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RESOLVENT ESTIMATES ON ASYMPTOTICALLY CYLINDRICAL MANIFOLDS AND ON THE HALF LINE

BY TANYA J. CHRISTIANSEN AND KIRIL DATCHEV

ABSTRACT. – Manifolds with infinite cylindrical ends have continuous spectrum of increasing multiplicity as energy grows, and in general embedded resonances (resonances on the real line, embedded in the continuous spectrum) and embedded eigenvalues can accumulate at infinity. However, we prove that if geodesic trapping is sufficiently mild, then the number of embedded resonances and eigenvalues is finite, and moreover the cutoff resolvent is uniformly bounded at high energies. We obtain as a corollary the existence of resonance free regions near the continuous spectrum.

We also obtain improved estimates when the resolvent is cut off away from part of the trapping, and along the way we prove some resolvent estimates for repulsive potentials on the half line which may be of independent interest.

RÉSUMÉ. – Les variétés à bouts infinis cylindriques ont du spectre continu dont la multiplicité est croissante en fonction de l'énergie, et en général les résonances plongées (les résonances sur l'axe réel, plongées dans le spectre continu) et les valeurs propres plongées peuvent s'accumuler à l'infini. Cependant, on démontre que si les géodésiques sont suffisamment peu captées, alors le nombre de résonances plongées et de valeurs propres plongées est fini, et en plus la résolvante tronquée est uniformément bornée en hautes énergies. On obtient comme corollaire l'existence de certaines régions sans résonance près du spectre continu.

On obtient aussi des estimations améliorées lorsque la résolvante est tronquée loin de certaines géodésiques captées, et, en chemin, on démontre des estimations de la résolvante pour des potentiels répulsifs sur la demi-droite, qui peuvent avoir leur intérêt propre.

1. Introduction

1.1. Resolvent estimates for manifolds with infinite cylindrical ends

The high energy behavior of the Laplacian on a manifold of infinite volume is, in many situations, well known to be related to the geometry of the *trapped set*; this is the set of bounded maximally extended geodesics. In the best understood cases, such as when the manifold has asymptotically Euclidean or hyperbolic ends (see [57, §3] for a recent survey),

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the trapped set is compact. Some results have been obtained for more general trapped sets (e.g., manifolds with cusps were studied in [7]) but less detailed information is available.

In this paper we study manifolds with infinite asymptotically cylindrical ends, which have noncompact trapped sets. A motivation for this study comes from waveguides and quantum dots connected to leads. The spectral geometry of these is closely related to that of asymptotically cylindrical manifolds, and they appear in certain models of electron motion in semiconductors and of propagation of electromagnetic and sound waves. We give just a few pointers to the physics and applied math literature here [34, 46, 47, 25, 2]. In [9], we prove analogues of some of the results below for suitable (star-shaped) waveguides.

The fundamental example of a manifold with cylindrical ends is the Riemannian product $\mathbb{R} \times \mathbb{S}^1$, which has an unbounded trapped set consisting of the circular geodesics. We are interested in the behavior of the resolvent of the Laplacian (and its meromorphic continuation, when this exists) for perturbations of such cylinders and their generalizations. As we discuss below, this behavior can sometimes be very complicated, but we show that if some geometric properties of the manifold are favorable, then the resolvent is uniformly bounded at high energy. In the companion paper [10], we study the closely related problem of long time wave asymptotics on such manifolds.

We begin with an illustration of a more general theorem to follow, by stating a high energy resolvent estimate for two kinds of *mildly trapping* manifolds (X, g) with infinite cylindrical ends.

Example 1. – Let (r, θ) be polar coordinates in $\mathbb{R}^d$ for some $d \geq 2$, and let

$$X = \mathbb{R}^d, \qquad g_0 = dr^2 + F(r)dS,$$

where dS is the usual metric on the unit sphere, $F(r) = r^2$ near $r = 0$, and F' is compactly supported on some interval $[0, R]$ and positive on $(0, R)$; see Figure 1.1.

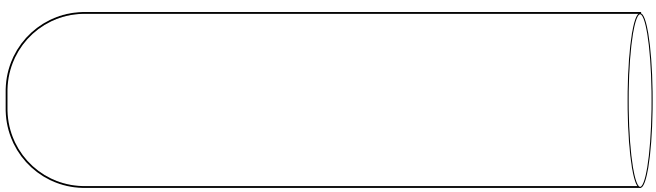

Figure 1.1. A cigar-shaped warped product.

Then for $r(t) > 0$ all g_0-geodesics obey

$$\ddot{r}(t) := \frac{d^2}{dt^2} r(t) = 2|\eta|^2 F'(r(t))F(r(t))^{-2} \geq 0,$$

where $r(t)$ is the r coordinate of the geodesic at time t and η is the angular momentum. Consequently, the only trapped geodesics are the ones with $\dot{r}(t) \equiv F'(r(t)) \equiv 0$, that is the circular ones in the cylindrical end. This is the smallest amount of trapping a manifold with a cylindrical end can have.

Let g be any metric such that $g - g_0$ is supported in $\{(r, \theta) \mid r < R\}$, and such that g and g_0 have the same trapped geodesics. For example we may take $g = g_0 + cg_1$, where g_1 is any symmetric two-tensor with support in $\{(r, \theta) \mid r < R\}$, and $c \in \mathbb{R}$ is chosen sufficiently small depending on g_1. Alternatively, we may take $g = dr^2 + g_S(r)$, where $g_S(r)$ is a smooth family of metrics on the sphere such that $g_S(r) = r^2 dS$ near $r = 0$ and $g_S(r) = F(r)dS$ near $r \geq R$, and such that $\partial_r g_S(r) > 0$ on $(0, R)$. This way we can construct examples where $g - g_0$ is not small.

EXAMPLE 2. – Let (X, g_H) be a convex cocompact hyperbolic surface, such as the symmetric hyperbolic 'pair of pants' surface with three funnels depicted in Figure 1.2.

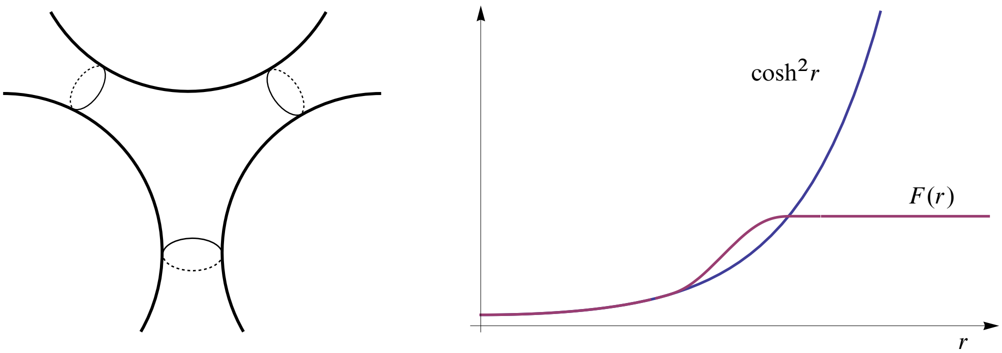

FIGURE 1.2. A hyperbolic surface (X, g_H) with three funnels, and a modification of the metric which changes the funnel ends to cylindrical ends.

In particular, there is a compact set $N \subset X$ (the convex core of X) such that

$$X \setminus N = (0, \infty)_r \times Y_y, \qquad g_H|_{X \setminus N} = dr^2 + \cosh^2 r \, dy^2,$$

where Y is a disjoint union of $k \geq 1$ geodesic circles (possibly having different lengths).

We modify the metric in the funnel ends so as to change them into cylindrical ends in the following way. Take g such that

$$g|_N = g_H|_N, \qquad g|_{X \setminus N} = dr^2 + F(r) dy^2,$$

where $F(r) = \cosh^2 r$ near $r = 0$, and F' is compactly supported and positive on the interior of the convex hull of its support.

To obtain higher dimensional examples, we can take (X, g_H) to be a conformally compact manifold of constant negative curvature, with dimension $d \geq 3$, but in this case we need the additional assumption that the dimension of the limit set is less than $(d-1)/2$. The construction of g now becomes more complicated and we give it in §3.3 below.

Our first result concerns only the above examples.

THEOREM 1.1. – *Let (X, g) be as in Example 1 or 2 above, and let $\Delta \leq 0$ be its Laplacian. There is $z_0 > 0$ such that for any $\chi \in C_c^\infty(X)$ there is $C > 0$ such that*

$$\|\chi(-\Delta - z)^{-1}\chi\|_{L^2(X) \to L^2(X)} \leq C \tag{1.1}$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq z_0$ *and* $\operatorname{Im} z \neq 0$.