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A PERSISTENTLY SINGULAR MAP OF $\mathbb{T}^n$ THAT IS C^1 ROBUSTLY TRANSITIVE

BY JUAN C. MORELLI

ABSTRACT. — We exhibit a C^1 robustly transitive endomorphism displaying critical points on the n-dimensional torus.

RÉSUMÉ (*Une application C^1 robustement transitive dans $\mathbb{T}^n$ avec singularités persistantes*). — Nous présentons un endomorphisme avec points critiques sur le tore à n dimensions qui est C^1 robustement transitif.

1. Introduction

Whenever we think about dynamical systems' properties the concepts of *stability* and *robustness* almost inevitably come to mind. Loosely speaking, we can say that stability implies the same dynamics for maps sufficiently close to each other, and robustness implies the same behavior relative to a specific property for maps sufficiently close to each other.

This work in particular is focused on the study of robust transitivity, by *transitive* meaning the existence of a forward dense orbit of a point. This may at first sight seem an unexciting topic since a fair amount of results concerning robust transitivity are known. Nonetheless, the class of maps aimed for, the

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singular endomorphisms about which little to nothing is known, as well as taking on the high dimensional context, are undoubtedly a fresh approach to the subject.

To set ideas in order we list up the most relevant known results about the topic.

We begin by summing up the most studied case: robust transitivity of diffeomorphisms. The image provided by known results is fairly complete. In the setting of surfaces, it is shown in [15] that robust transitivity implies the diffeomorphism to be Anosov and that the only surface that supports them is $\mathbb{T}^2$. Later on, in the arbitrary dimensional setting it is proven in [5] that robust transitivity implies a dominated splitting on the tangent spaces (i.e., weak hyperbolicity).

Going further, next comes robust transitivity of regular endomorphisms (not globally but locally invertible). The image we have about these is somewhat less complete; yet we know that volume expansion is a necessary but not sufficient condition for C^1 robust transitivity according to [11], who also give a sufficient condition for the case of manifold $\mathbb{T}^n$.

Carrying on, finally, there is the least studied case, robust transitivity of singular maps (nonempty critical sets). Until 2013 nothing had ever been written on the topic. Then, [2] showed the first example of a C^1 transitive singular map of $\mathbb{T}^2$. The second example was given only in 2016 by [9], who exhibited a C^1 robustly transitive map of $\mathbb{T}^2$ with a persistent critical set. Nothing more than these two examples was known at the time. Even so, there have been recent further advances on robust transitivity of singular surface endomorphisms: in 2018 [10] presented an example on $\mathbb{T}^2$, whose robust transitivity depends on the class of differentiability, and in 2019 [12] and [13] set the *state of the art* proving that partial hyperbolicity is a necessary condition, that the only surfaces that support them are $\mathbb{T}^2$ and the Klein bottle, and that they belong to the homotopy class of a linear map with an eigenvalue of modulus larger than 1. Finally, with respect to singular endomorphisms in high dimensions, the only known result was given by [14] who extended the result that appeared in [10] to $\mathbb{T}^n$.

In the spirit of generalizing known results in low to higher dimensions, the survey contained in our paper shows that the example exhibited in [9] can also be extended to $\mathbb{T}^n$, resulting in the first known example of a persistently singular endomorphism that is robustly transitive in the C^1 topology and supported on a manifold of dimension larger than 2.

The main result can be stated as:

THEOREM 1.1. — *Given $n \geq 2$, there exists a persistently singular endomorphism supported on $\mathbb{T}^n$ that is C^1 robustly transitive.*

By persistently singular endomorphism we mean a map f satisfying that there exists a C^1 neighborhood $\mathcal{U}_f$ of f such that every map g belonging to $\mathcal{U}_f$ displays critical points.

1.1. Sketch of the construction. — Start from an endomorphism induced by a diagonal expanding matrix with integer coefficients, with all but one direction strongly unstable and one central direction. Perturb the map to add a blending region that mixes everything getting the transitivity and then introduce artificially the critical points preserving the transitivity property. This entire construction is done in a robust way.

The author wants to remind the readers that the contents to follow are an adaptation of the surfaces' construction exhibited by [9, Section 2.2] to arbitrary dimensions. The proofs to some of the claims in our lemmas and theorems were, consequently, also inspired by [9]. Moreover, many of them can be adapted in a straightforward manner cleverly enough, but for the sake of a self-contained article all proofs will be explicitly provided here. Finally, if readers wish to get a lighter approach to our construction by considering the low-dimensional context first, they are invited to read the cited article.

2. Preliminaries

Some basic definitions are recalled at the beginning. If readers wish to gain more insight on the geometrical or dynamical background they can refer to [6] or [7].

Let M be a differentiable manifold of dimension m and $f : M \to M$ a differentiable endomorphism. The *orbit* of $x \in M$ is $\mathcal{O}(x) = \{f^n(x), n \in \mathbb{N}\}$. The map f is **transitive** if there exists a point $x \in M$ such that $\overline{\mathcal{O}(x)} = M$ and f is C^k**-robustly transitive** if there exists a neighborhood $\mathcal{U}_f$ of f in the C^k topology such that g is transitive for all g belonging to $\mathcal{U}_f$.

The proposition ahead is well known and of most practical use.

Proposition 2.1. — *If f is continuous then are equivalent:*

1. *f is transitive.*
2. *For all U, V open sets in M, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.*
3. *There exists a residual set R (countable intersection of open and dense sets) such that for all points $x \in R : \overline{\mathcal{O}(x)} = M$.*

2.1. Normally Hyperbolic (sub)manifolds. — We continue defining normally hyperbolic submanifolds in the sense of [1]. These kind of submanifolds for a given map are persistently invariant under perturbation, it allows defining dynamical systems within them. This will be the main usage we will make of them ahead in the paper. Their formal definition is as follows.

Let $f : M \to M$ be a C^1 diffeomorphism, $N \subset M$ a C^1 closed submanifold such that $f(N) = N$ (we say that N is *invariant*).

Definition 2.2. — We say that f is *normally hyperbolic at N* if there exists a splitting of the tangent bundle of M over N into three Df-invariant subbundles