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ON BLOW-ANALYTIC EQUIVALENCE

by

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Abstract. — We study function and map germs which become real analytic after composing with a locally finite number of blowing-ups. The main purpose of this article is to give a reasonably self-contained survey on this topic, including historical details concerning the development of this area, motivation, recent results, and important open problems.

***Résumé* (Sur l'équivalence par éclatements).** — Nous étudions les fonctions et applications qui deviennent analytiques après composition avec un nombre localement fini d'éclatements. Le but principal de cet article est de donner de cette théorie un panorama indépendant contenant l'historique, des motivations, ainsi que des résultats récents et des questions ouvertes.

Blowing-up is a fundamental notion in singularity theory, algebraic geometry and analytic geometry. In this article we study function and map germs which become real analytic when composed with a locally finite number of blowing-ups. For example, the real function germ

$$f(x,y) = x^3/(x^2+y^2) : \mathbb{R}^2, 0 \longrightarrow \mathbb{R}, 0$$

becomes real analytic when composed with the blowing-up of $\mathbb{R}^2$ at the origin. As we shall see in §2, a number of classical examples in calculus are also blow-analytic.

Motivated by the classification problem of analytic function germs, T.-C. Kuo [**30**] introduced the notions of blow-analytic map and blow-analytic equivalence. He discovered a finite classification theorem for analytic function germs with isolated singularities and also found some important triviality theorems. Ever since, several people have been working in this field and have obtained new results and gained deeper insight. The purpose of this article is to give a reasonably self-contained survey on this topic, including some open problems.

The article is organized as follows. In §1, we present the motivation of the classification problem. We give a naive definition of blow-analytic map in §2, and discuss

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another variant in the rest of the section and also in §4. A number of blow-analytic triviality theorems are stated and discussed in §3. The arc lifting property, which is of fundamental importance for us, is defined in §5. Some blow-analytic invariants for analytic function germs are defined and discussed in §6. In §§7–9, we investigate properties of blow-analytic maps themselves. Relations between Lipschitz maps and blow-analytic maps are also discussed in §7. In particular, a blow-analytic homeomorphism (and even a blow-analytic isomorphism which we define in §8) need not be bilipschitz. This fact was first discovered by S. Koike [**27**]. A blow-analytic homeomorphism may have exotic pathologies; this is illustrated by the examples in §8. We then introduce a strengthened notion, called blow-analytic isomorphism, and discuss the behaviour of their jacobians. In §9.1, we present a version of the inverse mapping theorem for blow-analytic isomorphisms. Several open problems are stated in the last section.

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1. Motivations

The notion of blow-analytic equivalence arises from attempts to classify analytic function germs. To begin with, one is tempted to use the following equivalence relations.

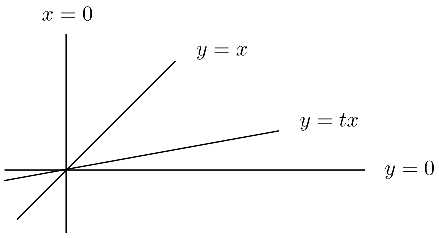
Definition 1.1. — Let $k = 0, 1, 2, \dots, \infty, \omega$. We say that two analytic function germs $f, g : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ are C^k*-equivalent* if there is a C^k-diffeomorphism germ $h : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ so that $f = g \circ h$.

However, the following example, due to H. Whitney, shows that the C^1-equivalence is already too fine for the purpose of classification.

***Example 1.2* (see [41])**. — Consider the functions

$$f_t : \mathbb{R}^2, 0 \longrightarrow \mathbb{R}, \quad 0 < t < 1, \quad f_t(x, y) = xy(y - x)(y - tx).$$

Then f_t is C^1-equivalent to $f_{t'}$, if and only if $t = t'$.



Proof. — If f_t and $f_{t'}$ are C^1-equivalent, then there is a C^1-diffeomorphism

$$h : \mathbb{R}^2, 0 \longrightarrow \mathbb{R}^2, 0$$

with $f_{t'} = f_t \circ h$. We may assume that h is a linear isomorphism of $\mathbb{R}^2$; replace h by its linear approximation at the origin, the tangent map $dh_0 : T_0\mathbb{R}^2 \to T_0\mathbb{R}^2$. The zero set of $f_{t'}$ is sent isomorphically onto the zero set of f_t. Let

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ t \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

be the corresponding four direction (column) vectors for f_t. The classical *cross ratio* of v_1, v_2, v_3, v_4 is defined by

$$\sigma_{ijk\ell} = \frac{\det(v_i v_k) \cdot \det(v_j v_\ell)}{\det(v_i v_\ell) \cdot \det(v_j v_k)}, \quad \text{where } \{i, j, k, \ell\} = \{1, 2, 3, 4\}.$$

It is clear that the cross ratio is invariant under the action of the general linear group and by multiplying each vector by a nonzero constant.

Similarly we consider

$$v'_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v'_2 = \begin{pmatrix} 1 \\ t' \end{pmatrix}, \quad v'_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v'_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for $f_{t'}$ and the corresponding cross ratios denoted by $\sigma'_{ijk\ell}, \{i, j, k, \ell\} = \{1, 2, 3, 4\}$.

Cross ratio may take only six values, corresponding to all permutations of v'_1, v'_2, v'_3, v'_4, in our situation

$$t', \quad \frac{1}{t'}, \quad 1 - t', \quad \frac{1}{1 - t'}, \quad -\frac{1 - t'}{t'}, \quad -\frac{t'}{1 - t'},$$

and similarly for f_t. It follows that $\sigma'_{1423} = t'$ should equate one of the corresponding values from f_t. Two of them are negative and two are bigger than one, so we remain with only two possibilities $t' = t$ or $t' = 1 - t$. The value $1 - t$ is realised by one of the following σ_{1243}, σ_{4312}, σ_{2134} and σ_{3421}. Observe that the region $f_t \geq 0$ should also be preserved.

These imply $t = t'$. □

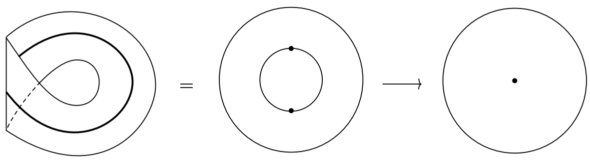
On the other hand, if we set $x = y'/a$, $y = ((t - 1)x' + y')/a$, where a is a constant with $a^4 + (1 - t)^2 = 0$, then

$$xy(y - x)(y - tx) = x'y'(y' - x')\big(y' - (1 - t)x'\big)$$

and thus f_t and f_{1-t} are analytically equivalent as complex functions.

As for the C^0-equivalence, the functions $(x, y) \mapsto x^2 + y^{2k+1}$, $k \geq 1$, for instance, are C^0-equivalent to the regular function $(x, y) \mapsto y$. Hence it is hopeless to expect a decent classification theory.

Now we consider the blowing-up $\pi : M \to \mathbb{R}^2$ at 0. This map is illustrated by the following picture:



The anti-podal points of the inner circle of the annulus in the middle figure are identified to obtain the Möbius strip in the left figure. Collapsing the inner circle to a point yields a mapping from the Möbius strip to the disk at the right. This is called the blowing-up of the disk at its centre. One can introduce local coordinates on the Möbius strip and express the above as a real analytic map, as follows. Let

$$M = \big\{(x,y) \times [\xi : \eta] \in D^2 \times P^1 : x\eta = y\xi\big\},$$

where D^2 is a 2-dimensional disk and P^1 is the real projective line. The restriction of the projection $(x,y) \times [\xi : \eta] \mapsto (x,y)$ to M is the desired π. For the functions f_t in example 1.2, all $f_t \circ \pi$ are C^ω- equivalent to each other (see [**30**]).

2. Definition of blow-analytic map

2.1. A naive introduction

***Definition 2.1* (Blowing-up).** — Let U be a disk in $\mathbb{R}^n$ centered at 0 with analytic coordinates $x_1, \dots, x_n$, and let $C \subset U$ be the locus $x_1 = \cdots = x_k = 0$. Let $[\xi_1 : \cdots : \xi_k]$ be homogeneous coordinates of the real projective space P^{k-1} and let $\widetilde{U} \subset U \times P^{k-1}$ be the nonsingular manifold defined by

$$\widetilde{U} = \big\{(x_1, \dots, x_n) \times [\xi_1 : \cdots : \xi_k] : x_i\xi_j = x_j\xi_i,\ 1 \leq i,j \leq k\big\}.$$

The projection $\pi : \widetilde{U} \to U$ on the first factor is clearly an isomorphism away from C. The manifold $\widetilde{U}$, together with the map $\pi : \widetilde{U} \to U$, is called the *blowing-up* with nonsingular center C. It is well-known that the blowing-up $\pi : \widetilde{U} \to U$ is independent of the coordinates chosen in U. This allows us to globalize the definition. Let M be a real analytic manifold of dimension n and C a submanifold of codimension k. Let $\{U_\alpha\}$ be a collection of disks in M covering C such that in each disc U_α the submanifold $C \cap U_\alpha$ may be given as the locus $(x_1 = \cdots = x_k = 0)$, and let $\pi_\alpha : \widetilde{U}_\alpha \to U_\alpha$ be the blowing-up with center $C \cap U_\alpha$. We then have isomorphisms

$$\pi_{\alpha\beta} : \pi_\alpha^{-1}(U_\alpha \cap U_\beta) \longrightarrow \pi_\beta^{-1}(U_\alpha \cap U_\beta),$$

and we can patch together the $\widetilde{U}_\alpha$ to form a manifold $\widetilde{U} = \bigcup_{\pi_{\alpha\beta}} \widetilde{U}_\alpha$ with map $\pi : \widetilde{U} \to \bigcup U_\alpha$. Since π is an isomorphism away from C, we can take

$$\widetilde{M} = \widetilde{U} \cup_\pi (M - C), \quad \widetilde{M}$$

together with the map $\pi : \widetilde{M} \to M$ extending π on $\widetilde{U}$ and the identity on $M - C$, is called the *blowing-up* of M with center C. We call $E = \pi^{-1}(C)$ the *exceptional divisor* of the blowing-up π.

Let M be a real analytic manifold. Take a function f defined on M except possibly on some nowhere dense subset of M. We often denote this function by

$$f : M \dashrightarrow \mathbb{R},$$

and say that f is defined almost everywhere.

Definition 2.2. — Let $\pi : \widetilde{M} \to M$ be a locally finite composition of blowing-ups with nonsingular centers. We say that $f : M \dashrightarrow \mathbb{R}$ is *blow-analytic via* π if $f \circ \pi$ has an analytic extension on $\widetilde{M}$. We say that f is *blow-analytic* if there is $\pi : \widetilde{M} \to M$, a locally finite composition of blowing-ups with nonsingular centers, so that f is blow-analytic via π.

Many functions, used as counterexamples in Calculus, are blow-analytic. Some of them are as follows.

Example 2.3. — (i) $f(x,y) = \frac{xy}{x^2+y^2}$, $(x,y) \neq (0,0)$. This function f is not continuously extendable at the origin. It is clearly blow-analytic via the blowing-up at the origin (for instance $f(xy,y) = \frac{x}{x^2+1}$ becomes analytic).

(ii) $f(x,y) = \frac{x^2y}{x^4+y^2}$, $(x,y) \neq (0,0)$. This function is not continuously extendable at the origin, although all directional derivatives exist, if we define $f(0,0) = 0$. This function f is also blow-analytic.

(iii) $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$, $(x,y) \neq (0,0)$. This function is continuously extendable at the origin, but the second order derivatives depend on the order of differentiation:

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

This function f is also blow-analytic via the blowing-up at the origin.

***Example 2.4* (see [2]).** — Another typical example of blow-analytic function is $f(x,y) = \sqrt{x^4+y^4}$. The zero set of $z^3 + (x^2+y^2)z + x^3$ is also the graph of a blow-analytic function $z = g(x,y)$. In fact, by applying Cardano's formulas, we obtain

$$g(x,y) = -\frac{2^{\frac{1}{3}}(x^2+y^2)}{h(x,y)} + \frac{h(x,y)}{3 \cdot 2^{\frac{1}{3}}}, \quad h(x,y) = \left(-27x^3 + \sqrt{729x^6 + 108(x^2+y^2)^3}\,\right)^{\frac{1}{3}},$$

and g is blow-analytic via the blow-up at the origin. It is also possible to show that g is blow-analytic using theorem 4.9.

The notion of blow-analytic map between real analytic manifolds is defined using local coordinates.

Definition 2.5. — Let X, Y be real analytic manifolds. We say that $f : X \to Y$ is a *blow-analytic homeomorphism* (bah, for short) if f is a homeomorphism and both f and f^{-1} are blow-analytic.

Definition 2.6. — Let $f, g : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be analytic functions. We say that f and g are *blow-analytically equivalent* if there is a blow-analytic homeomorphism $h : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ so that $f = g \circ h$.

Note that h preserves the zero sets of f and g. The equivalence relation determined by the above relation on the set of analytic function-germs $\mathbb{R}^n, 0 \to \mathbb{R}, 0$ will be called *blow-analytic equivalence.*