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BEHAVIOUR OF SOME HODGE INVARIANTS BY MIDDLE CONVOLUTION

by Nicolas Martin

Abstract. — Following a paper of Dettweiler and Sabbah, this article studies the behaviour of various Hodge invariants by middle additive convolution with a Kummer module. The main result gives the behaviour of Hodge numerical data at infinity. We also give expressions for Hodge numbers and degrees of some Hodge bundles without making the hypothesis of scalar monodromy at infinity, which generalizes the results of Dettweiler and Sabbah.

Résumé (*Comportement d'invariants de Hodge par convolution intermédiaire*). — Suivant les travaux de Dettweiler et Sabbah, cet article s'intéresse au comportement d'invariants de Hodge par convolution intermédiaire additive par un module de Kummer. Le résultat principal précise le comportement de données numériques de Hodge à l'infini. Nous explicitons également le comportement des nombres de Hodge et des degrés de certains fibrés de Hodge sans faire l'hypothèse de monodromie scalaire à l'infini, généralisant ainsi les résultats de Dettweiler et Sabbah.

The initial motivation to study the behaviour of various Hodge invariants by middle additive convolution is Katz's algorithm [5], which makes it possible to reduce a rigid irreducible local system $\mathscr{L}$ on a punctured projective line to a rank-one local system. This algorithm is a successive application of tensor products with a rank-one local system and middle additive convolutions with a Kummer local system and terminates with a rank-one local system. We assume

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that the monodromy at infinity of $\mathscr{L}$ is scalar, so this property is preserved throughout the algorithm.

If we assume that the eigenvalues of the local monodromies of $\mathscr{L}$ have absolute value 1, such a local system underlies a variation of polarized complex Hodge structure unique up to a shift of the Hodge filtration [12, 1], and this property is preserved at each step of Katz's algorithm. The work of Dettweiler and Sabbah [2] is devoted to computing the behaviour of Hodge invariants at each step of the algorithm.

Our purpose in this article is to complement the previous work of Dettweiler and Sabbah without assuming that the monodromy at infinity is scalar, and to do that we take up the notation introduced in [2, §2.2] and recalled in §1.1. More precisely, our main result consists in making explicit the behaviour of the nearby cycle local Hodge numerical data at infinity by middle additive convolution with the Kummer module $\mathcal{K}_{\lambda_0}$. Considering a regular holonomic $\mathscr{D}_{\mathbb{A}^1}$-module M verifying various assumptions, whose singularities at finite distance belong to $\boldsymbol{x} = \{x_1, \dots, x_r\}$, we denote by $\mathrm{MC}_{\lambda_0}(M) = M *_{\mathrm{mid}} \mathcal{K}_{\lambda_0}$ this convolution and show the following theorem (see §1.1 for the notation and assumptions). In the following, we set $\gamma_0 \in (0,1)$ such that $\exp(-2i\pi\gamma_0) = \lambda_0 \neq 1$.

Theorem 1. — *Let $\mathscr{M}^{\min}$ be the $\mathscr{D}_{\mathbb{P}^1}$-module minimal extension of M at infinity. Given $\gamma \in [0,1)$ and $\lambda = \exp(-2i\pi\gamma)$, we have:*

$$\nu^p_{\infty,\lambda,\ell}(\mathrm{MC}_{\lambda_0}(M)) = \begin{cases} \nu^{p-1}_{\infty,\lambda\lambda_0,\ell}(M) & \text{if } \gamma \in (0, 1-\gamma_0) \\ \nu^{p}_{\infty,\lambda\lambda_0,\ell}(M) & \text{if } \gamma \in (1-\gamma_0, 1) \\ \nu^{p}_{\infty,\lambda_0,\ell+1}(M) & \text{if } \lambda = 1 \\ \nu^{p-1}_{\infty,1,\ell-1}(M) & \text{if } \lambda = \overline{\lambda_0},\ \ell \geq 1 \\ h^p H^1(\mathbb{P}^1, \mathrm{DR}\,\mathscr{M}^{\min}) & \text{if } \lambda = \overline{\lambda_0},\ \ell = 0. \end{cases}$$

This result has applications beyond Katz's algorithm since it enables us to give another proof of a theorem of Fedorov [3], which completely determines the Hodge numbers of the variations of Hodge structures corresponding to hypergeometric differential equations; this work is developed in [7].

In addition, we get general expressions for Hodge numbers h^p of the variation and degrees δ^p of some Hodge bundles (recalled in §1.1), which generalize those of Dettweiler and Sabbah. The results are the following.

Theorem 2. — *The local invariants $h^p(\mathrm{MC}_{\lambda_0}(M))$ are given by:*

$$h^p(\mathrm{MC}_{\lambda_0}(M)) = \sum_{\gamma\in[0,\gamma_0)} \nu^p_{\infty,\lambda}(M) + \sum_{\gamma\in[\gamma_0,1)} \nu^{p-1}_{\infty,\lambda}(M) + h^p H^1(\mathbb{A}^1, \mathrm{DR}\,M) - \nu^{p-1}_{\infty,\lambda_0,\mathrm{prim}}(M).$$

THEOREM 3. — *The global invariants $\delta^p(\mathrm{MC}_{\lambda_0}(M))$ are given by:*

$$\delta^p(\mathrm{MC}_{\lambda_0}(M)) = \delta^p(M) + \sum_{\gamma\in[\gamma_0,1)} \nu^p_{\infty,\lambda}(M) - \sum_{i=1}^{r} \Big(\mu^p_{x_i,1}(M) + \sum_{\gamma\in(0,1-\gamma_0)} \mu^{p-1}_{x_i,\lambda}(M)\Big).$$

1. Hodge numerical data and modules of normal crossing type

1.1. Hodge invariants. — In this section, we recall the definition of local and global invariants introduced in [2, §2.2] (all references to [2] are made to the published paper). Let Δ be a disc centred at 0 with coordinate t and let $(V, F^\bullet V, \nabla)$ be a variation of the polarizable Hodge structure on $\Delta^* = \Delta \setminus \{0\}$ of weight 0. We denote by M the corresponding $\mathscr{D}_\Delta$-module minimal extension at 0.

Nearby cycles. — For $a \in (-1, 0]$ and $\lambda = e^{-2i\pi a}$, the nearby cycle space at the origin $\psi_\lambda(M)$ is equipped with the nilpotent endomorphism $\mathrm{N} = -2i\pi(t\partial_t - a)$, and the Hodge filtration is such that $\mathrm{N}F^p\psi_\lambda(M) \subset F^{p-1}\psi_\lambda(M)$. The monodromy filtration induced by N enables us to define the spaces $\mathrm{P}_\ell\psi_\lambda(M)$ of primitive vectors, equipped with a polarizable Hodge structure (see [9, §3.1.a] for more details). The nearby cycle local Hodge numerical data are defined by

$$\nu^p_{\lambda,\ell}(M) := h^p(\mathrm{P}_\ell\psi_\lambda(M)) = \dim \mathrm{gr}^p_F \mathrm{P}_\ell\psi_\lambda(M),$$

with the relation $\nu^p_\lambda(M) := h^p\psi_\lambda(M) = \sum_{\ell\geq 0}\sum_{k=0}^{\ell} \nu^{p+k}_{\lambda,\ell}(M)$. We set

$$\nu^p_{\lambda,\mathrm{prim}}(M) := \sum_{\ell\geq 0} \nu^p_{\lambda,\ell}(M) \quad\text{and}\quad \nu^p_{\lambda,\mathrm{coprim}}(M) := \sum_{\ell\geq 0} \nu^{p+\ell}_{\lambda,\ell}(M).$$

Vanishing cycles. — For $\lambda \neq 1$, the vanishing cycle space at the origin is given by $\phi_\lambda(M) = \psi_\lambda(M)$ and comes with N and F^p, as before. For $\lambda = 1$, the Hodge filtration on $\phi_1(M)$ is such that $F^p\mathrm{P}_\ell\phi_1(M) = \mathrm{N}(F^p\mathrm{P}_{\ell+1}\psi_1(M))$. Similarly to nearby cycles, the vanishing cycle local Hodge numerical data is defined by

$$\mu^p_{\lambda,\ell}(M) := h^p(\mathrm{P}_\ell\phi_\lambda(M)) = \dim \mathrm{gr}^p_F \mathrm{P}_\ell\phi_\lambda(M).$$

Degrees δ^p. — For a variation of polarizable Hodge structure $(V, F^\bullet V, \nabla)$ on $\mathbb{A}^1 \setminus \boldsymbol{x}$, we denote by M the underlying $\mathscr{D}_{\mathbb{A}^1}$-module minimal extension at each point of $\boldsymbol{x}$. The Deligne extension V^0 of (V, ∇) on $\mathbb{P}^1$ is contained in M, and we set

$$\delta^p(M) = \deg \mathrm{gr}^p_F V^0.$$

In this paper, we are mostly interested in the behaviour of the nearby cycle local Hodge numerical data at infinity by middle convolution with the Kummer module $\mathcal{K}_{\lambda_0} = \mathscr{D}_{\mathbb{A}^1}/\mathscr{D}_{\mathbb{A}^1}\cdot(t\partial_t - \gamma_0)$, with $\gamma_0 \in (0, 1)$ such that $\exp(-2i\pi\gamma_0) = \lambda_0$.