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MULTIVARIABLE (φ, Γ)-MODULES AND REPRESENTATIONS OF PRODUCTS OF GALOIS GROUPS: THE CASE OF THE IMPERFECT RESIDUE FIELD

by Jishnu Ray, Feng Wei & Gergely Zábrádi

Abstract. — Let K be a complete discretely valued field with mixed characteristic $(0, p)$ and imperfect residue field k_α. Let Δ be a finite set. We construct an equivalence of categories between finite dimensional $\mathbb{F}_p$-representations of the product of Δ copies of the absolute Galois group of K and multivariable étale (φ, Γ)-modules over a multivariable Laurent series ring over k_α.

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Résumé (*(φ,Γ)-modules multivariables et représentations du produit du groupe de Galois: le cas des corps résiduels imparfaits*). — Soit K un corps discrètement valué à charactéristique mixte $(0,p)$ et un corps résiduel imparfait k_α. Soit Δ un ensemble fini. Nous établissons une équivalence de catégories entre des représentations de dimensions finies sur $\mathbb{F}_p$ du produit de Δ copies du groupe absolu de Galois de K et des (φ,Γ)-modules étales multivariables sur un anneau multivariable des séries Laurent sur k_α.

1. Introduction

1.1. Motivation of this work. — Fontaine's theory of (φ,Γ)-modules is a fundamental tool to describe and classify continuous representations of the Galois group of a finite extension of $\mathbb{Q}_p$ on a finite-dimensional $\mathbb{Q}_p$-vector space. With the help of Fontaine's theory of (φ,Γ)-modules, one can understand the p-adic and mod-p Langlands correspondence in the case of the general linear group GL_2 over the field $\mathbb{Q}_p$ of p-adic numbers, see [9, 10, 11, 13, 20, 21, 22, 41]. By invoking the theory of (φ,Γ)-modules, the p-adic and mod-p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ can be connected with p-adic and mod-p Galois representations of $\mathbb{Q}_p$. To extend the correspondence to other p-adic reductive groups beyond $\mathrm{GL}_2(\mathbb{Q}_p)$, one naturally wants to generalize Fontaine's theory of (φ,Γ)-modules. There has been conjectural progress in attempts to generalize p-adic Langlands beyond $\mathrm{GL}_2(\mathbb{Q}_p)$ along these lines; two kinds of multivariable versions of (φ,Γ)-modules can be found in the literature. Berger's multivariable (φ,Γ)-modules is an attempt to generalize p-adic Langlands for $\mathrm{GL}_2(F)$, where F is a finite extension of $\mathbb{Q}_p$ [6, 7]. The third author of this current work also defines multivariable (φ,Γ)-module over a m-variable Laurent series ring in an attempt to generalize p-adic Langlands for $\mathrm{GL}_m(\mathbb{Q}_p)$ [43, 49, 50]. One might also try to look at Zábrádi's multivariable (φ,Γ)-modules over Lubin–Tate extension to conjecturally understand p-adic Langlands for $\mathrm{GL}_m(F)$ [28]. It has become clear that essentially all of p-adic Hodge theory can be formulated in terms of (φ,Γ)-modules; moreover, this formulation has driven much recent progress in the subject and powered some notable applications in arithmetic geometry [17]. See [31] for a quick introduction to this circle of ideas or [42] for a more in-depth treatment. Multivariable (φ,Γ)-modules are also related [19, 32] to Scholze's theory of perfectoid spaces.

This paper can be considered as a complement to the third author's independent work [49] in which he shows that the category of continuous representations of the m^{th} direct product of the absolute Galois group of $\mathbb{Q}_p$ on finite dimensional $\mathbb{F}_p$-vector spaces (or $\mathbb{Z}_p$-modules and $\mathbb{Q}_p$-vector spaces, respectively) is equivalent to the category of étale multivariable (φ,Γ)-modules over a certain m-variable Laurent series ring over $\mathbb{F}_p$ (or over $\mathbb{Z}_p$ and over $\mathbb{Q}_p$, respectively). In the current paper, we will extend this equivalence of categories for continuous $\mathbb{F}_p$-representations of the m^{th} direct product of the absolute Galois group of a complete discretely valued field K with mixed characteristic $(0,p)$ whose residue field k_α is imperfect and has a finite p-basis, i.e., $[k_\alpha : k_\alpha^p] = p^d$ (for some $d \geq 1$). We plan to come back to the question of p-adic representations in the future. We expect applications of our results to p-adic Hodge theory of products of varieties over p-adic fields. To state our main theorem (Theorem 5.13) precisely, we need to review the third author's work on multivariable (φ,Γ)-modules [49] and his main theorem.

1.2. Zábrádi's work [49]. — Let F be a finite extension of $\mathbb{Q}_p$ with residue field k_F (which is perfect). For a finite set Δ, let $\mathcal{G}_{\mathbb{Q}_p,\Delta} := \prod_{\alpha\in\Delta} \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ denote the direct power of the absolute Galois group of $\mathbb{Q}_p$ indexed by Δ. We denote by $\mathrm{Rep}_{k_F}(\mathcal{G}_{\mathbb{Q}_p,\Delta})$ the category of continuous representations of the profinite group $\mathcal{G}_{\mathbb{Q}_p,\Delta}$ on finite dimensional k_F-vector spaces. For independent commuting variables X_α $(\alpha \in \Delta)$, we write

$$E_{\Delta,k_F} := k_F\, [\![X_\alpha | \alpha\in\Delta]\!]\, [X_\Delta^{-1}],$$

where $X_\Delta = \prod_{\alpha\in\Delta} X_\alpha$. For each element $\alpha\in\Delta$, we have the partial Frobenius φ_α and group $G_{K_\alpha} \cong \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ acting on the variable X_α in the usual way and commuting with the other variables X_β $(\beta\in\Delta\setminus\{\alpha\})$ in the ring E_{Δ,k_F} (some authors also write G_{K_α} as Γ_α). A $(\varphi_\Delta,\Gamma_\Delta)$-*module* (or a $(\varphi_\Delta, G_\Delta)$-module) over E_{Δ,k_F} is a finitely generated E_{Δ,k_F}-module D together with commuting semilinear actions of the operators φ_α and groups G_{K_α} $(\alpha \in \Delta)$. We say that D is *étale* if the map $\mathrm{id}\otimes\varphi_\alpha : \varphi_\alpha^* D \longrightarrow D$ is an isomorphism for all $\alpha\in\Delta$. The third author shows independently that $\mathrm{Rep}_{k_F}(\mathcal{G}_{\mathbb{Q}_p,\Delta})$ is equivalent to the category of étale $(\varphi_\Delta, G_\Delta)$-modules over E_{Δ,k_F}.

1.3. Andreatta's work [1] **and Scholl's work** [44]. — Let us review Scholl's work [44] and parts of Andreatta's work [1], where they work with single variable classical (φ,Γ)-module but over an imperfect residue field. Let K be a complete discretely valued field (with uniformizer p) of mixed characteristic $(0,p)$ with imperfect residue field k_K having a p-basis, i.e., $[k_K : k_K^p] = p^d$. Let $t_1, t_2, \ldots, t_d \in K$ be a lift of a p-basis $\overline{t_1}, \overline{t_2}, \ldots, \overline{t_d}$ of k_K. Define $K_\infty = \bigcup_n K(\mu_{p^n}, t_1^{1/p^n}, \ldots, t_d^{1/p^n})$, $G_K = \mathrm{Gal}(K_\infty/K)$ and $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K)$. Note that, in contrast with the perfect residue field case, G_K is not abelian. Scholl [44] and Andreatta [1] defined a field of norms E_K for K, and have shown that $E_K \cong k_K((\overline{\pi}))$, where $\varepsilon = \overline{\pi}+1 \in E_K$ is a compatible system of p-power roots of unity in K_∞ (cf. [44, Section 2.3]). Finally, Andreatta [1, Theorem 7.11] showed that $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_K)$ is equivalent to the category of (single variable, i.e., classical) étale (φ, G_K)-module over E_K.