$ one obtains $\mathbb{Z}/2$ in the components $\mathrm{CH}^{\geq j+1}$. This shows that the graded factors of the gamma filtration do not give exact bounds for the topological filtration even in small codimensions.

COROLLARY 8.16. – *Let q be a generalized anisotropic Albert form of dimension $6 \cdot 2^r$. Then* $\mathrm{Tors}\,\mathrm{CH}^j(Q) = 0$ *for all $j < 2^r + 1$.*

Proof. – Indeed, the Albert form lies in $I^2(F)$, and therefore, q lies in $I^{r+2}(F)$. Since $d \equiv 2 \mod (2^r - 1)$, Theorem 8.14 implies the claim. □

Vishik has communicated to the authors that one can show the above corollary using techniques of [53].

9. Morava K-theory and cohomological invariants

In this section we relate the Morava K-theory with cohomological invariants of algebraic group; see also Proposition 6.18.

THEOREM 9.1. – *Let p be a prime number. Let G be a simple algebraic group over F and let X be the variety of Borel subgroups of G. Then*

1. *G is of inner type iff the $K(0)^*$-motive of X is split.*
2. *(Panin). Assume that G is of inner type. All Tits algebras of G are split iff the K^0-motive with integral coefficients of X is split.*
3. *Assume that G is of inner type and the p-components of the Tits algebras of G are split. Then the p-component of the Rost invariant of G is zero iff the $K(2)^*$-motive of X is split.*
4. *Let $p = 2$. Assume that G is of type E_8. Then G is split by an odd degree field extension iff the $K(m)^*$-motive of X is split for some $m \geq 4$ iff the $K(m)^*$-motive of X is split for all $m \geq 4$.*

Proof. – (1) Recall that $K(0)^* = \mathrm{CH}^* \otimes \mathbb{Q}$ by definition. If G is of inner type, then it is well-known that the Chow motive of X with rational coefficients is split (e.g., this follows from [39, Theorems 2.2 and 4.2], since K^0 and CH^* are isomorphic theories with rational coefficients). On the other hand, if G is of outer type, then the absolute Galois group of F acts non-trivially on the Chow group of $X_{F_{\mathrm{sep}}}$ (see [35, Section 2.1] for the description of the action on the Picard group of $X_{F_{\mathrm{sep}}}$). Therefore, the Chow motive of X with any coefficients cannot be split in this case.

(2) Follows from [39]; see also Section 3.4.

(3) First we make several standard reductions. Since all prime numbers coprime to p are invertible in the coefficient ring of the Morava K-theory, by transfer argument we are free to take finite field extensions of the base field of degree coprime to p. Hence we can assume that not only the p-components of the Tits algebras are split, but that the Tits algebras are completely split (and the same for the Rost invariant).

Types A *and* C. – If G is a group of inner type A or C with trivial Tits algebras, then G is split and the statement follows. Indeed, by [23, §26] the group G is isogenous to $\mathrm{SL}_1(A)$ for a central simple algebra A or, respectively, to $\mathrm{Sp}(B, \sigma)$ for a central simple algebra B with a symplectic involution σ. By [23, §27.B] the algebra A, respectively, the algebra B is a Tits algebra of G. Therefore, if A, respectively, B is split, then G is split, and the statement of the proposition is obvious.

Types B *and* D. – If G is a group of inner type B or D, then G is isogenous to $\mathrm{Spin}(V, q)$ or, respectively, to $\mathrm{Spin}(D, \tau)$ for an odd-dimensional quadratic space (V, q) or, respectively, for an algebra D with an orthogonal involution τ with trivial discriminant. By [23, §27.B] the even Clifford algebra $C_0(V, q)$, respectively, the algebra D and the Clifford algebras $C^{\pm}(D, \tau)$ are Tits algebras of G. Therefore, if the Tits algebras of G are split, we are in the situation of quadratic forms.

Now the statement of the proposition follows from Proposition 6.18 (in the even-dimensional case) and from Proposition 6.21 (in the odd-dimensional case). Indeed, an even-dimensional quadratic form with trivial discriminant lies in I^4 (resp. an odd-dimensional quadratic form lies in $I^4 + \langle c\rangle$ for some $c \in F^\times$) iff its Clifford and its Rost invariants are zero (see [23, §31.B] for a description of the Rost invariant in the case of quadratic forms).

Exceptional types. – Let now G be a group of an exceptional type. Taking coprime to p field extensions we assume that our base field is p-special. Assume that the $K(2)^*$-motive of X is split, but the Rost invariant of G is not trivial.

There is a field extension K of F such that the Rost invariant of G_K is a non-zero pure symbol. Indeed, for groups of types E_6, F_4, G_2, E_7 with $p = 3$ and E_8 with $p = 5$ this is already the case for $K = F$ (see [11, Part II]).

If G is of type E_7 with $p = 2$, then by [41, Theorem 5.7] the variety Y of maximal parabolic subgroups of G of type 6 (enumeration of simple roots follows Bourbaki) is not generically split. Over its function field $K = F(Y)$ the anisotropic kernel of G_K is of type D_4 and, thus, the Rost invariant of G_K is a non-zero pure symbol.

If G is of type E_8 with $p = 2$, then by [41, Theorem 5.7] one can take $K = F(Y)$, where Y is the variety of maximal parabolic subgroups of G of type 6 (the anisotropic kernel of G_K will be again of type D_4), and if G is of type E_8 with $p = 3$, then one can take $K = F(Y)$, where Y is the variety of maximal parabolic subgroups of G of type 7 (the anisotropic kernel of G_K will be of type E_6).

In all cases the motive of X_K is a direct sum of Rost motives corresponding to this non-zero symbol of degree 3 (see [42]). This gives a contradiction with Proposition 6.2.

Conversely, if the Rost invariant of G is zero and G is not of type E_8 with $p = 2$, then by [8, Theorem 0.5] (for exceptional groups different from E_8), [3] and [11, Proposition 15.5] (for E_8 at the prime 5), [4] and [10, Section 10c] (for E_8 at the prime 3) the group G is split and the statement of the proposition follows.

Therefore, it remains to consider the case when G is a group of type E_8 with trivial Rost invariant. By [52, Theorem 8.7] G has an invariant $u \in H^5_{\mathrm{et}}(F, \mathbb{Z}/2)$ such that for every field extension K/F the invariant $u_K = 0$ iff G_K splits over a field extension of K of odd degree. Exactly as in the proof of Proposition 6.18 (note that we can represent u by a quadratic form from I^5) we can pass to a splitting field $\widetilde{F}$ of u such that the restriction homomorphism $K(2)^*(X \times X) \to K(2)^*((X \times X)_{\widetilde{F}})$ is surjective. Therefore, by Rost nilpotence the $K(2)^*$-motive of X is split.

(4) If G is split by an odd degree field extension, then the $K(m)^*$-motives of X are split for all m, since $p = 2$. Conversely, if G does not split over an odd degree field extension of F and the even component of the Rost invariant of G is non-trivial, then by item (3) the $K(2)^*$-motive of X is not split and, hence, by Proposition 7.10 the $K(m)^*$-motives are not split for all $m \geq 2$.

Besides, if G does not split over an odd degree field extension of F and the even component of the Rost invariant of G is trivial, then the invariant u is defined and is non-zero. By [38, Theorem 2.10] there is field extension K of F such that u_K is a non-zero pure symbol. Over K the motive of X is a direct sum of Rost motives corresponding to u_K. By Proposition 6.2 the $K(m)^*$-Rost motives for a symbol of degree 5 are not split, if $m \geq 4$. □

Finally we remark that sequence (3.6) can be used to define the Rost invariant in general, the invariant f_5 for groups of type F_4 (see [23, §40]) and an invariant of degree 5 for groups of type E_8 with trivial Rost invariant (see [52]). Namely, for the Rost invariant let G be a simple simply-connected algebraic group over F. Let Y be a G-torsor and set $n = 3$. Then sequence (3.6) gives an exact sequence

$$0 \to H^{3,2}_{\mathcal{M}}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Ker}\big(H^3_{\mathrm{et}}(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{\mathrm{et}}(F(Y), \mathbb{Q}/\mathbb{Z}(2))\big) \to 0.$$

But by sequence (3.8) $\operatorname{Ker}\big(H^3_{\mathrm{et}}(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{\mathrm{et}}(F(Y), \mathbb{Q}/\mathbb{Z}(2))\big)$ is a finite cyclic group. Therefore, $H^{3,2}_{\mathcal{M}}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z})$ is a finite cyclic group and the Rost invariant of Y is the image of $1 \in H^{3,2}_{\mathcal{M}}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z})$ in $H^3_{\mathrm{et}}(F, \mathbb{Q}/\mathbb{Z}(2))$.

To construct invariants of degree 5 for F_4 (resp. for E_8) one takes $n = 5$ and Y to be the variety of parabolic subgroups of type 4 for F_4 (the enumeration of simple roots follows Bourbaki) and resp. the variety of parabolic subgroups of any type for E_8. In both cases $H^{5,4}_{\mathcal{M}}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z})$ is cyclic of order 2 and the invariant is the image of the only non-zero element of $H^{5,4}_{\mathcal{M}}(\mathcal{X}_Y, \mathbb{Q}/\mathbb{Z})$ in $H^5_{\mathrm{et}}(F, \mathbb{Q}/\mathbb{Z}(4))$; see [52].

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ON THE DYNAMICS OF MINIMAL HOMEOMORPHISMS OF $\mathbb{T}^2$ WHICH ARE NOT PSEUDO-ROTATIONS

BY ALEJANDRO KOCSARD

ABSTRACT. – We prove that any minimal 2-torus homeomorphism which is isotopic to the identity and whose rotation set is not just a point exhibits uniformly bounded rotational deviations on the perpendicular direction to the rotation set. As a consequence of this, we show that any such homeomorphism is topologically mixing and we prove Franks-Misiurewicz conjecture under the assumption of minimality.

RÉSUMÉ. – Soit f un homéomorphisme minimal du tore $\mathbb{T}^2$ qui est isotope à l'identité. Nous montrons que si son ensemble de rotation $\rho(f)$ n'est pas trivial (i.e., il n'est pas un singleton), alors les déviations rotationnelles dans la direction perpendiculaire à l'ensemble de rotation sont uniformément bornées. Par conséquent, nous prouvons qu'un tel homéomorphisme f est topologiquement mélangeant et on donne une démonstration de la conjecture de Franks et Misiurewicz pour homéomorphismes minimaux.

1. Introduction

The study of the dynamics of orientation preserving circle homeomorphisms has a long and well established history that started with the celebrated work of Poincaré [27]. If $f\colon \mathbb{T} = \mathbb{R}/\mathbb{Z} \circlearrowleft$ denotes such a homeomorphism and $\tilde{f}\colon \mathbb{R} \circlearrowleft$ is a lift of f to the universal cover, he showed that there exists a unique $\rho \in \mathbb{R}$, the so called *rotation number* of $\tilde{f}$, such that

$$\frac{\tilde{f}^n(z) - z}{n} \to \rho, \quad \text{as } n \to \infty,\ \forall z \in \mathbb{R},$$

where the convergence is uniform in z. Moreover, in this case a stronger (and very useful, indeed) condition holds: every orbit exhibits *uniformly bounded rotational deviations*, i.e.,

$$\left| \tilde{f}^n(z) - z - n\rho \right| \leq 1, \quad \forall n \in \mathbb{Z},\ \forall z \in \mathbb{R}.$$

In this setting, the homeomorphism f has no periodic orbit if and only if the rotation number is irrational; and any minimal circle homeomorphism is topologically conjugate to a rigid irrational rotation.

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However, in higher dimensions the situation dramatically changes. If $f\colon\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d\circlearrowleft$ is a homeomorphism homotopic to the identity and $\tilde{f}\colon\mathbb{R}^d\circlearrowleft$ is a lift of f, then one can define its *rotation set* by

$$\rho(\tilde{f}):=\left\{\rho\in\mathbb{R}^d:\exists n_k\uparrow+\infty,\ z_k\in\mathbb{R}^d,\ \rho=\lim_{k\to+\infty}\frac{\tilde{f}^{n_k}(z_k)-z_k}{n_k}\right\}.$$

This set is always compact and connected, and as we mentioned above, it reduces to a point when $d=1$. But for $d\geq 2$ some examples with larger rotation sets can be easily constructed.

In the two-dimensional case, which is the main scenario of this work, Misiurewicz and Ziemian showed in [24] that the rotation set is not just connected but convex. So, when $d=2$ all torus homeomorphisms of the identity isotopy class can be classified according to the geometry of their rotation sets: they can either have non-empty interior, or be a non-degenerate line segment, or be just a point. In the last case, such a homeomorphism is called a *pseudo-rotation*.

Regarding the boundedness of rotational deviations, this property has been shown to be very desirable in the study of the dynamics of pseudo-rotations (see for instance the works of Jäger and collaborators [9, 10, 11]). However, it has been proved in [12] and [15] that, in general, pseudo-rotations do not exhibit bounded rotational deviations in any direction of $\mathbb{R}^2$, i.e., it can hold

$$\sup_{z\in\mathbb{R}^2,n\in\mathbb{Z}}\left\langle\tilde{f}^n(z)-z-n\rho(\tilde{f}),v\right\rangle=+\infty,\quad\forall v\in\mathbb{S}^1.$$

When $\rho(\tilde{f})$ is a (non-degenerate) line segment, of course there exist points with different rotation vectors, so we cannot expect to have any boundedness at all for rotational deviations on the plane. However, in such a case there exists a unit vector $v\in\mathbb{S}^1$ and a real number α such that $\rho(\tilde{f})$ is contained in the line $\{z\in\mathbb{R}^2:\langle z,v\rangle=\alpha\}$, so one can analyze the boundedness of *rotational v-deviations,* i.e., whether there exist constants $M(z)\in\mathbb{R}$ such that

$$\left|\left\langle\tilde{f}^n(z)-z-n\rho,v\right\rangle\right|=\left|\left\langle\tilde{f}^n(z)-z,v\right\rangle-n\alpha\right|\leq M(z),\quad\forall n\in\mathbb{Z},$$

and any $\rho\in\rho(\tilde{f})$.

Unlike the case of pseudo-rotations, when $\rho(\tilde{f})$ is a non-degenerate line segment in general it is expected to have uniformly bounded rotational v-deviations, i.e., the constant $M(z)$ can be taken independently of z. This result has been already proved by Dávalos [3] in the case where $\rho(\tilde{f})$ has rational slope and intersects $\mathbb{Q}^2$, extending a previous result of Guelman, Koropecki and Tal [7]. In those works periodic orbits of f play a key role.

However, the situation is considerably subtler when dealing with periodic point free homeomorphisms. So far there did not exist any a priori boundedness of rotational deviations of torus homeomorphisms which are not pseudo-rotations and with no periodic points. In fact, it had been conjectured that any periodic point free homeomorphism should be a pseudo-rotation. More precisely, Franks and Misiurewicz had proposed in [5] the following

CONJECTURE 1.1 (Franks-Misiurewicz Conjecture). – *Let $f\colon\mathbb{T}^2\circlearrowleft$ be a homeomorphism homotopic to the identity and $\tilde{f}\colon\mathbb{R}^2\circlearrowleft$ be a lift of f such that $\rho(\tilde{f})$ is a non-degenerate line segment.*

Then, the following dichotomy holds:

(i) *either* $\rho(\tilde{f})$ *has irrational slope and one of its extreme points belongs to* $\mathbb{Q}^2$*;*

(ii) *or* $\rho(\tilde{f})$ *has rational slope and contains infinitely many rational points.*

Recently Avila has announced the existence of a minimal smooth diffeomorphism whose rotation set is an irrational slope segment containing no rational point, providing in this way a counter-example to the first case of Franks-Misiurewicz Conjecture. On the other hand, Le Calvez and Tal have proved in [21] that if $\rho(\tilde{f})$ has irrational slope and contains a rational point, then this point is an extreme one.

The second case of Conjecture 1.1 remains open, i.e., whether there exists a homeomorphism f such that $\rho(\tilde{f})$ has rational slope and $\rho(\tilde{f}) \cap \mathbb{Q}^2 = \emptyset$, and in fact this is one of the main motivations of our work.

The main result of this paper is the following *a priori boundedness* for rotational deviations of minimal homeomorphisms:

THEOREM A. – *Let* $f\colon \mathbb{T}^2 \circlearrowleft$ *be a minimal homeomorphism homotopic to the identity which is not a pseudo-rotation. Then there exists a unit vector* $v \in \mathbb{R}^2$ *and a real number* $M > 0$ *such that for any lift* $\tilde{f}\colon \mathbb{R}^2 \circlearrowleft$*, there is* $\alpha \in \mathbb{R}$ *so that*

$$\left| \left\langle \tilde{f}^n(z) - z, v \right\rangle - n\alpha \right| \le M, \quad \forall z \in \mathbb{R}^2,\ \forall n \in \mathbb{Z}. \tag{1}$$

As a consequence of Theorem A and a recent result due to Koropecki, Passeggi and Sambarino [14], we get a proof of the second case of Franks-Misiurewicz Conjecture (Conjecture 1.1) under minimality assumption. More precisely we get the following:

THEOREM B. – *There is no minimal homeomorphism of* $\mathbb{T}^2$ *in the identity isotopy class such that its rotation set is a non-degenerate rational slope segment.*

As a consequence of Theorem B, some results of [13] and a recent generalization of a theorem of Kwapisz [18] due to Beguin, Crovisier and Le Roux [2], we have the following

THEOREM C. – *If* $f\colon \mathbb{T}^2 \circlearrowleft$ *is a minimal homeomorphism homotopic to the identity and is not a pseudo-rotation, then* f *is topologically mixing.*

Moreover, in such a case the rotation set of f *is a non-degenerate irrational slope line segment and its supporting line does not contain any point of* $\mathbb{Q}^2$*.*

1.1. Strategy of the proof of Theorem A

Theorem A is certainly the most important result of the paper and its proof is rather long and technical. So, for the sake of readability, here we summarize the main steps of the proof in a rather informal way.

We proceed by contradiction. First of all one can observe that there is no loss of generality assuming the rotation set $\rho(\tilde{f})$ is transversal to the horizontal axes, i.e., it intersects the upper and lower horizontal semi-planes (see Propositions 2.5 and 2.16 for details). This means there exist points with a positive asymptotic vertical mean speed and others with a negative one.

Then we define the *stable sets at infinity* $\Lambda_h^+, \Lambda_h^- \subset \mathbb{R}^2$ as the unbounded connected components of the maximal $\tilde{f}$-invariant sets of the upper and lower semi-plane, respectively. More precisely,

$$\Lambda_h^+ := \operatorname{cc}\left(\left\{z \in \mathbb{R}^2 : \operatorname{pr}_2\big(\tilde{f}^n(z)\big) \geq 0,\ \forall n \in \mathbb{Z}\right\}, \infty\right)$$

and

$$\Lambda_h^- := \operatorname{cc}\left(\left\{z \in \mathbb{R}^2 : \operatorname{pr}_2\big(\tilde{f}^n(z)\big) \leq 0,\ \forall n \in \mathbb{Z}\right\}, \infty\right),$$

where $\operatorname{pr}_2 \colon \mathbb{R}^2 \to \mathbb{R}$ denotes the projection on the second coordinate and $\operatorname{cc}(\cdot, \infty)$ the union of the unbounded connected components of the corresponding set. In §5, we study the geometry of these stable sets at infinity, showing in particular that they are non empty (Theorem 5.1), and they are in fact the union of "infinitely long hairs". Then, assuming estimate (1) is false, we show in Theorem 5.5 that these "hairs" exhibit arbitrarily large oscillations in the horizontal direction.

Then, in §6 we define the *stable sets at infinity* but this time with respect to the direction determined by the rotation set. At this point some new important technical problems appear. In fact, the number α in (1) represents the mean asymptotic speed of every point with respect to the perpendicular direction to the rotation set, and we know a posteriori, by Theorem C, that it is always irrational, and in particular, non-zero. That means if we just define these sets analogously to what we did above for Λ_h^+ and Λ_h^-, we shall just get empty sets. Nevertheless, if we modify the definition writing

$$\Lambda_v^+ := \operatorname{cc}\left(\left\{z \in \mathbb{R}^2 : \left\langle \tilde{f}^n(z), v\right\rangle - n\alpha \geq 0,\ \forall n \in \mathbb{Z}\right\}, \infty\right)$$

and

$$\Lambda_v^- := \operatorname{cc}\left(\left\{z \in \mathbb{R}^2 : \left\langle \tilde{f}^n(z), v\right\rangle - n\alpha \leq 0,\ \forall n \in \mathbb{Z}\right\}, \infty\right),$$

we get non-empty sets, but they are not dynamically defined. So, this is the reason why we have to introduce the *fiber-wise Hamiltonian skew-products* in §6.1 in order to get these sets Λ_v^+ and Λ_v^- as dynamical ones (see §6.2 for details).

Then, always assuming that (1) does not hold, we use these sets to show that the induced fiber-wise Hamiltonian skew-product exhibits a certain form of topologically mixing behavior along the fibers (Theorem 6.7). Then we use this dynamical information to show the sets $\Lambda_v^+, \Lambda_v^-, \Lambda_h^+, \Lambda_h^-$ are pairwise disjoint (Proposition 7.1). Finally, we finish the proof showing that this disjointedness is incompatible with the large horizontal oscillation of the connected components of the sets Λ_h^+ and Λ_h^- we proved in Theorem 5.5.

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2. Preliminaries and notations

2.1. Maps, topological spaces and groups

Given any map $f\colon X \circlearrowleft$, we write $\mathrm{Fix}(f)$ for its set of fixed points and

$$\mathrm{Per}(f) := \bigcup_{n\geq 1} \mathrm{Fix}(f^n)$$

for the set of periodic ones. If $A \subset X$ denotes an arbitrary subset, we define its positively maximal f-invariant subset by

$$\mathscr{I}_f^+(A) := \bigcap_{n\geq 0} f^{-n}(A).$$

When f is bijective, we can also define its maximal f-invariant subset by

$$\mathscr{I}_f(A) := \mathscr{I}_f^+(A) \cap \mathscr{I}_{f^{-1}}^+(A) = \bigcap_{n\in\mathbb{Z}} f^n(A). \tag{2}$$

When X is a topological space and $A \subset X$ is any subset, we write $\operatorname{int} A$ for the interior of A and $\bar{A}$ for its closure. When A is connected, we write $\mathrm{cc}(X, A)$ for the connected component of X containing A. As usual, $\pi_0(X)$ denotes the set of connected components of X. When X is connected and $A \subset X$, we say that A *disconnects* X when $X \setminus A$ is not connected. Given two connected sets $U, V \subset X$, we say that A *separates* U and V when $\mathrm{cc}(X \setminus A, U) \neq \mathrm{cc}(X \setminus A, V)$.

The space X is said to be a *continuum* when it is compact, connected and non-trivial, i.e., it is neither empty nor a singleton.

A homeomorphism $f\colon X \circlearrowleft$ is said to be *non-wandering* when given any non-empty open set $U \subset X$, there exists a positive integer n such that $f^n(U) \cap U \neq \emptyset$.

We say that f is *topologically mixing* when for every pair of non-empty open sets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$, for every $n \geq N$.

The homeomorphism f is said to be *minimal* when it does not exhibit any proper f-invariant closed set, i.e., X and $\emptyset$ are the only closed f-invariant sets.

If (X, d) is a metric space, the open ball of radius $r > 0$ and center at $x \in X$ will be denoted by $B_r(x)$. Given an arbitrary set $A \subset X$ and a point $x_0 \in X$, we write

$$d(x_0, A) := \inf_{y\in A} d(x_0, y).$$

For any $\varepsilon > 0$, the *ε-neighborhood of A* is given by

$$A_\varepsilon := \{x \in X : d(x, A) < \varepsilon\} = \bigcup_{x\in A} B_\varepsilon(x). \tag{3}$$

The diameter of $A \subset X$ is defined by $\operatorname{diam} A := \sup_{x,y\in A} d(x, y)$ and we say A is *unbounded* whenever $\operatorname{diam} A = +\infty$. Making a slight abuse of notation, we shall write $\mathrm{cc}(A, \infty)$ to denote the union of the unbounded connected components of A.

The space of (non-empty) compact subsets of X will be denoted by

$$\mathscr{K}(X) := \{K \subset X : K \text{ is compact, } K \neq \emptyset\}$$

and we endow this space with its Hausdorff distance d_H defined by

$$d_H(K_1, K_2) := \max\left\{\max_{x\in K_1} d(x, K_2), \max_{y\in K_2} d(y, K_1)\right\},$$

for every $K_1, K_2 \in \mathscr{K}(X)$.

Whenever $M_1, M_2, \dots M_n$ are n arbitrary sets, we shall use the generic notation $\mathrm{pr}_i : M_1 \times M_2 \times \dots \times M_n \to M_i$ to denote the i^{th}-coordinate projection map.

Finally, when X is a compact topological space, we shall always consider the vector space of continuous functions $C^0(X, \mathbb{R}^d)$ endowed with the uniform norm given by

$$\|\phi\|_{C^0} := \max_{1 \leq i \leq d} \max_{x \in X} \left|\mathrm{pr}_i\big(\phi(x)\big)\right|, \quad \forall \phi \in C^0(X, \mathbb{R}^d).$$

2.2. Topological factors and extensions

Let (X, d_X) and (Y, d_Y) be two compact metric spaces. We say that a homeomorphism $f : X \circlearrowleft$ is a *topological extension* of a homeomorphism $g : Y \circlearrowleft$ when there exists a continuous surjective map $h : X \to Y$ such that $h \circ f = g \circ h$; and we say g is a *topological factor* of f. In such a case, h is called a *semi-conjugacy.*

As usual, when h is a homeomorphism, f and g are said to be *topologically conjugate,* and h is said to be a *conjugacy.*

2.3. Euclidean spaces, tori and the annulus

We consider $\mathbb{R}^d$ endowed with its usual Euclidean structure, which is denoted by $\langle \cdot, \cdot \rangle$. We write $\|v\| := \langle v, v \rangle^{1/2}$ for its induced norm and $d(v, w) := \|v - w\|$ for its induced distance function.

The unit $(d-1)$-sphere is denoted by $\mathbb{S}^{d-1} := \{v \in \mathbb{R}^d : \|v\| = 1\}$. For any $v \in \mathbb{R}^d \setminus \{0\}$ and any $r \in \mathbb{R}$ we define the (open) *half-space*

$$\mathbb{H}_r^v := \Big\{ z \in \mathbb{R}^d : \langle z, v \rangle > r \Big\}. \tag{4}$$

Given any $\alpha \in \mathbb{R}^d$, we write T_α for the translation $T_\alpha : z \mapsto z + \alpha$ on $\mathbb{R}^d$.

The d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ will be denoted by $\mathbb{T}^d$ and we write $\pi : \mathbb{R}^d \to \mathbb{T}^d$ for the canonical quotient projection. We will always consider $\mathbb{T}^d$ endowed with the distance

$$d_{\mathbb{T}^d}(x, y) := \min \big\{ d(\tilde{x}, \tilde{y}) : \tilde{x} \in \pi^{-1}(x),\ \tilde{y} \in \pi^{-1}(y) \big\}, \quad \forall x, y \in \mathbb{T}^d$$

Given any $\alpha \in \mathbb{T}^d$, we write T_α for the torus translation $T_\alpha : \mathbb{T}^d \ni z \mapsto z + \alpha$. A point $\alpha \in \mathbb{R}^d$ is said to be *totally irrational* when $T_{\pi(\alpha)}$ is minimal on $\mathbb{T}^d$.

In several places along this paper the symbol "$\pm$" shall have the following meaning: given $v \in \mathbb{R}^d$, we write $\pm v$ to denote either the vector v or $-v$.

2.3.1. *The plane* $\mathbb{R}^2$. – In the particular case of $d = 2$, given any $v = (a, b) \in \mathbb{R}^2$, we define $v^\perp := (-b, a)$. For any $\alpha \in \mathbb{R}$ and any $v \in \mathbb{S}^1$, we shall use the following notation for the straight line through the point αv and perpendicular to v:

$$\ell_\alpha^v := \alpha v + \mathbb{R}v^\perp = \{\alpha v + t v^\perp : t \in \mathbb{R}\}. \tag{5}$$

We say a vector $v \in \mathbb{R}^2 \setminus \{0\}$ has *rational slope* when there exists $\alpha > 0$ such that $\alpha v \in \mathbb{Z}^2$; and it is said to have *irrational slope* otherwise.

We will also need the following notation for strips on $\mathbb{R}^2$: given any $v \in \mathbb{S}^1$ and $r < s$, we define the (closed) strip

$$\mathbb{A}_{r,s}^v := \overline{\mathbb{H}_r^v} \setminus \mathbb{H}_s^v = \{z \in \mathbb{R}^2 : r \leq \langle z, v\rangle \leq s\}. \tag{6}$$

A *Jordan curve* is any subset of $\mathbb{R}^2$ which is homeomorphic to $\mathbb{S}^1$. A *Jordan domain* is any bounded open subset of $\mathbb{R}^2$ whose boundary is a Jordan curve.

We shall need the following theorem due to Janiszewski (see for instance, [17, Chapter X, Theorem 2]):

Theorem 2.1. – *If $X_1, X_2 \subset \mathbb{S}^2$ are two continua such that $X_1 \cap X_2$ is not connected, then $X_1 \cup X_2$ disconnects $\mathbb{S}^2$, i.e., $\mathbb{S}^2 \setminus (X_1 \cup X_2)$ is not connected.*

2.3.2. *The annulus*. – The *open annulus* is given by $\mathbb{A} := \mathbb{T} \times \mathbb{R}$. Its universal covering map will be denoted by $P\colon \mathbb{R}^2 \to \mathbb{A}$ and is defined by

$$P(x, y) := \big(\pi(x), y\big) = (x + \mathbb{Z}, y), \quad \forall (x, y) \in \mathbb{R}^2. \tag{7}$$

We will always consider the annulus endowed by the distance

$$d_{\mathbb{A}}(x, y) := \min\left\{\|\tilde{x} - \tilde{y}\| : \tilde{x} \in P^{-1}(x),\ \tilde{y} \in P^{-1}(y)\right\}, \quad \forall x, y \in \mathbb{A}.$$

We write $\hat{\mathbb{A}} := \mathbb{A} \sqcup \{-\infty, +\infty\}$ for the two-end compactification of the annulus $\mathbb{A}$. Observe that $\hat{\mathbb{A}}$ is homeomorphic to the 2-sphere $\mathbb{S}^2$.

We will need the following elementary result about unbounded connected subsets of $\mathbb{A}$:

Lemma 2.2. – *Let $C \subset \mathbb{A}$ be a closed connected unbounded set. If P is the covering map given by* (7)*, then every connected component of $P^{-1}(C)$ is unbounded in $\mathbb{R}^2$.*

Proof. – Reasoning by contradiction, let us suppose there is a bounded connected component K of $P^{-1}(C) \subset \mathbb{R}^2$.

If we write $\overline{C}$ for the closure of C in $\hat{\mathbb{A}}$, we get that $\overline{C}$ is compact and connected as well, and contains at least one of the two ends. Without loss of generalization, we can assume the upper end $+\infty$ belongs to $\overline{C}$.

Then we consider a sequence of open connected subsets $(U_n)_{n\geq 1}$ of $\hat{\mathbb{A}}$ satisfying the following properties: $\overline{C} \subset U_n \subset \hat{\mathbb{A}}$ and $\overline{U_{n+1}} \subset U_n$, for every $n \geq 1$; and

$$\overline{C} = \bigcap_{n\geq 1} U_n = \bigcap_{n\geq 1} \overline{U_n}.$$

Such a sequence of nested open sets can be constructed as follows: one considers a distance function $d_{\hat{\mathbb{A}}}$ on $\hat{\mathbb{A}}$ which is compatible with its topology and then defines $U_n := \overline{C}_{1/n}$, for every $n \geq 1$, where $\overline{C}_{1/n}$ denotes the $1/n$-neighborhood of $\overline{C}$ with respect to the distance $d_{\hat{\mathbb{A}}}$ as defined by (3).

Now let z_0 be an arbitrary point of $K \subset \mathbb{R}^2$. So we have $P(z_0) \in C \subset U_n$, for every $n \geq 1$. Since U_n is open and connected, there is a continuous curve $\gamma_n\colon [0,1] \to U_n$ such that $\gamma_n(0) = P(z_0)$, $\gamma_n(1) = +\infty$ and $\gamma_n(t) \in \mathbb{A}$, for each $n \in \mathbb{N}$ and every $t \in [0,1)$.

Then observe that, since P is a covering map, there exists a unique continuous curve $\tilde{\gamma}_n\colon [0,1) \to \mathbb{R}^2$ such that $\tilde{\gamma}_n(0) = z_0$ and $P \circ \tilde{\gamma}_n = \gamma_n$.

If $\hat{\mathbb{R}}^2 := \mathbb{R}^2 \sqcup \{\infty\}$ denotes the one-point compactification of $\mathbb{R}^2$, one sees that each $\tilde{\gamma}_n$ has a unique continuous extension from $[0,1]$ to $\hat{\mathbb{R}}^2$ just defining $\tilde{\gamma}_n(1) := \infty$. In this way, each $\tilde{\gamma}_n\big([0,1]\big)$ is a compact subset of $\hat{\mathbb{R}}^2$, and by compactness of the Hausdorff space $\mathscr{K}(\hat{\mathbb{R}}^2)$, there exists a sub-sequence $n_j \to \infty$ and a non-empty compact subset $L \subset \hat{\mathbb{R}}^2$ such that $\tilde{\gamma}_{n_j}\big([0,1]\big) \to L$, as $n_j \to \infty$, where the convergence is considered with respect to the Hausdorff distance. One can easily verify that L is connected, both points z_0 and ∞ belong to L and $P\big(L \setminus \{\infty\}\big) \subset C$. In particular, $L \setminus \{\infty\}$ is a closed, connected, unbounded subset of $P^{-1}(C) \subset \mathbb{R}^2$ and $K \cap (L \setminus \{\infty\}) \neq \emptyset$, contradicting the supposition that K is a bounded connected component of $P^{-1}(C)$. $\square$

2.4. Ergodic theory and cocycles

Given a topological space X, we write $\mathscr{B}_X$ to denote its Borel σ-algebra.

The Haar (probability) measure on $(\mathbb{T}^d, \mathscr{B}_{\mathbb{T}^d})$, also called Lebesgue measure, will be denoted by Leb_d. By a slight abuse of notation, we will also write Leb_d for the Lebesgue measure on $\mathbb{R}^d$; and for the sake of simplicity of notation, we shall just write Leb instead of Leb_1.

Given an arbitrary σ-finite measure space $(X, \mathscr{B}, \mu)$, a map $f\colon (X,\mathscr{B}) \circlearrowleft$ is said to be *non-singular* (respect to μ) when it is measurable and, for every $B \in \mathscr{B}$, it holds $\mu(f^{-1}(B)) = 0$ if and only if $\mu(B) = 0$. A non-singular map $f\colon (X,\mathscr{B}) \circlearrowleft$ is said to be *conservative* (with respect to μ) when for every $B \in \mathscr{B}$ such that $\mu(B) > 0$, there exists $n \geq 1$ satisfying $\mu\big(B \cap f^{-n}(B)\big) > 0$.

As usual, we say that a measurable map $f\colon (X,\mathscr{B}) \circlearrowleft$ preserves μ when $f_\star\mu = \mu$, where $f_\star\mu(B) := \mu(f^{-1}(B))$, for every $B \in \mathscr{B}$; and f is said to be an *automorphism* of $(X,\mathscr{B},\mu)$ when it is bijective and its inverse is measurable and preserves μ, too.

Given an invertible map $f\colon X \circlearrowleft$, a function $\phi\colon X \to \mathbb{R}$ and any $n \in \mathbb{Z}$, one defines the *Birkhoff sum*

$$\mathcal{S}_f^n(\phi) := \begin{cases} \sum_{j=0}^{n-1} \phi \circ f^j, & \text{if } n \geq 1; \\ 0, & \text{if } n = 0; \\ -\sum_{j=1}^{-n} \phi \circ f^{-j}, & \text{if } n < 0. \end{cases} \tag{8}$$

Putting together two classical results of Atkinson [1, Theorem] and Schmidt [28, Proposition 6], we get the following

THEOREM 2.3. – *Let $(X,\mathscr{B},\mu)$ be a probability space, $f\colon (X,\mathscr{B},\mu) \circlearrowleft$ be an ergodic automorphism and $\phi \in L^1(X,\mathscr{B},\mu)$ be a real function such that $\int_X \phi \,\mathrm{d}\mu = 0$. Then, the skew-product automorphism $F\colon X \times \mathbb{R} \circlearrowleft$ given by*

$$F(x,t) := \big(f(x), t + \phi(x)\big), \quad \forall (x,t) \in X \times \mathbb{R}, \tag{9}$$

is conservative with respect to the F-invariant σ-finite measure $\mu \otimes \mathrm{Leb}$.

2.5. Groups of homeomorphisms

From now on and until the end of this section, M will denote an arbitrary topological manifold. We write $\mathrm{Homeo}(M)$ for the group of homeomorphisms from M onto itself. The subgroup formed by those homeomorphisms which are homotopic to the identity id_M will be denoted by $\mathrm{Homeo}_0(M)$.

2.5.1. *Torus homeomorphisms and their lifts*. – The group of lifts of torus homeomorphisms which are homotopic to the identity will be denoted by

$$\widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d) := \left\{ \tilde{f} \in \mathrm{Homeo}_0(\mathbb{R}^d) : \tilde{f} - id_{\mathbb{R}^d} \in C^0(\mathbb{T}^d, \mathbb{R}^d) \right\}.$$

Notice that in this definition, as it is usually done, we are identifying the elements of $C^0(\mathbb{T}^d, \mathbb{R}^d)$ with those $\mathbb{Z}^d$-periodic continuous functions from $\mathbb{R}^d$ to itself.

Making some abuse of notation, we also write $\pi\colon \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d) \to \mathrm{Homeo}_0(\mathbb{T}^d)$ for the map that associates to each $\tilde{f}$ the only torus homeomorphism $\pi\tilde{f}$ such that $\tilde{f}$ is a lift of $\pi\tilde{f}$. Notice that with our notations, it holds $\pi T_\alpha = T_{\pi(\alpha)} \in \mathrm{Homeo}_0(\mathbb{T}^d)$, for every $\alpha \in \mathbb{R}^d$.

Given any $\tilde{f} \in \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d)$, we define its *displacement function* by

$$\Delta_{\tilde{f}} := \tilde{f} - id_{\mathbb{R}^d} \in C^0(\mathbb{T}^d, \mathbb{R}^d). \tag{10}$$

Observe that this function can be naturally considered as a cocycle over $f := \pi\tilde{f}$ because

$$\Delta_{\tilde{f}^n} = \sum_{j=0}^{n-1} \Delta_{\tilde{f}} \circ f^j, \quad \forall n \geq 1. \tag{11}$$

For the sake of readability, we shall use the usual notation for cocycles defining

$$\Delta_{\tilde{f}}^{(n)} := \Delta_{\tilde{f}^n}, \quad \forall n \in \mathbb{Z}.$$

The map $\mathbb{R}^d \ni \alpha \mapsto T_\alpha \in \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d)$ defines an injective group homomorphism, and hence, $\mathbb{R}^d$ naturally acts on $\widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d)$ by conjugacy. However, since every element of $\widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d)$ commutes with $T_{\boldsymbol{p}}$, for all $\boldsymbol{p} \in \mathbb{Z}^d$, we conclude $\mathbb{T}^d$ itself acts on $\widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d)$ by conjugacy, i.e., the map $\mathrm{Ad}\colon \mathbb{T}^d \times \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d) \to \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d)$ given by

$$\mathrm{Ad}_t(\tilde{f}) := T_{\tilde{t}}^{-1} \circ \tilde{f} \circ T_{\tilde{t}}, \quad \forall (t, \tilde{f}) \in \mathbb{T}^d \times \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d),\ \forall \tilde{t} \in \pi^{-1}(t), \tag{12}$$

is well-defined.

2.5.2. *Invariant measures*. – We write $\mathfrak{M}(M)$ for the space of Borel probability measures on M. A measure $\mu \in \mathfrak{M}(M)$ is said to have *total support* when $\mu(A) > 0$ for every non-empty open set $A \subset M$. We say μ is a *topological measure* if it has total support and no atoms.

For every $\mu \in \mathfrak{M}(M)$, we consider the group of homeomorphisms

$$\mathrm{Homeo}_\mu(M) := \{f \in \mathrm{Homeo}(M) : f_\star\mu = \mu\}.$$

Given $f \in \mathrm{Homeo}(M)$, we write $\mathfrak{M}(f) := \{\nu \in \mathfrak{M}(M) : f_\star\nu = \nu\}$.

The following classical result is due to Oxtoby and Ulam [26]:

THEOREM 2.4. – *Let M be a compact topological manifold and $\mu, \nu \in \mathfrak{M}(M)$ two topological measures. Then, there exists $h \in \mathrm{Homeo}(M)$ such that $h_\star \mu = \nu$.*

For the sake of simplicity of notation, on the two-dimensional torus we define the group of *symplectomorphisms* (also called *area-preserving homeomorphisms*) by

$$\mathrm{Symp}(\mathbb{T}^2) := \left\{f \in \mathrm{Homeo}(\mathbb{T}^2) : \mathrm{Leb}_2 \in \mathfrak{M}(f)\right\}.$$

It is well known that its connected component containing the identity, which will be denoted by $\mathrm{Symp}_0(\mathbb{T}^2)$, coincides with $\mathrm{Symp}(\mathbb{T}^2) \cap \mathrm{Homeo}_0(\mathbb{T}^2)$. We write

$$\widetilde{\mathrm{Symp}_0}(\mathbb{T}^2) := \pi^{-1}\left(\mathrm{Symp}_0(\mathbb{T}^2)\right) < \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^2).$$

2.6. Rotation set and rotation vectors

Let $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ be an arbitrary homeomorphism and $\tilde{f} \in \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d)$ be a lift of f. The *rotation set of* $\tilde{f}$ is given by

$$\rho(\tilde{f}) := \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} \left\{\frac{\Delta_{\tilde{f}}^{(n)}(z)}{n} : z \in \mathbb{R}^d\right\}}. \tag{13}$$

It can be easily shown that $\rho(\tilde{f})$ is non-empty, compact and connected.

When $d = 1$, by classical Poincaré theory of circle homeomorphisms [27] we know that $\rho(\tilde{f})$ reduces to a point, but in general this does not hold in higher dimensions.

We summarized some elementary facts about rotation sets which are due to Misiurewicz and Ziemian [24, Proposition 2.1]:

PROPOSITION 2.5. – *Given any $\tilde{f} \in \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^d)$, the following properties hold:*

(i) $\rho(\tilde{f}^n) = n\rho(\tilde{f}) := \{n\rho \in \mathbb{R}^d : \rho \in \rho(\tilde{f})\}$, *for any $n \in \mathbb{Z}$;*

(ii) $\rho(T_{\boldsymbol{p}} \circ \tilde{f}) = T_{\boldsymbol{p}}\big(\rho(\tilde{f})\big)$, *for any $\boldsymbol{p} \in \mathbb{Z}^d$.*

As a consequence of *(ii)* of Proposition 2.5, we see that given any $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and any lift $\tilde{f} : \mathbb{R}^d \circlearrowleft$ of f, we can define

$$\rho(f) := \pi(\rho(\tilde{f})) \subset \mathbb{T}^d.$$

We say that $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ is a *pseudo-rotation* when $\rho(f)$ is a singleton.

By (11) and (13), we know the rotation set is formed by accumulation points of Birkhoff averages of the displacement function. So given any $\mu \in \mathfrak{M}(f)$, one can define its *rotation vector* by

$$\rho_\mu(\tilde{f}) := \int_{\mathbb{T}^d} \Delta_{\tilde{f}} \, \mathrm{d}\mu.$$

Thus, by Birkhoff ergodic theorem we get $\rho_\mu(\tilde{f}) \in \rho(\tilde{f})$, for every f-invariant ergodic probability measure μ. Moreover, the following holds:

THEOREM 2.6 (Theorem 2.4 in [24]). – *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and $\tilde{f}\colon\mathbb{R}^d \circlearrowleft$ be a lift of f. Then, for every extreme point $w \in \rho(\tilde{f})$, there exists an ergodic measure $\mu \in \mathfrak{M}(f)$ such that $\rho_\mu(\tilde{f}) = w$. Consequently, it holds*

$$\mathrm{Conv}\big(\rho(\tilde{f})\big) = \Big\{\rho_\nu(\tilde{f}) : \nu \in \mathfrak{M}(\pi\tilde{f})\Big\},$$

where $\mathrm{Conv}(\cdot)$ *denotes the convex hull operator.*

However, in the two-dimensional case rotation sets are always convex:

THEOREM 2.7 (Theorem 3.4 in [24]). – *For every $\tilde{f} \in \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^2)$, we have*

$$\rho(\tilde{f}) = \Big\{\rho_\nu(\tilde{f}) : \nu \in \mathfrak{M}\left(\pi\tilde{f}\right)\Big\}.$$

2.7. Hamiltonian homeomorphisms

In the symplectic setting, that is when $\tilde{f} \in \widetilde{\mathrm{Symp}_0}(\mathbb{T}^2)$, the rotation vector of Leb_2 is also called the *flux of* $\tilde{f}$ and is usually denoted by $\mathrm{Flux}(\tilde{f}) := \rho_{\mathrm{Leb}_2}(\tilde{f})$. In this case, it can be easily shown that the flux map $\mathrm{Flux}\colon\widetilde{\mathrm{Symp}_0}(\mathbb{T}^2) \to \mathbb{R}^2$ is indeed a group homomorphism. Since

$$\mathrm{Flux}(T_{\boldsymbol{p}} \circ \tilde{f}) = T_{\boldsymbol{p}}(\mathrm{Flux}(\tilde{f})), \quad \forall \boldsymbol{p} \in \mathbb{Z}^2,\ \forall f \in \widetilde{\mathrm{Symp}_0}(\mathbb{T}^2),$$

this homomorphism clearly induces a map $\mathrm{Symp}_0(\mathbb{T}^2) \to \mathbb{T}^2$ which, by some abuse of notation, will be denoted by Flux, too.

The kernel of this homomorphism $\mathrm{Flux}\colon\mathrm{Symp}_0(\mathbb{T}^2) \to \mathbb{T}^2$ is denoted by

$$\mathrm{Ham}(\mathbb{T}^2) := \{f \in \mathrm{Symp}_0(\mathbb{T}^2) : \mathrm{Flux}(f) = 0\} \lhd \mathrm{Symp}_0(\mathbb{T}^2),$$

i.e., it is a normal subgroup of $\mathrm{Symp}_0(\mathbb{T}^2)$. The elements of $\mathrm{Ham}(\mathbb{T}^2)$ are called *Hamiltonian homeomorphisms.*

Analogously, the kernel of $\mathrm{Flux}\colon\widetilde{\mathrm{Symp}_0}(\mathbb{T}^2) \to \mathbb{R}^2$ is denoted by

$$\widetilde{\mathrm{Ham}}(\mathbb{T}^2) := \Big\{\tilde{f} \in \widetilde{\mathrm{Symp}_0}(\mathbb{T}^2) : \mathrm{Flux}(\tilde{f}) = 0\Big\}.$$

REMARK 2.8. – Notice that $\mathrm{Ham}(\mathbb{T}^2)$ and $\widetilde{\mathrm{Ham}}(\mathbb{T}^2)$ can be naturally identified. In fact, the restriction $\pi|_{\widetilde{\mathrm{Ham}}(\mathbb{T}^2)}\colon\widetilde{\mathrm{Ham}}(\mathbb{T}^2) \to \mathrm{Ham}(\mathbb{T}^2)$ is a continuous group isomorphism.

Observe the following short exact sequence splits:

$$0 \to \mathrm{Ham}(\mathbb{T}^2) \hookrightarrow \mathrm{Symp}_0(\mathbb{T}^2) \xrightarrow{\mathrm{Flux}} \mathbb{T}^2 \to 0.$$

In fact, the map $\mathbb{T}^2 \ni \alpha \mapsto T_\alpha$ is a section of Flux, and thus, the group $\mathrm{Symp}_0(\mathbb{T}^2)$ can be decomposed as a semi-direct product $\mathrm{Symp}_0(\mathbb{T}^2) = \mathbb{T}^2 \ltimes \mathrm{Ham}(\mathbb{T}^2)$. In other words, given $\alpha, \beta \in \mathbb{T}^2$ and $h, g \in \mathrm{Ham}(\mathbb{T}^2)$, we have

$$(T_\alpha \circ h) \circ (T_\beta \circ g) = T_{\alpha+\beta} \circ \big(\mathrm{Ad}_\beta(h) \circ g\big),$$

where the $\mathbb{T}^2$-action Ad is given by (12).

This elementary fact about the group structure of $\mathrm{Symp}_0(\mathbb{T}^2)$ is our main inspiration for the construction of the fiber-wise Hamiltonian skew-product we will perform in §6.1.

2.8. Rotation set, periodic points and minimality

The following result due to Handel asserts that the rotation set of a periodic point free homeomorphism has empty interior:

THEOREM 2.9 (Handel [8]). – *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be such that* $\mathrm{Per}(f) = \emptyset$ *and $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$ be a lift of f. Then, there exist $v \in \mathbb{S}^1$ and $\alpha \in \mathbb{R}$ so that*

$$\frac{\left\langle \tilde{f}^n(z) - z, v \right\rangle}{n} \to \alpha, \quad \text{as } n \to \infty,$$

where the convergence is uniform for $z \in \mathbb{R}^2$. In other words, the rotation set $\rho(\tilde{f}) \subset \ell_\alpha^v$, where the straight line ℓ_α^v is given by (5).

The following result due to Franks will play a fundamental role in our work:

THEOREM 2.10 (Franks [4]). – *If $\tilde{f} \in \widetilde{\mathrm{Symp}}_0(\mathbb{T}^2)$ and* $\mathrm{Flux}(\tilde{f}) = (p_1/q, p_2/q) \in \mathbb{Q}^2$, *then there exists $z \in \mathbb{R}^2$ such that*

$$\tilde{f}^q(z) = z + (p_1, p_2).$$

In particular, $\pi(z) \in \mathrm{Per}(\pi \tilde{f})$.

Every probability measure which is invariant under a minimal homeomorphism is necessarily a topological measure. Hence, as a straightforward consequence of Theorems 2.4 and 2.10, we get the following

COROLLARY 2.11. – *If $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is minimal and $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$ is a lift of f, then*

$$\rho(\tilde{f}) \cap \mathbb{Q}^2 = \emptyset.$$

2.9. Classification of plane fixed points

Let $V, V' \subset \mathbb{R}^2$ be two non-empty open sets and let $f: V \to V'$ be a homeomorphism. Following the terminology of Le Calvez [20], a fixed point $z_0 \in \mathrm{Fix}(f)$ is said to be:

- *isolated* when it is an isolated point of the set $\mathrm{Fix}(f)$;
- *accumulated* when every neighborhood of z_0 contains a periodic orbit of f different from z_0;
- *dissipative* when z_0 admits a local basis $(U_n)_{n\geq 0}$ of neighborhoods such that $f(\partial U_n) \cap \partial U_n = \emptyset$, for every $n \geq 0$, i.e., each neighborhood is either attractive or repulsive;
- *indifferent* when there exists a neighborhood W of z_0 such that $\overline{W} \subset V$ and for every Jordan domain $U \subset W$ which is a neighborhood of z_0 it holds

  $$\mathrm{cc}\left(\mathscr{I}_f(\overline{U}), z_0\right) \cap \partial U \neq \emptyset,$$

  where $\mathscr{I}_f(\overline{U})$ denotes the maximal f-invariant subset of U given by (2).

2.10. Fixed point indexes

If $f : V \to V'$ is as in §2.9, and $\gamma : \mathbb{S}^1 \to V$ is a Jordan curve such that $\gamma(\mathbb{S}^1) \cap \mathrm{Fix}(f) = \emptyset$, then one defines the *index of f along γ* as the integer

$$i(f,\gamma) := \deg\left(\mathbb{S}^1 \ni t \mapsto \frac{f(\gamma(t)) - \gamma(t)}{\|f(\gamma(t)) - \gamma(t)\|} \in \mathbb{S}^1\right),$$

where $\deg(\cdot)$ denotes the topological degree.

When $z_0 \in \mathrm{Fix}(f)$ is isolated, then one can define the *index of f at z_0* as

$$i(f,z_0) := i(f,\partial U),$$

where U denotes any Jordan domain satisfying $\overline{U} \subset V$ and $\overline{U} \cap \mathrm{Fix}(f) = \{z_0\}$. Since this index does not depend on the choice of U, this notion is well-defined.

We will need the following topological version of Leau-Fatou's flower theorem due to Le Calvez [19], that has been lately improved by Le Roux [23]:

THEOREM 2.12. – *Let us suppose $f : V \to V'$ is an orientation-preserving homeomorphism and $z_0 \in \mathrm{Fix}(f)$ is an isolated fixed point such that $i(f,z_0) \geq 2$. Then there exist two open non-empty subsets $W^+, W^- \subset V \setminus \{z_0\}$ such that*

(i) *$f^n(W^+)$ is well-defined for every $n \geq 0$, $f^m(W^+) \cap f^n(W^+) = \emptyset$ whenever m and n are different non-negative integers and $\omega_f(z) = \{z_0\}$, for every $z \in W^+$;*

(ii) *$f^{-n}(W^-)$ is well-defined for every $n \geq 0$, $f^{-m}(W^-) \cap f^{-n}(W^-) = \emptyset$ whenever m and n are different non-negative integers and $\alpha_f(z) = \{z_0\}$, for every $z \in W^-$;*

where α_f and ω_f denote the α-and ω-limit sets, respectively.

The following result about indexes of iterates of non-accumulated fixed points is due to Le Calvez and Yoccoz [22] but its proof has never been published (see for instance [20, Proposition 3.3]):

THEOREM 2.13. – *If f is an orientation-preserving homeomorphism and $z_0 \in \mathrm{Fix}(f)$ is isolated, non accumulated, non indifferent and non dissipative, then there exist integers $q \geq 1$ and $r \geq 1$ such that*

$$\begin{cases} i(f^k, z_0) = 1, & \text{if } k \notin q\mathbb{Z}; \\ i(f^k, z_0) = 1 - rq, & \text{if } k \in q\mathbb{Z}. \end{cases}$$

2.11. Minimal homeomorphisms

In this paragraph we recall some classical and elementary results about minimal homeomorphisms that we shall frequently use all along the paper.

We say that a subset $A \subset \mathbb{Z}$ has *bounded gaps* if there exists $N \in \mathbb{N}$ such that

$$A \cap \{n, n+1, \dots, n+N\} \neq \emptyset, \quad \forall n \in \mathbb{Z}. \tag{14}$$

The minimum natural number N such that (14) holds shall be denoted by $\mathscr{G}(A)$.

The following three results are very well-known, but we decided to include them here just for the sake of reference:

PROPOSITION 2.14. – *If (X, d) is a compact metric space and $f: X \circlearrowleft$ is a minimal homeomorphism, then for every non-empty open set $U \subset X$ and any $x \in X$, the* visiting time set

$$\tau(x, U, f) := \{n \in \mathbb{Z} : f^n(x) \in U\}$$

has bounded gaps.

As a consequence of this result, one can easily show the following:

COROLLARY 2.15. – *For every $\alpha \in \mathbb{T}^d$, any $x \in \mathbb{T}^d$ and any neighborhood $V \subset \mathbb{T}^d$ of x, the visiting time set $\tau(x, V, T_\alpha)$ has bounded gaps.*

PROPOSITION 2.16. – *If (X, d) is a connected compact metric space and $f: X \circlearrowleft$ is a minimal homeomorphism, then f^n is minimal for every $n \in \mathbb{Z} \setminus \{0\}$.*

The last result we recall here is due to Gottschalk and Hedlund and deals with cohomological equations:

THEOREM 2.17 (Gottschalk, Hedlund [6]). – *Let X be a compact metric space and $f: X \circlearrowleft$ be a minimal homeomorphism. Let $\phi: X \to \mathbb{R}$ be a continuous function and assume there exists $x_0 \in X$ such that*

$$\sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n-1} \phi\big(f^j(x_0)\big) \right| < \infty.$$

Then, there is a continuous function $u: X \to \mathbb{R}$ such that $u \circ f - u = \phi$. In particular,

$$\sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n-1} \phi\big(f^j(x)\big) \right| \leq 2 \|u\|_{C^0} < \infty, \quad \forall x \in X.$$

3. An ergodic deviation result

This section is devoted to prove an abstract ergodic deviation theorem that will play a key role in §7. Even though this result might be already known to some experts, we were not able to find any reference in the literature and thus we have decided to include its proof here.

THEOREM 3.1. – *Let $(X, \mathscr{B}, \mu)$ be a probability space, $f: (X, \mathscr{B}, \mu) \circlearrowleft$ an ergodic automorphism and $\phi \in L^1(X, \mathscr{B}, \mu)$ such that $\int_X \phi \, \mathrm{d}\mu = 0$. Let us suppose that*

$$\sup_{n \geq 0} \mathcal{S}_f^n \phi(x) = +\infty, \quad \textit{and} \quad \inf_{n \geq 0} \mathcal{S}_f^n \phi(x) > -\infty, \tag{15}$$

for μ-a.e. $x \in X$, where $\mathcal{S}_f^n \phi$ denotes the Birkhoff sum given by (8).

Then, it holds

$$\sup_{n \leq 0} \mathcal{S}_f^n \phi(x) = +\infty, \quad \textit{and} \quad \inf_{n \leq 0} \mathcal{S}_f^n \phi(x) > -\infty,$$

for μ-a.e. $x \in X$.

To prove Theorem 3.1, first we need a lemma which is a simple consequence of Theorem 2.3:

LEMMA 3.2. – *Let $(X, \mathscr{B}, \mu)$, f and ϕ be as in Theorem 3.1 and let us assume there exists $M > 0$ such that*

$$\sup_{n\in\mathbb{N}} \mathcal{S}_f^n \phi(x) < M, \quad \textit{for } \mu\textit{-a.e. } x \in X. \tag{16}$$

Then, it holds

$$\inf_{n\in\mathbb{N}} \mathcal{S}_f^n \phi(x) \geq -M, \quad \textit{for } \mu\textit{-a.e. } x \in X. \tag{17}$$

Proof of Lemma 3.2. – Let us suppose (17) is false. So, if we define

$$A_n^m := \left\{x \in X : \mathcal{S}_f^m \phi(x) \leq -M - 1/n\right\},$$

for each $m, n \geq 1$, we have $\mu\Big(\bigcup_{m,n\geq 1} A_{m,n}\Big) > 0$. Then, there exist $N, n \geq 1$ such that the set $A := A_n^N$ satisfies $\mu(A) > 0$ and

$$\mathcal{S}_f^N \phi(x) \leq -M - \frac{1}{n}, \quad \forall x \in A. \tag{18}$$

Now, let us consider the skew-product $F\colon X \times \mathbb{R} \circlearrowleft$ given by (9) and define the set $\tilde{A} := A \times [-1/2n, 1/2n] \subset X \times \mathbb{R}$. Since $\mu \otimes \mathrm{Leb}(\tilde{A}) = n^{-1}\mu(A) > 0$, by Theorem 2.3 we know there exists $k \geq 1$ such that $\mu \otimes \mathrm{Leb}\big(\tilde{A} \cap F^{-k}(\tilde{A})\big) > 0$. By classical arguments in ergodic theory, this implies that there exists a sequence $k_j \to +\infty$ such that

$$\mu \otimes \mathrm{Leb}\big(\tilde{A} \cap F^{-k_j}(\tilde{A})\big) > 0, \quad \forall j \in \mathbb{N}.$$

This is equivalent to say that the set

$$B_j := \left\{x \in A : f^{k_j}(x) \in A,\ \left|\mathcal{S}_f^{k_j} \phi(x)\right| \leq \frac{1}{n}\right\}$$

has positive μ-measure, for every j.

Now, choosing j large enough in order to have $k_j > N$, and combining this with (18), we get

$$\mathcal{S}_f^{k_j - N}\big(f^N(x)\big) = \mathcal{S}_f^{k_j}(x) - \mathcal{S}_f^N(x) > -\frac{1}{n} + M + \frac{1}{n} = M,$$

for μ-a.e. $x \in B_j$. Since $\mu(B_j) > 0$, this contradicts (16). □

Proof of Theorem 3.1. – Of course we can assume $|\phi(x)| < \infty$, for every $x \in X$. By (15), there exist $K > 0$ and $A \in \mathscr{B}$ with $\mu(A) > 0$ such that

$$\mathcal{S}_f^n \phi(x) > -K, \quad \forall n \geq 0, \tag{19}$$

for μ-a.e. $x \in A$.

Consider the functions $\tau_A^+, \tau_A^-\colon X \to \mathbb{N}_0 \cup \{\infty\}$ given by

$$\tau_A^+(x) := \min\{n \geq 0 : f^n(x) \in A\},$$
$$\tau_A^-(x) := \min\{n \geq 0 : f^{-n}(x) \in A\}, \quad \forall x \in X.$$

Since f is ergodic, τ_A^+ and τ_A^- are finite μ-a.e. We also define the *first return time* to A by $r_A := \tau_A^+ \circ f\big|_A + 1$.

Now, we consider the probability space $(A, \mathscr{B}_A, \mu_A)$ given by $\mathscr{B}_A := \{B \in \mathscr{B} : B \subset A\}$ and $\mu_A := \mu(A)^{-1}\mu\big|_A$ and the ergodic automorphism $f_A\colon (A, \mathscr{B}_A, \mu_A) \circlearrowleft$ given by the first return map:

$$f_A(x) := f^{\tau_A^+(f(x))}\big(f(x)\big) = f^{r_A(x)}(x),$$

for μ-a.e. $x \in A$. We also define the function $\phi_A(x) := \mathcal{S}_f^{r_A(x)}\phi(x)$. Then it holds $\phi_A \in L^1(A, \mathscr{B}_A, \mu_A)$ and $\int_A \phi_A \, \mathrm{d}\mu = 0$. On the other hand, by (19) we know

$$\mathcal{S}_{f_A}^n \phi_A(x) > -K, \quad \forall n \geq 0, \tag{20}$$

for μ_A-a.e. $x \in A$.

Then, applying Lemma 3.2 in this context, we conclude that for μ_A-a.e. $x \in A$ it holds

$$\mathcal{S}_{f_A}^n \phi_A(x) \leq K, \quad \forall n \geq 0. \tag{21}$$

Now, let us consider the measurable functions $M \colon A \to \mathbb{R}$ and $N \colon A \to \mathbb{N}_0$ given by

$$M(x) := \sup_{1 \leq n \leq r_A(x)} \mathcal{S}_f^{n-1}\phi(x), \tag{22}$$

and

$$N(x) := \inf\left\{n \geq 0 : n < r_A(x),\ \mathcal{S}_f^n \phi(x) = M(x)\right\}, \tag{23}$$

and notice they are well defined μ-a.e. $x \in A$.

For each pair $(m, n) \in \mathbb{N} \times \mathbb{N}_0$, let us consider the set

$$A_m^n := \{x \in A : m \leq M(x) < \infty,\ N(x) = n\}.$$

Putting together (15) and (21) it follows that

$$\mu\Bigg(\bigcup_{n \geq 0} A_m^n\Bigg) > 0, \quad \forall m \in \mathbb{N}. \tag{24}$$

By (24), for each $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ so that $\mu(A_m^{n_m}) > 0$. So, let us define the set

$$B := \bigcap_{m \in \mathbb{N}} \bigcap_{i \geq 0} \bigcup_{j \geq i} f^j(A_m^{n_m}). \tag{25}$$

Since f is an ergodic automorphism, B has full μ-measure.

Now, let us consider an arbitrary point $x \in B$ and any $m \in \mathbb{N}$. Since $A_m^{n_m} \subset A$, it clearly holds $\tau_A^-(x) < \infty$. By (25), there exists a natural number $j = j(x, m) > n_m$ such that

$$f^{-j}\big(f^{-\tau_A^-(x)}(x)\big) \in A_m^{n_m}. \tag{26}$$

Since both points $f^{-\tau_A^-(x)}(x)$ and $f^{-j}\big(f^{-\tau_A^-(x)}(x)\big)$ belong to A, there exists $j_A \in \mathbb{N}$ such that

$$f_A^{-j_A}\big(f^{-\tau_A^-(x)}(x)\big) = f^{-j}\big(f^{-\tau_A^-(x)}(x)\big).$$

Now invoking (20), (22), (23) and (26), we get

$$\begin{aligned}
\mathcal{S}_f^{-\tau_A^-(x)-j+n_m}\phi(x) &= \mathcal{S}_f^{-\tau_A^-(x)}\phi(x) + \mathcal{S}_f^{-j+n_m}\phi\big(f^{-\tau_A^-(x)}(x)\big) \\
&= \mathcal{S}_f^{-\tau_A^-(x)}\phi(x) + \mathcal{S}_{f_A}^{-j_A}\phi_A\big(f^{-\tau_A^-(x)}(x)\big) + \mathcal{S}_f^{n_m}\phi\Big(f_A^{-j_A}\big(f^{-\tau_A^-(x)}(x)\big)\Big) \\
&\geq \mathcal{S}_f^{-\tau_A^-(x)}\phi(x) - K + m.
\end{aligned} \tag{27}$$

Since m is arbitrary in (27), $\mu(B) = 1$ and $j > n_m$, we have proved that

$$\sup_{n \leq 0} \mathcal{S}_f^n \phi(x) = +\infty, \quad \text{for } \mu\text{-a.e. } x \in X.$$

On the other hand, let us consider the set

$$C := \bigcap_{i\geq 0}\bigcup_{j\geq i} f^j(A).$$

Since f is ergodic, it holds $\mu(C) = 1$.

Now, consider any $x \in C$ and any $n \in \mathbb{N}$. Thus, $\tau_A^-(x)$ and $\tau_A^-\big(f^{-n}(x)\big)$ are both finite, and both points $f^{-\tau_A^-(x)}(x)$ and $f^{-n-\tau_A^-(f^{-n}(x))}(x)$ belong to A. So, there exists $l_A = l_A(x,n) \in \mathbb{N}$ such that

$$f_A^{l_A}\big(f^{-n-\tau_A^-(f^{-n}(x))}(x)\big) = f^{-\tau_A^-(x)}(x).$$

Then, we have

$$\begin{aligned}
\mathcal{S}_f^{-n}\phi(x) &= \mathcal{S}_f^{-n-\tau_A^-(f^{-n}(x))}\phi(x) + \mathcal{S}_f^{\tau_A^-(f^{-n}(x))}\phi\big(f^{-n-\tau_A^-(f^{-n}(x))}(x)\big)\\
&> \mathcal{S}_f^{-n-\tau_A^-(f^{-n}(x))}\phi(x) - K\\
&= \mathcal{S}_f^{-\tau_A^-(x)}\phi(x) - \mathcal{S}_f^{n+\tau_A^-(f^{-n}(x))-\tau_A^-(x)}\phi\big(f^{-n-\tau_A^-(f^{-n}(x))}(x)\big) - K\\
&= \mathcal{S}_f^{-\tau_A^-(x)}\phi(x) - \mathcal{S}_{f_A}^{l_A}\phi_A\big(f^{-n-\tau_A^-(f^{-n}(x))}(x)\big) - K\\
&\geq \mathcal{S}_f^{-\tau_A^-(x)}\phi(x) - 2K,
\end{aligned}$$

where the first inequality follows from (19) and the second one from (21).

From this last estimate, and since $\mu(C) = 1$, it follows that

$$\inf_{n\leq 0} \mathcal{S}_f^n\phi(x) > -\infty, \quad \text{for } \mu\text{-a.e. } x \in X. \qquad \square$$

4. Rotational deviations

In this section we enter into the core of this work: the study of rotational deviations for 2-torus homeomorphisms in the identity isotopy class.

Let us start by recalling some definitions we introduced in [13]. Let $f\colon \mathbb{T}^2 \circlearrowleft$ be a homeomorphism homotopic to the identity and $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$ be a lift of f. Let us suppose that there exist $v \in \mathbb{S}^1$ and $\alpha \in \mathbb{R}$ such that

$$\rho(\tilde{f}) \subset \ell_\alpha^v = \{\alpha v + t v^\perp : t \in \mathbb{R}\}. \tag{28}$$

Observe that, by Theorem 2.9, this hypothesis is always satisfied when f is periodic point free.

We say that a point $z_0 \in \mathbb{T}^2$ exhibits *bounded v-deviations* when there exists a real constant $M = M(z_0, f) > 0$ such that

$$\Big\langle \tilde{f}^n(\tilde{z}_0) - \tilde{z}_0 - n\rho, v\Big\rangle = \Big\langle \Delta_{\tilde{f}}^{(n)}(z_0), v\Big\rangle - n\alpha \leq M, \quad \forall n \in \mathbb{Z}, \tag{29}$$

for any $\tilde{z}_0 \in \pi^{-1}(z_0)$, any $\rho \in \rho(\tilde{f})$ and where $\Delta_{\tilde{f}}$ denotes the displacement cocycle of $\tilde{f}$ given by (10).

Moreover, we say that f exhibits *uniformly bounded v-deviation* when there exists $M = M(f) > 0$ such that

$$\Big\langle \Delta_{\tilde{f}}^{(n)}(z), v\Big\rangle - n\alpha \leq M, \quad \forall z \in \mathbb{T}^2,\ \forall n \in \mathbb{Z}.$$

Even though the straight lines ℓ_α^v and $\ell_{-\alpha}^{-v}$ coincide as subsets of $\mathbb{R}^2$, there is no a priori obvious relation between boundedness of v-deviation and $(-v)$-deviation, because in our definition of "bounded v-deviations" given by (29) we are just considering boundedness from above.

However, we got the following result that relates both v-and $(-v)$-deviations:

THEOREM 4.1 (Corollary 3.2 in [13]). – *If $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and $\tilde{f}\colon\mathbb{R}^2 \circlearrowleft$ is a lift of f such that condition* (28) *holds, then f exhibits uniformly bounded v-deviations if and only if $\tilde{f}$ exhibits uniformly bounded $(-v)$-deviations.*

As a particular case of our definition of boundedness rotational deviations, let us recall that a homeomorphism $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is said to be *annular* (see for instance [16, 11]) when there exist a lift $\tilde{f} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$, $M > 0$ and $v \in \mathbb{S}^1$ with rational slope such that

$$\left|\left\langle \Delta_{\tilde{f}}^{(n)}(z), v\right\rangle\right| \leq M, \quad \forall z \in \mathbb{T}^2,\ \forall n \in \mathbb{Z}. \tag{30}$$

Observe that in such a case, the rotation set $\rho(\tilde{f})$ is contained in the line ℓ_0^v, i.e., the straight line parallel to v and passing through the origin.

On the other hand, a homeomorphism $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is said to be *eventually annular* when there exists $k \in \mathbb{N}$ such that f^k is annular.

In [13] we proved that boundedness of v-deviations is equivalent to the existence of a certain invariant topological object called torus *pseudo-foliation.*

4.1. Pseudo-foliations

In this paragraph we recall the notions of *plane and torus pseudo-foliations* we introduced in [13].

4.1.1. *Plane pseudo-foliations*. – Let $\mathscr{F}$ be a partition of $\mathbb{R}^2$. We shay that $\mathscr{F}$ is a *plane pseudo-foliation* when every atom of $\mathscr{F}$ is closed, connected, has empty interior and disconnects $\mathbb{R}^2$ in exactly two connected components.

Given any $z \in \mathbb{R}^2$, we write $\mathscr{F}_z$ for the atom of the partition $\mathscr{F}$ containing the point z. If $h\colon\mathbb{R}^2 \circlearrowleft$ is an arbitrary map, we say that $\mathscr{F}$ is h-invariant when

$$h\big(\mathscr{F}_z\big) = \mathscr{F}_{h(z)}, \quad \forall z \in \mathbb{R}^2.$$

Let us recall the following result of [13, Proposition 5.1]:

PROPOSITION 4.2. – *If $\mathscr{F}$ is a plane pseudo-foliation, then both connected components of $\mathbb{R}^2 \setminus \mathscr{F}_z$ are unbounded, for every $z \in \mathbb{R}^2$.*

4.1.2. *Torus pseudo-foliations.* – A partition $\mathscr{F}$ of $\mathbb{T}^2$ is said to be a *toral pseudo-foliation* whenever there exists a plane pseudo-foliation $\widetilde{\mathscr{F}}$, called the *lift of* $\mathscr{F}$, satisfying

$$\pi\left(\widetilde{\mathscr{F}}_z\right) = \mathscr{F}_{\pi(z)}, \quad \forall z \in \mathbb{R}^2.$$

Notice that such a plane pseudo-foliation is $\mathbb{Z}^2$-invariant, i.e., $\widetilde{\mathscr{F}}$ is $T_{\boldsymbol{p}}$-invariant for every $\boldsymbol{p} \in \mathbb{Z}^2$.

In [13] we have gotten the following result that guarantees the existence of an asymptotic homological direction for torus pseudo-foliations:

THEOREM 4.3. – *If $\widetilde{\mathscr{F}}$ is the lift of torus pseudo-foliation, then there exist $v \in \mathbb{S}^1$ and $M > 0$ such that*

$$|\langle w - z, v\rangle| \leq M, \quad \forall z \in \mathbb{R}^2,\ \forall w \in \widetilde{\mathscr{F}}_z.$$

The vector v given by Theorem 4.3 is unique up to multiplication by (-1). So, we will call it the *asymptotic direction* of either the torus pseudo-foliation $\mathscr{F}$ or its lift $\widetilde{\mathscr{F}}$.

One of the main results of [13] is the following:

THEOREM 4.4 (Theorem 5.5 in [13]). – *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a periodic point free, area-preserving, non-wandering and non-eventually annular homeomorphism. If f exhibits uniformly bounded v-deviations, for some $v \in \mathbb{S}^1$, then there exists an f-invariant pseudo-foliation whose asymptotic direction is given by $v^{\perp}$.*

4.2. Rotational deviations for minimal homeomorphisms

In this paragraph we present some simple results about rotational deviations of minimal homeomorphisms. So, from now on let us assume that $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is minimal and $\tilde{f}\colon \mathbb{R}^2 \circlearrowleft$ is a lift of f. By Theorem 2.9 we know there exist v and α such that the rotation set of $\tilde{f}$ is contained in the line ℓ_α^v, i.e., inclusion (28) holds.

The following result is an improvement of Theorem 4.1 under the minimality assumption:

PROPOSITION 4.5. – *If f is minimal, $\tilde{f}$ is a lift of f and v and α are such that condition* (28) *holds, then the following properties are equivalent:*

(i) *f does not exhibit uniformly bounded v-deviations;*

(ii) *f does not exhibit uniformly bounded $(-v)$-deviations;*

(iii) *for every $z \in \mathbb{T}^2$ it holds*

$$\sup_{n\geq 0}\left|\left\langle \Delta_{\tilde{f}}^{(n)}(z), v\right\rangle - n\alpha\right| = \sup_{n\leq 0}\left|\left\langle \Delta_{\tilde{f}}^{(n)}(z), v\right\rangle - n\alpha\right| = \infty.$$

Proof. – This is a straightforward consequence of Theorems 2.17 and 4.1. □

For the proof of Theorem B we shall need the following

PROPOSITION 4.6. – *If $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is a minimal homeomorphism, then it is not eventually annular.*

Proof. – By Proposition 2.16, f^k is minimal for any $k \in \mathbb{N}$. So, it is enough to show that f is not annular.

Reasoning by contradiction, let us suppose f is annular. Then, there exist a lift $\tilde{f}\colon\mathbb{R}^2 \circlearrowleft$ and $v \in \mathbb{S}^1$ with rational slope such that

$$\sup_{n\in\mathbb{Z},\ z\in\mathbb{T}^2} \left|\left\langle \Delta_{\tilde{f}}^{(n)}(z), v\right\rangle\right| < \infty.$$

Since v has rational slope, by Proposition 2.5 there is no loss of generality assuming $v = (1,0)$. So, by Theorem 2.17, there exists $u \in C^0(\mathbb{T}^2,\mathbb{R})$ satisfying

$$\mathrm{pr}_1 \circ \Delta_{\tilde{f}} = u \circ f - u. \tag{31}$$

Now, let us consider the continuous maps $\tilde{g}, \tilde{h}\colon \mathbb{R}^2 \circlearrowleft$ given by

$$\begin{aligned}\tilde{g}(x,y) &:= \big(x, y + \mathrm{pr}_2 \circ \Delta_{\tilde{f}}(x,y)\big),\\ \tilde{h}(x,y) &:= \big(x - u(x,y), y\big),\end{aligned}$$

for every $(x,y) \in \mathbb{R}^2$.

As consequence of (31), we know $\tilde{h} \circ \tilde{f} = \tilde{g} \circ \tilde{h}$, and hence, $h \circ f = g \circ h$, where g and h are the continuous torus maps whose lifts are $\tilde{g}$ and $\tilde{h}$, respectively. However, since h is homotopic to the identity, it is surjective and g is clearly not minimal, contradicting the minimality of f.

So, f is not annular. □

Even though our next result is rather simple, it may be useful in future works:

THEOREM 4.7. – *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be a minimal homeomorphism, $\tilde{f}\colon\mathbb{R}^2 \circlearrowleft$ a lift of f and ρ any point in $\rho(\tilde{f})$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $n \in \mathbb{Z}$ satisfying $d(n\rho, \mathbb{Z}^2) < \delta$, there is $z \in \mathbb{R}^2$ such that*

$$\left\| \tilde{f}^n(z) - z - n\rho \right\| < \varepsilon.$$

Proof. – By Theorem 2.7, there exists $\mu \in \mathfrak{M}(f)$ such that $\rho_\mu(\tilde{f}) = \rho$. Since f is minimal, μ is a topological measure (i.e., has total support and no atoms). So, by Theorem 2.4, there exists $h \in \mathrm{Homeo}(\mathbb{T}^2)$ such that $h_\star \mathrm{Leb}_2 = \mu$. Moreover, after precomposing with a linear automorphism of $\mathbb{T}^2$ if necessary, we can assume that h is isotopic to the identity. Then, if $\tilde{h} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$ is a lift of h and we write $\tilde{g} := \tilde{h}^{-1} \circ \tilde{f} \circ \tilde{h}$, we have $\tilde{g} \in \widetilde{\mathrm{Symp}}_0(\mathbb{T}^2)$ and $\mathrm{Flux}(\tilde{g}) = \rho$.

Observing the displacement function $\Delta_{\tilde{h}}$ is $\mathbb{Z}^2$-periodic, and hence, uniformly continuous, so given any $\varepsilon > 0$ there exists $\delta > 0$ such that $\left\|\Delta_{\tilde{h}}(x) - \Delta_{\tilde{h}}(y)\right\| < \varepsilon$ whenever $d_{\mathbb{T}^2}(x,y) < \delta$.

On the other hand, we have

$$\mathrm{Flux}\left(T_\rho^{-n} \circ \tilde{g}^n\right) = \mathrm{Flux}\left(T_\rho^{-n}\right) + \mathrm{Flux}(g) = -n\rho + n\rho = 0.$$

So, $T_\rho^{-n} \circ \tilde{g}^n \in \widetilde{\mathrm{Ham}}(\mathbb{T}^2)$ for every $n \in \mathbb{Z}$. By Theorem 2.10, for each n there exists $z_n \in \mathbb{R}^2$ such that $T_\rho^{-n} \circ \tilde{g}^n(z_n) = z_n$. Then,

$$\tilde{f}^n\big(\tilde{h}(z_n)\big) = \tilde{h}(z_n + n\rho) = z_n + n\rho + \Delta_{\tilde{h}}(z_n + n\rho),$$

and consequently, defining $w_n := \tilde{h}(z_n)$, we get

$$\left\| \tilde{f}^n(w_n) - w_n - n\rho \right\| = \left\| \Delta_{\tilde{h}}(z_n + n\rho) - \Delta_{\tilde{h}}(z_n) \right\| < \varepsilon,$$

whenever $d(n\rho, \mathbb{Z}^2) < \delta$. □

5. Stable sets at infinity: transverse direction

All along this section $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ will continue to denote a minimal homeomorphism and $\tilde{f} \in \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^2)$ a lift of f. Here we start the study of *stable sets at infinity* associated to (certain lifts of) f. We begin considering stable sets at infinity with respect to the horizontal direction assuming there exists a lift $\tilde{f}$ such that the rotation set $\rho(\tilde{f})$ intersects the horizontal axis.

We call this "a transverse direction" because, under the hypotheses of Theorem A, there is no loss of generality assuming $\rho(\tilde{f})$ intersects transversely the horizontal axis, modulo finite iterates, conjugacy and appropriate choice of the lift.

To simplify our notation, in this section we will write u to denote either the vector $(0, 1)$ or $(0, -1)$.

THEOREM 5.1. – *Let f and $\tilde{f}$ be as above, and assume $\rho(\tilde{f})$ intersects the horizontal axis, i.e.,*

$$\rho(\tilde{f}) \cap \ell_0^{(0,1)} \neq \emptyset. \tag{32}$$

For each $r \in \mathbb{R}$ and $u \in \{(0,1), (0,-1)\}$, consider the set

$$\Lambda_r^u := \mathrm{cc}\left(\mathscr{I}_{\tilde{f}}(\overline{\mathbb{H}_r^u}), \infty\right), \tag{33}$$

where $\mathscr{I}_{\tilde{f}}\left(\overline{\mathbb{H}_r^u}\right)$ denotes the maximal invariant set given by (2).

Then, it holds:

(i) $\Lambda_r^u = T_{(1,0)}(\Lambda_r^u)$, $T_{(0,1)}(\Lambda_r^u) = \Lambda_{r+1}^u$, *and* $\Lambda_r^u \subset \Lambda_s^u$, *for any $r \in \mathbb{R}$ and any $s < r$;*

(ii) $\Lambda_r^u \cap \ell_r^u \neq \emptyset$, *for every $r \in \mathbb{R}$;*

(iii) $\mathscr{I}_{\tilde{f}}(\overline{\mathbb{H}_r^u}) \cap \mathscr{I}_{\tilde{f}}(\overline{\mathbb{H}_{r'}^{-u}}) = \emptyset$, *for every $r, r' \in \mathbb{R}$;*

(iv) *given any r and any connected unbounded closed subset $\Gamma \subset \Lambda_r^u$, it holds*

$$\Gamma \cap \ell_s^u \neq \emptyset, \quad \forall s > \inf\{|\mathrm{pr}_2(z)| : z \in \Gamma\};$$

(see Figure 1);

(v) $\overline{\bigcup_{r\in\mathbb{R}} \Lambda_r^u} = \mathbb{R}^2$;

(vi) *For each $r \in \mathbb{R}$, the set $\mathscr{I}_{\tilde{f}}(\overline{\mathbb{H}_r^u})$ has empty interior and does not disconnect $\mathbb{R}^2$, i.e., $\mathbb{R}^2 \setminus \mathscr{I}_{\tilde{f}}(\overline{\mathbb{H}_r^u})$ is connected. In particular, this implies that Λ_r^u has empty interior and does not disconnect $\mathbb{R}^2$ as well.*

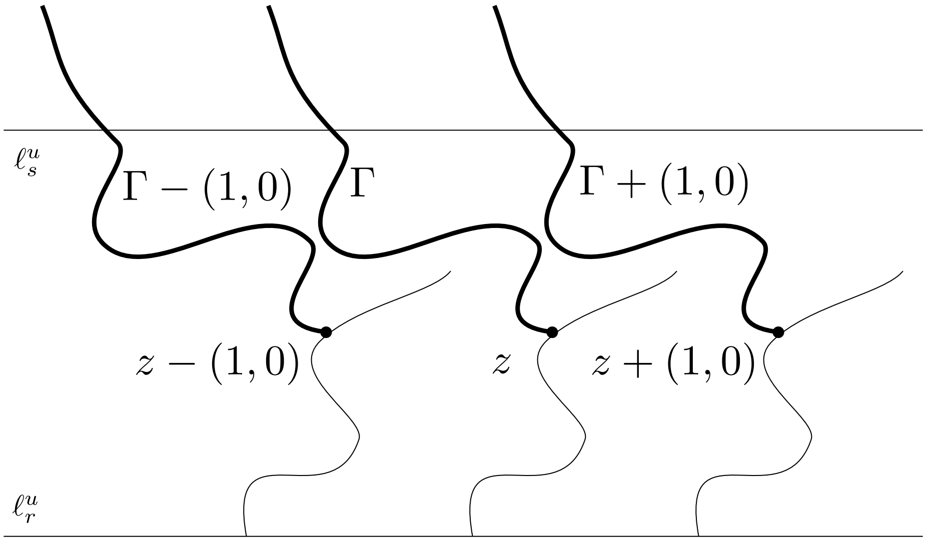

FIGURE 1. $\Gamma \subset \Lambda^u_r$ intersects ℓ^u_s for $u = (0,1)$ and $s > \mathrm{pr}_2(z)$.

Proof. – Statement ((i)) easily follows from the fact that $\mathbb{H}^u_r \subset \mathbb{H}^u_s$ whenever $r > s$ and recalling that $\tilde{f}$ commutes with every integer translation.

To show ((ii)), let $\mathbb{A}$ be the open annulus and $P\colon \mathbb{R}^2 \to \mathbb{A}$ the covering map as defined in §2.3.2. Since $\tilde{f}$ commutes with all deck transformations of P, it induces a homeomorphism on $\mathbb{A}$; and if we write $\hat{\mathbb{A}} := \mathbb{A} \sqcup \{-\infty, +\infty\}$ for the two-end compactification of the annulus $\mathbb{A}$, this homeomorphism admits a unique extension to $\hat{\mathbb{A}}$. More precisely, one can define $\hat{f} \in \mathrm{Homeo}(\hat{\mathbb{A}})$ by

$$\hat{f}(z) := \begin{cases} +\infty, & \text{if } z = +\infty; \\ -\infty, & \text{if } z = -\infty; \\ P\big(\tilde{f}(\tilde{z})\big), & \text{if } z \in \mathbb{A},\ \tilde{z} \in P^{-1}(z). \end{cases}$$

Now, we want to show both fixed points $-\infty$ and $+\infty$ are indifferent for $\hat{f}$, according to the classification of fixed points given in §2.9.

Since f is minimal, it holds $\mathrm{Fix}(\hat{f}) = \mathrm{Per}(\hat{f}) = \{-\infty, +\infty\}$. In particular, both fixed points are non-accumulated.

On the other hand, since $\hat{\mathbb{A}}$ is homeomorphic to $\mathbb{S}^2$ and $\hat{f}$ is isotopic to the identity, by Lefschetz fixed point theorem one gets

$$i\big(\hat{f}^n, -\infty\big) + i\big(\hat{f}^n, +\infty\big) = L(\hat{f}^n) = \chi(\hat{\mathbb{A}}) = 2, \quad \forall n \in \mathbb{Z} \setminus \{0\}. \tag{34}$$

Then, we make the following

CLAIM 5.2. – *Fixed point indexes at $+\infty$ and $-\infty$ satisfy*

$$i\big(\hat{f}^n, \pm\infty\big) \leq 1, \quad \forall n \in \mathbb{Z} \setminus \{0\}. \tag{35}$$

Since the statement is completely symmetric, we will just prove Claim 5.2 for the point $+\infty$. Let us proceed by contradiction. Suppose that $i(\hat{f}, +\infty) \geq 2$. Then, let W^+ and $W^- \subset \mathbb{A}$ denote the open sets given by Theorem 2.12.

Let $\nu \in \mathfrak{M}(f)$ be any ergodic f-invariant measure. We will consider the three possible cases: $\mathrm{pr}_2(\rho_\nu(\tilde{f}))$ is either positive, negative or zero. Let us start assuming $\mathrm{pr}_2(\rho_\nu(\tilde{f})) > 0$. Then by Birkhoff ergodic theorem, for ν-almost every $z \in \mathbb{T}^2$ and every $\hat{z} \in (\mathrm{id} \times \pi)^{-1}(z)$, it holds

$$\hat{f}^{-n}(\hat{z}) \to -\infty \in \hat{\mathbb{A}}, \quad \text{as } n \to +\infty, \tag{36}$$

where $\mathrm{id} \times \pi \colon \mathbb{A} = \mathbb{T} \times \mathbb{R} \to \mathbb{T}^2$ denotes the natural covering map. Since f is minimal, ν is a topological measure and hence, taking into account W^- is open, there exists a point $\hat{z} \in W^-$ satisfying (36). This contradicts the fact that $\alpha_{\hat{f}}(\hat{z}) = \{+\infty\}$, for every $\hat{z} \in W^-$.

Analogously, one gets a contradiction assuming $\mathrm{pr}_2\big(\rho_\nu(\tilde{f})\big) < 0$.

So, it just remains to consider the case $\mathrm{pr}_2\big(\rho_\nu(\tilde{f})\big) = 0$. In such a case, as a consequence of Theorem 2.3, we know that $\hat{f} \colon \hat{\mathbb{A}} \circlearrowleft$ is non-wandering, i.e., $\Omega(\hat{f}) = \hat{\mathbb{A}}$. In fact, let $\hat{V} \subset \mathbb{A}$ be a non-empty open set and suppose $\operatorname{diam} \hat{V} < 1/4$; let us define $V := (\mathrm{id} \times \pi)(\hat{V}) \subset \mathbb{T}^2$. Since $\mathrm{id} \times \pi$ is a covering map, V is open and thus, $\nu(V) > 0$. On the other hand, since we are assuming $\mathrm{pr}_2\big(\rho_\nu(\tilde{f})\big) = 0$, we have

$$\int_{\mathbb{T}^2} \mathrm{pr}_2 \circ \Delta_{\tilde{f}} \, \mathrm{d}\nu = 0.$$

Then, invoking Theorem 2.3 we know that there exist $z \in V$ and $n \geq 1$ such that $f^n(z) \in V$ and $\left|\mathcal{S}_f^n\big(\mathrm{pr}_2 \circ \Delta_{\tilde{f}}\big)(z)\right| = \left|\mathrm{pr}_2 \circ \Delta_{\tilde{f}}^{(n)}(z)\right| < 1/4$. This implies $\hat{V} \cap \hat{f}^{-n}(\hat{V}) \neq \emptyset$. So, $\mathbb{A} \subset \Omega(\hat{f})$, then clearly we have $\Omega(\hat{f}) = \hat{\mathbb{A}}$. But the both sets W^+ and W^- given by Theorem 2.12 are wandering for $\hat{f}$, getting a contradiction.

So, $i(\hat{f}, +\infty) \leq 1$. By Proposition 2.16, one can easily adapt the previous reasoning for the general case, i.e., where $i(\hat{f}^n, +\infty) \geq 2$, with $|n| \geq 2$; and Claim 5.2 is proven.

Now, putting together (34) and (35) we conclude that

$$i\big(\hat{f}^k, -\infty\big) = i\big(\hat{f}^k, +\infty\big) = 1, \quad \forall k \in \mathbb{Z} \setminus \{0\}. \tag{37}$$

By an argument similar to the one we used to prove Claim 5.2, one can show both fixed points $-\infty$ and $+\infty$ are not dissipative (i.e., they are neither attractive nor repulsive). In fact, arguing by contradiction let us suppose, for instance, that there is a trapping neighborhood V of $+\infty$, i.e., $V \subset \hat{\mathbb{A}}$ is an open set such that $+\infty \in V$, $-\infty \notin V$ and $\hat{f}(\overline{V}) \subset V$. Following the very same reasoning we used to prove Claim 5.2 one can conclude in this case that

$$\mathrm{pr}_2\big(\rho_\nu(\tilde{f})\big) > 0, \quad \forall \nu \in \mathfrak{M}(f).$$

However this last estimate is incompatible with the convexity of $\rho(\tilde{f})$ and our hypothesis (32). So, $-\infty$ and $+\infty$ are non-dissipative.

Now, combining this last assertion, Theorem 2.13 and (37) we show that $+\infty$ and $-\infty$ are indifferent (according to the classification of fixed points given in §2.9). In other words, if we define

$$V_r^{\pm(0,1)} := P\left(\overline{\mathbb{H}_r^{\pm(0,1)}}\right) \sqcup \{\pm\infty\} \subset \hat{\mathbb{A}}, \tag{38}$$

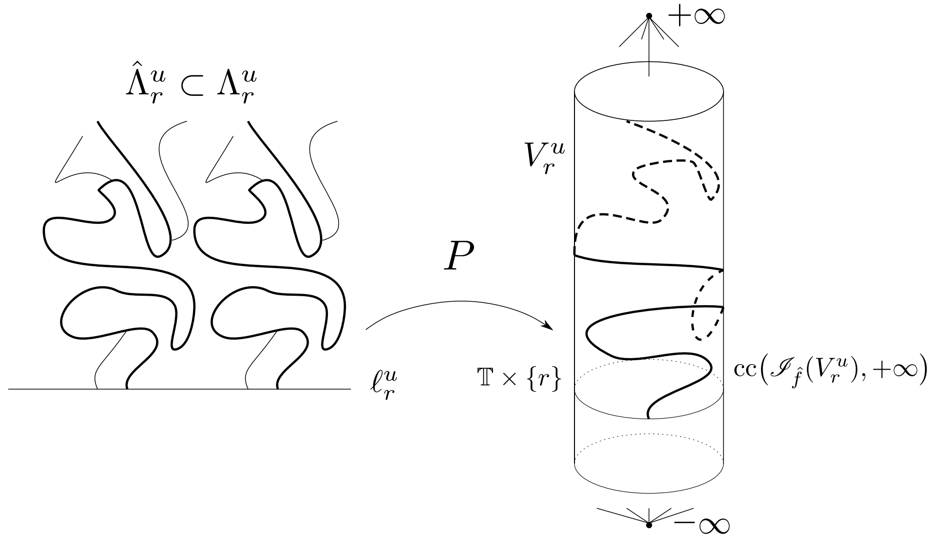

FIGURE 2. $\hat{\Lambda}_r^u := P^{-1}\big(\mathscr{I}_{\hat{f}}(V_r^u), +\infty\big) \subset \Lambda_r^u$, with $u = (0, 1)$.

it can be easily verified that $V_r^{\pm(0,1)}$ is a neighborhood of $\pm\infty$ and therefore,

$$\mathrm{cc}\left(\mathscr{I}_{\hat{f}}\big(V_r^{\pm(0,1)}\big), \pm\infty\right) \cap \partial V_r^{\pm(0,1)} \neq \emptyset, \quad \forall r \in \mathbb{R},$$

where $\partial V_r^{\pm(0,1)} = \mathbb{T} \times \{r\} \subset \mathbb{A}$. In particular, the set

$$\hat{\Lambda}_r^{\pm(0,1)} := P^{-1}\left(\mathrm{cc}\Big(\mathscr{I}_{\hat{f}}\big(V_r^{\pm(0,1)}\big), \pm\infty\Big) \setminus \{\pm\infty\}\right) \subset \mathbb{R}^2 \tag{39}$$

intersects the horizontal line $\ell_r^{\pm(0,1)}$, for every $r \in \mathbb{R}$. By Lemma 2.2, every connected component of $\hat{\Lambda}_r^{\pm(0,1)}$ is unbounded, so it holds

$$\hat{\Lambda}_r^u \subset \Lambda_r^u, \quad \forall r \in \mathbb{R}. \tag{40}$$

Hence, (ii) is proven (see Figure 2 for an illustrative representation of the construction we have performed).

Assertion (iii) easily follows from Proposition 4.5 and Proposition 4.6. In fact, let us assume there exists $z \in \mathscr{I}_{\tilde{f}}\Big(\overline{\mathbb{H}_r^{(0,1)}}\Big) \cap \mathscr{I}_{\tilde{f}}\Big(\overline{\mathbb{H}_{r'}^{(0,-1)}}\Big)$, for some $r, r' \in \mathbb{R}$. Then, this implies that

$$r \leq \mathrm{pr}_2\Big(\Delta_{\tilde{f}}^{(n)}(z)\Big) = \sum_{j=0}^{n-1} \mathrm{pr}_2 \circ \Delta_{\tilde{f}}\big(\tilde{f}^j(z)\big) \leq -r', \quad \forall n \in \mathbb{N}.$$

So, by Proposition 4.5, f should be annular and by Proposition 4.6, this is incompatible with minimality of f.

Then, let us prove (iv) reasoning by contradiction. Suppose there exists a connected closed unbounded set $\Gamma \subset \Lambda_r^u$ such that $\Gamma \cap \ell_s^u = \emptyset$, for some real number $s > \inf\{|\mathrm{pr}_2(z)| : z \in \Gamma\}$. This means Γ is contained in $\mathbb{A}_{r,s}^u$, where the strip $\mathbb{A}_{r,s}^u$ is given by (6).

By (i) we know that Λ_r^u is $T_{1,0}$-invariant. So,

$$\Gamma' := \bigcup_{n\in\mathbb{Z}} T_{1,0}^n(\Gamma) \subset \Lambda_r^u,$$

and Γ' is contained in $\mathbb{A}_{r,s}^u$ as well. Moreover, since $\mathscr{I}_{\tilde{f}}\big(\overline{\mathbb{H}_r^u}\big)$ is a closed $\tilde{f}$-invariant set, and $\Gamma' \subset \Lambda_r^u \subset \mathscr{I}_{\tilde{f}}\big(\overline{\mathbb{H}_r^u}\big)$, we conclude that

$$\overline{\Gamma'} \subset \mathscr{I}_{\tilde{f}}\big(\overline{\mathbb{H}_r^u}\big) \cap \mathbb{A}_{r,s}^u. \tag{41}$$

On the other hand, since Γ is unbounded, one sees that $\overline{\Gamma'}$ is contained in the interior of the strip $\mathbb{A}_{r-1,s+1}$ and separates both connected components of its boundary.

Then let us write $\hat{\Gamma}' := P(\overline{\Gamma'})$ and $\hat{\mathbb{A}}_{r-1,s+1}^u := P(\mathbb{A}_{r-1,s+1}^u)$, where $P\colon \mathbb{R}^2 \to \mathbb{A}$ denotes the covering map given by (7). Observe that, since $\overline{\Gamma'}$ is $T_{1,0}$-invariant and $T_{1,0}$ generates the group of deck transformations of P, $\hat{\Gamma}'$ is a compact subset of $\mathbb{A}$.

So $\hat{\Gamma}' \subset \hat{\mathbb{A}}_{r-1,s+1}^u$ and when $\hat{\Gamma}'$ is considered as a compact subset of $\hat{\mathbb{A}} = \mathbb{A} \sqcup \{-\infty, +\infty\}$, it separates the horizontal circle $P(\ell_{s+1}^u)$ and the point $-\operatorname{sign}(u)\infty \in \hat{\mathbb{A}}$, where $\operatorname{sign}(u) = 1$, for $u = (0,1)$ and $\operatorname{sign}(u) = -1$, for $u = (0,-1)$.

In the proof of (ii) we have shown that the set $\operatorname{cc}\Big(\mathscr{I}_{\hat{f}}\big(V_{-s-1}^{-u}\big), -\operatorname{sign}(u)\infty\Big)$ intersects the boundary of V_{-s-1}^{-u}, where the set V_{-s-1}^{-u} is given by (38), and so we have

$$\operatorname{cc}\Big(\mathscr{I}_{\hat{f}}\big(V_{-s-1}^{-u}\big), -\operatorname{sign}(u)\infty\Big) \cap \hat{\Gamma}' \neq \emptyset.$$

By (40), this implies that

$$\emptyset \neq \hat{\Lambda}_{-s-1}^{-u} \cap \overline{\Gamma'} \subset \mathscr{I}_{\tilde{f}}\big(\overline{\mathbb{H}_{-s-1}^{-u}}\big) \cap \mathscr{I}_{\tilde{f}}\big(\overline{\mathbb{H}_r^u}\big),$$

contradicting (iii).

In order to prove (v), first notice that, as a consequence of (i), the set $\bigcup_{r\in\mathbb{R}} \Lambda_r^u$ is $\mathbb{Z}^2$-invariant, i.e., it is $T_{\boldsymbol{p}}$-invariant, for every $\boldsymbol{p} \in \mathbb{Z}^2$.

On the other hand, since the set Λ_r^u is defined as the union of unbounded connected components of an $\tilde{f}$-invariant closed set, it is $\tilde{f}$-invariant itself.

So, the set $\bigcup_{r\in\mathbb{R}} \Lambda_r^u$ is invariant under the abelian subgroup $\langle \tilde{f}, \{T_{\boldsymbol{p}}\}_{\boldsymbol{p}\in\mathbb{Z}^2}\rangle < \operatorname{Homeo}_+(\mathbb{R}^2)$ which acts minimally on $\mathbb{R}^2$. Then, $\bigcup_{r\in\mathbb{R}} \Lambda_r^u$ is dense in $\mathbb{R}^2$, as desired.

Last assertion (vi) is a rather straightforward consequence of (iii), (iv) and (v).

In fact, first observe that, combining (iii) and (v) one easily shows that Λ_r^u has empty interior.

On the other hand, if $\mathbb{R}^2 \setminus \Lambda_r^u$ were not connected, then there should exist a connected component $V \in \pi_0(\mathbb{R}^2 \setminus \Lambda_r^u)$ such that $V \subset \mathbb{H}_r^u$.

By (v), there exists $r' \in \mathbb{R}$ such that $\Lambda_{r'}^{-u} \cap V \neq \emptyset$. If z_0 is any point in $\Lambda_{r'}^{-u} \cap V$, then by (iv) we know that $\operatorname{cc}\big(\Lambda_{r'}^{-u}, z_0\big)$ is not contained in the strip $\mathbb{A}_{r,-r'}^u$. Consequently, the connected set $\operatorname{cc}\big(\Lambda_{r'}^{-u}, z_0\big)$ is not contained in V. So it intersects the boundary of V which is contained in Λ_r^u. This contradicts (iii) and (vi) is proved. □

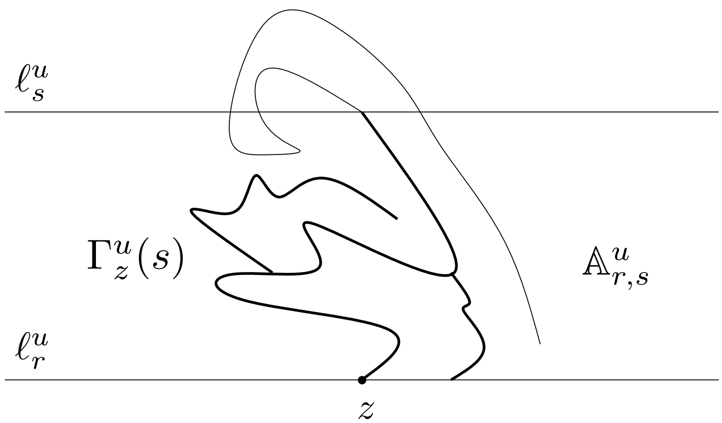

FIGURE 3. Definition of the set $\Gamma^u_z(s)$ sets for $u = (0,1)$

Let us continue assuming $\tilde{f} \in \widetilde{\mathrm{Homeo}_0}(\mathbb{T}^2)$ is a lift of a minimal homeomorphism f and condition (32) holds. Fixing a real number r, for each $z \in \Lambda^u_r \cap \ell^u_r$ and every $s > r$ we define the set

$$\Gamma^u_z(s) := \mathrm{cc}\left(\Lambda^u_r \cap \mathbb{A}^u_{r,s}, z\right), \tag{42}$$

where the strip $\mathbb{A}^u_{r,s}$ is given by (6) (see Figure 3).

As consequence of Theorem 5.1, we get the following result about the geometry of the sets $\Gamma^u_z(s)$:

COROLLARY 5.3. – *For every $r \in \mathbb{R}$ and $u \in \{(0,1); (0,-1)\}$ the following conditions are satisfied:*

(i) *for every $z \in \Lambda^u_r \cap \ell^u_r$ and any $s > r$,*

$$T^n_{1,0}\big(\Gamma^u_z(s)\big) \cap \Gamma^u_z(s) = \emptyset, \quad \forall n \in \mathbb{Z} \setminus \{0\}; \tag{43}$$

(ii) *for any $s > r$, there exists a real number $D = D(f,s,r) > 0$ such that*

$$\mathrm{diam}\left(\mathrm{pr}_1\big(\Gamma^u_z(s)\big)\right) \leq D, \quad \forall z \in \Lambda^u_r \cap \ell^u_r, \tag{44}$$

and so, $\Gamma^u_z(s)$ is compact;

(iii) *for every $U \in \pi_0\big(\mathbb{H}^u_r \setminus \Lambda^u_r\big)$,*

$$T^n_{1,0}(U) \cap U = \emptyset, \quad \forall n \in \mathbb{Z} \setminus \{0\}. \tag{45}$$

See Figure 4 for a graphical representation of these properties.

Proof. – Let us fix real numbers $s > r$ and let z denote an arbitrary point in $\Lambda^u_r \cap \ell^u_r$. Reasoning by contradiction, let us start supposing $\mathrm{diam}\left(\mathrm{pr}_1(\Gamma^u_z(s)\right)$ is infinite. Then, the set

$$\Gamma := \overline{\bigcup_{n\in\mathbb{Z}} T^n_{1,0}\big(\Gamma^u_z(s)\big)}$$

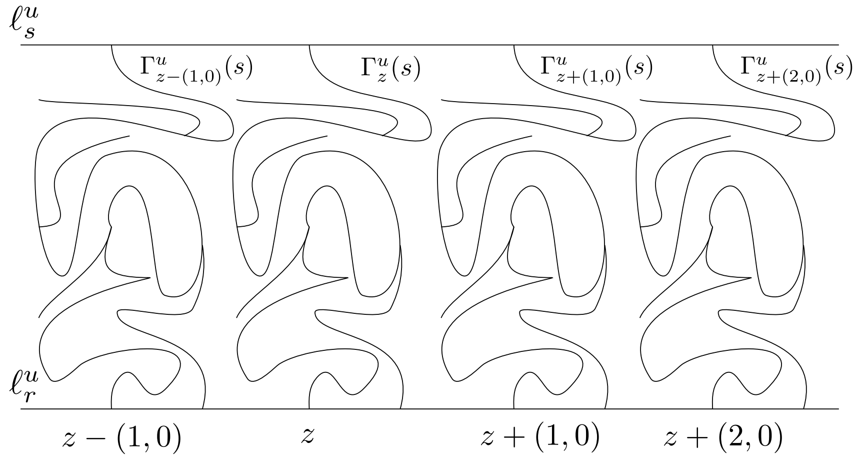

FIGURE 4. $\Gamma^u_z(s)$ and their horizontal translations for $u=(0,1)$

disconnects $\mathbb{R}^2$ and since $\mathscr{I}_{\tilde f}(\mathbb{H}^u_r)$ is $T_{1,0}$-invariant, $\Gamma\subset\mathscr{I}_{\tilde f}(\mathbb{H}^u_r)$. So, $\mathscr{I}_{\tilde f}(\mathbb{H}^u_r)$ disconnects $\mathbb{R}^2$ contradicting (vi) of Theorem 5.1. Thus it holds

$$\operatorname{diam}\Big(\mathrm{pr}_1\big(\Gamma^u_z(s)\big)\Big)<\infty,\quad \forall z\in\Lambda^u_r\cap\ell^u_r,\ \forall s>r. \tag{46}$$

Now suppose (43) is false, i.e., there exist z, s and n such that

$$T_{n,0}\big(\Gamma^u_z(s)\big)\cap\Gamma^u_z(s)\neq\emptyset,$$

with $n\neq 0$. Then, since Λ^u_r is $T_{1,0}$-invariant, the set

$$\bigcup_{m\in\mathbb{Z}}T^m_{n,0}\big(\Gamma^u_z(s)\big)\subset\Lambda^u_r$$

is connected, contains $\Gamma^u_z(s)$ and so, it coincides with $\Gamma^u_z(s)$. Since $n\neq 0$, we conclude that $\operatorname{diam}(\mathrm{pr}_1(\Gamma^u_z(s)))=\infty$, contradicting (46).

Property (44) is a straightforward consequence of (43) and (46). In fact, for fixed real numbers $s>r$ and any point $w\in\Lambda^u_r\cap\ell^u_r$, by (46) we know that $\operatorname{diam}(\mathrm{pr}_1(\Gamma_w(s)))$ is finite. Then, given any $z\in\Lambda^u_r\cap\ell^u_r$, there exists a unique $n\in\mathbb{Z}$ such that $\mathrm{pr}_1\circ T^n_{1,0}(w)\leq\mathrm{pr}_1(z)<\mathrm{pr}_1\circ T^{n+1}_{1,0}(w)$. By (43), $T^n_{1,0}\big(\Gamma^u_w(s)\big)$ and $T^{n+1}_{1,0}\big(\Gamma^u_w(s)\big)$ are disjoint. Thus, it holds

$$\operatorname{diam}\Big(\mathrm{pr}_1\big(\Gamma^u_z(s)\big)\Big)\leq\operatorname{diam}\Big(\mathrm{pr}_1\big(\Gamma^u_w(s)\big)\Big)+2,\quad\forall z\in\Lambda^u_r\cap\ell^u_r,$$

and (44) is proved.

To prove (45), we first need to introduce some definitions: for each connected component $U\in\pi_0\big(\mathbb{H}^u_r\setminus\Lambda^u_r\big)$ we define its *boundary at level* r by

$$\partial^u_r U:=(\partial U\cap\ell^u_r)\setminus\Lambda^u_r=(\overline{U}\cap\ell^u_r)\setminus\Lambda^u_r, \tag{47}$$

where ∂U denotes the boundary of U in $\mathbb{R}^2$ (see Figure 5).

So, the boundary at level r operator satisfies the following property:

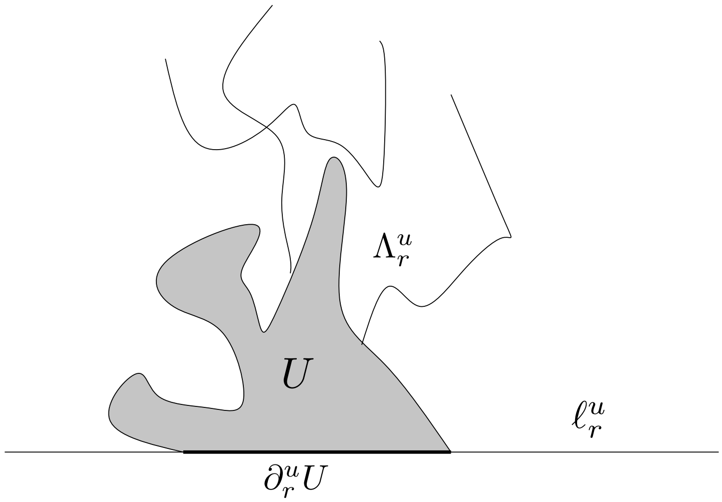

FIGURE 5. $\partial_r^u U$ definition for a $u = (0,1)$.

CLAIM 5.4. – *Every boundary at level r is connected and non-empty. In other words, the operator*

$$\partial_r^u : \pi_0\big(\mathbb{H}_r^u \setminus \Lambda_r^u\big) \to \pi_0\big(\ell_r^u \setminus \Lambda_r^u\big)$$

given by (47) *is a well-defined bijection.*

To prove our claim, let us consider an arbitrary point $z \in \Lambda_r^u \cap \ell_r^u$ whose existence is guaranteed by (ii) of Theorem 6.1 and define $\Gamma_z^u := \operatorname{cc}\big(\Lambda_r^u, z\big)$. Then, notice that Γ_z^u disconnects the half-plane $\mathbb{H}_r^u$. In fact, let us consider the one-point compactification of the plane $\widehat{\mathbb{R}^2} := \mathbb{R}^2 \sqcup \{\infty\}$. Then, the closures $\widehat{\ell_r^u}$ and $\widehat{\Gamma_z^u}$ in $\widehat{\mathbb{R}^2}$ are compact and connected, and the points z and ∞ belong to both continua. Since Λ_r^u does not disconnect $\mathbb{R}^2$ and is $T_{1,0}$-invariant, then the intersection $\widehat{\ell_r^u} \cap \widehat{\Gamma_z^u}$ cannot be connected. Thus, by Theorem 2.1, $\widehat{\ell_r^u} \cup \widehat{\Gamma_z^u}$ disconnects $\widehat{\mathbb{R}^2}$ and consequently, $\mathbb{H}_r^u \setminus \Gamma_z^u$ is not connected. This implies $\partial_r^u U$ is connected for every $U \in \pi_0\big(\mathbb{H}_r^u \setminus \Lambda_r^u\big)$, and Claim 5.4 is proved.

On the other hand, since $\mathbb{H}_r^u$, ℓ_r^u and Λ_r^u are $T_{1,0}$-invariant, we observe the translation $T_{1,0}$ naturally acts on $\pi_0\big(\mathbb{H}_r^u \setminus \Lambda_r^u\big)$ and, consequently, on $\pi_0\big(\ell_r^u \setminus \Lambda_r^u\big)$, too. Moreover, it can be easily seen that the following diagram commutes

$$\begin{tikzcd}
\pi_0\big(\mathbb{H}_r^u \setminus \Lambda_r^u\big) \arrow[r, "T_{1,0}"] \arrow[d, "\partial_r^u"'] & \pi_0\big(\mathbb{H}_r^u \setminus \Lambda_r^u\big) \arrow[d, "\partial_r^u"] \\
\pi_0\big(\ell_r^u \setminus \Lambda_r^u\big) \arrow[r, "T_{1,0}"] & \pi_0\big(\ell_r^u \setminus \Lambda_r^u\big)
\end{tikzcd},$$

the boundary at level r operator ∂_r^u being bijective. Hence, the actions of $T_{1,0}$ on both sets are conjugate and, clearly, there is no periodic orbit for $T_{1,0} : \pi_0\big(\ell_r^u \setminus \Lambda_r^u\big) \circlearrowleft$. So, there is no periodic orbit for $T_{1,0} : \pi_0\big(\mathbb{H}_r^u \setminus \Lambda_r^u\big) \circlearrowleft$ either, and (45) is proved. $\square$

The rest of this section is devoted to studying the geometry of sets $\Gamma_z^u(s)$ given by (42), assuming f is not a pseudo-rotation and does not exhibit uniformly bounded v-deviations.

We will show that the connected sets $\Gamma_z^u(s)$ exhibit unbounded oscillations in the v direction, as $s \to +\infty$:

THEOREM 5.5. – *Let us assume f is minimal and $\tilde{f}$ is a lift of f such that its rotation set $\rho(\tilde{f})$ intersects the upper and lower open semi-planes, i.e., with our notation it holds*

$$\rho(\tilde{f}) \cap \mathbb{H}_0^{(0,1)} \neq \emptyset \quad \textit{and} \quad \rho(\tilde{f}) \cap \mathbb{H}_0^{(0,-1)} \neq \emptyset. \tag{48}$$

On the other hand, we know there exist $v \in \mathbb{S}^1$ and $\alpha \in \mathbb{R}$ such that inclusion (28) *holds.*

If f does not exhibit uniformly bounded v-deviations, then for every $r \in \mathbb{R}$, any $z \in \Lambda_r^u \cap \ell_r^u$ and $s > r$, it holds

$$\lim_{s\to+\infty} \sup_{w\in\Gamma_z^u(s)} \langle w, v\rangle = +\infty, \tag{49}$$

and

$$\lim_{s\to+\infty} \inf_{w\in\Gamma_z^u(s)} \langle w, v\rangle = -\infty. \tag{50}$$

The proof of Theorem 5.5 will follow combining Theorem 3.1 and the following

LEMMA 5.6. – *Under hypotheses of Theorem 5.5, the following holds:*

$$\lim_{s\to+\infty} \sup_{w\in\Gamma_z^u(s)} |\langle w, v\rangle| = +\infty. \tag{51}$$

Proof of Lemma 5.6. – For the sake of simplicity of notation, all along this proof we shall just write Λ_r^+ and Λ_r^- instead of $\Lambda_r^{(0,1)}$ and $\Lambda_r^{(0,-1)}$, and do the same for any object that depends on the vectors $(0,1)$ or $(0,-1)$. Let us fix an arbitrary real number r. We will just prove (51) for Λ_r^+. The other case is completely analogous.

By our hypothesis (48) and Theorem 2.6, there exist two ergodic measures $\mu^+, \mu^- \in \mathfrak{M}(f)$ such that $\mathrm{pr}_2\big(\rho_{\mu^+}(\tilde{f})\big) > 0$ and $\mathrm{pr}_2\big(\rho_{\mu^-}(\tilde{f})\big) < 0$.

By Birkhoff ergodic theorem, for μ^+-almost every $x \in \mathbb{T}^2$ and any $\tilde{x} \in \pi^{-1}(x)$, it holds $\mathrm{pr}_2(\tilde{f}^n(\tilde{x})) \to +\infty$ and $\mathrm{pr}_2(\tilde{f}^{-n}(\tilde{x})) \to -\infty$, as $n \to +\infty$. Analogously, for μ^--almost every $x \in \mathbb{T}^2$ and any $\tilde{x} \in \pi^{-1}(x)$, it holds $\mathrm{pr}_2(\tilde{f}^n(\tilde{x})) \to -\infty$ and $\mathrm{pr}_2(\tilde{f}^{-n}(\tilde{x})) \to +\infty$, as $n \to +\infty$. In particular, this implies that

$$\mu^+\big(\pi(\Lambda_r^+) \cup \pi(\Lambda_r^-)\big) = \mu^-\big(\pi(\Lambda_r^+) \cup \pi(\Lambda_r^-)\big) = 0.$$

Then, given any μ^+-generic point $x^+ \in \mathbb{T}^2 \setminus \pi(\Lambda_r^+)$, we can find a point $z^+ \in \pi^{-1}(x^+)$ such that

$$\mathrm{pr}_2\big(\tilde{f}^n(z^+)\big) > r + 2\left\|\Delta_{\tilde{f}}\right\|_{C^0}, \quad \forall n \geq 0. \tag{52}$$

So we can define

$$U_n := \mathrm{cc}\big(\mathbb{H}_r^+ \setminus \Lambda_r^+, \tilde{f}^n(z^+)\big) \in \pi_0(\mathbb{H}_r^+ \setminus \Lambda_r^+), \quad \forall n \geq 0.$$

We claim that the sequence of boundaries at level r, i.e., $(\partial_r^+(U_n))_{n\geq 0}$ where ∂_r^+ is given by (47), exhibits bounded “rotational deviations”. More precisely, we make the following

CLAIM 5.7. – *There exists a constant $C > 0$ such that for any sequence of real numbers $(a_n)_{n\geq 0}$ satisfying*

$$(a_n, r) \in \partial_r^+(U_n), \quad \forall n \geq 0,$$

it holds

$$\left| a_n - n\frac{\alpha}{\mathrm{pr}_1(v)} \right| \leq C, \quad \forall n \geq 0.$$

To prove our claim, first observe that, since we are assuming condition (48), v is not vertical, i.e., its first coordinate $\mathrm{pr}_1(v)$ does not vanish, because the rotation set is not horizontal.

Since the measure μ^- has total support on $\mathbb{T}^2$ and the set $U_0 \subset \mathbb{R}^2$ is open, there is a point $w^- \in U_0$ such that $\pi(w^-)$ is μ^--generic and consequently, it holds that $\mathrm{pr}_2\big(\tilde{f}^n(w^-)\big) \to -\infty$ as $n \to +\infty$. Since $z^+, w^- \in U_0$ and U_0 is arc-wise connected, there is a continuous path $\gamma\colon [0,1] \to U_0$ connecting w^- and z^+. Then for every n sufficiently large, $\tilde{f}^n(w^-)$ belongs to semi-plane $\mathbb{H}_{-r}^-$ and so, there exists $t_n \in [0,1]$ such that $\tilde{f}^n\big(\gamma(t_n)\big) \in \partial_r^+(U_n)$. By inclusion (28) we know that

$$\frac{1}{n}\left\langle \tilde{f}^n\big(\gamma(t_n)\big) - \gamma(t_n), v \right\rangle \to \alpha, \quad \text{as } n \to +\infty.$$

By Claim 5.4 we know $\mathrm{diam}(\partial_r^+(U_n)) \leq 1$ and since both points $\tilde{f}^n\big(\gamma(t_n)\big)$ and (a_n, r) belong to $\partial_r^+(U_n)$ for n sufficiently big, we conclude that

$$\lim_{n\to+\infty} \frac{a_n}{n} = \frac{\alpha}{\mathrm{pr}_1(v)}. \tag{53}$$

To finish the proof of our claim, we use a classical sub-additive argument: let us show there exists $C > 0$ such that

$$|a_{m+n} - a_m - a_n| \leq C, \quad \forall m, n \geq 0. \tag{54}$$

To prove this, let us define the following total order on $\pi_0(\mathbb{H}_r^+ \setminus \Lambda_r^+)$: given any pair of connected components $V, V' \in \pi_0(\mathbb{H}_r^+ \setminus \Lambda_r^+)$, we write

$$V \prec V' \iff \mathrm{pr}_1(w) < \mathrm{pr}_1(w'), \quad \forall w \in \partial_r^+ V,\ \forall w' \in \partial_r^+ V'.$$

For each $n \geq 0$, let us write $p_n := \lfloor a_n - a_0 \rfloor \in \mathbb{Z}$, where $\lfloor\cdot\rfloor$ denotes the integer part operator. Then, observe that

$$T_{1,0}^{p_n - 1}(U_0) \prec U_n \prec T_{1,0}^{p_n+1}(U_0), \quad \forall n \geq 0.$$

Since $\tilde{f}$ commutes with every integer translation, preserves orientation and the point z^+ has been chosen such that (52) holds, we have

$$T_{1,0}^{p_n - 1}(U_m) \prec U_{m+n} \prec T_{1,0}^{p_n+1}(U_m), \quad \forall m, n \geq 0.$$

In particular, this implies that $a_m + p_n - 1 < a_{m+n} < a_m + p_n - 1$ and then,

$$|a_{m+n} - a_m - a_n| \leq |a_0| + 1, \quad \forall m, n \geq 0.$$

Then, Claim 5.7 easily follows from (53), (54) and an elementary fact about sub-additive sequences (see for instance [25, Lemma 2.2.1]).

Continuing with the notation we introduced in the proof of Claim 5.7 and since we are assuming f exhibits unbounded v-deviations, by ((iii)) of Proposition 4.5 we know that for every $M > 0$ there exists $n = n(M) \geq 0$ such that

$$\left|\left\langle \tilde{f}^n(z^+), v\right\rangle - n\alpha\right| > M.$$

Hence,

$$\begin{aligned} \left|\left\langle \tilde{f}^n(z^+) - (a_n, r), v\right\rangle\right| &\geq \left|\left\langle \tilde{f}^n(z^+), v\right\rangle - a_n \mathrm{pr}_1(v)\right| - |r\mathrm{pr}_2(v)| \\ &\geq \left|\left\langle \tilde{f}^n(z^+), v\right\rangle - n\alpha\right| - |\mathrm{pr}_1(v)C| - |r\mathrm{pr}_2(v)| \\ &\geq M - |\mathrm{pr}_1(v)C| - |r\mathrm{pr}_2(v)|, \end{aligned} \tag{55}$$

where C is the constant given by Claim 5.7.

Finally, estimate (51) easily follows from Corollary 5.3, (55) and the fact that M is arbitrary. □

Then, Theorem 5.5 will follow combining Theorem 3.1 and Lemma 5.6.

Proof of Theorem 5.5. – By Lemma 5.6 we know that, for each $u \in \{(0,1),(0,-1)\}$, either (49) or (50) holds.

Reasoning by contradiction and for the sake of concreteness, let us suppose that for every $r \in \mathbb{R}$ and every $z \in \Lambda_r^+ \cap \ell_r^+$ condition (50) does not hold. Notice here we continue using the notation we introduced in the proof of Lemma 5.6. Analyzing the argument we used in the proof of Lemma 5.6, this implies that

$$\sup_{n\geq 0}\left\langle \Delta_{\tilde{f}}^{(n)}(x), v\right\rangle - n\alpha = +\infty \quad \text{and} \quad \inf_{n\geq 0}\left\langle \Delta_{\tilde{f}}^{(n)}(x), v\right\rangle - n\alpha > -\infty, \tag{56}$$

for μ^+-almost every $x \in \mathbb{T}^2$.

On the other hand, by (iii) of Theorem 5.1 and invoking Lemma 5.6 for Λ_r^-, we conclude that (50) holds and (49) does not, for any $z \in \Lambda_r^- \cap \ell_r^-$ and every $r \in \mathbb{R}$.

However, applying Theorem 3.1 to the ergodic system (f, μ^+) and the real function $\phi := \left\langle \Delta_{\tilde{f}}, v\right\rangle - \alpha$, and taking into account (56), we conclude that

$$\sup_{n\geq 0}\left\langle \Delta_{\tilde{f}}^{(-n)}(x), v\right\rangle - n\alpha = +\infty \quad \text{and} \quad \inf_{n\geq 0}\left\langle \Delta_{\tilde{f}}^{(-n)}(x), v\right\rangle - n\alpha > -\infty,$$

for μ^+-a.e. $x \in \mathbb{T}^2$.

Now, taking into account that $\mathrm{pr}_2(\Delta_{\tilde{f}}^{(-n)}(x)) \to -\infty$, as $n \to +\infty$ and μ^+-a.e. $x \in \mathbb{T}^2$, we can repeat the argument we used to prove Lemma 5.6 for negative times and a μ^+-generic point to show that (49) holds for the set Λ_r^- as well. Then, we have gotten a contradiction and Theorem 5.5 is proved. □

Combining Corollary 5.3 and Theorem 5.5 one can easily strengthen this last result getting the following

Corollary 5.8. – *If f, $\tilde{f}$, v and r are as in Theorem 5.5 and Γ is a closed connected unbounded subset of Λ_r^u, then it holds*

$$-\inf_{z\in\Gamma}\langle z, v\rangle = \sup_{z\in\Gamma}\langle z, v\rangle = +\infty.$$

6. Stable sets at infinity: parallel direction

Our next purpose consists in defining stable sets at infinity with respect to the same direction of a supporting line of the rotation set. More precisely, if $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$, $\tilde{f}\colon \mathbb{R}^2 \circlearrowleft$ is a lift of f and we suppose there are $v \in \mathbb{S}^1$ and α such that the rotation set $\rho(\tilde{f})$ is contained in the line ℓ_α^v, we want to define stable sets at infinity with respect to v, i.e., associated to the families of semi-planes $\mathbb{H}_r^v$ and $\mathbb{H}_r^{-v}$, with $r \in \mathbb{R}$.

In such a case it might happen that there is no lift $\tilde{f}$ of f such that the supporting line of $\rho(\tilde{f})$ passes through the origin, and therefore, if we naively defined $\Lambda_r^v(\tilde{f}) = \mathscr{I}_{\tilde{f}}(\overline{\mathbb{H}_r^v})$, we would get $\Lambda_r^v(\tilde{f}) = \emptyset$ for every $\tilde{f}$ and every $r \in \mathbb{R}$.

To overpass this difficulty, we shall use the *fiber-wise Hamiltonian skew-product* to define such stable sets at infinity.

6.1. The fiber-wise Hamiltonian skew-product

Since we are assuming f is minimal, by Theorem 2.4 we do not lose any generality assuming f is area-preserving, i.e., $f \in \mathrm{Symp}_0(\mathbb{T}^2)$ and let $\tilde{f} \in \widetilde{\mathrm{Symp}_0}(\mathbb{T}^2)$ denote an arbitrary lift of f.

Then, we define the *fiber-wise Hamiltonian skew-product* associated to $\tilde{f}$, which can be considered as a particular case of the construction performed in [13]. In very rough words the main idea of this construction consists in splitting our homeomorphism f into a "rotational" part and a "Hamiltonian" or "rotationless" one. Doing that, the "Hamiltonian" part is responsible by rotational deviations. The main technical advantage of dealing with such skew-products is that an arbitrary point exhibits bounded rotational deviations if and only if its orbit is bounded.

This novel object is certainly the main character of this work and will play a fundamental role in the rest of the paper.

For the sake of simplicity, we fix some notations we shall use until the end of the paper: we write $\tilde{\rho} := \mathrm{Flux}(\tilde{f}) \in \mathbb{R}^2$ and $\rho := \pi(\tilde{\rho}) = \mathrm{Flux}(f) \in \mathbb{T}^2$.

Then we define the map $H\colon \mathbb{T}^2 \to \widetilde{\mathrm{Ham}}(\mathbb{T}^2)$ by

$$H_t := \mathrm{Ad}_t\left(T_{\tilde{\rho}}^{-1} \circ \tilde{f}\right) = T_{\tilde{t}}^{-1} \circ T_{\tilde{\rho}}^{-1} \circ \tilde{f} \circ T_{\tilde{t}}, \quad \forall t \in \mathbb{T}^2,\ \forall \tilde{t} \in \pi^{-1}(t),$$

where Ad denotes the $\mathbb{T}^2$-action given by (12).

Considering H as a cocycle over the torus translation $T_\rho\colon \mathbb{T}^2 \circlearrowleft$, one defines the *fiber-wise Hamiltonian skew-product* associated to f as the map $F\colon \mathbb{T}^2 \times \mathbb{R}^2 \circlearrowleft$ given by

$$F(t,z) := \big(T_\rho(t), H_t(z)\big), \quad \forall (t,z) \in \mathbb{T}^2 \times \mathbb{R}^2.$$

Notice that F depends just on f and not on the chosen lift $\tilde{f}$.

One can easily verify that

$$F(t,z) = \Big(t + \rho, z + \Delta_{\tilde{f}}\big(t + \pi(z)\big) - \tilde{\rho}\Big), \quad \forall (t,z) \in \mathbb{T}^2 \times \mathbb{R}^2,$$

where $\Delta_{\tilde{f}} \in C^0(\mathbb{T}^2, \mathbb{R}^2)$ is the displacement function given by (10). We will use the following usual notation for cocycles: given $n \in \mathbb{Z}$ and $t \in \mathbb{T}^2$, we write

$$H_t^{(n)} := \begin{cases} id_{\mathbb{T}^2}, & \text{if } n = 0; \\ H_{t+(n-1)\rho} \circ H_{t+(n-2)\rho} \circ \cdots \circ H_t, & \text{if } n > 0; \\ H_{t+n\rho}^{-1} \circ \cdots \circ H_{t-2\rho}^{-1} \circ H_{t-\rho}^{-1}, & \text{if } n < 0. \end{cases}$$

Then we have

$$F^n(t,z) = \left(T_\rho^n(t), H_t^{(n)}(z)\right) = \left(t + n\rho, \mathrm{Ad}_t\left(T_{\tilde{\rho}}^{-n} \circ \tilde{f}^n\right)(z)\right), \tag{57}$$

for every $(t,z) \in \mathbb{T}^2 \times \mathbb{R}^2$ and every $n \in \mathbb{Z}$.

Hence, if $\rho(\tilde{f})$ has empty interior, there exist $\alpha \in \mathbb{R}$ and $v \in \mathbb{S}^1$ such that inclusion (28) holds, and from (57) it easily follows that a point $z \in \mathbb{R}^2$ exhibits bounded v-deviations if and only if

$$\left\langle H_0^{(n)}(z) - z, v\right\rangle \leq M, \quad \forall n \in \mathbb{Z},$$

where $M = M(z, f)$ denotes the positive constant given by (29).

6.2. Fibered stable sets at infinity

Continuing with previous notation, let $\alpha \in \mathbb{R}$ and $v \in \mathbb{S}^1$ be such that property (28) holds. Notice that in such a case, $\langle \tilde{\rho}, v\rangle = \alpha$.

Then, for each $r \in \mathbb{R}$ and each $t \in \mathbb{T}^2$ we define the *fibered* (r,v)*-stable set at infinity* by

$$\Lambda_r^v(\tilde{f}, t) := \mathrm{pr}_2\left(\mathrm{cc}\left(\{t\} \times \mathbb{R}^2 \cap \mathscr{I}_F\left(\mathbb{T}^2 \times \overline{\mathbb{H}_r^v}\right), \infty\right)\right) \subset \mathbb{R}^2, \tag{58}$$

where $\mathbb{H}_r^v$ is the semi-plane given by (4), $\{t\} \times \mathbb{R}^2$ is naturally endowed with the euclidean distance of $\mathbb{R}^2$, $\mathrm{cc}(\cdot, \infty)$ denotes the union of unbounded components as defined in 2.1, and $\mathrm{pr}_2 \colon \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is the projection on the second coordinate.

Let us also define the (r,v)*-stable set at infinity* as

$$\Lambda_r^v(\tilde{f}) := \bigcup_{t \in \mathbb{T}^2} \{t\} \times \Lambda_r^v(\tilde{f}, t) \subset \mathbb{T}^2 \times \mathbb{R}^2.$$

For the sake of simplicity, if there is no risk of confusion we shall just write $\Lambda_r^v(t)$ and Λ_r^v instead of $\Lambda_r^v(\tilde{f}, t)$ and $\Lambda_r^v(\tilde{f})$, respectively.

Now we recall some results of [13]:

THEOREM 6.1 (Theorem 3.4 in [13]). – *Assuming inclusion* (28) *holds, for every $r \in \mathbb{R}$ the set Λ_r^v is non-empty, closed and F-invariant. Moreover, $\Lambda_r^v(t) \neq \emptyset$, for every $t \in \mathbb{T}^2$.*

Analogously, the same assertions hold for the $(r, -v)$-stable set at infinity.

The following result describes some elementary properties of (r,v)-stable sets at infinity:

PROPOSITION 6.2 (Proposition 3.6 in [13]). – *For each $t \in \mathbb{T}^2$ and any $r \in \mathbb{R}$, the following properties hold:*

(i) $\Lambda_r^v(t) \subset \Lambda_{r'}^v(t)$*, for every $r' < r$;*

(ii) $\Lambda_r^v(t) = \bigcap_{s<r} \Lambda_s^v(t)$*;*

(iii) $\Lambda_{r+\langle\tilde{t},v\rangle}^v\left(t' - \pi(\tilde{t})\right) = T_{\tilde{t}}\left(\Lambda_r^v(t')\right)$*, for all $\tilde{t} \in \mathbb{R}^2$ and every $t' \in \mathbb{T}^2$;*

(iv) $T_{\boldsymbol{p}}\big(\Lambda_r^v(t)\big) = \Lambda_{r+\langle \boldsymbol{p},v\rangle}^v(t)$, *for every* $\boldsymbol{p} \in \mathbb{Z}^2$.

We shall need the following additional regularity result:

PROPOSITION 6.3. – *Continuing with the same notation of Proposition 6.2, the map* $t \to \Lambda_r^v(t)$ *is* compactly upper semi-continuous, *i.e., if* $t_0 \in \mathbb{T}$ *and* $U \subset \mathbb{R}^2$ *is an open set such that* $\Lambda_r^v(t_0) \subset U$ *and* $\mathbb{R}^2 \setminus U$ *is compact, then there is a neighborhood* $W(t_0)$ *of* t_0 *in* $\mathbb{T}^2$ *such that*

$$\Lambda_r^v(t) \subset U, \quad \forall t \in W(t_0).$$

Proof. – This is a straightforward consequence of the very definition of (r, v)-stable sets at infinity given by (58).

In fact, arguing by contradiction, let us suppose there exists a sequence $\{t_n\}_{n\geq 1}$ of points of $\mathbb{T}$ such that $t_n \to t_0$ as $n \to +\infty$ and

$$\Lambda_r^v(t_n) \cap (\mathbb{R}^2 \setminus U) \neq \emptyset, \quad \forall n \geq 1.$$

For each $n \geq 1$, let us consider a point $z_n \in \Lambda_r^v(t_n) \cap (\mathbb{R}^2 \setminus U)$. Since the complement of U is compact, there exists a sub-sequence $\{z_{n_j}\}_{j\geq 1}$ converging to a point $z_\infty \in \mathbb{R}^2 \setminus U$. However, the whole set Λ_r^v is closed in $\mathbb{T}^2 \times \mathbb{R}^2$ and thus, $z_\infty \in \Lambda_r^v(t_0)$ as well, contradicting the fact that $\Lambda_r^v(t_0) \subset U$. □

We also need the following

THEOREM 6.4 (Theorem 4.1 in [13]). – *If* $f \in \mathrm{Symp}_0(\mathbb{T}^2)$ *is periodic point free, i.e.,* $\mathrm{Per}(f) = \emptyset$, *then for every* $t \in \mathbb{T}^2$ *the set*

$$\bigcup_{r\geq 0} \Lambda_{-r}^v(t)$$

is dense in $\mathbb{R}^2$.

As a rather straightforward consequence of Theorems 2.17 and 6.4 we get the following

COROLLARY 6.5. – *If* $f \in \mathrm{Symp}_0(\mathbb{T}^2)$ *is minimal and does not exhibit uniformly bounded* v*-deviations, then, for every* $r \in \mathbb{R}$ *and any* $t \in \mathbb{T}^2$, *the following assertions hold:*

(i) $\Lambda_r^v(t) \cap \Lambda_{r'}^{-v}(t) = \emptyset$, *for any* $r' \in \mathbb{R}$;

(ii) $\Lambda_r^v(t)$ *has empty interior;*

(iii) $\Lambda_r^v(t)$ *does not disconnect* $\mathbb{R}^2$, *i.e.,* $\mathbb{R}^2 \setminus \Lambda_r^v(t)$ *is connected.*

Proof. – To prove (i) let us start assuming there exists $z \in \Lambda_r^v(t) \cap \Lambda_{r'}^{-v}(t)$. Thus, putting together (57) and (58) we get

$$r \leq \left\langle T_{\tilde{t}}^{-1} \circ T_{\tilde{\rho}}^{-n} \circ \tilde{f}^n \circ T_{\tilde{t}}(z), v\right\rangle \leq -r', \quad \forall n \in \mathbb{Z},\ \forall \tilde{t} \in \pi^{-1}(t),$$

and consequently, if we define $\phi \in C^0(\mathbb{T}^2, \mathbb{R})$ by $\phi(x) := \left\langle \Delta_{\tilde{f}}(x), v\right\rangle$, then it holds

$$r \leq \mathcal{S}_f^n \phi\big(t + \pi(z)\big) \leq -r', \quad \forall n \in \mathbb{Z}.$$

Then, since f is minimal, by Theorem 2.17 we conclude that ϕ is a continuous coboundary for f, and hence, f exhibits uniformly bounded v-deviations, contradicting our assumption.

Property (ii) is a straightforward consequence of Theorem 6.4 and property (i).

Finally, in order to show (iii) let us suppose there exist $r \in \mathbb{R}$ and $t \in \mathbb{T}$ such that $\Lambda_r^v(t)$ disconnects $\mathbb{R}^2$. So, there exists $U \in \pi_0\big(\mathbb{R}^2 \setminus \Lambda_r^v(t)\big)$ such that $U \cap \overline{\mathbb{H}_{-r}^{-v}} = \emptyset$. Then one can easily check that the boundary of U is completely contained in $\Lambda_r^v(t)$ and thus, $\overline{U}$ is contained in $\Lambda_r^v(t)$ as well, contradicting property (ii). $\square$

6.3. Rotational deviations and the spreading property

From now on and until the end of this section, we shall assume $f \in \mathrm{Symp}_0(\mathbb{T}^2)$ is a minimal homeomorphism such that there is a lift $\tilde{f} \in \widetilde{\mathrm{Symp}_0}(\mathbb{T}^2)$ satisfying

$$\tilde{\rho} = \mathrm{Flux}(\tilde{f}) = (\tilde{\rho}_1, 0) \in \mathbb{R}^2 \tag{59}$$

$$\rho(\tilde{f}) \cap \mathbb{H}_0^{(0,1)} \neq \emptyset \quad \text{and} \quad \rho(\tilde{f}) \cap \mathbb{H}_0^{(0,-1)} \neq \emptyset. \tag{60}$$

Notice that, since f is minimal, by Corollary 2.11 we know that $\tilde{\rho}_1 \in \mathbb{R} \setminus \mathbb{Q}$.

Then, if $F\colon \mathbb{T}^2 \times \mathbb{R}^2 \circlearrowleft$ denotes the fiber-wise Hamiltonian skew-product induced by $\tilde{f}$, the closed set $\mathbb{T} \times \{0\} \times \mathbb{R}^2 \subset \mathbb{T}^2 \times \mathbb{R}^2$ is F-invariant. Making some abuse of notation and for the sake of simplicity, we shall write F to denote the restriction of the fiber-wise Hamiltonian skew-product to this set. More precisely, from now on we have $F\colon \mathbb{T} \times \mathbb{R}^2 \circlearrowleft$ where

$$F(t,z) = \Big(t + \rho_1, z + \Delta_{\tilde{f}}\big((t,0) + \pi(z)\big) - (\tilde{\rho}_1, 0)\Big), \quad \forall (t,z) \in \mathbb{T} \times \mathbb{R}^2,$$

and $\rho_1 := \pi(\tilde{\rho}_1)$.

In a joint work with Koropecki [12], we introduced the notion of topological *spreading*, which is stronger than topological mixing:

Definition 6.6. – A homeomorphism $h \in \mathrm{Homeo}(\mathbb{T}^2)$ is said to be *spreading* when for any lift $\tilde{h} \in \widetilde{\mathrm{Homeo}}(\mathbb{T}^2)$, any $R, \varepsilon > 0$ and any non-empty open set $U \subset \mathbb{R}^2$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, there exists a point $z_n \in \mathbb{R}^2$ such that $\tilde{h}^n(U)$ is ε-dense in the ball $B_R(z_n)$.

Motivated by this notion, we will prove the following theorem, which is the main result of this section:

Theorem 6.7. – *Let us suppose f does not exhibit uniformly bounded v-deviations. Then, for every pair of non-empty open sets $U, V \subset \mathbb{R}^2$, there exists $N \in \mathbb{N}$ such that for every $t \in \mathbb{T}$ it holds*

$$F^n\big(\{t\} \times U\big) \cap \mathbb{T} \times V \neq \emptyset, \quad \forall n \geq N.$$

We shall divide the proof of Theorem 6.7 in several lemmas. Notice that without loss of generality we can assume the open set V in Theorem 6.7 is bounded.

Lemma 6.8. – *There exists $r \in \mathbb{R}$ such that*

$$\Lambda_r^v(t) \cap V \neq \emptyset \quad \textit{and} \quad \Lambda_r^{-v}(t) \cap V \neq \emptyset, \quad \forall t \in \mathbb{T}.$$

Proof. – This is a straightforward consequence of Theorem 6.4, properties (i) and (iii) of Proposition 6.2, and compactness of $\mathbb{T}$. $\square$

From now on we fix a real number $r \in \mathbb{R}$ such that the conclusion of Lemma 6.8 holds.

Since we are assuming V is bounded and f does no exhibit uniformly bounded v-deviations, by Theorem 2.1 we know that the set $\Lambda_r^v(t) \cup \Lambda_r^{-v}(t) \cup V$ disconnects $\mathbb{R}^2$, for every $t \in \mathbb{T}$. For the sake of simplicity of notation, for each $t \in \mathbb{T}$ let us write

$$\Gamma_t := \Lambda_r^v(t) \cup \Lambda_r^{-v}(t) \cup V.$$

Now for each $\varepsilon > 0$, we define the following set:

$$\mathbb{Z}^2(v,\varepsilon) := \left\{ \boldsymbol{p} \in \mathbb{Z}^2 : |\langle \boldsymbol{p}, v\rangle| \leq \varepsilon \right\}.$$

Notice that by (60), the rotation set $\rho(\tilde{f})$ is not a horizontal segment, so v is not vertical, i.e., $\mathrm{pr}_1(v) \neq 0$. By classical arguments about approximations by rational numbers one can easily get the following

LEMMA 6.9. – *For each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{Z}$, there exist $p \in \{n, n+1, \dots, n+N\}$ and $\boldsymbol{p} \in \mathbb{Z}^2(v,\varepsilon)$ satisfying $\mathrm{pr}_2(\boldsymbol{p}) = p$.*

Proof. – This easily follows from Corollary 2.15 and the fact that v is not horizontal. □

LEMMA 6.10. – *For every $t \in \mathbb{T}^2$ there exist two connected components $W_t^+, W_t^- \in \pi_0(\mathbb{R}^2 \setminus \Gamma_t)$ such that the following property holds: for every $z \in \mathbb{R}^2 \setminus \Gamma_t$, there exist $\varepsilon > 0$ and $M \in \mathbb{N}$ such that*

$$T_{\boldsymbol{p}}(z) \in W_t^+, \quad \text{and} \quad T_{\boldsymbol{p}}^{-1}(z) \in W_t^-,$$

for every $\boldsymbol{p} \in \mathbb{Z}^2(v,\varepsilon)$ satisfying $\mathrm{pr}_2(\boldsymbol{p}) > M$.

Proof. – Let z be any point in $\mathbb{R}^2 \setminus \Gamma_t$. By statement (ii) of Proposition 6.2, there exists $\varepsilon > 0$ such that $z \notin \Lambda_{r-2\varepsilon}^v(t) \cup \Lambda_{r-2\varepsilon}^{-v}(t)$, and so we can consider the positive number

$$\delta := \frac{1}{2} d\big(z, \Lambda_{r-2\varepsilon}^v(t) \cup \Lambda_{r-2\varepsilon}^{-v}(t)\big).$$

Hence, we have

$$T_{\boldsymbol{p}}\big(B_\delta(z)\big) \cap \big(\Lambda_r^v(t) \cup \Lambda_r^{-v}(t)\big) = \emptyset, \quad \forall \boldsymbol{p} \in \mathbb{Z}^2(v,\varepsilon). \tag{61}$$

Now, since by Corollary 6.5 the set $\Lambda_{r-2\varepsilon}^v(t) \cup \Lambda_{r-2\varepsilon}^{-v}(t)$ has empty interior and does not disconnect $\mathbb{R}^2$, for each $\boldsymbol{m} \in \mathbb{Z}^2$ we can find a continuous path $\gamma_{\boldsymbol{m}} \colon [0,1] \to \mathbb{R}^2$ such that $\gamma_{\boldsymbol{m}}(0) = z$, $\gamma_{\boldsymbol{m}}(1) \in T_{\boldsymbol{m}}\big(B_\delta(z)\big)$ and

$$\gamma_{\boldsymbol{m}}(s) \notin \Lambda_{r-2\varepsilon}^v(t) \cup \Lambda_{r-2\varepsilon}^{-v}(t), \quad \forall s \in [0,1]. \tag{62}$$

Let N denote the natural number given by Lemma 6.9 for v and ε as above and consider the set

$$A := \left\{ \boldsymbol{p} \in \mathbb{Z}^2(v, 2\varepsilon) : 0 \leq \mathrm{pr}_2(\boldsymbol{p}) \leq N \right\}.$$

Since A is a non-empty finite set and V is bounded, we can define the real number

$$M := 1 + \sup_{\boldsymbol{m}\in A} \sup_{s\in[0,1]} \left|\mathrm{pr}_2\big(\gamma_{\boldsymbol{m}}(s)\big)\right| + \sup_{w\in V} |\mathrm{pr}_2(w)| < \infty. \tag{63}$$

Consider any two points $\boldsymbol{m}, \boldsymbol{p} \in \mathbb{Z}^2(v,\varepsilon)$ satisfying $0 \leq \mathrm{pr}_2(\boldsymbol{p}) - \mathrm{pr}_2(\boldsymbol{m}) \leq N$ and $\mathrm{pr}_2(\boldsymbol{m}) > M$. Thus, we have $T_{\boldsymbol{m}} \circ \gamma_{\boldsymbol{p}-\boldsymbol{m}}$ is a continuous path connecting $T_{\boldsymbol{m}}(z)$ and the ball $T_{\boldsymbol{p}}\big(B_\delta(z)\big)$; since $\boldsymbol{m} \in \mathbb{Z}^2(v,\varepsilon)$ and (62) holds, the image of $T_{\boldsymbol{m}} \circ \gamma_{\boldsymbol{p}-\boldsymbol{m}}$ does not intersect $\Lambda_r^v(t) \cup \Lambda_r^{-v}(t)$; and since $\boldsymbol{p} - \boldsymbol{m} \in A$ and $\mathrm{pr}_2(\boldsymbol{m}) > M$, invoking (63) we conclude that the image of $T_{\boldsymbol{m}} \circ \gamma_{\boldsymbol{p}-\boldsymbol{m}}$ does not intersect V either. So, by (61), $T_{\boldsymbol{m}}(z)$ and $T_{\boldsymbol{p}}(z)$ belong to the same connected component of $\mathbb{R}^2 \setminus \Gamma_t$.

Hence, choosing any $\boldsymbol{m} \in \mathbb{Z}^2(v,\varepsilon)$ satisfying $\mathrm{pr}_2(\boldsymbol{m}) > M$, we can define

$$W_t^+ := \mathrm{cc}\big(\mathbb{R}^2 \setminus \Gamma_t, T_{\boldsymbol{m}}(z)\big),$$

and combining the last argument with Lemma 6.9, one can show that $T_{\boldsymbol{p}}(z) \in W_t^+$, for any $\boldsymbol{p} \in \mathbb{Z}^2(v,\varepsilon)$ such that $\mathrm{pr}_2(\boldsymbol{p}) > M$.

To prove the uniqueness of W_t^+, let w be any other point in $\mathbb{R}^2 \setminus \Gamma_t$. Let $\varepsilon' \leq \varepsilon$ be any positive number such that $w \in \mathbb{R}^2 \setminus (\Lambda^v_{r-2\varepsilon'}(t) \cup \Lambda^{-v}_{r-2\varepsilon'}(t))$. So, since by Corollary 6.5 the set $\Lambda^v_{r-2\varepsilon'}(t) \cup \Lambda^{-v}_{r-2\varepsilon'}(t)$ does not disconnect $\mathbb{R}^2$, then there exists a continuous path $\gamma\colon [0,1] \to \mathbb{R}^2$ such that $\gamma(0) = w$, $\gamma(1) = z$ and

$$\gamma(s) \notin \Lambda^v_{r-2\varepsilon'}(t) \cup \Lambda^{-v}_{r-2\varepsilon'}(t), \quad \forall s \in [0,1].$$

So, the image of $T_{\boldsymbol{p}} \circ \gamma$ does not intersect $\Lambda^v_r(t) \cup \Lambda^{-v}_r(t)$, for any $\boldsymbol{p} \in \mathbb{Z}^2(v,\varepsilon')$, and does not intersect V either, provided $\mathrm{pr}_2(\boldsymbol{p})$ is sufficiently large. Thus, $T_{\boldsymbol{p}}(w) \in W_t^+$ for such a $\boldsymbol{p}$ and uniqueness of W_t^+ is proven.

Finally, defining $W_t^- := \mathrm{cc}\big(\mathbb{R}^2 \setminus \Gamma_t, T_{\boldsymbol{m}}^{-1}(z)\big)$ for $\boldsymbol{m}$ as above, one can easily show that analogous properties hold. $\square$

In order to finish the proof of Theorem 6.7, we fix a non-empty open set $U \subset \mathbb{R}^2$. Without loss of generality we can assume that U is bounded, connected and

$$\overline{U} \cap \big(\Lambda^v_r(0) \cup \Lambda^{-v}_r(0)\big) = \emptyset, \tag{64}$$

where r is the real number we fixed after Lemma 6.8. Since $\overline{U}$ is compact and $\Lambda^v_r \cup \Lambda^{-v}_r$ is contained in $\mathbb{R}^2 \setminus \overline{U}$, by Proposition 6.3 we know the maps $t \mapsto \Lambda^v_r(t)$ and $t \mapsto \Lambda^{-v}_r(t)$ are both compactly upper semi-continuous. Thus, there is $\eta > 0$ such that

$$B_\eta(0) \times \overline{U} \cap \big(\Lambda^v_r \cup \Lambda^{-v}_r\big) = \emptyset, \tag{65}$$

where $B_\eta(0)$ denotes the η-ball centered at $0 \in \mathbb{T}$ with respect to the distance $d_{\mathbb{T}}$.

Now, by minimality of f and recalling that $\rho_1 = \pi(\tilde{\rho}_1)$ where $\tilde{\rho}_1 \in \mathbb{R} \setminus \mathbb{Q}$, there exists $k \geq 1$ such that

$$\bigcup_{i=0}^{k} f^i\big(\pi(U)\big) = \mathbb{T}^2, \quad \text{and} \quad \bigcup_{i=0}^{k} T^i_{\rho_1}\big(B_\eta(0)\big) = \mathbb{T}. \tag{66}$$

Let us define

$$\mathscr{U} := \bigcup_{i=0}^{k} F^i(B_\eta(0) \times U) \subset \mathbb{T} \times \mathbb{R}^2,$$

and for every $t \in \mathbb{T}$, let us write

$$\mathscr{U}(t) := \mathrm{pr}_2\left(\mathscr{U} \cap \{t\} \times \mathbb{R}^2\right) \subset \mathbb{R}^2.$$

Notice that by (65), $\overline{\mathscr{U}} \cap (\Lambda^v_r \cup \Lambda^{-v}_r) = \emptyset$. So, by (ii) of Proposition 6.2, there exists $\varepsilon > 0$ such that

$$\overline{\mathscr{U}(t)_\varepsilon} \cap \big(\Lambda^v_{r-2\varepsilon}(t) \cup \Lambda^{-v}_{r-2\varepsilon}(t)\big) = \emptyset, \quad \forall t \in \mathbb{T},$$

where $(\cdot)_\varepsilon$ denotes the ε-neighborhood given by (3).

On the other hand, by our hypothesis (60) there exists $\tilde{\rho}^+ \in \rho(\tilde{f})$ such that $\mathrm{pr}_2(\tilde{\rho}^+) > 0$. So, let δ be a positive number given by Theorem 4.7 for f, $\tilde{\rho}^+$ and $\varepsilon/2$. Without loss of generality we can assume $\delta < \min\{\eta, \frac{\varepsilon}{4}\}$, where η was chosen in (65).

Now, consider the translation $T := T_{\rho_1, \pi\tilde{\rho}^+} : \mathbb{T} \times \mathbb{T}^2 = \mathbb{T}^3 \circlearrowleft$ and the visiting time set $\tau := \tau(0, B_\delta(0), T)$ defined in Corollary 2.15.

Then, by Theorem 4.7 and (66) we get that, for each $n \in \tau$ there exist $z_n \in U$, $j_n \in \{0, 1, \ldots, k\}$, $\boldsymbol{p}_n \in \mathbb{Z}^2$ and $q_n \in \mathbb{Z}$ such that $\|\boldsymbol{p}_n - n\tilde{\rho}^+\| < \delta$, $|q_n - n\tilde{\rho}_1| < \delta$ and

$$\left\| \tilde{f}^n\big(\tilde{f}^{j_n}(z_n)\big) - \tilde{f}^{j_n}(z_n) - n\tilde{\rho}^+ \right\| < \frac{\varepsilon}{2},$$

or equivalently,

$$F^n\big(F^{j_n}(0, z_n)\big) \in \{(j_n + n)\rho_1\} \times T_{q_n,0}^{-1} \circ T_{\boldsymbol{p}_n}(\mathscr{U}(j_n\rho_1)_\varepsilon). \tag{67}$$

Observe that

$$\begin{aligned} |\langle v, \boldsymbol{p}_n - (q_n, 0)\rangle| &= \big|\langle v, \boldsymbol{p}_n - n\tilde{\rho}^+ + n(\tilde{\rho}_1, 0) - (q_n, 0)\rangle\big| \\ &\leq \big|\langle v, \boldsymbol{p}_n - n\tilde{\rho}^+\rangle\big| + |\langle v, n(\tilde{\rho}_1, 0) - (q_n, 0)\rangle| \leq 2\delta < \frac{\varepsilon}{2}. \end{aligned} \tag{68}$$

So, in particular, this implies that $\boldsymbol{p}_n - (q_n, 0) \in \mathbb{Z}^2(v, \varepsilon)$.

Then observe that since we are assuming U is connected, $\mathscr{U}(t)_\varepsilon$ has finitely many connected components for every $t \in \mathbb{T}$, and then we can apply Lemma 6.10 to conclude that there exists $M > 0$ such that

$$F^i\Big(\{t\} \times T_{\boldsymbol{p}}\big(\mathscr{U}(t)_\varepsilon\big)\Big) \subset \{t + i\rho_1\} \times W^+_{t+i\rho_1} \tag{69}$$

for every $\boldsymbol{p} \in \mathbb{Z}^2(v, \varepsilon)$ satisfying $\mathrm{pr}_2(\boldsymbol{p}) > M$, any $t \in \mathbb{T}$ and every $0 \leq i \leq \max\{k, \mathscr{G}(\tau)\}$, where $\mathscr{G}(\tau)$ denotes the maximum length gap of τ, just defined after (14).

Putting together (67), (68) and (69), and observing $\mathscr{U}$ is open, we conclude that there is a positive number $\eta_0^+ > 0$ such that fixing any $N_0^+ \in \tau$ verifying $\mathrm{pr}_2(\boldsymbol{p}_{N_0^+}) > M$, where $\boldsymbol{p}_{N_0^+} \in \mathbb{Z}^2$ is chosen as above, it holds

$$F^m(\{t\} \times U) \subset \{t + m\rho_1\} \times W^+_{m\rho_1}, \quad \forall m \geq N_0^+, \ \forall t \in B_{\eta_0}(0). \tag{70}$$

Analogously, one may prove a similar statement for some $\tilde{\rho}^- \in \rho(\tilde{f}) \cap \mathbb{H}_0^-$, showing that there exist $\eta_0^- > 0$ and $N_0^- \in \mathbb{N}$ such that

$$F^m(\{t\} \times U) \subset \{t + m\rho_1\} \times W^-_{m\rho_1}, \quad \forall m \geq N_0^-, \ \forall t \in B_{\eta_0}(0). \tag{71}$$

Putting together, (64), (70) and (71) we can conclude that

$$F^m(\{t\} \times U) \cap \big(\{t + m\rho_1\} \times V\big) \neq \emptyset, \quad \forall m \geq N_0, \ \forall t \in B_{\eta_0}(0), \tag{72}$$

where $N_0 := \max\{N_0^+, N_0^-\}$ and $\eta_0 := \min\{\eta_0^-, \eta_0^+\}$.

Then, invoking property 57 one can repeat above argument to show that property (72) in fact holds for any s in $\mathbb{T}$, i.e., given any $s \in \mathbb{T}$, there exist $\eta_s > 0$ and $N_s \in \mathbb{N}$ such that

$$F^m(\{t\} \times U) \cap \big(\{t + m\rho_1\} \times V\big) \neq \emptyset, \quad \forall m \geq N_s, \ \forall t \in B_{\eta_s}(s).$$

Finally, by compactness of $\mathbb{T}$ there are points $s_1, s_2, \ldots s_r \in T$ such that

$$\bigcup_{j=1}^{r} B_{\eta_{s_j}}(s_j) = \mathbb{T}.$$

Defining $N := \max\{N_{s_j} : 1 \leq j \leq r\}$, one can easily verify that the conclusion of Theorem 6.7 holds for any $n \geq N$.

7. Proof of Theorem A

In this section we finish the proof of Theorem A. To do this let us suppose f does not exhibit uniformly bounded v-deviations. By Proposition 2.5, Theorem 2.4 and Proposition 2.16 there is no loss of generality if we assume that f is a minimal symplectic homeomorphism and admits a lift $\tilde{f} \in \widetilde{\mathrm{Symp}_0}(\mathbb{T}^2)$ whose rotation set $\rho(\tilde{f})$ is transversal to the horizontal axis and they intersect at the rotation vector of Lebesgue, i.e., it holds

$$\rho(\tilde{f}) \cap \mathbb{H}_0^{(0,1)} \neq \emptyset, \quad \text{and} \quad \rho(\tilde{f}) \cap \mathbb{H}_0^{(0,-1)} \neq \emptyset,$$

and where $\mathrm{Flux}(\tilde{f}) = (\tilde{\rho}_1, 0)$, for some $\tilde{\rho}_1 \in \mathbb{R}$. Notice that by Corollary 2.11, $\tilde{\rho}_1$ is irrational.

So, we can define the fiber-wise Hamiltonian skew-product $F\colon \mathbb{T} \times \mathbb{R}^2 \circlearrowleft$ as in §6.3.

By analogy with (33), for each $r \in \mathbb{R}$ and $u \in \{(0,1), (0,-1)\}$ we define the stable set at infinity with respect to horizontal direction (we called it the transversal direction in §5) by

$$\Lambda_r^u(t) := \mathrm{pr}_2\Big(\mathrm{cc}\Big(\{t\} \times \mathbb{R}^2 \cap \mathscr{I}_F\big(\mathbb{T}^2 \times \overline{\mathbb{H}_r^u}\big), \infty\Big)\Big) \subset \mathbb{R}^2. \tag{73}$$

One can easily see that stable sets at infinity defined by (33) and (73) are very close related and, in fact,

$$\Lambda_r^u(t) = T_{\tilde{t},0}^{-1}(\Lambda_r^u), \quad \forall \tilde{t} \in \pi^{-1}(t),\ \forall r \in \mathbb{R}.$$

In particular, this implies that all topological and geometric results we proved in Theorems 5.1 and 5.5, and Corollary 5.3 for the sets Λ_r^u continue to hold *mutatis mutandis* for the new ones $\Lambda_r^u(t)$.

Then we have the following

Proposition 7.1. – *If f does not exhibit uniformly bounded v-deviations, then*

$$\Lambda_r^{(0,1)}(t) \cap \Lambda_s^v(t) = \emptyset,$$

for every $r, s \in \mathbb{R}$ and every $t \in \mathbb{T}$.

Proof. – Arguing by contradiction, let us suppose there exist $r, s \in \mathbb{R}$ and $t \in \mathbb{T}$ such that $C := \Lambda_r^{(0,1)}(t) \cap \Lambda_s^v(t) \neq \emptyset$. We claim that, in such a case, every connected component of C is bounded in $\mathbb{R}^2$. In order to prove our claim, let us suppose there exists an unbounded closed connected component $\Gamma \in \pi_0(C)$.

Since Γ is contained in $\Lambda_r^{(0,1)}(t)$, invoking ((iv)) of Theorem 5.1 we know that Γ is "vertically unbounded", i.e., it is not contained in any horizontal strip. On the other hand, since $\Gamma \subset \Lambda_s^v(t) \subset \overline{\mathbb{H}_s^v}$, we get that

$$\langle z, v\rangle \geq s, \quad \forall z \in \Gamma,$$

which contradicts Corollary 5.8.

So, every connected component of C is bounded in $\mathbb{R}^2$. Invoking Theorem 2.1 and taking into account that $\Lambda_r^{(0,1)}(t) \cup \Lambda_s^v(t)$ is unbounded, we conclude that $\Lambda_r^{(0,1)}(t) \cup \Lambda_s^v(t)$ should disconnect $\mathbb{R}^2$. Now let us consider two different connected components V_1 and V_2

of $\mathbb{R}^2 \setminus \left(\Lambda_r^{(0,1)}(t) \cup \Lambda_s^{v}(t)\right)$, and let $U \subset \mathbb{R}^2$ be a non-empty connected open set and $\varepsilon > 0$ such that

$$U \cap \left(\Lambda_r^{u}(t') \cup \Lambda_s^{v}(t')\right) = \emptyset,$$

for $t' \in \mathbb{T}$ satisfying $d_{\mathbb{T}}(0,t') < \varepsilon$.

Then, invoking Theorem 6.7 we know that there is a natural number N such that

$$F^n\big(B_\varepsilon(0) \times U\big) \cap \mathbb{T} \times V_i \neq \emptyset,$$

for every $i = 1, 2$ and every $n \geq N$. In particular, there is some $n_0 \geq N$ and $t' \in B_\varepsilon(0)$ such that $R_{\rho_1}^{n_0}(t') = t$ and $F^{n_0}(\{t'\} \times U)$ intersects $\{t\} \times V_1$ and $\{t\} \times V_2$, and therefore, intersects $\{t\} \times \big(\Lambda_r^{(0,1)}(t) \cup \Lambda_s^{v}(t)\big)$ as well, getting a contradiction. $\square$

Now, by Proposition 7.1, given any $r \in \mathbb{R}$ and any $z \in \Lambda_r^v(0)$, we can define the set

$$U_s := \mathrm{cc}\Big(\mathbb{H}_s^{(0,1)} \setminus \Lambda_s^{(0,1)}(t), z\Big), \quad \forall s < \mathrm{pr}_2(z).$$

Since $\Lambda_r^v(t) \subset \mathbb{H}_r^v$ and is connected, combining Theorem 5.5 and Proposition 7.1 we conclude that

$$\Lambda_r^v(t) \cap \partial_s^{(0,1)}(U_s) \neq \emptyset, \quad \forall s < \mathrm{pr}_2(z), \tag{74}$$

where $\partial_s^{(0,1)}$ denotes the boundary operator as level s given by (47).

However, Theorem 5.5 also implies that there is some $s_0 < \mathrm{pr}_2(z)$ such that

$$\partial_{s_0}^{(0,1)}(U_{s_0}) \subset \mathbb{H}_{-r}^{-v}. \tag{75}$$

Since $\Lambda_r^v(t) \subset \mathbb{H}_r^v$, we see that (74) and (75) cannot simultaneously hold, and Theorem A is proved.

8. Proof of Theorem B

Let us suppose there exists a minimal homeomorphism $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ such that its rotation set is a non-degenerate rational slope segment. So, if $\tilde{f}\colon \mathbb{R}^2 \circlearrowleft$ is a lift of f, then there are $v \in \mathbb{S}^1$ and $\alpha \in \mathbb{R}$ such that inclusion (28) holds. We know that v has rational slope and, by Corollary 2.11, α is an irrational number.

Then, by Theorem A f exhibits uniformly bounded v-deviations, i.e., estimate (1) holds. As a straightforward consequence of Theorem 2.17 one can show that f is a topological extension of an irrational circle rotation (see [10, Proposition 2.1] for details). But this contradicts the following result due to Koropecki, Passaggi and Sambarino [14, Theorem I]:

THEOREM 8.1. – *If $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is a topological extension of an irrational circle rotation, then f is a pseudo-rotation.*

9. Proof of Theorem C

Let $v \in \mathbb{S}^1$ and $\alpha \in \mathbb{R}$ such that $\rho(\tilde{f}) \subset \ell_\alpha^v$. By Theorem A we know that f exhibits uniformly bounded v-deviations. On the other hand, invoking Theorem B we conclude v has irrational slope.

Then, the first step of proof consists in showing the existence of an f-invariant torus pseudo-foliation (see §4.1 for definitions). Since f is minimal, by Theorem 2.4 there is no loss of generality assuming it is area-preserving, and by Proposition 4.6, f is not eventually annular. So we can invoke Theorem 4.4 to conclude f leaves invariant a torus pseudo-foliation $\mathscr{F}$. Let $\widetilde{\mathscr{F}}$ denote its lift to $\mathbb{R}^2$.

In order to study some topological and geometric properties of $\widetilde{\mathscr{F}}$, let us recall some simple steps of its construction from [13]. Since f exhibits uniformly bounded v-deviations, by [13, Corollary 3.2], there exists a constant $C > 0$ such that every (r, v)-stable set at infinity given by (58) satisfies

$$\mathbb{H}^v_{r+C} \subset \Lambda^v_r(0), \quad \forall r \in \mathbb{R}.$$

So, for each $r \in \mathbb{R}$, we define the open set $U_r := \operatorname{cc}\Big(\operatorname{int}\big(\Lambda^v_r(0)\big), \mathbb{H}^v_{r+C}\Big)$; and then, we consider the function $H\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$H(z) := \sup\{r \in \mathbb{R} : z \in U_r\}, \quad \forall z \in \mathbb{R}^2. \tag{76}$$

In the proof of [13, Theorem 5.5], we showed that

$$H\big(\tilde{f}(z)\big) = H(z) + \alpha, \quad \forall z \in \mathbb{R}^2 \tag{77}$$

and then we defined the *pseudo-leaves* (i.e., the atoms of the partition $\widetilde{\mathscr{F}}$) by

$$\widetilde{\mathscr{F}}_z := H^{-1}\big(H(z)\big), \quad \forall z \in \mathbb{R}^2.$$

In general the function H is just semi-continuous, but under our minimality assumption, we will show it is indeed continuous. In fact, let $\phi\colon \mathbb{T}^2 \to \mathbb{R}$ be given by

$$\phi(z) := \Big\langle \Delta_{\tilde{f}}(z), v \Big\rangle - \alpha, \quad \forall z \in \mathbb{T}^2. \tag{78}$$

Since f exhibits uniformly bounded v-deviations, invoking Theorem 2.17 we know there is $u \in C^0(\mathbb{T}^2, \mathbb{R})$ satisfying

$$\phi = u \circ f - u. \tag{79}$$

However, putting together (58), (76) and (78) one can show that the function $u'\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$u'(z) := \langle z, v\rangle - H(z), \quad \forall z \in \mathbb{R}^2$$

is semi-continuous, $\mathbb{Z}^2$-periodic and satisfies $\phi = u' \circ f - u'$, as well. By minimality and classical arguments on semi-continuous functions, we have that $u - u'$ is a constant, and thus, u' is continuous. Consequently, function H given by (76) is continuous as well.

So, for simplicity from now on we can assume that functions H and u given by (76) and (79), respectively, satisfy

$$H(z) = \langle z, v\rangle - u(z), \quad \forall z \in \mathbb{R}^2, \tag{80}$$

where $u \in C^0(\mathbb{T}^2, \mathbb{R})$.

In order to complete the proof of Theorem C, let $U, V \subset \mathbb{T}^2$ be two non-empty open subsets. We want to show there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$, for all $n \geq N$. Without loss of generality we can assume both of them are connected and inessential. Let $\tilde{U}, \tilde{V} \subset \mathbb{R}^2$ be two connected components of $\pi^{-1}(U)$ and $\pi^{-1}(V)$, respectively.

Since pseudo-leaves of $\widetilde{\mathscr{F}}$ have empty interior, there exist two points $z_0, z_1 \in \tilde{V}$ such that $H(z_0) < H(z_1)$; let us write $\delta := 1/2\big(H(z_1) - H(z_0)\big)$. Notice that for every $z \in \mathbb{R}^2$ satisfying $H(z_0) < H(z) < H(z_1)$, the corresponding pseudo-leaves $\widetilde{\mathscr{F}}_z$ separate the points z_1 and z_2 in $\mathbb{R}^2$, i.e., the connected components $\mathrm{cc}\big(\mathbb{R}^2 \setminus \widetilde{\mathscr{F}}_z, z_0\big)$ and $\mathrm{cc}\big(\mathbb{R}^2 \setminus \widetilde{\mathscr{F}}_z, z_1\big)$ are different. In particular, the pseudo-leaves $\widetilde{\mathscr{F}}_z$ must intersect the connected set $\tilde{V}$.

On the other hand, by compactness, there exists a finite set $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_k\} \subset \mathbb{Z}^2$ satisfying the following property: for every $z \in [0,1]^2$, there is $1 \leq n_z \leq k$ such that the pseudo-leaf $H^{-1}\big(r\big)$ intersects $T_{\boldsymbol{p}_{n_z}}(\tilde{V})$, for every $r \in (H(z) - \delta, H(z) + \delta)$, and moreover, it separates the points $T_{\boldsymbol{p}_{n_r}}(z_0)$ and $T_{\boldsymbol{p}_{n_r}}(z_1)$ in $\mathbb{R}^2$.

Since every set of the form $H^{-1}(r - \delta, r + \delta)$ is arc-wise connected, by compactness there exists a real number $M > 0$ such that for every $z \in [0,1]^2$ and any continuous arc $\gamma\colon [0,1] \to \mathbb{R}^2$ such that

$$\gamma(t) \in H^{-1}\big(H(z) - \delta, H(z) + \delta\big), \quad \forall t \in [0,1] \tag{81}$$

and

$$\min_{t\in[0,1]} \mathrm{pr}_2 \circ \gamma(t) < -M < M < \max_{t\in[0,1]} \mathrm{pr}_2 \circ \gamma(t), \tag{82}$$

there exists $t_z \in [0,1]$ such that $\gamma(t_z) \in T_{\boldsymbol{p}_{n_z}}(\tilde{V})$.

Now, let z be an arbitrary point of $\tilde{U}$, and write $r := H(z)$ and

$$\tilde{U}_z := \mathrm{cc}\big(\tilde{U} \cap H^{-1}(r - \delta, r + \delta), z\big). \tag{83}$$

Since we are assuming the rotation set $\rho(\tilde{f})$ is an irrational slope segment, there exist two f-invariant Borel probability measures μ and ν such that $\mathrm{pr}_2\big(\rho_\mu(\tilde{f})\big) \neq \mathrm{pr}_2\big(\rho_\nu(\tilde{f})\big)$. By minimality of f, μ and ν have total support, and this means there exist two points $z_\mu, z_\nu \in U_z$ such that

$$\frac{\tilde{f}^n(z_\mu) - z_\mu}{n} \to \rho_\mu(\tilde{f}), \quad \text{and} \quad \frac{\tilde{f}^n(z_\nu) - z_\nu}{n} \to \rho_\nu(\tilde{f}), \tag{84}$$

as $n \to \infty$.

By (83), $\tilde{U}_z$ is arc-wise connected. So, there is a continuous curve $\eta \in [0,1] \to \tilde{U}_z$ such that $\eta(0) = z_\mu$ and $\eta(1) = z_\nu$. Then, by (84) we conclude there exists a natural number N_M such that

$$\Big|\mathrm{pr}_2 \circ \tilde{f}^n\big(\gamma(0)\big) - \mathrm{pr}_2 \circ \tilde{f}^n\big(\gamma(1)\big)\Big| > 2M, \quad \forall n \geq N_M, \tag{85}$$

where M is the positive real constant invoked in (82).

Observing that, by (77) and (83), one has

$$\tilde{f}^n\big(\tilde{U}_z\big) = \mathrm{cc}\Big(\tilde{f}^n(\tilde{U}) \cap H^{-1}(r + n\alpha - \delta, r + n\alpha + \delta), \tilde{f}^n(z)\Big), \quad \forall n \in \mathbb{Z}.$$

This implies that, putting together (81), (82) and (85) one can conclude that, for each $n \geq N_M$ there exists $\boldsymbol{q}_n \in \mathbb{Z}^2$ and $i_n \in \{1, \dots, k\}$ such that

$$T_{\boldsymbol{q}_n} \circ \tilde{f} \circ \gamma[0,1] \cap T_{\boldsymbol{p}_{i_n}}\big(\tilde{V}\big) \neq \emptyset, \quad \forall n \geq N_M$$

and this proves f is topologically mixing, because the image of γ is completely contained in $\tilde{U}$.

In order to show that $\ell_\alpha^v \cap \mathbb{Q}^2 = \emptyset$, we invoke a recent result of Beguin, Crovisier and Le Roux [2] which extends a previous one due to Kwapisz [18]. In fact, if there is any rational point on ℓ_α^v, one can show that f is flow equivalent to a homeomorphism $g \in \mathrm{Homeo}_0(\mathbb{T}^2)$ such that $\rho(\tilde{g})$ is a vertical line segment, for any lift $\tilde{g}\colon \mathbb{R}^2 \circlearrowleft$ of g (see [18, §§2,3] and [2] for details). Since minimality is preserved under flow equivalence, we conclude that g is a minimal homeomorphism and its rotation set is a non-degenerate rational slope segment, contradicting Theorem B.

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THE FRANKS-MISIUREWICZ CONJECTURE FOR EXTENSIONS OF IRRATIONAL ROTATIONS

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ABSTRACT. – We show that a toral homeomorphism which is homotopic to the identity and topologically semiconjugate to an irrational rotation of the circle is always a pseudo-rotation (i.e., its rotation set is a single point). In combination with recent results, this allows us to complete the study of the Franks-Misiurewicz conjecture in the minimal case.

RÉSUMÉ. – On montre qu'un homéomorphisme du tore homotope à l'identité et topologiquement semiconjugué à une rotation irrationnelle du cercle est une pseudo-rotation (c'est-à-dire, son ensemble de rotation se réduit à un point). À l'aide de résultats récents, ceci conclut l'étude de la conjecture de Franks-Misiurewicz pour les homéomorphismes minimaux.

1. Introduction

It is a general goal in mathematics to classify objects by means of simpler invariants associated to them. In the study of the dynamics of surface maps, the rotation set is a prototypical example of this approach. Being a natural generalization in different contexts of the Poincaré rotation number of orientation preserving circle homeomorphisms, it provides basic dynamical information for surface maps in the homotopy class of the identity [20, 23, 9].

In the two dimensional torus, it can be said that a theory has emerged supported on this invariant (see [20] for a wide exposition). If F is a lift of a torus homeomorphism f in the homotopy class of the identity, its rotation set is defined by

$$\rho(F) = \left\{ \lim_{i\to\infty} \frac{F^{n_i}(x_i) - x_i}{n_i} \,:\, \text{where } n_i \nearrow +\infty,\ x_i \in \mathbb{R}^2 \right\}.$$

In the seminal article [20] Misiurewicz and Ziemian proved the convexity and compactness of rotation sets. The finite nature for the possible geometries of a convex set in the plane given by points, non-trivial line segments, or convex sets with nonempty interior, allowed to start a systematic study based on these three cases.

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Results concerning this theory can be classified in two different directions. A first direction aims to obtain interesting dynamical information from knowing the geometry of the rotation set, where the list of results is huge. For instance it is known that when the rotation set has nonempty interior the map has positive topological entropy [19] and abundance of periodic orbits and ergodic measures [8, 7, 21]; bounded deviations properties are found both for the nonempty interior case and the non-trivial segment case [6, 1, 18, 15] (see also [2, 22] as possible surveys (1)).

A second direction aims to establish which kind of convex sets can be realized as rotation sets. Here we find fundamental problems which remain unanswered (compared with the first direction, it can be said, the state of art is considerably underdeveloped). For convex sets having nonempty interior, all known examples achieved as rotation sets have countably many extremal points [16, 5]. For rotation sets with empty interior, there is a long-standing conjecture due to Franks and Misiurewicz [10], which is the matter of this work. The conjecture aims to classify the possible rotation sets with nonempty interior, and it states that any such rotation set is either a singleton or a non-trivial line segment I which falls in one of the following cases:

(i) I has rational slope and contains rational points (2);

(ii) I has irrational slope and one of the endpoints is rational.

For case (ii) A. Avila has presented a counterexample in 2014 (3), where a non-trivial segment with irrational slope containing no rational points is obtained as rotation set. Moreover, the counterexample is minimal ($\mathbb{T}^2$ is the unique compact invariant set) and C^∞, among other interesting features.

Still concerning case (ii), P. Le Calvez and F. A. Tal showed that whenever the rotation set is a non-trivial segment with irrational slope and containing a rational point, this rational point must be an endpoint of the segment [18], so segments of irrational slope containing rational points obey the conjecture.

Item (i), however, remains open: is it true that the only nontrivial segments of rational slope realized as a rotation set are those containing rational points? Although partial progress has been made in recent years [11, 14, 13, 15], the question remained open even in the minimal case.

In this article we prove that, in contrast to Avila's counter example, case (i) in the conjecture is true for minimal homeomorphisms. As we see in the next paragraph, we prove that case (i) must hold in the family of *extensions of irrational rotations* which in particular provides the answer for minimal homeomorphisms.

1.1. Precise statement, context and scope.

The family of *extensions of irrational rotations* is given by those toral homeomorphisms in the homotopy class of the identity which are topologically semi-conjugate to an irrational rotation of the circle. The study of the conjecture in this particular family was introduced in [13], following a program by T. Jäger: supported in the ideas presented in [11], one may first

(1) Unfortunately both surveys are far from being up to date.
(2) I.e., points with both coordinates rational.
(3) Still unpublished.

aim to show that every possible counter example for the rational case (i) in the conjecture must be contained in this family, and as a second step one may study the conjecture in the class of extensions of irrational rotations. There is significant progress in the first step of the program under some recurrence assumptions [11, 14, 15]. On the other hand, for the second step the only known result states that if a counter-example exists, the fibers of the conjugation must be topologically complicated [13]. This sole fact does not lead to a contradiction, since such a fiber structure is possible for extensions of irrational rotations (see [3]). Our main result in this article completely solves the second step of Jäger's program: there are no counter examples to the Franks-Misiurewicz conjecture in the family of extensions of irrational rotations.

The rotation set of an extension of an irrational rotation in $\mathbb{T}^2$ contains no rational points, and it must be either a singleton or an interval of rational slope (see for instance [13]). In [14] it is proved that every area-preserving homeomorphism homotopic to the identity having a *bounded deviations* property is an extension of an irrational rotation (see also [11]). Recently A. Kocsard showed that minimal homeomorphisms having a non-trivial interval with rational slope as rotation set have the *bounded deviations* property [15], and as a consequence every minimal homeomorphism having a non-trivial interval with rational slope as rotation set must be an extension of an irrational rotation.

Our main result is the following:

Theorem 1. – *The rotation set of a lift of any extension of an irrational rotation is a singleton.*

Using the previously mentioned results we find that case (i) in the Franks-Misiurewicz conjecture is true for minimal homeomorphisms:

Theorem 2. – *The rotation set of a lift of any minimal homeomorphism of $\mathbb{T}^2$ homotopic to the identity is either:*

(i) *a single point of irrational coordinates, or*

(ii) *a segment with irrational slope containing no rational points.*

Note that both cases are realized; the first one by minimal rotations, and the second by Avila's example.

In the case of diffeomorphisms, J. Kwapisz has shown that the possible existence of an example whose rotation set is an interval contained in a line of irrational slope having a rational point outside the interval is equivalent to the existence of an example with a non-trivial segment of rational slope containing no rational points [17]. This was adapted to the C^0 setting by Béguin, Crovisier and Le Roux [4]. Using these results, we have the following:

Corollary. – *Case* (ii) *of the previous theorem can only hold if the supporting line of the segment contains no rational points.*

Finally, it should be mentioned that one of the main theorems in [17] contains the previous corollary, and moreover states that rotation sets which are intervals of rational slope having no rational points cannot be realized by diffeomorphisms. Unfortunately, there is a critical flaw in the proof [(4)], which uses some convoluted estimations relying in quasiconformality properties and extremal length. AK was supported by FAPERJ-Brasil and CNPq-Brasil, AP and MS were partially supported by CSIC group 618.

1.2. Comments on the proof of Theorem I

In [13] it is proved that an extension f of an irrational rotation having an interval as rotation set has a semiconjugacy to an irrational rotation so that every fiber is an essential annular continuum, and almost every fiber contains points realizing both extremal rotation vectors.

In order to prove Theorem I we develop some techniques concerning the geometry of essential loops which have the property of remaining under iteration close enough to two points having different rotation vectors (see Section 4). In Section 3 we show that one can choose a topological annulus A which contains at least two fibers of the semiconjugacy and whose "width" remains small enough after most iterations by f. Applying the results from Section 4, we are able to show that every essential loop in $f^n(A)$ will contain arcs whose *winding number* becomes arbitrarily large with n. This in turn will imply that A is increasingly *distorted*, and as a consequence of this distortion we show that the two boundary circles of $f^n(A)$ contain points arbitrarily close to each other. This leads to a contradiction since the (pointwise) distance between two different fibers remains bounded below by a constant under iterations, due to the semiconjugation to a rigid rotation.

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2. Preliminaries

We denote by $\mathrm{pr}_1: \mathbb{R}^2 \to \mathbb{R}$ and $\mathrm{pr}_2: \mathbb{R}^2 \to \mathbb{R}$ the projections onto the first and second coordinates, respectively, and we define $T_1(x, y) = (x + 1, y)$, $T_2(x, y) = (x, y + 1)$. We consider the open annulus $\mathbb{A}$ defined as $R/\langle T_1 \rangle$, and we let $\tau: \mathbb{R}^2 \to \mathbb{A}$ be the covering projection. The vertical translation T_2 induces a vertical translation on $\mathbb{A}$ which we still denote T_2, and we consider the torus $\mathbb{T}^2 = \mathbb{A}/\langle T_2 \rangle$ with covering projection $\theta: \mathbb{A} \to \mathbb{T}^2$. Note that the map $\pi = \tau \circ \theta$ is a universal covering of $\mathbb{T}^2$. All spaces are endowed by the metric induced by the euclidean metric in $\mathbb{R}^2$.

For a surface S, we denote by $\mathrm{Homeo}_0(S)$ the space of homeomorphisms of S isotopic to the identity. Any $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ can be lifted (by the covering θ) to a homeomorphism $\hat{f} \in \mathrm{Homeo}_0(\mathbb{A})$ which commutes with T_2, and $\hat{f}$ in turn lifts to a homeomorphism $F: \mathbb{R}^2 \to \mathbb{R}^2$, which commutes with both T_1 and T_2, and which is also a lift of f (by the covering π).

[(4)] As acknowledged by the author.

For convenience, let us denote by $\phi_F\colon \mathbb{A} \to \mathbb{R}$ the *horizontal displacement function* associated to F, defined on $\mathbb{A}$ by $\phi_F(x) = \mathrm{pr}_1(F(\tilde{x}) - \tilde{x})$ for any $\tilde{x} \in \theta^{-1}(x)$. Note that since $z \mapsto F(z) - z$ is $\mathbb{Z}^2$-periodic, this definition is independent on the choice of $\tilde{x}$, and moreover ϕ_F is T_2-periodic. In particular it is bounded. It is useful to note that $\phi_{F^n}(x) = \sum_{k=0}^{n-1} \phi_F(\hat{f}^k(x)) = \mathrm{pr}_1(F^n(\tilde{x}) - \tilde{x})$ for any $\tilde{x} \in \theta^{-1}(x)$.

2.1. Some topological definitions and facts

An arc in $\mathbb{A}$ from x to y is a continuous map $\sigma\colon [a,b] \to \mathbb{A}$ such that $\sigma(a) = x$ and $\sigma(b) = y$. Two arcs are equivalent if one is a reparametrization of the other (preserving the endpoints). We identify equivalent arcs. The arc is simple if the map σ may be chosen injective. In the case of a simple arc, we often use the same notation for σ and the image of σ. A loop is an arc γ whose two endpoints coincide. In that case we say that γ is simple if there is a parametrization $\gamma\colon [a,b] \to \mathbb{A}$ which is injective on $[a,b)$. A simple loop γ is essential if its complement in $\mathbb{A}$ has two unbounded components.

An *essential continuum* $E \subset \mathbb{A}$ is a continuum such that $\mathbb{A} \setminus E$ has two unbounded connected components, which we denote $\mathcal{U}^+(E)$ and $\mathcal{U}^-(E)$ (where $\mathcal{U}^+$ is the one unbounded above and $\mathcal{U}^-$ is the one unbounded below). We say that E is an *essential annular continuum* if $\mathbb{A} \setminus E$ has exactly two connected components, both of which are unbounded.

A continuum $C \subset \mathbb{T}^2$ is called a *horizontal (annular) continuum* if each connected component of $\theta^{-1}(C)$ is an essential (annular) continuum. Similarly, an open or closed (topological) annulus $A \subset \mathbb{T}^2$ is called horizontal if each connected component of $\theta^{-1}(A)$ contains an essential loop.

If X, Y are two sets in $\mathbb{A}$ or $\mathbb{T}^2$, we write $d(X,Y) = \inf\{d(x,y) : x \in X, y \in Y\}$. When X is a singleton we write $d(x,Y)$ instead of $d(\{x\},Y)$. By $B_r(X)$ we denote the r-neighborhood of X, i.e., the set $\{y : d(y,X) < r\}$. If X, Y are compact we denote by $d_H(X,Y)$ the Hausdorff distance between the two sets, i.e., the infimum of all numbers $\epsilon > 0$ such that $X \subset B_\epsilon(Y)$ and $Y \subset B_\epsilon(X)$. The Hausdorff distance is a complete metric.

Given two essential continua C_1, C_2 in $\mathbb{A}$, we write $C_1 \prec C_2$ if $C_1 \subset \mathcal{U}^-(C_2)$. This defines a partial order. The following lemma is contained in [12, Lemma 3.8].

LEMMA 2.1. – *If a sequence of essential continua $(C_k)_{k\in\mathbb{N}}$ is increasing and bounded from above in the partial order $\prec$, then there is an essential continuum C such that $d_H(C_k, C) \to 0$ as $k \to \infty$. Moreover, $C = \partial \bigcup_{k\in\mathbb{N}} \mathcal{U}^-(C_k)$. A similar property holds for a decreasing sequence.*

Given an essential annular continuum $A \subset \mathbb{A}$, we define its *upper width* as

$$\mathrm{uw}(A) = \sup\{d_H(C_1, C_2) : C_1, C_2 \text{ are essential continua in } A\}$$

and its *lower width* as

$$\mathrm{lw}(A) = \sup\{d(C_1, C_2) : C_1, C_2 \text{ are essential continua in } A\}.$$

If A is a closed topological annulus, one can easily verify that $\mathrm{lw}(A) = d(\partial^+ A, \partial^- A)$, where $\partial^+ A$ and $\partial^- A$ are the two boundary components of A, and $\mathrm{uw}(A) = d_H(\partial^+ A, \partial^- A)$. Note that we used the infimum distance in the first case and the Hausdorff distance for the second case.

We remark that an equivalent definition of $\mathrm{uw}(A)$ is as the smallest number $\varepsilon > 0$ such that for every essential continuum $C \subset A$ one has $A \subset B_\epsilon(C)$. Note also that if $A \subset A'$ then $\mathrm{uw}(A) \leq \mathrm{uw}(A')$ and $\mathrm{lw}(A) \leq \mathrm{lw}(A')$.

If $A \subset \mathbb{T}^2$ is a horizontal annular continuum, we define its upper and lower width as the upper and lower width of any lift of A to $\mathbb{A}$, respectively (and this is independent of the choice of the lift).

3. Extensions of irrational rotations

Let us say that $h: \mathbb{T}^2 \to \mathbb{T}^1$ is a *horizontal map* if h is continuous, surjective, and $h^{-1}(t)$ is a horizontal annular continuum for each $t \in \mathbb{T}^1$.

Given $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$, we say that f is a *horizontal extension of an irrational rotation* if there exists a horizontal map h such that $hf = Rh$, where R is an irrational rotation of $\mathbb{T}^1$. In this case, it follows from the main results of [13] that such h may be chosen to be homotopic to the projection $\mathbb{T}^2 \to \mathbb{T}^1$ onto the second coordinate; thus we will always assume this to be the case.

We will use the following result due to T. Jäger and the second author of this article [13]:

THEOREM 3.1. – *If f is an extension of an irrational rotation, then f is topologically conjugate to a horizontal extension of an irrational rotation.*

As mentioned in the introduction, the proof of Theorem I is based in showing that if we iterate certain annular neighborhood A of a fiber of the horizontal semiconjugacy, then the d-distance between its two boundary components becomes arbitrarily small. A proof of this fact would be easier if we knew that every fiber of the semiconjugacy has small upper width (bounded above by the continuity module of $\frac{1}{4}$ for f), which would be true for instance if every fiber was a *circloid*[(5)]. Unfortunately, we cannot ensure this fact; instead we will strongly use that only finitely many fibers can have large upper width, which in turn will imply that with a high frequency of iterations the boundary components of A are within a small Hausdorff distance from each other.

For the remainder of this section, fix a horizontal extension of an irrational rotation $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$, and let h be a horizontal map such that $hf = Rh$ where R is an irrational rotation and h is homotopic to the projection $\mathbb{T}^2 \to \mathbb{T}^1$ onto the second coordinate. We also fix a lift $\hat{f} \in \mathrm{Homeo}_0(\mathbb{A})$ of f and a lift $F: \mathbb{R}^2 \to \mathbb{R}^2$ of $\hat{f}$.

Our main purpose in this section is to show the following result, which enumerates the key properties which will be used in the proof of our main theorem. Recall that the lower density of a set $G \subset \mathbb{N}$ is defined as $\liminf_{n\to\infty} \#\{k \in G : k \leq n\}/n$. We state the lemma in the annulus $\mathbb{A}$ since we will work in that setting later.

LEMMA 3.2. – *Suppose that $\rho(F) = [\rho^-, \rho^+] \times \{\alpha\}$. Then, given $\delta > 0$ and $\epsilon > 0$, there exists a closed essential topological annulus $A \subset \mathbb{A}$, an essential simple loop $\gamma \subset \mathbb{A} \setminus A$, two points $x, y \in \mathbb{A} \setminus A$, a set $G \subset \mathbb{Z}$ and $B = B(f) > 0$ such that*

(1) *$\phi_{F^n}(x)/n \to \rho^-$ and $\phi_{F^n}(y)/n \to \rho^+$ as $n \to \infty$;*

(5) A minimal annular continuum with respect to the inclusion.

(2) *If* $n \in G$, *then* $d(\hat{f}^n(x), \hat{f}^n(\gamma)) < \epsilon$ *and* $d(\hat{f}^n(y), \hat{f}^n(\gamma)) < \epsilon$*;*

(3) *The lower density of* G *is at least* $1 - \delta$*;*

(4) $\operatorname{diam}(\operatorname{pr}_2(\hat{f}^n(A) \cup \{x, y\} \cup \gamma)) \leq B$ *for all* $n \in \mathbb{N}$*;*

(5) A *separates* $\{x, y\}$ *from* γ *in* $\mathbb{A}$*, and*

$$\inf_{n \in \mathbb{N}} \operatorname{lw}(\hat{f}^n(A)) > 0.$$

Before proceeding to its proof, we need some results about the fibers of the map h. Note that the family $\mathcal{F}$ of all fibers of h is a decomposition of $\mathbb{T}^2$ into horizontal annular continua. From the continuity of h follows that $\mathcal{F}$ is an *upper semicontinuous decomposition*: if $C_n \in \mathcal{F}$ is a sequence of fibers such that $C_n \to C$ in the Hausdorff topology, then $C \subset C'$ for some $C' \in \mathcal{F}$. We also note that h lifts to a map $H: \mathbb{A} \to \mathbb{R}$ whose fibers are the lifts of fibers of h, and choosing the orientation of $\mathbb{R}$ adequately we have that $H^{-1}(x) \prec H^{-1}(y)$ if and only if $x < y$. Finally we remark that due to the fiber structure of h, whenever $I \subset \mathbb{T}^1$ is an open interval, its preimage $A = h^{-1}(I)$ is a horizontal open topological annulus. This follows from the analogous claim for H: if $I \subset \mathbb{R}$ is an open interval then $H^{-1}(I)$ is an essential topological annulus in $\mathbb{A}$. The latter claim is true because, since the fibers of H are compact and connected, $H^{-1}(I)$ is open and connected, and since the fibers are essential and connected, $H^{-1}(I)$ is "filled" (its complement has no inessential components). From these observations we also see that when $I \subset \mathbb{T}^1$ is a closed interval, $h^{-1}(I)$ is a horizontal annular continuum, as it can be written as a decreasing intersection of horizontal topological annuli.

LEMMA 3.3. – *For each* $\epsilon > 0$ *and* $t \in \mathbb{T}^1$ *there exists a neighborhood* I_t *of* t *such that whenever* $I \subset I_t \setminus \{t\}$ *is a closed interval one has* $\operatorname{uw}(h^{-1}(I)) < \epsilon$.

Proof. – It suffices to prove the analogous claim on $\mathbb{A}$, i.e., for each $t \in \mathbb{R}$ there exists a neighborhood I_t of t such that whenever $I \subset I_t \setminus \{t\}$ is a closed interval one has $\operatorname{uw}(H^{-1}(I)) < \epsilon$. Suppose this is not the case. Then there exists a sequence J_n of closed intervals disjoint from t, converging to t, such that $\operatorname{uw}(H^{-1}(J_n)) \geq \epsilon$. For each J_n we may find two essential continua $C_n^1, C_n^2 \subset J_n$ such that $d_H(C_n^1, C_n^2) \geq \epsilon$. Passing to a subsequence we may assume that the intervals J_n are either increasing or decreasing (in the linear order of $\mathbb{R}$). We assume the former case, as the other case is analogous. This implies that both sequences $(C_n^i)_{n \in \mathbb{N}}$ are increasing in the order $\prec$. Thus by Lemma 2.1 we have $d_H(C_n^i, C^i) \to 0$ where $C^i := \partial U_i^-$ and $U_i^- = \bigcup_{k \in \mathbb{N}} \mathcal{U}^-(C_k^i)$. But one easily verifies that $U_i^- = H^{-1}((-\infty, t])$, so $C^1 = C^2$. This implies that $d_H(C_n^1, C_n^2) \to 0$ as $n \to \infty$, a contradiction. □

REMARK 3.4. – We note that as a consequence of the previous lemma, the set of all fibers of h whose upper width is greater than a given $\epsilon > 0$ must be finite. Indeed the lemma implies that the set $\{t \in \mathbb{T}^1 : \operatorname{uw}(h^{-1}(t)) \geq \epsilon\}$ has no accumulation points.

LEMMA 3.5. – *Given* $\epsilon > 0$ *and* $\delta > 0$*, there exists* $\eta > 0$ *such that for any closed interval* $I \subset \mathbb{T}^1$ *of length smaller than* η*, if* $A = h^{-1}(I)$ *the set* $\{n \in \mathbb{N} : \operatorname{uw}(f^n(A)) < \epsilon\}$ *has lower density at least* $1 - \delta$.

Proof. – Consider a cover of $\mathbb{T}^1$ by finitely many neighborhoods $I_{t_1},\dots,I_{t_k}$ as in Lemma 3.3, and let $0<\eta<\delta/k$ be such that whenever a closed interval I has length smaller than η one has $I\subset I_{t_i}$ for some i. This means that any such I satisfies $\mathrm{uw}(h^{-1}(I))<\epsilon$ unless it contains t_i for some i. If $G_i(I)\subset\mathbb{N}$ denotes the set of all $n\in\mathbb{N}$ such that $t_i\notin R^n(I)$, we have from the unique ergodicity of the irrational rotation R that $G_i(I)$ has lower density $1-\ell(I)>1-\eta$, where $\ell(I)$ denotes the length of I. Hence the set $G(I)=\bigcap_{i=1}^k G_i(I)$ has lower density at least $1-k\eta>1-\delta$. Note that since $R^n(I)$ has the same length as I, we have $\mathrm{uw}(h^{-1}(R^n(I)))<\epsilon$ whenever $n\in G(I)$. The proof is concluded noting that if $A=h^{-1}(I)$ then $f^n(A)=h^{-1}(R^n(I))$. □

LEMMA 3.6. – *If $A=h^{-1}(I)$ for some nontrivial closed interval $I\subset\mathbb{T}^1$, then*

$$\inf\{\mathrm{lw}(f^n(A)) : n\in\mathbb{Z}\}>0.$$

Proof. – Let a,b be the endpoints of I, and $\epsilon=d(a,b)$. Choose $\delta>0$ such that whenever $d(x,y)<\delta$ for $x,y\in\mathbb{T}^2$ one has $d(h(x),h(y))<\epsilon$. Note that this means that $d(h^{-1}(a),h^{-1}(b))\geq\delta$. Moreover, since $d(R^n(a),R^n(b))=d(a,b)=\epsilon$, we also have for any $n\in\mathbb{N}$

$$d(f^n(h^{-1}(a)),f^n(h^{-1}(b)))=d(h^{-1}(R^n(a)),h^{-1}(R^n(b)))\geq\delta.$$

Thus $C_a=f^n(h^{-1}(a))$ and $C_b=f^n(h^{-1}(b))$ are two horizontal continua in $f^n(A)$ such that $d(C_a,C_b)\geq\delta$, and it follows easily that $\mathrm{lw}(f^n(A))\geq\delta$ for all $n\in\mathbb{N}$. □

LEMMA 3.7. – *There exists $B>0$ such that for every interval $I\subset\mathbb{T}^1$, if $A\subset\mathbb{A}$ is a lift of $A_0=h^{-1}(I)$ then* $\mathrm{diam}(\mathrm{pr}_2(\hat{f}^n(A)))\leq B$ *for all $n\in\mathbb{Z}$.*

Proof. – Fix $t\in\mathbb{T}^1$, let C be a lift to $\mathbb{A}$ of $h^{-1}(t)$. If $\mathscr{A}\subset\mathbb{A}$ is the annulus bounded by C and its vertical translation by two, i.e., $T_2^2(C)$, then for each n there is $i\in\mathbb{Z}$ such that $\hat{f}^n(A)\subset T^i(\mathscr{A})$. Since T is an isometry, $B=\mathrm{diam}(\mathrm{pr}_2(\mathscr{A}))$ satisfies the required property. □

3.1. Proof of Lemma 3.2

Let $\eta<1$ be as in Lemma 3.5, let $\mathscr{A}_0=h^{-1}(I_0)$ where $I_0\subset\mathbb{T}^1$ is some closed (nondegenerate) interval of length smaller than η, and choose any lift $A_0\subset\mathbb{A}$ of $\mathscr{A}_0$. Note that A_0 is fibered by the fibers of H, i.e., $A_0=H^{-1}(I_0')$ for some interval $I_0'\subset\mathbb{R}$ (which is a lift of I_0). Noting also that the preimage by H of an open interval is an open topological annulus, and in particular contains an essential simple loop, by an easy argument one obtains a loop γ and two disjoint closed topological annuli $A, A'\subset A_0$ such that:

- $\gamma\prec A\prec A'$;
- $H^{-1}(I)\subset A$ for some nontrivial closed interval $I\subset I_0$;
- $H^{-1}(J)\subset A'$ for some nontrivial closed interval $J\subset I_0$.

Since extremal points of the rotation set are realized by ergodic measures, there exist nonempty f-invariant sets S^+ and S^- in $\mathbb{T}^2$ with the following property (see [20])

$$\lim_{n\to\infty}(F^n(x)-x)/n = (\rho^\pm, \alpha) \text{ for all } x \in \pi^{-1}(S^\pm).$$

Recalling that $\theta(A')$ is the projection of A' into $\mathbb{T}^2$, we know that $h^{-1}(J_0) \subset \theta(A')$ for some nontrivial interval J_0 (the projection of $J \subset \mathbb{R}$ into $\mathbb{T}^1$). Since R is an irrational rotation, $\bigcup_{n\in\mathbb{Z}} R^n(J_0) = \mathbb{T}^1$, thus $\bigcup_{n\in\mathbb{Z}} f^n(\theta(h^{-1}(J_0))) = \mathbb{T}^2$ (from the fact that $hf = Rh$). Hence $h^{-1}(J_0)$ intersects the invariant set S^+, and therefore A' contains some point x which projects into S^+, which implies that $\phi_{F^n}(x)/n \to \rho^+$. The point $y \in A'$ is obtained similarly. Since x, y were chosen in A' and $\gamma \prec A \prec A'$ we deduce that A separates $\{x, y\}$ from γ. In addition, $\operatorname{lw}(\hat{f}^n(A)) \geq \operatorname{lw}(\hat{f}^n(H^{-1}(I)))$ which is uniformly bounded below by Lemma 3.6 (which is stated on $\mathbb{T}^2$ but clearly implies this), so (5) holds (and (1) as well).

Lemma 3.5 implies that the set $G = \{n \in \mathbb{Z} : \operatorname{uw}(\hat{f}^n(A_0)) < \epsilon\}$ has density at least $1-\delta$. Thus, since $\hat{f}^n(\gamma)$ is an essential loop in $\hat{f}^n(A_0)$, for any $n \in G$ one has $\hat{f}^n(A_0) \subset B_\epsilon(\hat{f}^n(\gamma))$, and in particular (2) and (3) hold, since $\{x, y\} \subset A' \subset A_0$. Finally, part (4) follows from Lemma 3.7 applied to A_0. □

4. Topological lemmas in the annulus

In this section we develop some results concerning essential loops in the annulus which under iteration remain close enough to two points having different rotation vectors. This allows to find in the sequence of iterations of the loop a sequence of arcs with increasingly large winding numbers. This will be the key point for proving Theorem I.

The *winding number* of an arc $\sigma: [a, b] \to \mathbb{A}$ is the number $\mathrm{W}(\sigma) = \mathrm{pr}_1(\tilde{\sigma}(b) - \tilde{\sigma}(a))$ where $\tilde{\sigma}: [a, b] \to \mathbb{R}^2$ is a lift of σ and pr_1 denotes the projection onto the first coordinate. This number is independent of the choice of the lift. The *homotopical diameter* $\mathrm{D}(\sigma)$ is the diameter of the projection of $\tilde{\sigma}$ onto the first coordinate, which again is independent of the lift. The following simple remarks will be used:

— if σ_1, σ_2 are two arcs which can be concatenated, then

$$\mathrm{W}(\sigma_1 * \sigma_2) = \mathrm{W}(\sigma_1) + \mathrm{W}(\sigma_2);$$

— if σ_1 and σ_2 are homotopic with fixed endpoints, then $\mathrm{W}(\sigma_1) = \mathrm{W}(\sigma_2)$;

— $\mathrm{D}(\sigma) = \sup_{\sigma'} \left|\mathrm{W}(\sigma')\right|$, where the supremum runs over all subarcs σ' of σ;

— if γ is a simple loop, then $\left|\mathrm{W}(\gamma)\right| \leq 1$.

Recall the definition of the horizontal displacement function $\phi_F: \mathbb{A} \to \mathbb{R}$ from Section 2.

LEMMA 4.1. – *If $\hat{f} \in \mathrm{Homeo}_0(\mathbb{A})$ is a homeomorphism isotopic to the identity with a lift $F: \mathbb{R}^2 \to \mathbb{R}^2$, for any arc σ in $\mathbb{A}$ joining x to y,*

$$\mathrm{W}(\hat{f}(\sigma)) = \mathrm{W}(\sigma) + \phi_F(y) - \phi_F(x).$$

Proof. – It suffices to note that if $\tilde{\sigma}$ is a lift of σ to $\mathbb{R}^2$ joining $\tilde{x}$ to $\tilde{y}$, then $F(\tilde{\sigma})$ is a lift of $\hat{f}(\sigma)$ and its endpoints are $F(\tilde{x})$ and $F(\tilde{y})$, so $\mathrm{W}(\hat{f}(\sigma)) = \mathrm{pr}_1(F(\tilde{y})) - \mathrm{pr}_1(F(\tilde{x})) = \mathrm{pr}_1(\tilde{y}) - \mathrm{pr}_1(\tilde{x}) + \phi_F(y) - \phi_F(x)$ and the claim follows. □

LEMMA 4.2. – *Suppose α is a simple arc, and σ is any arc disjoint from α except at their two endpoints, which coincide. Then $\left|\mathrm{W}(\alpha)\right| \leq \mathrm{D}(\sigma)+1$.*

Proof. – We may assume that σ is a simple arc by choosing a simple arc in its image joining the same two endpoints (which can be chosen with a homotopical diameter smaller than or equal to that of σ). Since σ and α are simple arcs intersecting only at their endpoints, after a change in orientation of σ if necessary we have that $\sigma^{-1} * \alpha$ is a simple loop. This means that $\left|\mathrm{W}(\alpha)-\mathrm{W}(\sigma)\right| = \left|\mathrm{W}(\sigma^{-1} * \alpha)\right| \leq 1$. Hence $\left|\mathrm{W}(\alpha)\right| \leq 1 + \left|\mathrm{W}(\sigma)\right| \leq 1 + \mathrm{D}(\sigma)$ as claimed. □

LEMMA 4.3. – *Suppose α, β are two disjoint simple arcs. Let σ_1 be an arc joining the initial point of α to the initial point of β and otherwise disjoint from α and β, and σ_2 an arc joining the final point of α to the final point of β and otherwise disjoint from α and β. Then*

$$\left|\mathrm{W}(\alpha)-\mathrm{W}(\beta)\right| \leq 2\,\mathrm{D}(\sigma_1) + 2\,\mathrm{D}(\sigma_2) + 2.$$

Proof. – We may assume that σ_1 and σ_2 are simple arcs by choosing a simple arc in their images joining the same two endpoints. Suppose first that σ_1 intersects σ_2. In that case, we may choose an arc σ in the union of their images, joining the final point of α to its initial point. Since σ is disjoint from α except at its two endpoints, the previous lemma implies

$$\left|\mathrm{W}(\alpha)\right| \leq \mathrm{D}(\sigma)+1 \leq \mathrm{D}(\sigma_1)+\mathrm{D}(\sigma_2)+1.$$

A similar argument shows that $\left|\mathrm{W}(\beta)\right| \leq \mathrm{D}(\sigma_1)+\mathrm{D}(\sigma_2)+1$, and the claim follows.

Now assume that σ_1 and σ_2 are disjoint. Then since they are also disjoint from α and β except at their endpoints, it follows that $\alpha * \sigma_2 * \beta * \sigma_1^{-1}$ is a simple loop, hence $\left|\mathrm{W}(\alpha * \sigma_2 * \beta^{-1} * \sigma_1^{-1})\right| \leq 1$. This implies that

$$\left|\mathrm{W}(\alpha)+\mathrm{W}(\sigma_2)-\mathrm{W}(\beta)-\mathrm{W}(\sigma_1)\right| \leq 1,$$

and so

$$\left|\mathrm{W}(\alpha)-\mathrm{W}(\beta)\right| \leq 1 + \left|\mathrm{W}(\sigma_1)\right| + \left|\mathrm{W}(\sigma_2)\right| \leq 1 + \mathrm{D}(\sigma_2) + \mathrm{D}(\sigma_1),$$

which implies the claim of the lemma. □

The following is a key lemma. Although we give a general statement, we will be interested in the case where an essential loop remains close to two points having different rotation vectors.

LEMMA 4.4 (Dragging lemma). – *Suppose that $\hat{f}\colon \mathbb{A} \to \mathbb{A}$ is isotopic to the identity, and let $\gamma \subset \mathbb{A}$ be a simple loop. Given $x, y \in \mathbb{A}$, let σ_x, σ_y be two simple arcs joining x, y to γ and disjoint from γ except at their endpoints x_0, y_0, respectively. Similarly let $\sigma_{\hat{f}(x)}, \sigma_{\hat{f}(y)}$ be two simple arcs joining $\hat{f}(x)$, $\hat{f}(y)$ to $\hat{f}(\gamma)$ and disjoint from $\hat{f}(\gamma)$ except at their endpoints x_1, y_1, respectively. Let I be a simple arc in γ joining x_0 to y_0 and I' a simple arc in $\hat{f}(\gamma)$ joining x_1 to y_1. Then*

$$\left|\mathrm{W}(I')-\mathrm{W}(\hat{f}(I))\right| \leq 3 + \mathrm{D}(\sigma_{\hat{f}(x)}) + \mathrm{D}(\hat{f}(\sigma_x)) + \mathrm{D}(\sigma_{\hat{f}(y)}) + \mathrm{D}(\hat{f}(\sigma_y)).$$

Proof. – Fix a point z in $\hat{f}(\gamma)$ disjoint from $\hat{f}(I)$. Let α_x be a simple arc in $\hat{f}(\gamma)$ joining x_1 to $\hat{f}(x_0)$ not containing z, and α_y a simple arc in $\hat{f}(\gamma)$ joining y_1 to $\hat{f}(y_0)$ not containing z. Note that the endpoints of α_x are connected by the arc $\sigma^{-1}_{\hat{f}(x)} * \hat{f}(\sigma_x)$, which is disjoint from α_x except at its endpoints.

Thus from Lemma 4.2, we have $\mathrm{W}(\alpha_x) \leq \mathrm{D}(\sigma^{-1}_{\hat{f}(x)} * \hat{f}(\sigma_x)) + 1 \leq \mathrm{D}(\sigma_{\hat{f}(x)}) + \mathrm{D}(\hat{f}(\sigma_x)) + 1$. A similar argument shows that $\mathrm{W}(\alpha_y) \leq \mathrm{D}(\sigma_{\hat{f}(y)}) + \mathrm{D}(\hat{f}(\sigma_x)) + 1$. Note that $\alpha_y^{-1} * \hat{f}(I) * \alpha_x$ is an arc contained in the simple loop $\hat{f}(\gamma)$ and does not contain the point z, so it is homotopic (with fixed endpoints) to a simple arc J in $\hat{f}(\gamma)$ joining x_1 to y_1, and we have $\mathrm{W}(J) = \mathrm{W}(\alpha_y) + \mathrm{W}(\hat{f}(I)) + \mathrm{W}(\alpha_x)$. Note that the arc I' from the statement and J are both simple subarcs of the simple loop $\hat{f}(\gamma)$ joining the same points, so there are two possibilities: $I' = J$ or I' is the complementary arc of J in γ. In the first case, we have $\mathrm{W}(I') = \mathrm{W}(J)$, and in the latter case $I' * J^{-1}$ is a simple loop, so $\left|\mathrm{W}(I') - \mathrm{W}(J)\right| = \left|\mathrm{W}(I' * J^{-1})\right| \leq 1$. In both cases, we have

$$\left|\mathrm{W}(I') - \mathrm{W}(\hat{f}(I))\right| \leq \left|\mathrm{W}(I') - \mathrm{W}(J)\right| + \left|\mathrm{W}(J) - \mathrm{W}(\hat{f}(I))\right| \leq 1 + \left|\mathrm{W}(\alpha_x)\right| + \left|\mathrm{W}(\alpha_y)\right|$$

and the desired inequality follows. □

4.1. Distortion of loops and annuli

If $\gamma \subset \mathbb{A}$ is an essential loop, we define its distortion as

$$\mathrm{dist}(\gamma) = \sup\{\mathrm{D}(\sigma) : \sigma \text{ is a simple arc in } A\}.$$

If $A \subset \mathbb{A}$ is an essential closed topological annulus we define its distortion as

$$\mathrm{dist}(A) = \inf\{\mathrm{dist}(\gamma) : \gamma \text{ is an essential loop in } A\}.$$

The lower width of an annulus in a compact region of $\mathbb{A}$ is related to its distortion by the next lemma.

LEMMA 4.5. – *If $A \subset \mathbb{A}$ is an essential closed topological annulus and* $\mathrm{dist}(A) > 1$*, then*

$$\mathrm{lw}(A) \leq \frac{\mathrm{diam}(\mathrm{pr}_2(A))}{\mathrm{dist}(A) - 1}.$$

REMARK 4.6. – With some additional work, one may improve the bound on the right hand side to $\mathrm{diam}(\mathrm{pr}_2(A))/(2\mathrm{dist}(A) - 1)$, but we leave the details to the reader as we will not need this fact.

Proof. – Let $M = \mathrm{dist}(A)$, and fix a vertical line L in $\mathbb{A}$. Let $\mathcal{J}$ be the family of all connected components of $A \cap L$ which connect two points from different boundary components of A. We claim that the number of elements of $\mathcal{J}$ is bounded below by $\mathrm{dist}(A) - 1$. To show this, let m be the number of elements of $\mathcal{J}$ (which we assume finite, otherwise there is nothing to be done). Choose an essential loop γ in A intersecting each element of $\mathcal{J}$ exactly once. If α is any simple subarc of γ with $\mathrm{D}(\alpha) \geq 1$, then α is a concatenation of arcs $\alpha_0 * \alpha_1 * \cdots * \alpha_k$ such that each α_i is disjoint from L except perhaps at its endpoints, and when $0 < i \leq k$ the initial point of α_i belongs to some element of $\mathcal{J}$. Since α_i is simple and contained in γ, each element of $\mathcal{J}$ appears as the initial point of at most one α_i. This implies that $k \leq m$. From the fact that α_i is disjoint from L except at most at its endpoints, we deduce that $\mathrm{D}(\alpha_i) \leq 1$. Since α is the concatenation of the arcs α_i, we have $\mathrm{D}(\alpha) \leq k+1 \leq m+1$, and taking the supremum among all such arcs α we obtain $\operatorname{dist}(\gamma) \leq m+1$. Thus $m \geq \operatorname{dist}(\gamma)-1 \geq \operatorname{dist}(A)-1$ as claimed.

Finally, since the elements of $\mathcal{J}$ are pairwise disjoint intervals in the vertical line L, their total length is at most $\operatorname{diam}(\operatorname{pr}_2(A))$, and since there are at least $\operatorname{dist}(A)-1$ such elements, there must exist some $I \in \mathcal{J}$ of length at most $\ell = \operatorname{diam}(\operatorname{pr}_2(A))/(\operatorname{dist}(A)-1)$. Since the endpoints of I are in different connected components of ∂A, it follows from the definition of lower width that $\operatorname{lw}(A) \leq \ell$, as claimed. □

5. Proof of the main theorem

Let $f \in \operatorname{Homeo}_0(\mathbb{T}^2)$ be a horizontal extension of an irrational rotation, let $\hat{f}\colon \mathbb{A} \to \mathbb{A}$ be a lift of f and let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of $\hat{f}$ (which also lifts f). Suppose for a contradiction that $\rho(F)$ is not a singleton, so it is an interval of the form $[\rho^-, \rho^+] \times \{\alpha\}$ where $\rho^+ > \rho^-$. Since $\rho(F^n) = n\rho(F)$, replacing f by some power of f if necessary we may assume that $\rho^+ - \rho^- \geq 10$.

Fix $0 < \epsilon < 1/4$ such that whenever $z_1, z_2 \in \mathbb{R}^2$ satisfy $d(z_1, z_2) < \epsilon$ one has $d(F(z_1), F(z_2)) < 1/4$. Note that this implies that for $z_1, z_2 \in \mathbb{A}$,

$$\text{if } d(z_1, z_2) < \epsilon \text{ then } \left|\phi_F(z_1) - \phi_F(z_2)\right| < 1/4 + \epsilon. \tag{5.1}$$

Moreover, we remark that for any arc σ in $\mathbb{A}$,

$$\text{if } \mathrm{D}(\sigma) < \epsilon \text{ then } \mathrm{D}(\hat{f}(\sigma)) < 1/4. \tag{5.2}$$

Fix $\delta > 0$, and let $A \subset \mathbb{A}$, $x, y \in \mathbb{A} \setminus A$, $G \subset \mathbb{N}$, and the essential loop $\gamma \subset \mathbb{A} \setminus A$ be as in Lemma 3.2. Let $K = \max_{z \in \mathbb{A}} \left|\phi_F(z)\right|$, which is finite since ϕ_F is continuous and T_2-periodic. Note that for any arc σ in $\mathbb{A}$ one has

$$\mathrm{D}(\hat{f}(\sigma)) \leq \mathrm{D}(\sigma) + 2K. \tag{5.3}$$

We will show that $\operatorname{dist}(\hat{f}^n(A)) \to \infty$ as $n \to \infty$. For each $n \geq 0$ fix a geodesic arc $\sigma_{x,n}$ such that $\sigma_{x,n}$ joins $\hat{f}^n(x)$ to a point x_n of $\hat{f}^n(\gamma)$ minimizing the distance from $\hat{f}^n(x)$ to $\hat{f}^n(\gamma)$, and similarly let $\sigma_{y,n}$ be a geodesic arc joining $\hat{f}^n(y)$ to a point y_n of $\hat{f}^n(\gamma)$ minimizing the distance from $\hat{f}^n(y)$ to $\hat{f}^n(\gamma)$. Note that both arcs are disjoint from $\hat{f}^n(\gamma)$ except for their endpoints x_n, y_n, and $\mathrm{D}(\sigma_{x,n}) \leq d(\hat{f}^n(x), \hat{f}^n(\gamma))$ (and similarly for $\sigma_{y,n}$).

For each $n \geq 0$, let I_n be a simple arc in $\hat{f}^n(\gamma)$ joining x_n to y_n. Note that if $n \in G$, from Lemma 3.2 we have $\mathrm{D}(\sigma_{x,n}) < \epsilon < 1/4$, so by (5.2) we have $\mathrm{D}(\hat{f}(\sigma_{x,n})) < 1/4$. An analogous estimate holds for $\sigma_{y,n}$. Thus from Lemma 4.4 we have

$$\mathrm{W}(I_{n+1}) \geq \mathrm{W}(\hat{f}(I_n)) - 4$$

and so from Lemma 4.1,

$$\mathrm{W}(I_{n+1}) \geq \mathrm{W}(I_n) + \phi_F(y_n) - \phi_F(x_n) - 4.$$

Noting that $d(x_n, \hat{f}^n(x)) < \epsilon$ when $n \in G$, from (5.1) we have

$$\left|\phi_F(x_n) - \phi_F(\hat{f}^n(x))\right| < 1/4 + \epsilon < 1/2,$$

and similarly for y. Thus,

$$\mathrm{W}(I_{n+1}) \geq \mathrm{W}(I_n) + \phi_F(\hat{f}^n(y)) - \phi_F(\hat{f}^n(x)) - 5. \tag{5.4}$$

On the other hand, if $n \notin G$ we may obtain a rougher estimate: Lemma 3.2(4) implies $\mathrm{D}(\sigma_{x,n}) \le B$, so by (5.3) we have $\mathrm{D}(\hat{f}(\sigma_{x,n})) \le 2K + B$, hence again from Lemma 4.4

$$\mathrm{W}(I_{n+1}) \ge \mathrm{W}(\hat{f}(I_n)) - (3 + 4B + 4K)$$

and from Lemma 4.1,

$$\mathrm{W}(I_{n+1}) \ge \mathrm{W}(I_n) + \phi_F(y_n) - \phi_F(x_n) - (3 + 4B + 4K).$$

Since $|\phi_F(x_n) - \phi_F(\hat{f}^n(x))| \le 2K$, we conclude

$$\mathrm{W}(I_{n+1}) \ge \mathrm{W}(I_n) + \phi_F(\hat{f}^n(y)) - \phi_F(\hat{f}^n(x)) - (3 + 4B + 6K). \tag{5.5}$$

Combining (5.4) and (5.5) we obtain

$$\mathrm{W}(I_n) \ge \mathrm{W}(I_0) - 5r_n - (3 + 4B + 6K)(n - r_n) + \sum_{k=0}^{n-1} \phi_F(\hat{f}^k(y)) - \phi_F(\hat{f}^k(x)),$$

where r_n is the cardinality of $G \cap \{1, 2, \dots, n\}$. Note that the summation above is the same as $\phi_{F^n}(y) - \phi_{F^n}(x)$. Thus, using the fact that $\lim_{n\to\infty} \phi_{F^n}(y)/n - \phi_{F^n}(x)/n = \rho^+ - \rho^- \ge 10$ and that the density of G is at last $1 - \delta$, we have

$$\liminf_{n\to\infty} \mathrm{W}(I_n)/n \ge -5(1 - \delta) - (3 - 4B + 6K)\delta + 10.$$

Recalling that the constants K and B depend only on f and not on δ, we may fix $\delta < (3 - 4B + 6K)^{-1}$ to conclude that

$$\liminf_{n\to\infty} \mathrm{W}(I_n)/n \ge 4.$$

In particular, $\mathrm{W}(I_n) \to \infty$ as $n \to \infty$. Now let $\gamma' \subset \hat{f}^n(A)$ be any essential simple loop. Recall from Lemma 3.2(5) that A separates γ from $\{x, y\}$, so $\hat{f}^n(A)$ separates $\hat{f}^n(\gamma)$ from $\{\hat{f}^n(x), \hat{f}^n(y)\}$. This implies that any arc joining $\hat{f}^n(x)$ or $\hat{f}^n(y)$ to a point of $\hat{f}^n(\gamma)$ intersects every essential loop in $\hat{f}^n(A)$. In particular, the arcs $\sigma_{x,n}$ and $\sigma_{y,n}$ must intersect γ'. Let $\sigma'_{x,n}$ and $\sigma'_{y,n}$ be simple subarcs of $\sigma_{x,n}$ and $\sigma_{y,n}$ joining x_n to a point x' of γ' and y_n to a point y' of γ', and otherwise disjoint from γ'. Denoting by I' a simple subarc of γ' joining x' to y', we have from Lemma 4.3 that

$$\left|\mathrm{W}(I_n) - \mathrm{W}(I')\right| \le 2\,\mathrm{D}(\sigma'_{x,n}) + 2\,\mathrm{D}(\sigma'_{y,n}) + 2.$$

Since $\mathrm{D}(\sigma'_{x,n}) \le B$ and $\mathrm{D}(\sigma'_{y,n}) \le B$, we conclude that

$$\mathrm{W}(I') \ge \mathrm{W}(I_n) - 4B - 2.$$

Thus

$$\mathrm{D}(\gamma') \ge \left|\mathrm{W}(I')\right| \ge \left|\mathrm{W}(I_n) - 4B - 2\right|.$$

Since this estimate is independent of the choice of the loop γ' in $\hat{f}^n(A)$, we conclude that

$$\mathrm{dist}(\hat{f}^n(A)) \ge \left|\mathrm{W}(I_n) - 4B - 2\right| \to \infty$$

as $n \to \infty$. But then, recalling that $\mathrm{diam}(\mathrm{pr}_2(\hat{f}^n(A))) \le B$, Lemma 4.5 implies that $\mathrm{lw}(\hat{f}^n(A)) \to 0$ as $n \to \infty$, contradicting Lemma 3.2(5). This completes the proof. $\square$

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RESOLVENT ESTIMATES ON ASYMPTOTICALLY CYLINDRICAL MANIFOLDS AND ON THE HALF LINE

BY TANYA J. CHRISTIANSEN AND KIRIL DATCHEV

ABSTRACT. – Manifolds with infinite cylindrical ends have continuous spectrum of increasing multiplicity as energy grows, and in general embedded resonances (resonances on the real line, embedded in the continuous spectrum) and embedded eigenvalues can accumulate at infinity. However, we prove that if geodesic trapping is sufficiently mild, then the number of embedded resonances and eigenvalues is finite, and moreover the cutoff resolvent is uniformly bounded at high energies. We obtain as a corollary the existence of resonance free regions near the continuous spectrum.

We also obtain improved estimates when the resolvent is cut off away from part of the trapping, and along the way we prove some resolvent estimates for repulsive potentials on the half line which may be of independent interest.

RÉSUMÉ. – Les variétés à bouts infinis cylindriques ont du spectre continu dont la multiplicité est croissante en fonction de l'énergie, et en général les résonances plongées (les résonances sur l'axe réel, plongées dans le spectre continu) et les valeurs propres plongées peuvent s'accumuler à l'infini. Cependant, on démontre que si les géodésiques sont suffisamment peu captées, alors le nombre de résonances plongées et de valeurs propres plongées est fini, et en plus la résolvante tronquée est uniformément bornée en hautes énergies. On obtient comme corollaire l'existence de certaines régions sans résonance près du spectre continu.

On obtient aussi des estimations améliorées lorsque la résolvante est tronquée loin de certaines géodésiques captées, et, en chemin, on démontre des estimations de la résolvante pour des potentiels répulsifs sur la demi-droite, qui peuvent avoir leur intérêt propre.

1. Introduction

1.1. Resolvent estimates for manifolds with infinite cylindrical ends

The high energy behavior of the Laplacian on a manifold of infinite volume is, in many situations, well known to be related to the geometry of the *trapped set*; this is the set of bounded maximally extended geodesics. In the best understood cases, such as when the manifold has asymptotically Euclidean or hyperbolic ends (see [57, §3] for a recent survey),

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the trapped set is compact. Some results have been obtained for more general trapped sets (e.g., manifolds with cusps were studied in [7]) but less detailed information is available.

In this paper we study manifolds with infinite asymptotically cylindrical ends, which have noncompact trapped sets. A motivation for this study comes from waveguides and quantum dots connected to leads. The spectral geometry of these is closely related to that of asymptotically cylindrical manifolds, and they appear in certain models of electron motion in semiconductors and of propagation of electromagnetic and sound waves. We give just a few pointers to the physics and applied math literature here [34, 46, 47, 25, 2]. In [9], we prove analogues of some of the results below for suitable (star-shaped) waveguides.

The fundamental example of a manifold with cylindrical ends is the Riemannian product $\mathbb{R} \times \mathbb{S}^1$, which has an unbounded trapped set consisting of the circular geodesics. We are interested in the behavior of the resolvent of the Laplacian (and its meromorphic continuation, when this exists) for perturbations of such cylinders and their generalizations. As we discuss below, this behavior can sometimes be very complicated, but we show that if some geometric properties of the manifold are favorable, then the resolvent is uniformly bounded at high energy. In the companion paper [10], we study the closely related problem of long time wave asymptotics on such manifolds.

We begin with an illustration of a more general theorem to follow, by stating a high energy resolvent estimate for two kinds of *mildly trapping* manifolds (X, g) with infinite cylindrical ends.

EXAMPLE 1. – Let (r, θ) be polar coordinates in $\mathbb{R}^d$ for some $d \geq 2$, and let

$$X = \mathbb{R}^d, \qquad g_0 = dr^2 + F(r)dS,$$

where dS is the usual metric on the unit sphere, $F(r) = r^2$ near $r = 0$, and F' is compactly supported on some interval $[0, R]$ and positive on $(0, R)$; see Figure 1.1.

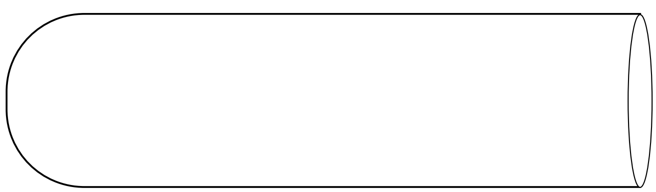

FIGURE 1.1. A cigar-shaped warped product.

Then for $r(t) > 0$ all g_0-geodesics obey

$$\ddot{r}(t) := \frac{d^2}{dt^2} r(t) = 2|\eta|^2 F'(r(t)) F(r(t))^{-2} \geq 0,$$

where $r(t)$ is the r coordinate of the geodesic at time t and η is the angular momentum. Consequently, the only trapped geodesics are the ones with $\dot{r}(t) \equiv F'(r(t)) \equiv 0$, that is the circular ones in the cylindrical end. This is the smallest amount of trapping a manifold with a cylindrical end can have.

Let g be any metric such that $g - g_0$ is supported in $\{(r, \theta) \mid r < R\}$, and such that g and g_0 have the same trapped geodesics. For example we may take $g = g_0 + cg_1$, where g_1 is any symmetric two-tensor with support in $\{(r, \theta) \mid r < R\}$, and $c \in \mathbb{R}$ is chosen sufficiently small depending on g_1. Alternatively, we may take $g = dr^2 + g_S(r)$, where $g_S(r)$ is a smooth family of metrics on the sphere such that $g_S(r) = r^2 dS$ near $r = 0$ and $g_S(r) = F(r)dS$ near $r \geq R$, and such that $\partial_r g_S(r) > 0$ on $(0, R)$. This way we can construct examples where $g - g_0$ is not small.

EXAMPLE 2. – Let (X, g_H) be a convex cocompact hyperbolic surface, such as the symmetric hyperbolic 'pair of pants' surface with three funnels depicted in Figure 1.2.

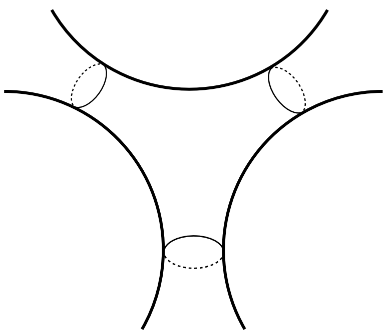
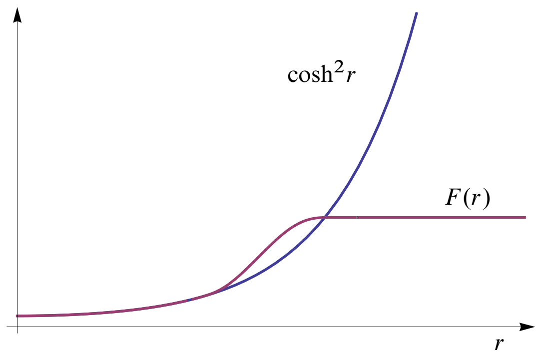

FIGURE 1.2. A hyperbolic surface (X, g_H) with three funnels, and a modification of the metric which changes the funnel ends to cylindrical ends.

In particular, there is a compact set $N \subset X$ (the convex core of X) such that

$$X \setminus N = (0, \infty)_r \times Y_y, \qquad g_H|_{X \setminus N} = dr^2 + \cosh^2 r \, dy^2,$$

where Y is a disjoint union of $k \geq 1$ geodesic circles (possibly having different lengths).

We modify the metric in the funnel ends so as to change them into cylindrical ends in the following way. Take g such that

$$g|_N = g_H|_N, \qquad g|_{X \setminus N} = dr^2 + F(r)dy^2,$$

where $F(r) = \cosh^2 r$ near $r = 0$, and F' is compactly supported and positive on the interior of the convex hull of its support.

To obtain higher dimensional examples, we can take (X, g_H) to be a conformally compact manifold of constant negative curvature, with dimension $d \geq 3$, but in this case we need the additional assumption that the dimension of the limit set is less than $(d-1)/2$. The construction of g now becomes more complicated and we give it in §3.3 below.

Our first result concerns only the above examples.

THEOREM 1.1. – *Let (X, g) be as in Example 1 or 2 above, and let $\Delta \leq 0$ be its Laplacian. There is $z_0 > 0$ such that for any $\chi \in C_c^\infty(X)$ there is $C > 0$ such that*

$$\|\chi(-\Delta - z)^{-1}\chi\|_{L^2(X) \to L^2(X)} \leq C \tag{1.1}$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq z_0$ and $\operatorname{Im} z \neq 0$.

Here $(-\Delta - z)^{-1}$ denotes the standard resolvent which maps $L^2(X) \to L^2(X)$, and not its meromorphic continuation. Below, in Theorem 5.6, we also obtain bounds for the meromorphic continuation, but these are more complicated to state.

The bound (1.1) is optimal in the sense that we cannot replace the right hand side by a function of z which tends to 0 as $\operatorname{Re} z \to \infty$. Indeed, taking the case of Example 1 with $d = 2$ for definiteness, we have $(-\Delta - k^2)v(r)e^{ik\theta} = -v''(r)e^{ik\theta}$ for any $v \in C_c^\infty((R,\infty))$ and $k \in \mathbb{Z}$.

Note also that the resolvent in these examples is better behaved than it is for the (geometrically simpler) Riemannian product $(X, g) = (\mathbb{R} \times Y, g = dr^2 + g_Y)$, where (Y, g_Y) is a compact Riemannian manifold. Indeed, take $\chi \in C_c^\infty(X)$ a function of r such that $\chi \geq 0$ and $\chi \not\equiv 0$, and take $\chi_0 \in C_c^\infty(X)$ such that $\chi_0\chi = \chi$, and let ϕ be an eigenfunction of the Laplacian on (Y, g_Y) with $-\Delta\phi = \sigma^2\phi$. Then, by separation of variables,

$$\|\chi(-\Delta - z)^{-1}\chi\chi_0\phi\|_{L^2(X)} = \|\phi\|_{L^2(Y)}\|\chi(-\partial_r^2 - z + \sigma^2)^{-1}\chi\|_{L^2(\mathbb{R})} \xrightarrow{z\to\sigma^2} +\infty, \tag{1.2}$$

where we take the limit using the explicit formula for the resolvent [23, (2.2.1)]. For our proof of Theorem 1.1 it will be crucial that $F' > 0$ near the cylindrical ends in Examples 1 and 2, and this is what is missing in the Riemannian product just discussed.

We will deduce Theorem 1.1 from Theorem 3.1 below, which gives a stronger result (allowing χ to be replaced by a noncompactly supported weight) and also applies to Schrödinger operators on more general manifolds with asymptotically cylindrical ends. We will further prove in Theorem 3.2 that we can obtain stronger resolvent bounds by suitably refining the cutoffs χ.

An estimate like (1.1) has well-known implications for the spectrum of $-\Delta$. In particular, by [48, Theorem XIII.20], the spectrum is purely absolutely continuous on (z_0, ∞), which rules out any embedded eigenvalues there, and we will see below, in §5, that embedded resonances (resonances on the real line, embedded in the continuous spectrum) are also ruled out.

To our knowledge ours is the first result ruling out the presence of infinitely many embedded eigenvalues or resonances for a large class of examples of manifolds with infinite cylindrical ends.

The situation can be very different for other manifolds with cylindrical ends. For example, let $X = \mathbb{R}\times Y$ and $g = dr^2 + F(r)g_Y$, where (Y, g_Y) is a compact Riemannian manifold and $F \in C^\infty(\mathbb{R}; (0,\infty))$, $1 - F$ is compactly supported, and $\max F > 1$. Then $-\Delta$ has infinitely many embedded eigenvalues converging to $+\infty$ ([11, §3], [45, (3.6)]).

The study of the spectral and scattering theory of the Laplacian on manifolds with cylindrical ends, and their perturbations, goes back to Guillopé [28] and Melrose [36] and is an active and wide-ranging area of research: see for example [31, 38, 49] for some recent results and more references. There is also a large body of literature on the closely related study of the Laplacian on waveguides: something of a survey can be found in [33], and let us also mention the older result [26], and that there is a nonexistence result for eigenvalues in [20]. In a slightly different direction, weighted resolvent estimates up to the spectrum and limiting absorption principles have been investigated using Mourre theory [37, 1, 21], and this has been applied to geometric situations such as ours in [40].

Our results also have implications for the distribution of *resonances*; these are the poles of the meromorphic continuation of the resolvent, and their study in this context also goes back to [28, 36]. An existence result for resolvent poles (in the presence of appropriate quasimodes, and which may be embedded in the real line or complex) on waveguides can be found in [24], and for more such results see [33]. Upper bounds on the number resonances for manifolds with infinite cylindrical ends are given in [8].

In Theorem 5.6, we will use an identity due to Vodev [52] to prove that (1.1) (or a more general resolvent estimate up to the spectrum) implies the existence of a resonance free region near the continuous spectrum. In a companion paper to this one, [10], we use these results to prove an asymptotic expansion for solutions to the wave equation.

1.2. Repulsive potentials on the half line

In this paper we also obtain some resolvent estimates for Schrödinger operators on the half line which we need in the course of the proofs of our main results, and which may be of independent interest. We state them here.

Let V_D be a bounded, nonnegative, nonincreasing potential on the half line, such that

$$V_D'(r) \leq -\delta_V (1+r)^{-1} V_D(r) \leq 0 \tag{1.3}$$

for some $\delta_V > 0$ and for all $r \geq 0$, where if V_D is not everywhere differentiable then (1.3) is meant in the sense of measures. Note that in particular the potential is repulsive in the sense of classical mechanics, since $V_D'(r) < 0$ except where $V_D(r) = 0$.

For $h > 0$ and $\zeta \in \mathbb{C} \setminus [0, \infty)$ let

$$(-h^2 \partial_r^2 + V_D - \zeta)^{-1}$$

denote the Dirichlet resolvent. In this paper we prove the following semiclassical resolvent estimates:

THEOREM 1.2. – *For all s, s_1, $s_2 > 1/2$ with $s_1 + s_2 > 2$ there is $C > 0$ such that for all $\zeta \in \mathbb{C} \setminus [0, \infty)$ and $h > 0$ we have*

$$\|(1+r)^{-s}(-h^2\partial_r^2 + V_D(r) - \zeta)^{-1}(1+r)^{-s}\| \leq \frac{C}{h\sqrt{|\zeta|}}, \tag{1.4}$$

$$\|(1+r)^{-s_1}(-h^2\partial_r^2 + V_D(r) - \zeta)^{-1}(1+r)^{-s_2}\| \leq \frac{C}{h^2}, \tag{1.5}$$

and

$$\|V_D(r)^{1/2}(1+r)^{-1/2}(-h^2\partial_r^2 + V_D(r) - \zeta)^{-1}(1+r)^{-s}\| \leq \frac{C}{h}, \tag{1.6}$$

where the norms are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$.

Recall that, in the case $V_D \equiv 0$, (1.4) and (1.5) are well known to be sharp as $\operatorname{dist}(\zeta, [0,\infty)) \to 0$; this can be checked from the explicit formula for the resolvent in that case, which we give below in (5.4).

In fact, we will deduce these estimates from some uniform estimates for Schrödinger operators with repulsive potentials, replacing C by an explicit constant. To state them, let

$$P_D := -\partial_r^2 + V_D(r),$$

regarded as a self-adjoint operator on $L^2(\mathbb{R}_+)$ with domain $\{u \in H^2(\mathbb{R}_+) \mid u(0) = 0\}$.

THEOREM 1.3. – *For all $\delta > 0$, $\theta \in [0, 1]$, and $z \in \mathbb{C} \setminus [0, \infty)$, we have*

$$\|(1+r)^{-\frac{1+\delta}{2}}(P_D - z)^{-1}(1+r)^{-\frac{1+\delta}{2}}\| \le \frac{1+\sqrt{2}}{\sqrt{|z|}}\left(\frac{1}{\delta} + \frac{1}{\delta_V}\right), \tag{1.7}$$

$$\|(1+r)^{-\frac{1+\delta}{2}-\theta}(P_D - z)^{-1}(1+r)^{-\frac{1+\delta}{2}-(1-\theta)}\| \le (1+\sqrt{2})\left(\frac{1}{\delta} + \frac{1}{\delta_V}\right), \tag{1.8}$$

and

$$\|V_D(r)^{\frac{\theta}{2}}(1+r)^{-\frac{1+(1-\theta)\delta}{2}}(P_D - z)^{-1}V_D(r)^{\frac{1-\theta}{2}}(1+r)^{-\frac{1+\theta\delta}{2}}\| \le \frac{2\sqrt{2}}{\delta_V}\sqrt{1 + \frac{\delta_V}{\delta}}, \tag{1.9}$$

where the norms are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$.

Note that Theorem 1.3 implies Theorem 1.2.

If $V_D \in C^1([0, \infty))$ is compactly supported and has $V_D' < 0$ on the interior of the support of V_D, then (1.3) is satisfied for some $\delta_V > 0$ (because $\log V_D$ and $(\log V_D)'$ tend to $-\infty$ at the boundary of the support). Moreover the class of potentials satisfying (1.3) for a given $\delta_V > 0$ is closed under nonnegative linear combinations and contains all functions of the form $(1+r)^{-m}$ with $m \ge \delta_V$. The same proof could also handle potentials V_D satisfying (1.3) and such that $V_D(r) \to \infty$ as $r \to 0$, provided $V_D(r)|u(r)|^2 \to 0$ as $r \to 0$ for all u in the domain of P_D.

The bounds (1.4) and (1.7) are best when the spectral parameter is not too close to 0, and (1.5) and (1.8) are best when the spectral parameter is close to 0. We can think of (1.6) and (1.9) as being a kind of Agmon or elliptic estimate in the limit $|z| \to 0$ (see also (4.14) below); they give an improvement when we are looking at the resolvent in the elliptic and classically forbidden range in the interior of the support of V_D. When $V_D(r) \sim (1+r)^{-m}$ as $r \to \infty$ for some $m > 0$, the weights in (1.9) are also to be compared to the weights in [55, 39]; see in particular [39, Theorem 1.3].

If we do not demand explicit constants in the estimates, then Theorem 1.3 is essentially well-known if either $V_D(0)$ (which we can think of as a coupling constant) is not large (see [55, Chapter 4] for a more general discussion of scattering on the half line, and [32] for some more recent results and references), or if $V_D(0)$ and $|z|$ are large (this is the semiclassical, nontrapping regime: see [55, Chapter 7, Theorem 1.6] for a similar result). The main novelty here is that we cover all values of $V_D(0)$ and $|z|$ uniformly, and for our applications in §3 we will especially need the case where $V_D(0)$ is large compared to $|z|$: this corresponds to a low-energy semiclassical problem.

We prove Theorem 1.3 in §2 below.

1.3. Notation

Throughout the paper C is a large constant which can change from line to line, and all estimates are uniform for $h \in (0, h_1]$, where h_1 can change from line to line. It will sometimes be convenient to write derivatives with respect to r using the notation $D_r := -i\partial_r$. We use

$$\|u\|_{H_h^m(X)} := \|(-h^2\Delta + 1)^{m/2}u\|_{L^2(X)}$$

and similarly define $\|u\|_{H_h^m(\mathbb{R})}$ and $\|u\|_{H_h^m(\mathbb{R}_+)}$ (in the latter case we will only be concerned with u vanishing near $r = 0$, so the boundary condition on the Laplacian implicit in the notation in this case is immaterial).

The energy level $E_0 > 0$ is fixed in §3.1, along with the rest of the notation needed for our general abstract setup of a mildly trapping Schrödinger operator on a manifold with asymptotically cylindrical ends. The auxiliary notations E_j and E_* are defined in §4.2 in terms of this setup. The notation E without a subscript is used in §2 and §5 to denote a variable positive energy, not related in any particular way to E_0 or E_j or E_*.

The radial variable r on the cylindrical end has the same meaning in §3.1, in §4, and in §5. The usage in §2 is consistent with this usage, if we separate variables to write Schrödinger operator on an asymptotically cylindrical end as a sum of Schrödinger operators on $\mathbb{R}_+$. For example, if Δ is the Laplacian on $((0,\infty)\times Y, dr^2 + g_Y)$ we write

$$-\Delta = \sum_{j=0}^{\infty}(-\partial_r^2 + \sigma_j^2)\phi_j \otimes \phi_j, \text{ to mean } -\Delta u = \sum_{j=0}^{\infty}\phi_j \int_Y (-\partial_r^2 + \sigma_j^2)u(r,y)\phi_j(y)d\operatorname{vol}(y),$$

where $\{\phi_j\}_{j=0}^{\infty}$ is a complete set of real-valued orthonormal eigenfunctions of the Laplacian on Y and $-\Delta_Y\phi_j = \sigma_j^2\phi_j$.

Of course the results of §2 also apply to more general Schrödinger operators on $\mathbb{R}_+$.

The variable r is used a little differently in §1.1, §3.3, and §3.4. To convert the r in one of these sections to the r in the rest of the paper, use the affine map

$$r \mapsto 6(r - R_1)/(R_2 - R_1), \tag{1.10}$$

for suitably chosen R_1 and R_2, and then multiply g by $(R_2 - R_1)^2/36$ to remove the factor that appears in front of dr^2. For Example 1, take R_1 such that $\inf\{r > 0 \mid g(r,y) = g_0(r,y) \text{ for all } y\} < R_1 < R$ and use $R_2 = R$. For Example 2, let $R_2 = \max\operatorname{supp} F'$, and take $R_1 \in (0, R_2)$. For §3.3, let $R_1 = R + 1$ and $R_2 = \max\operatorname{supp} F'$. For §3.4, let $R_1 = R/2$ and $R_2 = R$.

2. Resolvent estimates on the half line

In this section we prove Theorem 1.3. All function norms and inner products in this section are in $L^2(\mathbb{R}_+)$, and operator norms are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$.

Proof of (1.7). – Let $E := \operatorname{Re} z$ and $\varepsilon := |\operatorname{Im} z|$. We begin by proving an a priori estimate when $E > 0$ and $\varepsilon > 0$. Roughly speaking, the idea is to exploit the fact that, since $V_D' \le 0$, we have the positive commutator $[P_D, r\partial_r] = -2\partial_r^2 - rV_D'(r) \ge 0$. However, to be able to control the remainder terms in our positive commutator argument, we must replace $r\partial_r$ with $w(r)\partial_r$ where w grows more slowly. Such commutants have been used by many authors (see [48, §XIII.7] and references therein); below we take an approach inspired by [52, 15] and papers cited therein.

Take $w \in C^1([0,\infty);[0,1])$ such that $w'(r) > 0$ for all $r \geq 0$, and take $u \in H^2(\mathbb{R}_+)$ such that $u(0) = 0$ and $(w')^{-1/2}(P_D - z)u \in L^2$; in particular, $u(r)$ and $u'(r)$ tend to 0 as $r \to \infty$. Adding together the integration by parts identities

$$-\langle (w(V_D - E))' u, u\rangle = 2\,\mathrm{Re}\langle w(V_D - E)u, u'\rangle$$

and

$$\langle w'u', u'\rangle + w(0)|u'(0)|^2 = -2\,\mathrm{Re}\langle wu'', u'\rangle$$

gives

$$E\|\sqrt{w'}u\|^2 + \|\sqrt{w'}u'\|^2 - \langle (wV_D)'u, u\rangle + w(0)|u'(0)|^2 \\ = 2\,\mathrm{Re}\langle w(P_D - z)u, u'\rangle - 2\,\mathrm{Im}\, z\, \mathrm{Im}\langle wu, u'\rangle.$$

Since $0 \leq w \leq 1$, this implies

$$E\|\sqrt{w'}u\|^2 + \|\sqrt{w'}u'\|^2 - \langle (wV_D)'u, u\rangle \leq 2\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\| \|\sqrt{w'}u'\| + 2\varepsilon\|u\|\|u'\|. \tag{2.1}$$

Later we will choose w so that $(wV_D)' \leq 0$, but first we estimate the second term on the right, which we think of as a remainder term. Since $V_D \geq 0$, integrating by parts gives

$$\|u'\|^2 \leq \mathrm{Re}\langle (P_D - z)u, u\rangle + E\|u\|^2 \leq \left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\| \|\sqrt{w'}u\| + E\|u\|^2$$

and we also have

$$\varepsilon\|u\|^2 = |\,\mathrm{Im}\langle (P_D - z)u, u\rangle| \leq \left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\| \|\sqrt{w'}u\|.$$

Combining these gives

$$\varepsilon^2\|u\|^2\|u'\|^2 \leq (E + \varepsilon)\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\|^2 \|\sqrt{w'}u\|^2$$

and then plugging this into (2.1) gives

$$E\|\sqrt{w'}u\|^2 + \|\sqrt{w'}u'\|^2 - \langle (wV_D)'u, u\rangle \leq 2\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\| \left(\|\sqrt{w'}u'\| + \sqrt{E+\varepsilon}\|\sqrt{w'}u\|\right).$$

Completing the square gives

$$\left(\sqrt{E}\|\sqrt{w'}u\| - \frac{\sqrt{E+\varepsilon}}{\sqrt{E}}\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\|\right)^2 + \left(\|\sqrt{w'}u'\| - \left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\|\right)^2 \\ - \langle (wV_D)'u, u\rangle \leq \frac{2E+\varepsilon}{E}\left\|\frac{1}{\sqrt{w'}}(P_D - z)u\right\|^2. \tag{2.2}$$

We now take

$$w(r) := 1 - \frac{\delta_V}{\delta_V + \delta}(1+r)^{-\delta}, \tag{2.3}$$

so that, by (1.3), we have

$$(wV_D)'(r) = \frac{\delta\delta_V V_D(r)}{(\delta_V+\delta)(1+r)^{1+\delta}} + w(r)V_D'(r) \le \frac{\delta_V V_D(r)}{1+r}\Big((1+r)^{-\delta}-1\Big) \le 0, \tag{2.4}$$

where, as with (1.3), we understand (2.4) in the sense of measures in the case that V_D is not differentiable everywhere. We may now drop the second and third terms from the left hand side of (2.2), giving

$$\sqrt{E}\|\sqrt{w'}u\| \le \frac{\sqrt{E+\varepsilon}+\sqrt{2E+\varepsilon}}{\sqrt{E}}\left\|\frac{1}{\sqrt{w'}}(P_D-z)u\right\|. \tag{2.5}$$

From (2.5) we can deduce a weighted resolvent estimate when $\operatorname{Re} z > 0$, $\operatorname{Im} z \neq 0$. To obtain an estimate for all $z \in \mathbb{C} \setminus [0,\infty)$, we use the Phragmén–Lindelöf principle in the following way. For $u,\ v \in L^2(\mathbb{R}_+)$, put

$$U(z) := \langle (1+r)^{-\frac{1+\delta}{2}}(P_D-z)^{-1}(1+r)^{-\frac{1+\delta}{2}}u, v\rangle\sqrt{z}, \tag{2.6}$$

and for $\alpha > 0$ put

$$\Omega_\alpha := \{z \in \mathbb{C} \mid \alpha \operatorname{Re} z < |\operatorname{Im} z|\}.$$

Then U is holomorphic in Ω_α, where it obeys

$$|U(z)| \le \frac{|\sqrt{z}|\|u\|\|v\|}{\operatorname{dist}(z,[0,\infty))} \le \frac{\sqrt{1+\alpha^{-2}}\|u\|\|v\|}{|\sqrt{z}|}.$$

Moreover, by (2.5), for $z \in \partial\Omega_\alpha \setminus \{0\}$, we have

$$|U(z)| \le \Big(\sqrt{1+\alpha}+\sqrt{2+\alpha}\Big)\left(\delta^{-1}+\delta_V^{-1}\right)\|u\|\|v\|. \tag{2.7}$$

Then the Phragmén–Lindelöf principle (see e.g., [48, p. 236]) implies (2.7) for all $z \in \Omega_\alpha$. Taking $\alpha \to 0$ gives (1.7). □

Proof of (1.8). – We begin by following the proof of (1.7), but we drop the first term, rather than the second, from the left hand side of (2.2), so that in place of (2.5) we have

$$\|\sqrt{w'}u'\| \le \Big(1+\sqrt{2+\varepsilon E^{-1}}\Big)\left\|\frac{1}{\sqrt{w'}}(P_D-z)u\right\|.$$

We now integrate by parts to obtain a weighted version of the Poincaré inequality:

$$\left\|(1+r)^{\frac{-3-\delta}{2}}u\right\|^2 = \frac{2}{2+\delta}\operatorname{Re}\Big\langle (1+r)^{-2-\delta}u', u\Big\rangle \le \left\|(1+r)^{\frac{-1-\delta}{2}}u'\right\|\left\|(1+r)^{\frac{-3-\delta}{2}}u\right\|,$$

giving

$$\left\|(1+r)^{\frac{-3-\delta}{2}}u\right\| \le \sqrt{\delta_V^{-1}+\delta^{-1}}\Big(1+\sqrt{2+\varepsilon E^{-1}}\Big)\left\|\frac{1}{\sqrt{w'}}(P_D-z)u\right\|. \tag{2.8}$$

We now apply the Phragmén–Lindelöf principle as in the proof of (1.7), with the difference that in place of (2.6) we use

$$U(z) := \langle (1+r)^{-\frac{3+\delta}{2}}(P_D-z)^{-1}(1+r)^{-\frac{1+\delta}{2}}u, v\rangle,$$

to obtain (1.8) when $\theta = 1$. Then taking the adjoint gives the result for $\theta = 0$, and interpolating (that is to say, applying the Phragmén–Lindelöf principle with respect to $\theta \in \mathbb{C}$ such that $\operatorname{Re}\theta \in [0,1]$) gives the result for $\theta \in (0,1)$. □

Proof of (1.9). – We again proceed as in the proof of (1.7), but this time we replace (2.3) by

$$w(r) := 1 - \frac{\delta_V}{2(\delta_V + \delta)}(1+r)^{-\delta},$$

so that (2.4) is replaced by

$$(wV_D)'(r) \leq -\frac{\delta_V V_D(r)}{2(1+r)}.$$

Now dropping the first two terms on the left hand side of (2.2) gives

$$\left\langle \frac{\delta_V V_D(r)}{2(1+r)} u, u \right\rangle \leq \frac{2E+\varepsilon}{E} \left\| \frac{1}{\sqrt{w'}}(P_D - z)u \right\|^2,$$

or

$$\| V_D(r)^{\frac{1}{2}}(1+r)^{-\frac{1}{2}}(P_D - z)^{-1}(1+r)^{-\frac{1+\delta}{2}} \| \leq \frac{2\sqrt{2+\varepsilon E^{-1}}}{\sqrt{\delta_V}} \sqrt{\delta^{-1} + \delta_V^{-1}}.$$

We now proceed as in the proof of (1.8), applying the Phragmen–Lindelöf principle to obtain (1.9) for $\theta = 1$, and then taking the adjoint and interpolating to obtain (1.9) for $\theta \in [0, 1)$. □

3. Resolvent estimates for mildly trapping manifolds

In §3.1 we state our main resolvent estimates for mildly trapping manifolds with asymptotically cylindrical ends, under suitable abstract assumptions. In the remainder of §3 we give examples which satisfy the assumptions, and then in §4 we prove the estimates.

3.1. Resolvent estimates for asymptotically cylindrical manifolds

Let (X, g) be a smooth Riemannian manifold of dimension $d \geq 2$, with or without boundary, with the following kind of asymptotically cylindrical ends: we assume there is an open set $X_e \subset X$ such that $\partial X \cap X_e = \emptyset$, $X \setminus X_e$ is compact, and

$$X_e = (0,\infty)_r \times Y, \qquad g|_{X_e} = dr^2 + f(r)^{4/(d-1)} g_Y.$$

Here Y is a compact, not necessarily connected, manifold without boundary of dimension $d-1$, g_Y is a fixed smooth metric on Y and $f \in C^\infty([0,\infty); (0,1])$. We suppose further that there is $\delta_0 > 0$ such that

$$|(f-1)^{(k)}(r)| \leq C_k(1+r)^{-k-\delta_0} \text{ for all } k \in \mathbb{N}_0 \text{ and } r \geq 0 \tag{3.1}$$

and

$$f'(r) \geq \delta_0(1+r)^{-1}(1-f) \geq 0 \text{ for all } r \geq 0. \tag{3.2}$$

Suppose finally that $f(r) < 1$ for $r < 6$. Note that if we replace $r < 6$ by $r < r_0$ in this last condition, we can reduce to the case $r_0 = 6$ by multiplying g by a constant and rescaling r (i.e., using (1.10) with $R_1 = 0$ and $R_2 = r_0$).

We briefly discuss the assumptions (3.1) and (3.2). Note that the class of functions f such that (3.1) and (3.2) hold for a given $\delta_0 > 0$ is convex, and contains all functions of the form $f(r) = 1 - (1+r)^{-m}$ whenever $m \geq \delta_0$. Moreover, all functions f, such that f' is compactly supported and positive on the interior of the support of $(1 - f)$, obey (3.1) and (3.2) for some $\delta_0 > 0$; indeed, letting $R_f := \max \operatorname{supp}(1 - f)$, we have

$$\lim_{r \uparrow R_f} \log(1 - f(r)) = \lim_{r \uparrow R_f} \frac{d}{dr} \log(1 - f(r)) = -\infty.$$

If f' is compactly supported then the ends are cylindrical, rather than just asymptotically cylindrical.

For notational convenience let us extend r to be a continuous function on X with $-1/2 \le r < 0$ on $X \setminus \overline{X_e}$, and extend f to be constant for $r \le 0$.

Let $\Delta \le 0$ be the Laplacian on X. Let

$$P = P_h := -h^2 \Delta + V,$$

where $h \in (0, h_0]$ for some $h_0 > 0$, and:

— $V = V_h \in C^\infty(X \times (0, h_0]; \mathbb{R})$ is bounded, together with all derivatives, uniformly in $h \in (0, h_0]$.

— $V|_{X_e}$ is a function of r and h only, and has a decomposition $V|_{X_e} = V_L + hV_S$, where V_L and V_S may also depend on h, and $V_S = 0$ for $r \ge 5$ and $|V_S^{(k)}(r)| + |V_L^{(k)}(r)| \le C_k(1 + r)^{-k-\delta_0}$ for all $k \ge 0$, uniformly in h.

— $V_L'(r) \le -\delta_0(1 + r)^{-1} V_L(r) \le 0$ for all $r \ge 0$.

Note that the assumptions allow $V \equiv 0$ but not $f \equiv 1$. Such a restriction is necessary to obtain a resolvent bound which is uniform up to the spectrum, in light of the computation in (1.2), which rules out such a bound in the case $(X, g) = (\mathbb{R} \times Y, dr^2 + dS)$ and $P = -h^2 \Delta$.

Fix $E_0 > 0$. We suppose that E_0 is a "mildly trapping" energy level for P in the sense that adding a complex absorbing barrier supported on X_e gives a polynomial resolvent bound. More specifically, suppose that for some $W_K \in C^\infty(\mathbb{R}; [0, 1])$ with $W_K = 0$ near $(-\infty, 5]$ and $W_K = 1$ near $[6, \infty)$, there is $N \in \mathbb{R}$ such that

$$\|(P - iW_K(r) - E_0)^{-1}\|_{L^2(X) \to L^2(X)} =: a(h)h^{-1} \le h^{-N}, \tag{3.3}$$

for all $h \in (0, h_0]$.

We have the following weighted resolvent bound up to the spectrum.

THEOREM 3.1. – *Let (X, g), P, E_0, and $a(h)$ be as above. Fix s_1, $s_2 > 1/2$ such that $s_1 + s_2 > 2$. There are $C > 0$ and $h_1 > 0$ such that*

$$\|(1 + r)^{-s_1}(P - E_0 - i\varepsilon)^{-1}(1 + r)^{-s_2}\|_{L^2(X) \to L^2(X)} \le C(a(h) + h^{-1})h^{-1}, \tag{3.4}$$

for all $\varepsilon \in \mathbb{R} \setminus 0$ and for all $h \in (0, h_1]$.

Note that the condition on s_1 and s_2 is the same as the one in §1.2 above, see in particular (1.5) and (1.8). This is the resolvent weighting needed to have a low energy bound for scattering on the half line (and for more general Euclidean scattering problems).

To deduce Theorem 1.1 from Theorem 3.1, in Examples 1 and 2 we let X_e be the part of X where $r \ge r_1$, for any $r_1 > 0$ such that $F'(r_1) > 0$, and put $V \equiv 0$. Then, after redefining r as in the remark following (3.2), we see that g has the desired form in X_e, and it remains to check that (3.3) holds with $N \le 2$. Below in §3.2 and §3.3 we will show this for some examples which generalize Examples 1 and 2 above.

We also have an improved bound when we cut off away from the trapping in the end. To state it, let $\chi_\Pi \in C^\infty(\mathbb{R};[0,1])$ be 0 near $(-\infty,0]$ and 1 near $[1,\infty)$. Let $\Delta_Y \le 0$ be the Laplacian on (Y,g_Y), and let $\{\phi_j\}_{j=0}^\infty$ be a complete real-valued orthonormal set of its eigenfunctions, with $-\Delta_Y\phi_j = \sigma_j^2\phi_j$, where $0=\sigma_0\le\sigma_1\le\cdots$. For any $\mathcal{J}\subset\{0,\ 1,\ \ldots\}$, we denote the orthogonal projection onto modes corresponding to $\mathcal{J}$ by $\Pi_{\mathcal{J}}\colon L^2(X_e)\to L^2(X_e)$, so that

$$(\Pi_{\mathcal{J}}u)(r,y) := \sum_{j\in\mathcal{J}}\phi_j(y)\int_Y u(r,y')\phi_j(y')d\,\mathrm{vol}(y'),$$

where y and y' denote points in Y. Then $\|\Pi_{\mathcal{J}}\chi_\Pi(r)\|_{L^2(X)\to L^2(X)} = 1$, unless $\mathcal{J}$ is empty.

THEOREM 3.2. – *Fix $s>1/2$ and $c_{\mathcal{J}}>0$. Let*

$$\mathcal{J} := \{j \mid E_j := E_0 - h^2\sigma_j^2 \notin [-c_{\mathcal{J}}h, c_{\mathcal{J}}]\}.$$

Define a microlocal cutoff $\chi_{\mathcal{J}}\colon L^2(X)\to L^2(X)$ by putting

$$\chi_{\mathcal{J}}u := \begin{cases}\Big(\Pi_{\mathcal{J}}\chi_\Pi(r) + \sqrt{V_L(r)+f(r)^{-4/(d-1)}-1}\Big)u, & u\in L^2(X_e),\\ u, & u\in L^2(X\setminus X_e),\end{cases} \tag{3.5}$$

and then extending to general $u\in L^2(X)$ by linearity. There are $C>0$ and $h_1>0$ such that

$$\|(1+r)^{-s}\chi_{\mathcal{J}}(P-E_0-i\varepsilon)^{-1}(1+r)^{-s}\|_{L^2(X)\to L^2(X)} \le C(1+a(h))h^{-1}, \tag{3.6}$$

for all $\varepsilon\in\mathbb{R}\setminus 0$ and for all $h\in(0,h_1]$.

By taking the adjoint, we see that (3.6) implies

$$\|(1+r)^{-s}(P-E_0-i\varepsilon)^{-1}\chi_{\mathcal{J}}(1+r)^{-s}\|_{L^2(X)\to L^2(X)} \le C(1+a(h))h^{-1}. \tag{3.7}$$

Note that the statement is strongest when $c_{\mathcal{J}}$ is chosen very small, much smaller than E_0. We think of $\chi_{\mathcal{J}}$ as cutting off away from (or, almost, projecting away from)

$$T_{\mathcal{J}} = \{u\in L^2(X_e) \mid fu=u,\ V_Lu=0,\ \Pi_{\mathcal{J}}u=0\}\subset L^2(X).$$

Observe that the condition $E_j\in[-c_{\mathcal{J}}h,c_{\mathcal{J}}]$ corresponds, when $V_L=0$ and $f=1$, to the condition that $\rho^2\in[-c_{\mathcal{J}}h,c_{\mathcal{J}}]$, where ρ is the dual variable to r. In this sense $T_{\mathcal{J}}$ corresponds to a neighborhood of the bicharacteristics in T^*X_e along which r is constant, that is to say bicharacteristics trapped in the cylindrical ends. In this sense $\chi_{\mathcal{J}}$ cuts off away from the trapping in the cylindrical ends. The asymmetry in the interval $[-c_{\mathcal{J}}h,c_{\mathcal{J}}]$ is due to the fact that our estimates are much easier when $E_j\le -Ch$ for any $C>0$ (see in particular the sentence following (4.41) below); we do not expect this form of the interval to be optimal.

To simplify matters, in our discussion of the interpretation and context of this result we focus on the special case of the following corollary, although most of the statements could be adapted to apply to the more general case.

COROLLARY 3.3. – *Let $(X,g)=(\mathbb{R}^d,g)$ be as in Example 1. In the notation of that example, fix $\chi\in C_c^\infty(X)$ with $\operatorname{supp}\chi\subset\{(r,\theta)\in\mathbb{R}^d \mid r<R\}$, and fix $s>1/2$. Then there are $z_0>0$ and $C>0$ such that*

$$\|(1+r)^{-s}(-\Delta-z)^{-1}\chi\|_{L^2(X)\to L^2(X)} + \|\chi(-\Delta-z)^{-1}(1+r)^{-s}\|_{L^2(X)\to L^2(X)} \le C/\sqrt{\operatorname{Re} z}, \tag{3.8}$$

for all $z\in\mathbb{C}$ with $\operatorname{Re} z\ge z_0$ and $\operatorname{Im} z\ne 0$.

Note that this χ is local, in contrast to the microlocal $\chi_{\mathcal{J}}$ of Theorem 3.2. Recall that R is the radius at which the cylindrical end begins; hence χ is a cut off away from the trapping on the cylindrical end, and in this example there is no other trapping. The right hand side of (3.8) is the usual nontrapping upper bound, cf. (1.7) and the bound of Ch^{-1} in (1.4). There have been many results in asymptotically Euclidean, conic, and hyperbolic scattering proving that such nontrapping bounds hold when one cuts off away from trapping on *both* sides of the resolvent: these go back to work of Cardoso and Vodev [7], refining an earlier result of Burq [4]. Intriguingly, in (3.8) we get a nontrapping bound by applying a spatial cutoff away from trapping on only *one* side of the resolvent; to our knowledge no such result is known in asymptotically Euclidean, conic, and hyperbolic scattering, although a related weaker bound can be found in [6, 12, 19] (and note that the weaker bound is shown to be optimal in a special example in [22]). A possible interpretation is the following: unlike in any of the examples studied in [6, 19], in Example 1 the set K of bicharacteristics trapped as $t \to +\infty$ *and* $t \to -\infty$ is the same as the set $\Gamma_\pm$ of bicharacteristics trapped as $t \to +\infty$ *or* $t \to -\infty$, and one expects resolvent estimate losses due to mild trapping to be concentrated on $\Gamma_\pm$.

On the other hand, in [17] it is shown that for a "well in an island" semiclassical Schrödinger operator (in which case incidentally K does equal $\Gamma_\pm$), losses due to trapping extend beyond $\Gamma_\pm$ and cutting off on one side only is not enough to give nontrapping bounds; as discussed in that paper, this is closely related to the fact that the trapping in this case is stable (so that tunneling can produce losses away from $\Gamma_\pm$), unlike in Example 1 or in the examples in [19]. It is then natural to ask: when is cutting off a resolvent away from trapping on one side sufficient to give nontrapping bounds, and when is it necessary to cut off on both sides?

3.2. Examples with no trapping away from the ends

Let X have no boundary and let K_{E_0} be the set of bicharacteristics of P at energy E_0 which do not intersect T^*X_e. If $K_{E_0} = \emptyset$, then it is essentially well-known that

$$\|(P - iW_K(r) - E_0)^{-1}\|_{L^2(X)\to L^2(X)} \leq Ch^{-1}; \tag{3.9}$$

the proof of (3.9) follows from the proof of [23, Theorem 6.11] or that of [16, Proposition 3.2]. In the case that $|V| \leq Ch$, demanding that $K_{E_0} = \emptyset$ is equivalent to demanding that all maximally extended geodesics on X intersect X_e; specific examples are given in Example 1.

3.3. Hyperbolic and normally hyperbolic trapped sets.

If $K_{E_0} \neq \emptyset$ we cannot hope to have (3.9), but if K_{E_0} is hyperbolic or normally hyperbolic then we may have

$$\|(P - iW_K(r) - E_0)^{-1}\|_{L^2(X)\to L^2(X)} \leq C \log(h^{-1})h^{-1}. \tag{3.10}$$

In the case of a closed hyperbolic orbit, such bounds are due to Burq [5] and Christianson [12]. For hyperbolic trapped sets satisfying a pressure condition they are due to Nonnenmacher and Zworski [42], and for normally hyperbolic trapped sets to Wunsch and Zworski [54] and to Nonnenmacher and Zworski [43] (and see also [22]). Some recent surveys of the substantial wider literature concerning estimates like (3.10) can be found in [41, 57].

To deduce (3.10) from [42] or [43], note that the difference between (3.10) and [42, (2.7)] or [43, (1.18)] lies in the assumptions in the region where $W_K = 1$. But in this region $P - iW_K$ is semiclassically elliptic, so the discrepancy can be removed using a parametrix G' analogous to the one in (4.1) below, and rather than having to go through a procedure like that in §4.5 we just have $(P - iW_K(r) - E_0)G' = I + O(h^\infty)$.

Rather than discussing the general dynamical assumptions further, we now specialize to more concrete examples.

Let (X, g_H) be a conformally compact manifold of constant negative curvature. We recall that this means that the metric g_H is asymptotically hyperbolic in the sense of [35] (see also [23, §5.1]), so there is an open set X_e' and $R \in \mathbb{R}$ such that $X \setminus X_e'$ is compact and

$$X_e' = (R, \infty)_r \times Y, \qquad g_H|_{X_e'} = dr^2 + e^{2r} g_Y(e^{-r}),$$

where Y is a compact, not necessarily connected, manifold without boundary and $g_Y(x)$ is a family of metrics on Y depending smoothly on x up to $x = 0$. Such a 'normal form' of the metric was first found in [27], and it is also in [23, §5.1.1].

We modify the metric to obtain a manifold with cylindrical ends in the following way. We first observe that, denoting points in T^*X_e' by (r, y, ρ, η), where $y \in Y$, ρ is dual to r, and η is dual to y, along g_H-geodesics we have

$$\frac{d^2}{dt^2} r =: \ddot{r} = -2\partial_r(e^{-2r}|\eta|^2_{r,y}) = 4e^{-2r}|\eta|^2_{r,y}(1 + O(e^{-r})),$$

where the length $|\eta|_{r,y}$ is taken with respect to the dual metric to $g_Y(e^{-r})$. Hence, after possibly redefining R to be larger, we may suppose that $\ddot{r} \geq 2e^{-2r}|\eta|^2_{r,y}$ for $r \geq R$, and in particular that no bounded g_H-geodesics intersect $\overline{X_e'}$. Indeed, since $E_0 := \rho^2 + e^{-2r}|\eta|^2_{r,y}$ is conserved and $\dot{r} = 2\rho$, in X_e' we have

$$\ddot{r} \geq 2e^{-2r}|\eta|^2_{r,y} = 2E_0 - \dot{r}^2/2,$$

which means r is not bounded for all t.

Fix $\chi_H \in C^\infty(\mathbb{R}; [0,1])$ such that $\chi_H(r) = 1$ near $(-\infty, R]$ and $\chi_H(r) = 0$ near $[R+1, \infty)$, and fix $F \in C^\infty([R, \infty), (0, \infty))$ such that F' is compactly supported, positive on the interior of its support, and such that $F'(r) > 0$ for $r \leq R + 2$. Take g such that $g|_{X \setminus X_e'} = g_H|_{X \setminus X_e'}$, and

$$g|_{X_e'} = \chi_H(r) g_H + C_g(1 - \chi_H(r))\left(dr^2 + F(r) g_Y(0)\right).$$

We claim that if C_g is large enough, then $\ddot{r} \geq 0$ along g-geodesics in $\overline{X_e}$. Indeed,

$$\ddot{r}/2 = -\chi_H(r)\partial_r(e^{-2r}|\eta|^2_{r,y}) + C_g(1 - \chi_H(r))F'(r)|\eta|_0^2 - \chi_H'(r)(e^{-2r}|\eta|^2_{r,y} - C_g F(r)|\eta|_0^2),$$

so it is enough to take C_g large enough that on $T^* \operatorname{supp} \chi_H'(r)$ we have

$$e^{-2r}|\eta|^2_{r,y} \leq C_g F(r)|\eta|_0^2.$$

Now we may take X_e to be the part of X_e' in which $r > R + 1$, and, after redefining r by (1.10), we see that it remains only to check (3.3).

We take $W_K \in C^\infty(\mathbb{R}; [0,1])$ which is 1 near $[R+2, \infty)$ and 0 near $(-\infty, R+1]$, and suppose $|V| \leq Ch$ and $E_0 = 1$. Let K denote the set of trapped unit speed geodesics of (X, g_H), regarded as a subset of T^*X. We see that K is also the set of the bicharacteristics of P at energy E_0 which do not intersect T^*X_e, and that $g_H = g$ near the projection of K onto X.

Let d_K be the Hausdorff dimension of K. If $d_K < d$, then the assumptions of [42] are satisfied, and (3.10) holds.

If $d = 2$ and $V \equiv 0$, then we can dispense with the requirement that $d_K < d$ thanks to a recent result of Bourgain and Dyatlov [3, Theorem 2] (this is the case presented in Example 2 above). To do this we use the fact (see [5, Lemma 4.7] or e.g., [23, Proof of (6.3.10)]) that [3, (1.1)] implies

$$\|\chi(-h^2\Delta_0 - E_0 - i0)^{-1}\chi\|_{L^2(X)\to L^2(X)} \leq C\log(h^{-1})h^{-1}$$

for any $\chi \in C_c^\infty(X)$. Then the gluing result of [18, Theorem 2.1] together with the semiclassically outgoing property of $(-h^2\Delta_0 - E_0 - i0)^{-1}$ (established by Vasy in [50] and see also [23, Theorem 5.34]) implies (3.10). In the interest of brevity we do not discuss this further here.

3.4. Warped products with embedded eigenvalues

Let $X := \mathbb{R}\times Y$ and $g := dr^2 + f(r)^{4/(d-1)}g_Y$ for some $f \in C^\infty(\mathbb{R};(0,1])$ which is 1 on $\mathbb{R}\setminus(-R,R)$ for some $R > 0$ and has a nondegenerate minimum as its only critical point in $(-R,R)$: see Figure 3.1.

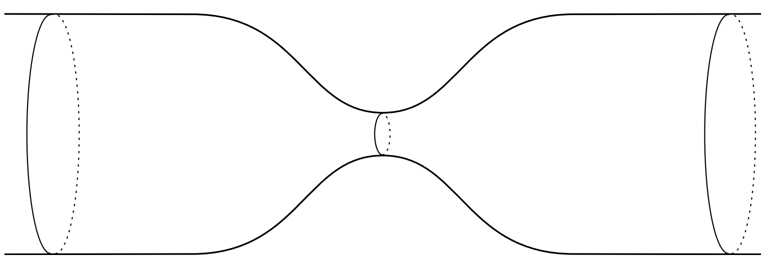

FIGURE 3.1. An hourglass shaped surface of revolution.

Suppose $V = h^2V_W$, with $V_W = V_W(r) \in C_c^\infty((-R,R))$. Then the part of the trapped set away from the cylindrical ends is normally hyperbolic and we have (3.10) (see [23, (6.3.10)], and see also [14, 13] for the case of a degenerate minumum where incidentally we also have (3.3)). Consequently, by Theorem 3.1, there is $z_0 > 0$ such that for all $s_1,\ s_2 > 1/2$ such that $s_1 + s_2 > 2$, there is $C > 0$ such that

$$\|(1+|r|)^{-s_1}(-\Delta + V_W - z)^{-1}(1+|r|)^{-s_2}\|_{L^2(X)\to L^2(X)} \leq C,$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq z_0$ and $\operatorname{Im} z \neq 0$. In particular the spectrum of $-\Delta + V_W$ is absolutely continuous on (z_0,∞).

But if f and V_W are suitably chosen, then $\Delta + V_W$ has an eigenvalue embedded in the spectrum in $[0, z_0]$. Indeed, we have

$$\Delta = f(r)^{-1}\left(\sum_{j=0}^{\infty}\left(\partial_r^2 - f''(r)f(r)^{-1} - \sigma_j^2 f(r)^{-4/(d-1)}\right)\phi\otimes\phi\right)f(r),$$

where $\{\phi_j\}_{j=0}^{\infty}$ is a complete set of real-valued orthonormal eigenfunctions of the Laplacian on Y and $-\Delta_Y \phi_j = \sigma_j^2 \phi_j$. For $J \in \mathbb{N}_0$, consider the effective potential

$$V_J(r) := f''(r) f(r)^{-1} + \sigma_J^2 (f(r)^{-4/(d-1)} - 1) + V_W(r).$$

Then $D_r^2 + V_J$ has an eigenvalue as long as $\int V_J(r) dr \leq 0$ by [48, Theorem XIII.110], and this corresponds to an embedded eigenvalue for $-\Delta + V_W$ as long as it is positive, for which it suffices to have $\min V_J(r) > -\sigma_J^2$. For example, we may take f such that $\int (f(r)^{-4/(d-1)} - 1) \leq 1/4$ and $V_W \in C_c^{\infty}((-R, R); [-\sigma_J^2/2, 0])$ such that $V_W(r) = -\sigma_J^2/2$ on $[-R/2, R/2]$, and then J sufficiently large.

By elaborating the above constuction one can also find examples with any finite number of embedded eigenvalues.

It is not clear whether there are examples of manifolds with cylindrical ends such that $-\Delta$ has a finite but nonzero number of eigenvalues. For all known examples where eigenvalues occur, the existence of infinitely many eigenvalues is either also established [11, 45] or at the least it is not ruled out [33]. On the other hand 0 is always a resonance of $-\Delta$ on a manifold with cylindrical ends, with the constant functions as resonant states, unless there is a boundary condition somewhere that eliminates them.

4. Proof of Theorems 3.1 and 3.2

4.1. Outline of proof

The idea of the proofs is to define a parametrix for $P - z$ by

$$G := \chi_K(r-1)(P - iW_K(r) - z)^{-1} \chi_K(r) + \chi_e(r+1)(P_e - z)^{-1} \chi_e(r), \tag{4.1}$$

where $\chi_e, \chi_K \in C^{\infty}(\mathbb{R})$ obey $\chi_e + \chi_K = 1$, $\operatorname{supp} \chi_e \subset (3, \infty)$, and $\operatorname{supp} \chi_K \subset (-\infty, 4)$, and P_e is a suitably chosen differential operator such that $P_e = P$ on the part of X where $r > 2$. Then

$$(P-z)G = I + [h^2 D_r^2, \chi_K(r-1)](P - iW_K(r) - z)^{-1} \chi_K(r) + [h^2 D_r^2, \chi_e(r+1)](P_e - z)^{-1} \chi_e(r)$$

and we will construct an inverse for $(P - z)$ by removing this remainder using a Neumann series; although the remainder above need not be small, we will see that powers of it are. We call the part of X where $r \in (2, 5)$ the *resolvent gluing region*, because the functions in the range of the remainder are supported in that region. To prove that powers of the remainder are small, we will need to know that:

1. The resolvents of $P - iW_K(r)$ and P_e obey estimates analogous to (3.4) and (3.6). This is the case for $P - iW_K(r)$ thanks to the assumption (3.3), and we will prove it for a suitable choice of P_e in §4.3 and §4.4.
2. The resolvents of $P - iW_K(r)$ and P_e obey improved estimates when multiplied by cutoffs with suitable support properties in the resolvent gluing region, corresponding to a (special case of a) *semiclassically outgoing condition* so that we are able to remove the remainders. The needed estimates are proved in [18] for $P - iW_K(r)$ and in §4.3 and §4.4 for P_e.

We combine these estimates to prove Theorems 3.1 and 3.2 in §4.5. There we follow a procedure analogous to that in [18], but with some finer analysis of remainders to remove the losses due to trapping in the cylindrical end (see also [16, §3] for another, in some ways related, variation on this resolvent gluing procedure).

4.2. Model operators for X_e

On X_e, Δ can be written as a direct sum of one-dimensional Schrödinger operators:

$$\Delta|_{X_e} = f(r)^{-1}\left(\sum_{j=0}^{\infty}\left(\partial_r^2 - f''(r)f(r)^{-1} - \sigma_j^2 f(r)^{-4/(d-1)}\right)\phi\otimes\phi\right)f(r),$$

where $\{\phi_j\}_{j=0}^{\infty}$ is a complete set of real-valued orthonormal eigenfunctions of the Laplacian on Y and $-\Delta_Y\phi_j = \sigma_j^2\phi_j$. We will introduce model operators P_j obeying

$$P_j|_{[2,\infty)} = -h^2\partial_r^2 + V_j(r), \qquad V_j(r) := V(r) + h^2 f''(r)f(r)^{-1} + h^2\sigma_j^2(f(r)^{-4/(d-1)} - 1), \tag{4.2}$$

and we will be studying them near the energy levels

$$E_j := E_0 - h^2\sigma_j^2.$$

We will study two ranges of j separately, and the model operators P_j will act on different spaces depending on j. These two ranges correspond to a different behavior in the resolvent gluing region, which is the part of X where $r \in (2,5)$ (see §4.1). To define the ranges, fix $E_* \in \mathbb{R}$, independent of h, such that

$$0 < E_* \le c_{\mathcal{J}},$$

where $c_{\mathcal{J}}$ is as in the statement of Theorem 3.2, and

$$E_j \le E_* \Longrightarrow h^2\sigma_j^2 f(5)^{-4/(d-1)} \ge E_0; \tag{4.3}$$

note that the conditions are compatible because $E_j = 0$ when $E_0 = h^2\sigma_j^2$ and $f(5) < 1$.

The first range we consider is $E_j \le E_*$; in this range the set where $r < 5$ is classically forbidden because $V_j > E_j$, and we control remainders in the gluing region using Agmon estimates, taking care to prove that our estimates are uniform as $j \to \infty$ (although the effective potentials V_j become unbounded as $j \to \infty$, they are nonnegative, so the relevant estimates actually get better in this limit). The second range is $E_j \ge E_*$; in this range the set where $r < 5$ is not classically forbidden, but the energy levels E_j are bounded below by a positive constant and the effective potentials V_j are repulsive, so nontrapping propagation of singularities estimates hold, which we can use to control the remainders in the gluing region (once again we take care to prove that the estimates are uniform in j).

For the first range of j we define the operators P_j to act on $L^2(\mathbb{R}_+)$, with a Dirichlet boundary condition at 0, in order to be able to use Theorem 1.3 (the Dirichlet boundary condition makes it easier to analyze the behavior of the resolvent when $|E_j|$ is small). For the second range of j it is more convenient to work over $\mathbb{R}$ than $\mathbb{R}_+$, in order to avoid reflection phenomena when studying propagation of singularities.

4.3. Analysis when $E_j \le E_*$

In §4.3 all function norms and inner products are in $L^2(\mathbb{R}_+)$, and operator norms are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, unless otherwise specified.

For this range of j, we put

$$P_j := h^2 D_r^2 + V_j(r), \tag{4.4}$$

regarded as a self-adjoint operator on $L^2(\mathbb{R}_+)$ with a Dirichlet boundary condition at $r = 0$.

We first prove resolvent estimates for P_j analogous to (3.4) and (3.6).

PROPOSITION 4.1. – *Fix $s_1,\ s_2,\ s > 1/2$ such that $s_1 + s_2 > 2$. Then*

$$\|(1+r)^{-s_1}(P_j - E_j - i\varepsilon)^{-1}(1+r)^{-s_2}\| \le Ch^{-2} \tag{4.5}$$

and

$$\|(1+r)^{-s}\chi(r)(P_j - E_j - i\varepsilon)^{-1}(1+r)^{-s}\| + \|(1+r)^{-s}(P_j - E_j - i\varepsilon)^{-1}\chi(r)(1+r)^{-s}\| \le Ch^{-1} \tag{4.6}$$

for all $\varepsilon \in \mathbb{R}\setminus 0,\ j \in \mathbb{N}$ such that $E_j \le E_*$, where

$$\chi(r) = \sqrt{V_L(r) + f(r)^{-4/(d-1)} - 1}.$$

Proof. – The idea of the proof is to apply Theorem 1.3; more precisely (4.5) corresponds to (1.8) (see also (1.5)), and (4.6) corresponds to (1.9) (see also (1.6)).

Before beginning the proof proper, by way of outline let us briefly discuss the terms in V_j, and explain how they each do or do not satisfy (1.3). The term $h^2\sigma_j^2(f(r)^{-4/(d-1)} - 1)$ does satisfy it thanks to (3.2) and (4.3), and moreover those bounds and $f(r) < 1$ for $r < 6$ imply that the term is nontrivial when $r < 6$. The term V_L satisfies it, and we think of it as being harmless. The term V_S does not satisfy it, but we will show that its effect is compensated by that of the $h^2\sigma_j^2(f(r)^{-4/(d-1)}-1)$ term. The most difficult term to treat is the $h^2 f''(r)f(r)^{-1}$ term. This term may prevent $h^{-2}V_j$ from satisfying (1.3), but we will show that thanks to (4.3) we can treat it as a small perturbation.

More precisely, let

$$V_M(r) := V_j(r) - h^2 f''(r)f(r)^{-1}$$

and observe that for h sufficiently small V_M obeys (1.3) for some $\delta_V > 0$, since V_L and $f^{-4/(d-1)} - 1$ obey it and $|V_S| + |V_S'| \le C(f^{-4/(d-1)} - 1)$ thanks to (4.3). Indeed, to see that $f^{-4/(d-1)} - 1$ obeys it we write, using $\alpha := 4/(d-1)$ and (3.2),

$$-(f(r)^{-\alpha} - 1)' = \alpha f'(r)f(r)^{-\alpha-1} \ge \alpha\delta_0 \frac{f(r)^{-\alpha-1} - f(r)^{-\alpha}}{1+r} \ge \frac{f(r)^{-\alpha} - 1}{C(1+r)},$$

where we also used the fact that if $a < b$ then

$$C^{-1}(1-f) \le f^a - f^b \le C(1-f). \tag{4.7}$$

Hence by (1.8) with $V_D = h^{-2}V_M$, we have

$$\|(1+r)^{-s_1}(h^2D_r^2 + V_M - E_j - i\varepsilon)^{-1}(1+r)^{-s_2}\| \le Ch^{-2}. \tag{4.8}$$

Note that by the resolvent identity

(4.9)
$$(1+r)^{-s_1}(P_j - E_j - i\varepsilon)^{-1}(1+r)^{-s_2} = (1+r)^{-s_1}(h^2D_r^2 + V_M - E_j - i\varepsilon)^{-1}(1+r)^{-s_2} \times \sum_{k=0}^{\infty}\left[-(1+r)^{s_2}h^2 f''(r)f(r)^{-1}(h^2D_r^2 + V_M - E_j - i\varepsilon)^{-1}(1+r)^{-s_2}\right]^k,$$

the proof of (4.5) is reduced to the proof of

$$\|(1+r)^{s_2}h^2 f''(r)f(r)^{-1}(h^2D_r^2 + V_M - E_j - i\varepsilon)^{-1}(1+r)^{-s_2}\| \le 1/2. \tag{4.10}$$

But by (1.9), with $\theta = 1$ and $V_D = h^{-2}V_M \ge h^{-2}(f^{-4/(d-1)} - 1)/C$ (again using (4.3)), we have

$$\|(f(r)^{-4/(d-1)} - 1)^{\frac{1}{2}}(1+r)^{-\frac{1}{2}}(h^2D_r^2 + V_M - E_j - i\varepsilon)^{-1}(1+r)^{-s_2}\| \le Ch^{-1}$$

and interpolating this with (4.8) gives

$$\|(f(r)^{-4/(d-1)} - 1)^{\frac{1}{4}}(1+r)^{-\frac{s_1}{2}-\frac{1}{4}}(h^2D_r^2 + V_M - E_j - i\varepsilon)^{-1}(1+r)^{-s_2}\| \le Ch^{-3/2}.$$

Hence to prove (4.10), and consequently also (4.5), it is enough to show that

$$(1+r)^{s_2}|f''(r)| \le C(f(r)^{-4/(d-1)} - 1)^{\frac{1}{4}}(1+r)^{-\frac{s_1}{2}-\frac{1}{4}}. \tag{4.11}$$

To prove (4.11) we will use the fact that any bounded $\varphi \in C^2([r,\infty);[0,\infty))$ satisfies

$$|\varphi'(r)|^2 \le 2\sup\varphi \sup|\varphi''|, \tag{4.12}$$

where the suprema are taken over $[r,\infty)$. Indeed, by Taylor's theorem, for every $t \ge 0$ there is $t_0 \in [r, r+t]$ such that

$$t|\varphi'(r)| = |\varphi(r+t) - \varphi(r) - t^2\varphi''(t_0)/2| \le \sup\varphi + t^2\sup|\varphi''|/2,$$

and taking $t = |\varphi'(r)|/\sup|\varphi''|$ gives (4.12). Applying (4.12) once with $\varphi = f'$ and once with $\varphi = 1 - f$ gives

$$\begin{aligned}|f''(r)|^4 &\le 4\sup|f'|^2\sup|f'''|^2 \le 8\sup(1-f)\sup|f''|\sup|f'''|^2 \\ &= 8(1-f(r))\sup|f''|\sup|f'''|^2,\end{aligned}$$

where the suprema are still all taken over $[r,\infty)$. Applying (3.1) gives

$$|f''(r)| \le C(1-f(r))^{\frac{1}{4}}(1+r)^{-2-\frac{3\delta_0}{4}}.$$

By (4.7) this implies (4.11) as long as $s_1 + 2s_2 \le (7 + 3\delta_0)/2$, which we may suppose without loss of generality. This completes the proof of (4.5).

The proof of (4.6) proceeds along similar lines. Applying (4.9) with $s_1 = s_2 = s$ allows us to reduce the proof of the bound on the first term in (4.6) to the proof of

$$\|(1+r)^{-s}\chi(r)(h^2D_r^2 + V_M - E_j - i\varepsilon)^{-1}(1+r)^{-s}\| \le Ch^{-1}. \tag{4.13}$$

But (4.13) follows from (1.9) with $\theta = 1$ and $V_D = h^{-2}V_M \ge h^{-2}(V_L + f^{-4/(d-1)} - 1)/C = h^{-2}\chi^2/C$. The bound on the second term of (4.6) follows from the bound on the first term after taking the adjoint. □

We will also need the following Agmon estimates:

PROPOSITION 4.2. – *Let $R \in (0,5]$, $\chi_- \in C_c^\infty((0,R))$, $\chi_+ \in C_c^\infty((R,\infty))$, and $s > 1/2$. Then*

$$\|\chi_-(P_j - E_j - i\varepsilon)^{-1}(1+r)^{-s}\|_{L^2(\mathbb{R}_+)\to H_h^1(\mathbb{R}_+)} + \|(1+r)^{-s}(P_j - E_j - i\varepsilon)^{-1}\chi_-\| \le C, \tag{4.14}$$

$$\|\chi_-(P_j - E_j - i\varepsilon)^{-1}\chi_+\| \le e^{-1/(Ch)} \tag{4.15}$$

for all $\varepsilon \in \mathbb{R}\setminus 0$, and $j \in \mathbb{N}$ such that $E_j \le E_*$.

Recall that the norms without subscripts are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ here, and that χ_- is supported in the classically forbidden region for $P_j - E_j$.

Proof. – These are similar to the usual Agmon estimates as in [56, §7.1] but we keep track of the j dependence.

Let $v \in L^2(\mathbb{R}_+)$, and let $u := (P_j - E_j - i\varepsilon)^{-1}(1+r)^{-s}v$. Fix $\varphi_0 \in C_c^\infty((0,R);[0,1])$ which is identically 1 on a neighborhood I of $\operatorname{supp}\chi_-$, and let $\varphi(r) := m\varphi_0(r)$, for a constant m to be chosen later. Then define

$$\begin{aligned} P_\varphi &:= e^{\varphi/h}(P_j - E_j - i\varepsilon)e^{-\varphi/h} \\ &= h^2 D_r^2 + 2i\varphi' h D_r + V_j - \varphi'^2 + h\varphi'' - E_j - i\varepsilon. \end{aligned}$$

Put $w := \chi_0 e^{\varphi/h} u$, where $\chi_0 \in C_c^\infty((0,R))$ is 1 near $\operatorname{supp}\varphi$. Using $\operatorname{Re}\langle 2h\varphi' w', w\rangle = -h\langle \varphi'' w, w\rangle$, write

$$\operatorname{Re}\langle P_\varphi w, w\rangle = \|hw'\|^2 + \langle (V_j - \varphi'^2 - E_j)w, w\rangle.$$

We now observe that, using (4.3) and the fact that $1 - f(r)^{-4/(d-1)} > 1 - f(5)^{-4/(d-1)} > 0$ for $r \in (0,5)$, we can choose $m > 0$ small enough, independent of h and j, such that there is $c_0 > 0$ independent of h and j for which $V_j - \varphi'^2 - E_j > c_0$ on $\operatorname{supp} w$ for h small enough. This implies

$$\|w\| \le C\|P_\varphi w\| \le C\|e^{\varphi/h}\chi_0 v\| + C\|[P,\chi_0]u\|,$$

where we used $\varphi\chi_0' = 0$ to deduce $[P_\varphi, \chi_0]e^{\varphi/h}u = [P,\chi_0]u$. We use an elliptic estimate to bound the commutator term: for $\chi_1 \in C_c^\infty((0,R))$ we have, using $V_j - E_j \ge V_0 - E_0 \ge -C$,

$$\begin{aligned} C\|\chi_1 v\|\|\chi_1 u\| &\ge \operatorname{Re}\langle (1+r)^{-s}v, \chi_1^2 u\rangle = \operatorname{Re}\langle (P_j - E_j)u, \chi_1^2 u\rangle \\ &\ge \|\chi_1 h u'\|^2 - Ch\|\chi_1 h u' u\|_{L^1(\mathbb{R}_+)} - C\|\chi_1 u\|^2, \end{aligned} \tag{4.16}$$

from which it follows that, provided $\chi_2 = 1$ near $\operatorname{supp}\chi_0$,

$$\|[P,\chi_0]u\| \le Ch\|\chi_2 u\| + Ch\|\chi_2 v\| \le Ch^{-1}\|v\|,$$

where we used (4.5). Consequently

$$\int_I |u|^2 = e^{-2m/h}\int_I |w|^2 \le Ce^{-2m/h}\left(\|e^{\varphi/h}\chi_0 v\|^2 + h^{-2}\|v\|^2\right) \le C\|v\|^2, \tag{4.17}$$

where we used $\varphi \le m$.

To estimate u' we apply (4.16) with $\chi_1 \in C_c^\infty(I)$, giving

$$\|\chi_1 h u'\|^2 \le C\left(\int_I |u|^2 dr + \|\chi_1 h v\|^2\right),$$

which implies the bound on the first term of (4.14). The bound on the second term follows from taking the adjoint, and (4.15) follows from the fact that if $\operatorname{supp} v \subset (R, \infty)$, then $\chi_0 v = 0$ and we can improve (4.17) to

$$\int_I |u|^2 = e^{-2m/h} \int_I |w|^2 \leq C e^{-2m/h} h^{-2} \|v\|^2. \qquad \square$$

4.4. Analysis when $E_j > E_*$

In §4.4 all function norms and inner products are in $L^2(\mathbb{R})$, and operator norms are $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, unless otherwise specified.

For this range of j the Agmon estimate (4.15) must be replaced by a propagation of singularities estimate. It is convenient to introduce a complex absorbing barrier and to work over $\mathbb{R}$: let $W_e \in C^\infty(\mathbb{R}; [0,1])$ be 1 near $(-\infty, 1]$ and 0 near $[2, \infty)$, and let

$$V_{j,0} := \chi_0 V_j,$$

where $\chi_0 \in C^\infty(\mathbb{R}; [0,1])$ is 0 near $(-\infty, 0]$ and 1 near $[1, \infty)$. We now put

$$P_j := h^2 D_r^2 + V_{j,0}(r) - i W_e(r),$$

regarded as an unbounded operator on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. We will prove

Proposition 4.3. – *For any $s > 1/2$ we have*

$$\|(1 + r_+)^{-s} (P_j - E_j - i\varepsilon)^{-1} (1 + r_+)^{-s}\| \leq C h^{-1}, \tag{4.18}$$

where $r_+ := \max\{0, r\}$. For any $\chi_- \in C_c^\infty((0,3))$, $\chi_+ \in C_c^\infty((3,\infty))$, $\psi \in C_c^\infty((0,\infty))$, we have

$$\|\chi_-(r)(P_j - E_j - i\varepsilon)^{-1} \chi_+(r) \psi(hD_r)\| = O(h^\infty). \tag{4.19}$$

Both (4.18) *and* (4.19) *hold uniformly for all $\varepsilon > 0$, and for all $j \in \mathbb{N}_0$ such that $E_j > E_*$.*

Note that since E_j is bounded from below away from 0, we can think of (4.18) as the analogue of (1.7) in this setting; we do not need a weight for $r < 0$ because the $-iW_e$ term makes the operator $P_j - E_j - i\varepsilon$ semiclassically elliptic there. It is also similar to the usual nontrapping resolvent estimate as in [51] and in other papers cited therein, but we need an estimate which is uniform in j.

The propagation of singularities estimate (4.19) is a microlocalized version of (4.18). The improved bound is due to the fact that solutions to the classical equations of motion $\dot r = 2\rho$, $\dot\rho = -V_j'(r)$ with $r(0) > 3$ and $\rho(0) > 0$ cannot have $r(t) < 3$ for any $t > 0$.

Proof of (4.18). – We prove (4.18) using a microlocal positive commutator argument, rather than (as is probably possible) integration by parts arguments as in the proof of (1.7). We do this because the proof of (4.19) follows along very similar lines, and the latter estimate does not seem to be provable by integration by parts arguments. The idea is to construct a microlocal commutant, based on the $w(r)\partial_r$ of the proof of (1.7), but which is nonnegative. This will be obtained as the quantization of an escape function, defined in (4.26) below.

As in [51] we will use the semiclassical scattering calculus, and we begin by recalling its relevant properties. We use (r, ρ) to denote points in $T^*\mathbb{R}$, and for $l,\ m \in \mathbb{R}$ we define the symbol class S_l^m to be the set of $a \in C^\infty(T^*\mathbb{R})$ such that, for any $n_1,\ n_2 \in \mathbb{N}_0$ there is C_{n_1,n_2} such that

$$|\partial_r^{n_1}\partial_\rho^{n_2} a(r,\rho)| \le C_{n_1,n_2}(1+|r|)^{l-n_1}(1+|\rho|)^{m-n_2}, \tag{4.20}$$

for all $(r,\rho) \in T^*\mathbb{R}$. We also write $S_l^\infty := \bigcup_m S_l^m$, $S_l^{-\infty} := \bigcap_m S_l^m$, and similarly for S_∞^m and $S_{-\infty}^m$. Below we will consider symbols depending on h and j, and the constants C_{n_1,n_2} in (4.20) will always be uniform with respect to those parameters. For such a, we denote the semiclassical quantization by $\mathrm{Op}_h(a)$, which we define by

$$\mathrm{Op}_h(a)u := \frac{1}{2\pi h}\iint e^{i(r-r')\rho/h} a(r,\rho)u(r')dr'd\rho. \tag{4.21}$$

When a symbol is denoted by a lowercase letter (with possible subscripts and superscripts), we will denote its quantization by the corresponding uppercase letter (with the same subscripts and superscripts, if any).

We recall the composition and adjoint formulas. If $a \in S_{l_1}^{m_1}$ and $b \in S_{l_2}^{m_2}$, then there is $a \# b \in S_{l_1+l_2}^{m_1+m_2}$ such that

$$AB = \mathrm{Op}_h(a \# b),$$

and, for any $N \in \mathbb{N}$,

$$\begin{aligned} a \# b(r,\rho) &= e^{-ih\partial_{r'}\partial_{\rho'}}\left(a(r,\rho')b(r',\rho)\right)|_{(r,\rho)=(r',\rho')} \\ &= \sum_{k=0}^{N-1}\frac{(-ih)^k}{k!}\partial_\rho^k a(r,\rho)\partial_r^k b(r,\rho) + h^N z_N(r,\rho), \end{aligned} \tag{4.22}$$

where $z_N \in S_{l_1+l_2-N}^{m_1+m_2-N}$ is given by

$$z_N(r,\rho) := \frac{(-i)^N}{(N-1)!}\int_0^1 (1-t)^{N-1} e^{-ith\partial_{r'}\partial_{\rho'}}\left(\partial_{\rho'}^N a(r,\rho')\partial_{r'}^N b(r',\rho)\right)|_{(r,\rho)=(r',\rho')}dt. \tag{4.23}$$

Indeed, [56, Theorem 4.14] gives the formula for Schwartz symbols, and [56, Theorems 4.13 and 4.18] give it for a larger class of symbols than the ones we consider, but with weaker bounds on z_N. The statement that $z_N \in S_{l_1+l_2-N}^{m_1+m_2-N}$ follows from applying [56, Theorem 4.17] to (4.23). See also [23, Proposition E.8], [44, (3) and (9)], [53], and [30, §18.5] for similar expansions, and [29] for a much more general version.

Similarly, if $a \in S_l^m$ there is $a^* \in S_l^m$ such that the formal adjoint of A is given by

$$A^* = \mathrm{Op}_h(a^*)$$

and, for any $N \in \mathbb{N}$,

$$a^*(r,\rho) = e^{-ih\partial_r\partial_\rho}\bar{a}(r,\rho) = \sum_{k=0}^{N-1}\frac{(-ih)^k}{k!}\partial_r^k\partial_\rho^k\bar{a}(r,\rho) + h^N z_N(r,\rho), \tag{4.24}$$

where this time $z_N \in S_{l-N}^{m-N}$ is given by

$$z_N(r,\rho) := \frac{(-i)^N}{(N-1)!}\int_0^1 (1-t)^{N-1} e^{-ith\partial_r\partial_\rho}\partial_r^N\partial_\rho^N\bar{a}(r,\rho)dt.$$

Let

$$p_j := \rho^2 + V_{j,0}(r) - iW_e(r)$$

be the semiclassical symbol of P_j (in the sense that $p_j \in S^2_0$ and $P_j = \mathrm{Op}_h(p_j)$), let

$$R_j := \inf\{r > 0 \mid \text{both } V_j(r) = V_{j,0}(r) \text{ and } V_{j,0}(r) \le E_*/2\},$$

so that $R_0 \le R_1 \le \cdots$, and let

$$F_j := \{(r,\rho) \mid r \ge 1 \text{ and } \rho^2 \le 2E_0\} \setminus \{(r,\rho) \mid R_j < r \text{ and } \rho^2 < E_*/3\}.$$

Note that each F_j is a closed neighborhood of the energy surface $p_j = E_j$, and they have been chosen such that they form a nested sequence $F_0 \subset F_1 \subset \cdots$. Moreover, since we only consider j such that $E_j > E_*$, all of the F_j agree outside of a compact set: see Figure 4.1.

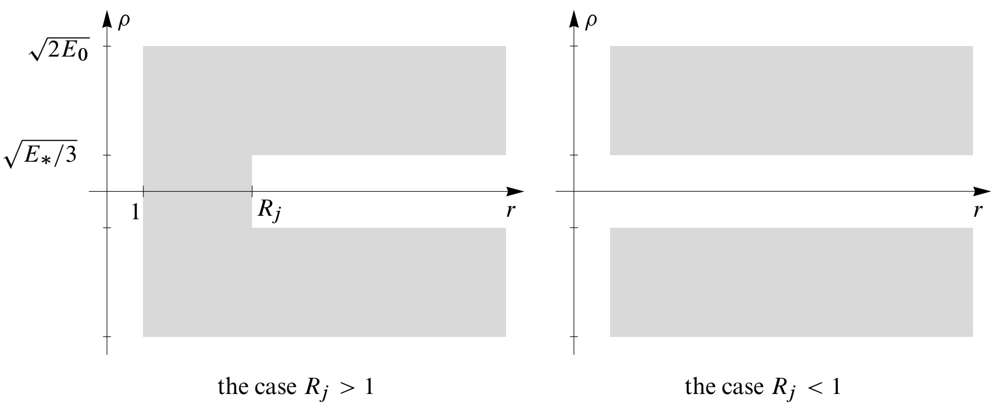

FIGURE 4.1. The shaded regions are the sets F_j. They are closed nested neighborhoods of the energy surfaces $p_j = E_j$ which agree outside of a compact set.

Observe that we have $|p_j - E_j - i\varepsilon| \ge c(1+\rho^2)$ on $T^*\mathbb{R} \setminus F_j$, for some $c > 0$, which implies the following elliptic estimate: for any $a \in S^m_l$, $a' \in S^{m-2}_l$ satisfying $\operatorname{supp} a \cap F_j = \emptyset$ and $|a'(r,\rho)| \ge (1+|r|)^l(1+|\rho|)^{m-2}$ for $(r,\rho) \in \operatorname{supp} a$, and for any $N \in \mathbb{R}$, we have

$$\|Au\| \le C\|A'(P_j - E_j - i\varepsilon)u\| + h^N\|Z_N u\|, \tag{4.25}$$

for some $z_N \in S^{m-N}_{l-N}$. This follows from (4.22) by the usual iterative elliptic parametrix construction as in [23, Theorem E.32].

To handle F_j, we define an escape function (based on the usual $-r\rho$ but modified to be nonnegative near F_j and more slowly growing) as follows. For $\delta \in (0, 1/4)$, take $\tilde{q}_\delta \in C^\infty(\mathbb{R})$ with $\tilde{q}_\delta(x) = x^\delta$ for $x \ge 2$, $\tilde{q}_\delta(x) = |x|^{-\delta}$ for $x \le -2$, and $\tilde{q}'_\delta(x) > 0$ for $|x| < 2$, and put

$$q(r,\rho) := \tilde{q}_\delta(-r\rho)\chi_q(r,\rho), \tag{4.26}$$

where $\chi_q \in S^{-\infty}_0$ is real valued, is 1 near all of the F_j, and vanishes in a neighborhood of

$$\{(r,\rho) \mid r \notin (-1, 1 + \max_j R_j) \text{ and } \rho = 0\},$$

whose boundary consists of two line segments and four half lines as in Figure 4.2.

Then $q \in S^{-\infty}_\delta$, and near F_j we have

$$\{\operatorname{Re} p_j, q^2\} = 2(-2\rho^2 + rV_j'(r))\tilde{q}'_\delta(-r\rho)\tilde{q}_\delta(-r\rho) \le -cr^{-1-2\delta}, \tag{4.27}$$

for some $c > 0$ (here we used $V_j \ge E_*/2 \Longrightarrow rV_j' \le -1/C$).

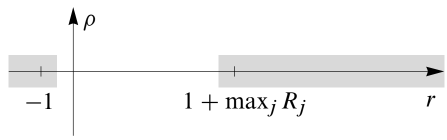

FIGURE 4.2. The kind of neighborhood where χ_q must vanish.

Consequently, there are real valued symbols $b \in S^{-\infty}_{-\frac{1}{2}+\delta}$ and $a_0 \in S^{-\infty}_{-1+2\delta}$ such that

$$b^2 = \{q^2, \operatorname{Re} p_j\} + a_0, \tag{4.28}$$

and such that $\operatorname{supp} a_0 \cap F_j = \emptyset$ and $b \geq c r^{-\frac{1}{2}-\delta} > 0$ near F_j; for example we can take $b := \{q^2, \operatorname{Re} p_j\}^{1/2}\chi_b$ for some $\chi_b \in S^{-\infty}_0$ with $\chi_b = 1$ near F_j and supported in the set where (4.27) holds. Note that q depends on δ, and b and a_0 depend on δ and j, although our notation does not reflect this.

Using (4.28), (4.22), and (4.24), we can write

$$B^*B = \frac{i}{h}[Q^*Q, \operatorname{Re} P_j] + A_0 + hA_1,$$

for some $a_1 \in S^{-\infty}_{-2+2\delta}$, giving

$$\|Bu\|^2 = \frac{i}{h}\langle [Q^*Q, \operatorname{Re} P_j]u, u\rangle + \langle A_0 u, u\rangle + h\langle A_1 u, u\rangle.$$

Combining this with (4.25) and the similar elliptic estimate

$$\|B'u\| \leq C\|Bu\| + h^N\|Z_N u\|, \tag{4.29}$$

which holds for all $b' \in S^{-\infty}_{-\frac{1}{2}-\delta}$ which is supported in a small enough neighborhood of F_j and for suitable $z_N \in S^{-\infty}_{-\frac{1}{2}-\delta-N}$, we have (since $\delta < 1/4$),

$$\|(1+r_+)^{-\frac{1}{2}-\delta}u\|^2 \leq C\frac{i}{h}\langle [Q^*Q, \operatorname{Re} P_j]u, u\rangle + C\|(P_j - E_j - i\varepsilon)u\|^2.$$

Next

$$i\langle [Q^*Q, \operatorname{Re} P_j]u, u\rangle = -2\operatorname{Im}\langle Q(P_j - E_j - i\varepsilon)u, Qu\rangle - 2\operatorname{Re}\langle Q(W_e(r)+\varepsilon)u, Qu\rangle,$$

giving

$$\|(1+r_+)^{-\frac{1}{2}-\delta}u\|^2 \leq \frac{C}{h^2}\|(1+r_+)^{\frac{1}{2}+3\delta}(P_j - E_j - i\varepsilon)u\|^2 - \frac{C}{h}\operatorname{Re}\langle Q(W_e(r)+\varepsilon)u, Qu\rangle.$$

But

$$-\operatorname{Re}\langle Q(W_e(r)+\varepsilon)u, Qu\rangle \leq |\operatorname{Re}\langle Q^*[Q, W_e(r)]u, u\rangle|,$$

thanks to $W_e + \varepsilon \geq 0$, and by (4.22) and (4.24) we have $\operatorname{Re} Q^*[Q, W_e(r)] = h^2 a_2$ for some $a_2 \in S^{-\infty}_{-\infty}$, giving

$$|\operatorname{Re}\langle Q^*[Q, W_e(r)]u, u\rangle| = h^2\langle A_2 u, u\rangle.$$

This proves (4.18) with $s = \frac{1}{2} + 3\delta$, and taking $\delta > 0$ small enough proves it for all $s > 1/2$. $\square$

Proof of (4.19). – Let

$$u := (P_j - E_j - i\varepsilon)^{-1}\chi_+(r)\psi(hD_r)v,$$

with $\|v\| = 1$, and fix $\delta \in (0, 1/4)$. We will use the following argument by induction to prove (4.19).

The inductive hypothesis is that for a given $k \in \mathbb{R}$ there is a neighborhood U of $F_j \setminus (3,\infty) \times (0,\infty)$ such that $\|Au\| \leq Ch^k$ for any $a \in S^{-\infty}_{k+\frac{1}{2}-\delta}$ which is supported in U.

The inductive step is that there is a (smaller) neighborhood U' of $F_j \setminus (3,\infty) \times (0,\infty)$ such that

$$\|A'u\| \leq Ch^{k+1/2} \tag{4.30}$$

for any $a' \in S^{-\infty}_{k+1+\delta}$ which is supported in U'.

Let us see first that (4.30) for arbitrary k implies (4.19). Indeed, by the elliptic estimate (4.25), the composition Formula (4.22), and the resolvent estimate (4.18), we see that

$$\|A''u\| \leq C_N h^N \tag{4.31}$$

for any $N \in \mathbb{R}$ and $a'' \in S^{\infty}_{-\infty}$ such that $\operatorname{supp} a'' \subset (0,3) \times \mathbb{R}$ and $\operatorname{supp} a'' \cap F_j = \emptyset$. Then we can write

$$\chi_-(r)u = \chi_-(r)\varphi_F(hD_r)u + \chi_-(r)(1 - \varphi_F(hD_r))u$$

for $\varphi_F \in C_c^\infty(\mathbb{R})$ chosen such that (4.30) applies to the first term on the right and (4.31) applies to the second.

We remark in passing that elaborating this argument we can actually show that u is semiclassically trivial everywhere away from the union of two sets (including uniformly as $|r| \to \infty$ and $|\rho| \to \infty$): the first is $\operatorname{supp} \chi_+ \times \operatorname{supp} \psi$, and the second is $F_j \cap (3,\infty) \times (0,\infty)$ which we can think of as a neighborhood of the forward bicharacteristic flowout of the first. Here we are focusing on a more concrete and narrower version of this conclusion which is sufficient for our purposes.

Next observe that the base case (the inductive hypothesis with $k = -1$ and $U = T^*\mathbb{R}$) follows from the resolvent estimate (4.18).

It remains to prove (4.30) under the inductive hypothesis. Roughly speaking, we use an escape function which on $F_j \setminus (3,\infty) \times (0,\infty)$ agrees with the one used in the proof of (4.18) above, but is adapted to vanish near $\operatorname{supp} \chi_+ \times \operatorname{supp} \psi$ and $F_j \setminus U$. (Note that $F_j \setminus U = \emptyset$ when $k = -1$ but that for $k > -1$ we expect $F_j \setminus U \neq \emptyset$ in general).

More specifically, to define the escape function, fix $\chi_k, \psi_k \in C^\infty(\mathbb{R})$ nondecreasing, and satisfying $\chi_k = 0$ near $(-\infty, 3]$, $\psi_k = 0$ near $(-\infty, 0]$, $\psi_k = 1$ near $[\sqrt{E_*/3}, \infty)$, and $\chi_k(r)\psi_k(\rho) = 1$ near $F_j \setminus U$. Then let

$$q_k(r,\rho) := \tilde{q}_{k+\frac{3}{2}-\delta}(-r\rho)\chi_q(r,\rho)(1 - \chi_k(r)\psi_k(\rho)),$$

where $\tilde{q}_{k+\frac{3}{2}-\delta}$ and χ_q are as in (4.26), so that $q_k \in S^{-\infty}_{k+\frac{3}{2}-\delta}$. Calculating as in (4.27), we see that near F_j we have

$$\{\operatorname{Re} p_j, q_k^2\} \leq 0,$$

and near $F_j \setminus (3,\infty) \times (0,\infty)$ we have $\chi_k(r)\psi_k(\rho) = 0$ and hence $\{\operatorname{Re} p_j, q_k^2\} \leq -cr^{2k+2-2\delta} < 0$ (this is slightly better than (4.27) because outside of a compact set we have $\rho < 0$ on $F_j \setminus (3,\infty)\times(0,\infty)$ and in particular we are staying away from the outgoing part of the energy surface).

Consequently, as before, we can write

$$b_k^2 = \{q_k^2, \operatorname{Re} p_j\} + a_{0,k},$$

where $b_k \in S^{-\infty}_{k+1-\delta}$, $a_{0,k} \in S^{-\infty}_{2k+2-2\delta}$, $\operatorname{supp} a_{0,k} \cap (F_j \cup \operatorname{supp}\chi_+ \times \operatorname{supp}\psi) = \emptyset$, $\operatorname{supp} b_k \subset \operatorname{supp} q_k$, and $b_k \geq c r^{k+1-\delta} > 0$ near $F_j \setminus (3,\infty)\times(0,\infty)$. Hence

$$B_k^* B_k = \frac{i}{h}[Q_k^* Q_k, \operatorname{Re} P_j] + A_{0,k} + hA_{1,k},$$

for some $a_{1,k} \in S^{-\infty}_{2k+1-2\delta}$. We refine this by using (4.22) and (4.24) to expand $a_{1,k}$ in powers of h up to h^N in terms of b_k, q_k, p_j, $a_{0,k}$, and their derivatives, which gives

$$B_k^* B_k = \frac{i}{h}[Q_k^* Q_k, \operatorname{Re} P_j] + A_{0,k} + hA'_{1,k} + h^N Z_N,$$

where $a'_{1,k} \in S^{-\infty}_{2k+1-2\delta}$ has $\operatorname{supp} a'_{1,k} \subset \operatorname{supp} q_k$ and $z_N \in S^{-\infty}_{2k+2-2\delta-N}$. Consequently

$$\|B_k u\|^2 = \frac{i}{h}\langle [Q_k^* Q_k, \operatorname{Re} P_j]u, u\rangle + \langle A_{0,k}u, u\rangle + h\langle A'_{1,k}u, u\rangle + h^N \langle Z_N u, u\rangle.$$

By the elliptic estimate (4.29) with b_k in place of b we see that to deduce (4.30) it is enough to show

$$\|B_k u\|^2 \leq C h^{2k+1}. \tag{4.32}$$

Now $\langle A_{0,k}u, u\rangle = O(h^\infty)$ by (4.25). Also, since q_k vanishes near $F_j \setminus U$, it follows that $a'_{1,k}$ vanishes near $F_j \setminus U$, so by (4.25), (4.22), and the inductive hypothesis, we have

$$|\langle A'_{1,k}u, u\rangle| \leq C h^{2k}.$$

Hence to show (4.32) it suffices to show that

$$i\langle [Q_k^* Q_k, \operatorname{Re} P_j]u, u\rangle \leq C h^{2k+2}. \tag{4.33}$$

As before we write, for any $N \in \mathbb{R}$,

$$\begin{aligned} i\langle [Q_k^* Q_k, \operatorname{Re} P_j]u, u\rangle &= -2\operatorname{Im}\langle Q_k(P_j - E_j - i\varepsilon)u, Q_k u\rangle - 2\operatorname{Re}\langle Q_k(W_e(r)+\varepsilon)u, Q_k u\rangle \\ &\leq 2|\operatorname{Re}\langle Q_k^*[Q_k, W_e(r)]u, u\rangle| + O(h^\infty), \end{aligned}$$

where we used $\operatorname{supp} q_k \cap \operatorname{supp}\chi_+ \times \operatorname{supp}\psi = \emptyset$. Now (4.33) follows from the inductive hypothesis together with the fact that (arguing as in the construction of $a'_{1,k}$ above) $\operatorname{Re} Q_k^*[Q_k, W_e(r)] = h^2 A_{2.k} + h^N Z_N$, with $a_{2,k}$, $z_N \in S^{-\infty}_{-\infty}$, and $\operatorname{supp} a_{2,k} \cap F_j \subset U$. ☐

4.5. Proof of Theorems 3.1 and 3.2

In this section all operator norms are $L^2(X) \to L^2(X)$. We implement the outline discussed in §4.1. We assume without loss of generality that $\varepsilon \in (0,1]$, as the statements with $\varepsilon > 1$ follow from self-adjointness and the statements with $\varepsilon < 0$ then follow by taking the adjoint.

We first explain the key dynamical property of the bicharacteristic flow in X_e which allows us to remove the remainders in the parametrix construction.

Let us denote points in T^*X_e by (r, y, ρ, η), where $y \in Y$, ρ is dual to r, and η is dual to y. The *energy surface* for P in T^*X_e at energy E_0 is the subset of T^*X_e defined by

$$p(r, y, \rho, \eta) := \rho^2 + |\eta|^2 f(r)^{-4/(d-1)} + V_L(r) = E_0,$$

and *bicharacteristics* in T^*X_e of this energy surface are solutions $\gamma(t) := ((r(t), y(t), \rho(t), \eta(t))$ to the Hamiltonian equation of motion $\dot\gamma(t) := \frac{d}{dt}\gamma(t) = \{p, \gamma(t)\}$. The *backward bicharacteristic flowout* in T^*X_e of a point $\gamma_0 \in T^*X_e$ is the set of points $\gamma' \in T^*X_e$ such that if $\gamma(t)$ is the bicharacteristic in T^*X_e with $\gamma(0) = \gamma_0$, then $\gamma(t) = \gamma'$ for some $t \le 0$; note that some bicharacteristics enter $T^*(X \setminus X_e)$ in finite time, and our definition only counts them while they stay in T^*X_e.

If $\gamma(t) := ((r(t), y(t), \rho(t), \eta(t))$ is a bicharacteristic, then

$$\dot r(t) = 2\rho(t), \qquad \dot\rho(t) = \frac{4}{d-1}|\eta|^2 f'(r(t)) f(r(t))^{-(d+3)/(d-1)} - V_L'(r(t)) \ge 0 \tag{4.34}$$

and hence $\ddot r = 2\dot\rho \ge 0$. Consequently no bicharacteristic can visit the sets $T^*((0,4))$, $T^*((4,5))$, and $T^*((2,3))$ in that order (here and below $T^*((a,b))$ denotes the subset of T^*X_e on which $a < r < b$), and this fact is exploited to prove the crucial remainder estimate in (4.38) below.

Fix $\chi_e, \chi_K \in C^\infty(\mathbb{R})$ such that $\chi_e + \chi_K = 1$, $\operatorname{supp}\chi_e \subset (3,\infty)$, and $\operatorname{supp}\chi_K \subset (-\infty, 4)$. Define a parametrix for $P - E - i\varepsilon$ by

$$G := \chi_K(r-1)R_K\chi_K(r) + \chi_e(r+1)R_e\chi_e(r).$$

Here

$$R_K = R_K(E_0 + i\varepsilon) := (-h^2\Delta - iW_K(r) - E_0 - i\varepsilon)^{-1}$$

and

$$R_e = R_e(E_0 + i\varepsilon) := f(r)\sum_{j=0}^{\infty}\big((P_j - i\varepsilon)^{-1}\phi_j \otimes \phi_j\big) f(r)^{-1},$$

and

$$\|R_K\| \le Ca(h)h^{-1}, \qquad \|(1+r)^{-s_1}\chi_e(r+1)R_e\chi_e(r)(1+r)^{-s_2}\| \le Ch^{-2}. \tag{4.35}$$

Indeed, R_K is well defined and obeys (4.35) thanks to (3.3); this follows from the resolvent identity for $\varepsilon > 0$ small enough and then from the bound $\operatorname{Im}(-h^2\Delta - iW_K(r) - E_0 - i\varepsilon) \le -\varepsilon$ for all $\varepsilon > 0$. Meanwhile $\chi_e(r+1)R_e\chi_e(r)$ acts on $L^2(X)$ thanks to (4.2) and the support property of χ_e, even though R_e acts on a funny space due to the way we defined the operators P_j differently depending on j; moreover R_e obeys (4.35) by (4.5) and (4.18).

Define operators A_K and A_e by

$$(P - E_0 - i\varepsilon)G = I + [h^2D_r^2, \chi_K(r-1)]R_K\chi_K(r) + [h^2D_r^2, \chi_e(r+1)]R_e\chi_e(r) =: I + A_K + A_e.$$

Our next step is to remove the remainders A_K and A_e. The idea of [18] is to do this using a *semiclassically outgoing* property of the resolvents R_K and R_e.

To explain this property, we use the following notation: if $U \subset T^*X_e$, then Γ_+U is the set of points in T^*X_e whose backward bicharacteristic flowout intersects U. Now in the case of R_K, the needed semiclassically outgoing property says (in the notation of (4.20) and (4.21)) that if $\tilde\chi \in C_c^\infty((0,\infty))$ and $a \in S_l^0$, then

$$\|\tilde\chi(r)\operatorname{Op}_h(a)A_K\| = O(h^\infty), \tag{4.36}$$

provided $|\partial_r^{n_1}\partial_\rho^{n_2} a(r,\rho)| = O(h^\infty)$ for every $n_1,\ n_2 \in \mathbb{N}_0$ and for every

$$(r,\rho) \in T^*((0,4)) \cup \Gamma_+ T^*((0,4)).$$

This property follows from [18, Lemma 5.1].

On the other hand, the resolvent R_e is only semiclassically outgoing for j such that $E_j \geq c > 0$ (the relevant statement for us is (4.19)); as $E_j \to 0$ this property fails, but then the gluing region (the part of X such that $r \in (2,5)$) becomes classically forbidden, and so we will be able to estimate and remove remainders using the Agmon estimates of §4.3.

More specifically, we observe that

$$\|A_K\| \leq C(1 + a(h)), \qquad \|A_e(1+r)^{-s_2}\| \leq C. \tag{4.37}$$

Indeed, A_K obeys the bound thanks to the corresponding bound on R_K in (4.35); note that $\|R_K\|_{L^2 \to H_h^2(X)} \leq C\|R_K\|$ since V, W, and ε are bounded, and E_0 is fixed. Meanwhile A_e obeys the bound by (4.14) and (4.18).

We refine the parametrix with some correction terms, observing that $A_K^2 = A_e^2 = 0$:

$$(P - E_0 - i\varepsilon)G(I - A_K - A_e + A_K A_e) = I - A_e A_K + A_e A_K A_e.$$

We will show that

$$\|A_e A_K\| = O(h^\infty). \tag{4.38}$$

Assuming (4.38) for the moment, we may write (using $R_e \chi_e(r) A_e = R_K \chi_K(r) A_K = 0$)

$$\begin{aligned}(P - E_0 - i\varepsilon)^{-1} &= G(I - A_K - A_e + A_K A_e)(I - A_e A_K + A_e A_K A_e)^{-1} \\ &= \chi_e(r+1) R_e \chi_e(r) + \chi_K(r-1) R_K \chi_K(r) - \chi_e(r+1) R_e A_K \\ &\quad - \chi_K(r-1) R_K A_e + \chi_e(r+1) R_e A_K A_e + O(h^\infty).\end{aligned} \tag{4.39}$$

Note that by (4.14), (4.18), and the bound on $\|R_K\|$ in (4.35), we have

$$\|(1+r)^{-s_1} \chi_e(r+1) R_e A_K\| \leq C a(h) h^{-1}. \tag{4.40}$$

Now multiplying (4.39) on the left by $(1+r)^{-s_1}$ and on the right by $(1+r)^{-s_2}$ and estimating the norm on the right term by term, we see that by (4.35) the first term on the right has norm bounded by Ch^{-2}, while by (4.35), (4.37), and (4.40), the next four terms have norm bounded by $Ca(h)h^{-1}$. This implies (3.4).

We similarly deduce (3.6) from (4.39), but rather than using the bound on R_e in (4.35), we use

$$\|(1+r)^{-s} \chi_e(r+1) \chi_{\mathcal{J}} R_e \chi_e(r) (1+r)^{-s}\| \leq C h^{-1}. \tag{4.41}$$

To prove (4.41), we use (4.6) when $E_j \in [-c_{\mathcal{J}} h, c_{\mathcal{J}}]$, we use (4.18) when $E_j \geq c_{\mathcal{J}}$, and we use the fact that P_j is almost nonnegative (more precisely, $P_j \geq -Ch^2$ by (4.2) and (4.4)) when $E_j \leq -c_{\mathcal{J}} h$.

To complete the proofs of Theorems 3.1 and 3.2, it remains to show (4.38). We have

$$A_e A_K = [\chi_e(r+1), h^2 D_r^2] R_e [\chi_K(r-1), h^2 D_r^2] R_K \chi_K(r).$$

Fix $\tilde{\chi} \in C_c^\infty((3,6))$ which is 1 on $[4,5]$, so that

$$A_e A_K = [\chi_e(r+1), h^2 D_r^2] R_e \tilde{\chi}(r) [\chi_K(r-1), h^2 D_r^2] R_K \chi_K(r).$$

For any $\psi \in C_c^\infty((0,\infty))$ we have

$$\|[\chi_e(r+1), h^2D_r^2]R_e\tilde{\chi}(r)\psi(hD_r)[\chi_K(r-1), h^2D_r^2]\| = O(h^\infty)$$

by (4.15) and (4.19), so it remains to show that there is $\psi \in C_c^\infty((0,\infty))$ such that

$$\|\tilde{\chi}(r)(I - \psi(hD_r))[\chi_K(r-1), h^2D_r^2]R_K\chi_K(r)\| = O(h^\infty).$$

We will deduce this from (4.36). Indeed, it is enough to check that there is $\rho_0 > 0$ such that if $\gamma(t)$ is a bicharacteristic at energy E_0 with $\gamma(0) \in T^* \operatorname{supp} \chi_K(r)$ and with $\gamma(T) \in T^* \operatorname{supp} \chi'_K(r-1)$ for some $T > 0$, then $\rho(T) \geq \rho_0$ (we already know that $\rho(T)^2 \leq E_0$, so we may then take ψ to be 1 near $[\rho_0, \sqrt{E_0}]$).

Thanks to (4.34) we know that $\rho(t)$ is nondecreasing, so we may assume that we have $\max\operatorname{supp}\chi_K(r) < r(t) < \min\operatorname{supp}\chi'_K(r-1)$ when $t \in (0,T)$, which implies in particular $\rho(0) \geq 0$. Then, for $t \in (0,T)$, we have $f(r(t)) \leq Cf'(r(t))$ and $V_L(r(t)) \leq -CV'_L(r(t))$, so that

$$\dot{\rho}(t) \geq (|\eta|^2 f(r)^{-4/(d-1)} + V_L(r))/C_0 = (E_0 - \rho(t)^2)/C_0.$$

If $\rho(0) = \sqrt{E_0}$, then $\rho(T) = \sqrt{E_0}$ and we are done; otherwise we can integrate and use $\rho(0) \geq 0$ to obtain

$$\frac{C_0}{\sqrt{E_0}}\tanh^{-1}\left(\frac{\rho(T)}{\sqrt{E_0}}\right) \geq T = \frac{r(T)-r(0)}{2\bar{\rho}} \geq \frac{r(T)-r(0)}{2\rho(T)},$$

where we used $\bar{\rho} := T^{-1}\int_0^T \rho(t)dt \leq \rho(T)$. This implies $\rho(T) \geq \rho_0$, for some $\rho_0 > 0$ depending on C_0, E_0, and χ_K.

5. Continuation of the resolvent

In this section we keep all of the assumptions of §3.1, and add the assumption that

$$r \geq 6 \Longrightarrow V_L(r) = f(r) - 1 = 0.$$

In §5.1 we briefly review how meromorphic continuation works in this setting, following [28] and [36, §6.7], and introduce the relevant notation. In §5.2 we prove some useful estimates for a model problem on the cylindrical end. In §5.3 we use an identity of Vodev from [52] to deduce the existence of a resonance free region.

Roughly speaking, writing $R(z)$ for the resolvent $(P-z)^{-1}$ and for its meromorphic continuation, we deduce from (3.4) that

$$\|\chi R(E_0 \pm i0)\chi\| \lesssim 1/\mu(h),$$

where $\chi \in C_c^\infty(X)$ and $0 < \mu(h) \leq h^2$. Then we use Vodev's identity to show that this implies

$$\|\chi R(z)\chi\| \lesssim 1/\mu(h),$$

as long as the distance from z to $E_0 \pm i0$ is small compared to $\mu(h)$. However some care is needed due to the complicated nature of the Riemann surface to which $R(z)$ continues (see §5.1), and due to the fact that our model resolvent obeys somewhat weaker bounds than the one used in [52] (see §5.2). The precise statement and proof are in §5.3.

Although we keep all of the assumptions of §3.1 in this section, strictly speaking they are not all needed once we have (3.4). Instead, as long as we had (3.4), we could allow X to be a more general manifold with cylindrical ends, or allow P to be a black-box perturbation of the Laplacian e.g., in the sense of [10, §2]. The proof could also be adapted to include the case of waveguides. We omit these generalizations here, to simplify the presentation and because all of our interesting examples satisfy the assumptions of §3.1.

5.1. Meromorphic continuation of the resolvent

In §5.1 we think of $h > 0$ as being fixed, until Lemma 5.2, in which we prove an estimate which is uniform as $h \to 0$.

The spectrum of P is given by $[0, \infty)$ together with a finite (possibly empty) set of negative eigenvalues. For z not in the spectrum we define the resolvent

$$R(z) := (P - z)^{-1} \colon L^2(X) \to L^2(X).$$

To define the Riemann surface onto which $R(z)$ meromorphically continues, for each $j \in \mathbb{N}_0$, and $z \in \mathbb{C} \setminus [h^2\sigma_j^2, \infty)$, we introduce the notation

$$\rho_j(z) := \sqrt{z - h^2\sigma_j^2},$$

with the branch of the square root chosen such that $\operatorname{Im} \rho_j(z) > 0$ for this range of z (recall that $0 = \sigma_0 \leq \sigma_1 \leq \cdots$ are the square roots of the eigenvalues of the nonnegative Laplacian on (Y, g_Y) included according to multiplicity).

For each $j \in \mathbb{N}_0$, there is a minimal Riemann surface $\hat{Z}_{h,j}$ onto which ρ_j continues analytically from $\mathbb{C} \setminus [h^2\sigma_j^2, \infty)$; this is a double cover of $\mathbb{C}$ ramified at the singular point $z = h^2\sigma_j^2$. By elaborating the construction of $\hat{Z}_{h,j}$, we see that there is a minimal Riemann surface $\hat{Z}_h$ onto which all the ρ_j extend simultaneously from $\mathbb{C} \setminus [0, \infty)$. This is a countable cover of $\mathbb{C}$, ramified at $z = h^2\sigma_j^2$ for each j, and for each $z \in \hat{Z}_h$ we have $\operatorname{Im} \rho_j(z) > 0$ for all but finitely many j. For more details, see [28] and [36, §6.7].

We use p to denote the projection $\hat{Z}_h \to \mathbb{C}$, we use the term *physical region* to refer to the sheet over $\mathbb{C} \setminus [0, \infty)$ on which $\operatorname{Im} \rho_j > 0$ for all j, and for notational convenience we identify the physical region with $\mathbb{C} \setminus [0, \infty)$. Then $R(z)$ continues meromorphically from the resolvent set in $\mathbb{C} \setminus [0, \infty)$ to all of $\hat{Z}_h$, as an operator from compactly supported L^2 functions to locally L^2 functions, and we have $(P - p(z))R(z) = I$. We refer to the poles of $R(z)$ as *resonances*.

For $E \geq 0$, we denote by $E \pm i0$ the points in $\hat{Z}_h$ on the boundary of the physical region which are obtained as limits $\lim_{\pm\delta \downarrow 0} E + i\delta$. Note that $\rho_j(E \pm i0) \in i\mathbb{R}_+$ if $E < h^2\sigma_j^2$, and $\pm\rho_j(E \pm i0) > 0$ if $h^2\sigma_j^2 < E$. Below we will only be concerned with points on $\hat{Z}_h$ which are quite close to the boundary of the physical region. To measure how far apart two points on $\hat{Z}_h$ are we use the following

LEMMA 5.1. – *The function* $d_h : \hat{Z}_h \times \hat{Z}_h \to [0, \infty]$ *given by*

$$d_h(z, z') := \sup_j |\rho_j(z) - \rho_j(z')| \tag{5.1}$$

takes only finite values and is a metric on $\hat{Z}_h$.

Proof. – To see that $|\rho_j(z) - \rho_j(z')|$ is bounded in j, note that

$$p(z) - p(z') = \rho_j^2(z) - \rho_j^2(z') = (\rho_j(z) - \rho_j(z'))(\rho_j(z) + \rho_j(z')). \tag{5.2}$$

Using that $\rho_j^2(z) = p(z) - h^2\sigma_j^2$, we find $\operatorname{Re}\rho_j^2(z) \to -\infty$ as $j \to \infty$. Since $\operatorname{Im}\rho_j(z) > 0$ if j is sufficiently large, $\operatorname{Im}\rho_j(z) \to \infty$ as $j \to \infty$ and we find, since the same is true for z', that for j large enough $|\rho_j(z) - \rho_j(z')| < |\rho_j(z) + \rho_j(z')|$. Since by (5.2), we have

$$\min\{|\rho_j(z) - \rho_j(z')|, |\rho_j(z) + \rho_j(z')|\} \leq |p(z) - p(z')|^{1/2},$$

we have for j sufficiently large, $|\rho_j(z) - \rho_j(z')| \leq |p(z) - p(z')|^{1/2}$.

That d_h is a metric is fairly straightforward; for completeness we check the triangle inequality. Let $z,\ z', w \in \hat{Z}_h$. Then

$$|\rho_j(z) - \rho_j(z')| \leq |\rho_j(z) - \rho_j(w)| + |\rho_j(w) - \rho_j(z')|.$$

But then

$$\begin{aligned} d_h(z, z') &= \sup_j |\rho_j(z) - \rho_j(z')| \leq \sup_j (|\rho_j(z) - \rho_j(w)| + |\rho_j(w) - \rho_j(z')|) \\ &\leq \sup_j |\rho_j(z) - \rho_j(w)| + \sup_j |\rho_j(w) - \rho_j(z')| = d_h(z, w) + d_h(w, z'). \end{aligned}$$

□

Later we will want to use $d_h(z, z')$ in a resolvent identity, and now we show that $d_h(z, z')$ controls $|p(z) - p(z')|$, at least when z' is on the boundary of the physical region:

LEMMA 5.2. – *Let $E > 0$, and let $E \pm i0$ denote one of the points on the boundary of the physical space in $\hat{Z}_h$ as described above. Then for any $\delta > 0$, if $h > 0$ is sufficiently small,*

$$|p(z) - E| \leq d_h(z, E \pm i0)[d_h(z, E \pm i0) + O(h^{1/2-\delta})]$$

for $z \in \hat{Z}_h$.

Proof. – We have, for any $j \in \mathbb{N}$,

$$\begin{aligned} |p(z) - E| &= |\rho_j^2(z) - \rho_j^2(E \pm i0)| \\ &= |\rho_j(z) - \rho_j(E \pm i0))||\rho_j(z) - \rho_j(E \pm i0) + 2\rho_j(E \pm i0)| \\ &\leq |\rho_j(z) - \rho_j(E \pm i0)| \left(|\rho_j(z) - \rho_j(E \pm i0)| + 2|\rho_j(E \pm i0)|\right). \end{aligned} \tag{5.3}$$

By the Weyl law, for any $\delta' > 0$ there is an $h_0 = h_0(\delta') > 0$ so that if $0 < h < h_0$, the interval $[Eh^{-2} - h^{-1-\delta'}, Eh^{-2} + h^{-1-\delta'}]$ contains an element of the spectrum of $-\Delta_Y$; call this $\sigma_{j_0}^2$. We note that j_0 depends on E and on h, but our notation does not reflect that dependence. Then

$$|\rho_{j_0}(E \pm i0)|^2 = |E - h^2\sigma_{j_0}^2| \leq h^{1-\delta'}.$$

Using this in (5.3) with $j = j_0$ proves the lemma, since

$$|\rho_{j_0}(z) - \rho_{j_0}(E \pm i0)| \leq d_h(z, E \pm i0).$$

□

5.2. Resolvent estimates for the model problem on the cylindrical end.

Let $X_0 = [0,\infty) \times Y$, let $\Delta_0 \le 0$ be the Laplacian on $(X_0, dr^2 + g_Y)$, and for $h > 0$ and $z \in \mathbb{C} \setminus [0,\infty)$, let

$$R_0(z) := (-h^2\Delta_0 - z)^{-1}$$

denote the semiclassical Dirichlet resolvent.

For $\operatorname{Im}\xi > 0$, let $R_D(\xi)$ be the resolvent for the Dirichlet Laplacian on the half-line with spectral parameter ξ^2 and Schwartz kernel given by

$$R_D(\xi, r, r') = \frac{i}{2h\xi}(e^{i\xi|r-r'|/h} - e^{i\xi(r+r')/h}). \tag{5.4}$$

Then, for z in the physical region of $\hat{Z}_h$ (see §5.1), we have

$$R_0(z) = \sum_{j=0}^{\infty} R_D(\rho_j(z))\phi_j \otimes \phi_j, \tag{5.5}$$

where $\{\phi_j\}_{j=0}^{\infty}$ is a complete set of real-valued orthonormal eigenfunctions of the Laplacian on Y and $-\Delta_Y\phi_j = \sigma_j^2\phi_j$.

Moreover, $R_0(z)$ continues holomorphically to $\hat{Z}_h$ as an operator from compactly supported L^2 functions to locally L^2 functions. In this section we prove some estimates for $R_0(z)$ which will be needed when we use a resolvent identity to find a neighborhood of the boundary of the physical region in which $R(z)$ has no poles.

Proposition 5.3. – *Let $\chi \in C_c^\infty([0,\infty))$ and fix $N > 0$. If $\operatorname{Im}\xi,\ \operatorname{Im}\xi' > -Nh$, then*

$$\|\chi R_D(\xi)\chi - \chi R_D(\xi')\chi\| \le Ch^{-3}|\xi - \xi'|. \tag{5.6}$$

If $\operatorname{Im}\xi,\ \operatorname{Im}\xi' > -Nh$ and $\alpha_1 + \alpha_2 = 1, 2$, then

$$\|\chi h^{\alpha_1} D_r^{\alpha_1} R_D(\xi) h^{\alpha_2} D_r^{\alpha_2}\chi - \chi h^{\alpha_1} D_r^{\alpha_1} R_D(\xi') h^{\alpha_2} D_r^{\alpha_2}\chi\| \le Ch^{-2}|\xi-\xi'|(|\xi|+|\xi'|+1)^{\alpha_1+\alpha_2-1}. \tag{5.7}$$

Fix $\delta > 0$ and suppose $\delta < \arg\xi, \arg\xi' < \pi - \delta$ and $|\xi|, |\xi'| \ge 1$. Then if $\alpha_1 + \alpha_2 \le 2$,

$$\|h^{\alpha_1} D_r^{\alpha_1} R_D(\xi) h^{\alpha_2} D_r^{\alpha_2}\chi - \chi h^{\alpha_1} D_r^{\alpha_1} R_D(\xi') h^{\alpha_2} D_r^{\alpha_2}\| \le C|\xi - \xi'|. \tag{5.8}$$

All the norms above are $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, and the constants depend on χ, N, and δ.

Proof. – We begin with (5.6). Note that $\chi \frac{d}{d\xi} R_D(\xi)\chi$ has Schwartz kernel

$$\frac{i\chi(r)}{2h^3(\xi/h)^2}\left((-1 + i|r-r'|\frac{\xi}{h})e^{i\xi|r-r'|/h} - (-1 + i(r+r')\frac{\xi}{h})e^{i\xi(r+r')/h}\right)\chi(r').$$

With $\operatorname{Im}\xi > -Nh$, this can be pointwise bounded by C/h^3, even when $\xi \to 0$, and hence since χ is compactly supported we have $\|\chi \frac{d}{d\tau} R_D(\tau)\chi\| \le \frac{C}{h^3}$. Integrating from ξ to ξ' gives (5.6). We note for future reference that if $|\xi| \ge h$, then we can improve the estimate to

$$\left|\frac{i\chi(r)}{2h^3(\xi/h)^2}\left(\left(-1 + i|r-r'|\frac{\xi}{h}\right)e^{i\xi|r-r'|/h} - \left(-1 + i(r+r')\frac{\xi}{h}\right)e^{i\xi(r+r')/h}\right)\chi(r')\right| \le C/(h^2|\xi|), \text{ when } |\xi| \ge h. \tag{5.9}$$

Next consider the operator $h\frac{\partial}{\partial r}R_D(\xi)$. It has Schwartz kernel

$$\frac{-1}{2h}\left(\operatorname{sgn}(r-r')e^{i\xi|r-r'|/h}-e^{i\xi(r+r')/h}\right).$$

Differentiating this with respect to ξ and proceeding as above gives $\left\|\chi\frac{d}{d\tau}h\frac{\partial}{\partial r}R_D(\tau)\chi\right\|\leq\frac{C}{h^2}$. Integrating in τ from ξ to ξ' gives (5.7) for $\alpha_1 = 1$, $\alpha_2 = 0$. To prove (5.7) for $\alpha_1 = 2$, $\alpha_2 = 0$, we can argue as before using the Schwartz kernel. Alternately, we can note that $h^2\frac{\partial^2}{\partial r^2}R_D(\xi) = I + \xi^2 R_D(\xi)$ and proceed as in the proof of the first inequality, using the improvement (5.9). Similar techniques give (5.7) when $\alpha_2 \neq 0$, if we consider the Schwartz kernel of $R_D(\xi)\frac{\partial}{\partial r}$.

When ξ, ξ' satisfy $\delta < \arg\xi, \arg\xi' < \pi-\delta$ they are both in the physical region and we can use the resolvent equation $R_D(\xi)-R_D(\xi') = (\xi^2-\xi'^2)R_D(\xi)R_D(\xi')$. If $|\xi|\geq 1$, using the bound on $\arg\xi$ we have $\|h^{\alpha_1}D_r^{\alpha_1}R_D(\xi)h^{\alpha_2}D_r^{\alpha_2}\| \leq C|\xi|^{\alpha_1+\alpha_2-2}$, where the constant depends on δ. The same inequality holds if ξ is replaced by ξ' everywhere. Using this in the resolvent equation proves (5.8). □

Proposition 5.4. – *Let $E > 0$ and consider one of the points $E \pm i0 \in \hat{Z}_h$ which lies on the boundary of the physical region. Fix $N > 0$ and $\chi \in C_c^\infty(X_0)$. Then*

$$\|\chi R_0(z)\chi - \chi R_0(E\pm i0)\chi\| \leq Ch^{-3}d_h(z, E\pm i0) \tag{5.10}$$

for all $z \in \hat{Z}_h$ such that $d_h(z, E\pm i0) < Nh$. If $\alpha_1+\alpha_2 = 1, 2$, then instead

$$\|\chi h^{\alpha_1}D_r^{\alpha_1}R_0(z)h^{\alpha_2}D_r^{\alpha_2}\chi - \chi h^{\alpha_1}D_r^{\alpha_1}R_0(E\pm i0)h^{\alpha_2}D_r^{\alpha_2}\chi\| \leq Ch^{-2}d_h(z, E\pm i0) \tag{5.11}$$

for all $z \in \hat{Z}_h$ such that $d_h(z, E\pm i0) < Nh$.

Proof. – We begin by noting that for any $j \in \mathbb{N}$, $\operatorname{Im}\rho_j(E\pm i0) \geq 0$, and for $h^2\sigma_j^2 > E$ we have $\rho_j(E\pm i0) \in i\mathbb{R}_+$. Hence if $d_h(z, E\pm i0) < Nh$, then $\operatorname{Im}\rho_j(z) \geq -Nh$ and $\operatorname{Im}\rho_j(z) \to \infty$ as $j\to\infty$.

Without loss of generality, we may assume χ is a function of r only, so that we may consider χ as a function defined on $[0,\infty)$. Using the expression (5.5), we find that

$$\begin{aligned}&\|\chi R_0(z)\chi - \chi R_0(E\pm i0)\chi\|_{L^2(X_0)\to L^2(X_0)}\\ &\qquad= \sup_j \|\chi R_D(\rho_j(z))\chi - \chi R_D(\rho_j(E\pm i0))\chi\|_{L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)}.\end{aligned}$$

Now (5.10) follows directly from (5.6) and the Definition (5.1) of $d_h(z, E\pm i0)$.

To prove (5.11), we note that for j sufficiently large we have $h^2\sigma_j^2 > E+5$, and $\pi/4 < \arg\rho_j(z), \arg\rho_j(E\pm i0) < 3\pi/4$. Using (5.7) when $h^2\sigma_j^2 \leq E+5$ and (5.8) when $h^2\sigma_j^2 > E+5$, along with the definition of $d_h(z, E\pm i0)$ proves (5.11). □

5.3. The resonance free region

Throughout §5.3, we keep all of the assumptions of §3.1, as well as the assumption that

$$r \geq 6 \Longrightarrow V_L(r) = f(r) - 1 = 0.$$

To show the existence of a resonance free region, we use an identity due to Vodev [52, (5.4)]. In [52] the identity is stated only for operators which are potential perturbations of the Laplacian on $\mathbb{R}^d$. However, it in fact holds in far greater generality for operators which are, in an appropriate sense, compactly supported perturbations of each other. Here we state a version adapted to our circumstance.

LEMMA 5.5 ([52, (5.4)]). – *Let $\chi_1 \in C_c^\infty(X;[0,1])$ be such that $r \geq 6$ near* supp $1-\chi_1$. *Choose $\chi \in C_c^\infty(X;[0,1])$ so that $\chi\chi_1 = \chi_1$. Then for $z, z_0 \in \hat{Z}_h$,*

$$\begin{aligned}\chi R(z)\chi - \chi R(z_0)\chi = (p(z) - p(z_0))\chi R(z)\chi\chi_1(2-\chi_1)\chi R(z_0)\chi \\ + (1-\chi_1 - \chi R(z)\chi[h^2\Delta, \chi_1])\,(\chi R_0(z)\chi - \chi R_0(z_0)\chi) \\ \times (1-\chi_1 + [h^2\Delta, \chi_1]\chi R(z_0)\chi).\end{aligned}$$

It is important to note in the identity above that $\chi R_0 \chi$ only appears where it is multiplied both on the left and right by an operator (either $1-\chi_1$ or $[h^2\Delta, \chi_1]$) supported in the set where $r \geq 6$. If we think of this set as a subset of $X_0 = [0,\infty) \times Y$, then the appearance of $\chi R_0 \chi$ makes sense.

We omit the proof of Lemma 5.5 because it is essentially the same as that of [52, (5.4)] (see also [23, Lemma 6.26] and, for another version in the setting of cylindrical ends, [10, Lemma 2.1]).

The proof we give of the following theorem follows the proof of [52, Theorem 1.5], but we write it out in detail because it is short and to highlight the role of the estimates we proved in §5.2.

THEOREM 5.6. – *With χ as in Lemma 5.5, using* (3.4) *take constants C and $\mu(h)$ such that*

$$\|\chi R(E \pm i0)\chi\|_{L^2(X)\to L^2(X)} \leq \frac{C}{\mu(h)},$$

where $E = E_0$ and $0 < \mu(h) \leq h^2$. Then there are constants C', $\tilde{C}$ so that for $h > 0$ sufficiently small, $\chi R(z)\chi$ is analytic in $\{z \in \hat{Z}_h : d_h(z, E \pm i0) < C'\mu(h)\}$. Moreover, in this region the cutoff resolvent satisfies the estimate

$$\|\chi R(z)\chi\|_{L^2(X)\to L^2(X)} \leq \frac{\tilde{C}}{\mu(h)},$$

with $\tilde{C}$ depending on χ.

Proof. – We use the identity from Lemma 5.5, with $z_0 = E \pm i0$. Rearranging, we find (all norms here are $L^2(X) \to L^2(X)$)

$$\begin{aligned}\|\chi R(z)\chi\| \leq \|\chi R(E \pm i0)\chi\| &+ 2|p(z) - E| \|\chi R(z)\chi\| \|\chi R(E \pm i0)\chi\| \\ &+ \|(1-\chi_1)(\chi R_0(z)\chi - \chi R_0(E \pm i0)\chi)(1-\chi_1)\| \\ &+ \|\chi R(z)\chi\| \|[h^2\Delta, \chi_1])(\chi R_0(z)\chi - \chi R_0(E \pm i0)\chi)(1-\chi_1)\| \\ &+ \|(1-\chi_1)\,(\chi(R_0(z)\chi - \chi R_0(E \pm i0)\chi)\,[h^2\Delta, \chi_1\| \|\chi R(E \pm i0)\chi\| \\ &+ \|\chi R(z)\chi\| \|\chi R(E \pm i0)\chi\| \|[h^2\Delta, \chi_1]\,(\chi R_0(z)\chi - \chi R_0(E \pm i0)\chi)\,[h^2\Delta, \chi_1]\|.\end{aligned}$$

By writing this bound in this detailed fashion we hope to indicate the importance of the improved estimate (5.11) as compared to (5.10), so that, for example,

$$\begin{aligned}(5.12)\quad &\|[h^2\Delta, \chi_1](\chi R_0(z)\chi - \chi R_0(E \pm i0)\chi)(1-\chi_1)\| \\ &\qquad = \|[h^2\Delta, \chi_1](R_0(z)\chi - R_0(E \pm i0)\chi)(1-\chi_1)\| \leq C d_h(z, E \pm i0)/h.\end{aligned}$$

Using the bound on $\|\chi R(E \pm i0)\chi\|$ from the assumptions along with bounds of Proposition 5.4, we find

$$\begin{aligned}\|\chi R(z)\chi\| \leq \frac{C}{\mu(h)} &+ \frac{C d_h(z, E \pm i0)}{\mu(h)} \|\chi R(z)\chi\| + \frac{C d_h(z, E \pm i0)}{h\mu(h)} \\ &+ C d_h(z, E \pm i0) \left(\frac{1}{h} + \frac{1}{\mu(h)}\right) \|\chi R(z)\chi\|.\end{aligned}$$

Here we have also bounded $|p(z) - E| \leq d_h(z, E \pm i0)$, which is weaker than the estimate from Lemma 5.2 since we will have $d_h(z, E \pm i0) = O(\mu(h))$. If we choose C' sufficiently small, the coefficients of $\|\chi R(z)\chi\|$ on the right hand side above will be small enough that the terms with $\|\chi R(z)\chi\|$ can be absorbed in the left hand side, proving the result. □

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