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SIEVE WEIGHTS AND THEIR SMOOTHINGS

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ABSTRACT. – We obtain asymptotic formulas for the $2k$th moments of partially smoothed divisor sums of the Möbius function. When $2k$ is small compared with A, the level of smoothing, then the main contribution to the moments comes from integers with only large prime factors, as one would hope for in sieve weights. However if $2k$ is any larger, compared with A, then the main contribution to the moments comes from integers with quite a few prime factors, which is not the intention when designing sieve weights. The threshold for "small" occurs when $A = \frac{1}{2k}\binom{2k}{k} - 1$.

One can ask analogous questions for polynomials over finite fields and for permutations, and in these cases the moments behave rather differently, with even less cancelation in the divisor sums. We give, we hope, a plausible explanation for this phenomenon, by studying the analogous sums for Dirichlet characters, and obtaining each type of behavior depending on whether or not the character is "exceptional".

RÉSUMÉ. – On obtient des formules asymptotiques pour les $2k$-ièmes moments de quelques sommes partiellement lissées de la fonction de Möbius sur les diviseurs d'un entier. Quand $2k$ est petit en comparaison avec A, qui est le niveau de lissage, alors la contribution principale aux moments provient des entiers n'ayant que de grands facteurs premiers, comme on l'espérait pour un poids de crible. Cependant, si $2k$ est plus grand en comparaison avec A, alors la contribution principale aux moments provient des entiers ayant beaucoup de facteurs premiers, ce qui n'est pas l'intention quand on crée des poids de crible. La valeur seuil pour « petit » est $A = \frac{1}{2k}\binom{2k}{k} - 1$.

On peut aussi poser des questions analogues pour les polynômes sur des corps finis et pour les permutations, et dans ces cas les moments se comportent de façon assez différente, avec moins d'annulations dans les sommes de diviseurs. On donne, on espère, une explication plausible pour ce phénomène, en étudiant les sommes analogues pour les caractères de Dirichlet, et en obtenant chaque type de comportement selon le caractère « exceptionnel » ou non.

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1. Introduction

Sieve methods are a set of techniques which give upper and lower bounds for the number of elements of a set of integers $\mathscr{A}$ which have no 'small' prime factors. Their key benefit is that they are very flexible - one can obtain bounds of the correct order of magnitude for many interesting sets $\mathscr{A}$, even though obtaining asymptotic formulae looks completely hopeless. In particular, they are typically very effective at obtaining upper bounds for the number of primes in sets $\mathscr{A}$ of interest which are only worse than the conjectured truth by a constant factor.

One of the most important sieves is the Selberg sieve. Selberg's approach [19] starts with the inequality

$$\Big(\sum_{\substack{d|n\\ P^+(d)\le z}} \lambda_d\Big)^2 \ge \sum_{\substack{d|n\\ P^+(d)\le z}} \mu(d) = \begin{cases}1, & P^-(n)\ge z,\\ 0, & \text{otherwise},\end{cases} \tag{1.1}$$

which is valid for any real numbers λ_d with $\lambda_1 = 1$. Here $P^+(n)$ and $P^-(n)$ are the largest and smallest prime factors of n respectively. Summing (1.1) over $n \in \mathscr{A}$ gives

$$\begin{aligned}\#\{n\in\mathscr{A}: P^-(n)\ge z\} &\le \sum_{n\in\mathscr{A}}\Big(\sum_{\substack{d|n\\ P^+(d)\le z}}\lambda_d\Big)^2\\ &= \sum_{P^+(d_1),P^+(d_2)\le z} \lambda_{d_1}\lambda_{d_2}\cdot\#\{n\in\mathscr{A}:[d_1,d_2]|n\},\end{aligned}$$

which is a quadratic form in the variables λ_d. Provided d_1 and d_2 are not too large, say at most R, one can hope to get a reasonable estimate for the coefficients $\#\{n\in\mathscr{A}:[d_1,d_2]|n\}$ of this quadratic form. The best upper bound stemming from this method then comes from minimizing the quadratic form over all choices of $\lambda_d\in\mathbb{R}$ with $\lambda_1=1$ and $\lambda_d=0$ for $d>R$.

For typical sets $\mathscr{A}$ that arise in arithmetic problems, one finds that the optimal choice for the λ_d takes the form

$$\lambda_d \approx \mu(d)\cdot\left(\frac{\log(R/d)}{\log R}\right)^A \qquad (d\le R),$$

where A is some positive constant. We note that the weights λ_d decay to 0, and the larger the value of A, the higher the level of smoothness at the truncation point R. In the optimal choice, the exponent A is taken to be κ, the *dimension* of the sieve problem. However, for a given dimension κ, it is known [20, pg. 154] that any exponent $A>\kappa-1/2$ yields weights λ_d whose dominant contribution comes from numbers that have very few prime factors smaller than z, whereas this fails to be true for smaller A. See [8, ch. 10] for further discussion.

More generally, one can consider the smoothed sieve weight

$$M_f(n;R) := \sum_{d|n}\mu(d) f\left(\frac{\log d}{\log R}\right),$$

where $f:\mathbb{R}\to\mathbb{R}$ is a function supported on $(-\infty,1]$, which corresponds to taking $\lambda_d=\mu(d)f(\log d/\log R)$ for $d\le R$. In Selberg sieve arguments one typically chooses f to be a polynomial in $[0,1]$, perhaps of high degree. Such an example is offered by the 'GPY sieve' of Goldston-Pintz-Yıldırım [9, 24]. In more recent developments on gaps between primes by the third author [15] and Tao [21] one works with general smooth functions f.

The main motivation of this paper is to understand the exact role of the smoothing in the structure of the Selberg sieve weights. To this end, we consider their moments

$$\sum_{n\le x} M_f(n;R)^k$$

as a tool of gaining additional insight on the distribution of the values of $M_f(n;R)$. On the practical side, higher moments naturally appear when applying Hölder's inequality, so it would be useful to know their behavior (1).

From the discussion above, in the case $f(x)=\max(1-x,0)^A$ and $k=2$, we have seen that if A is sufficiently large, then $M_f(n;R)^2$ 'behaves like a sieve weight' in the sense that the sum $\sum_{n<x} M_f(n;R)^2$ is $O_f(x/\log R)$ and the main contribution to this comes from numbers with few prime factors less than R. If A is too small and so f is not smooth enough, however, then $M_f(n;R)$ exhibits a qualitatively different behavior; the sum is larger than $x/\log R$, and the main contribution is no longer from numbers with few prime factors $\le R$.

How smooth should f be so that $M_f(n;R)^{2k}$ behaves like a sieve weight when k varies, that is to say the main contribution to the $2k$-th moment (2) of $M_f(n;R)$ comes from integers a that have very few prime factors $\le R$? What happens in the extreme case where f is the discontinuous function $\mathbf{1}_{(-\infty,1]}$? These are the types of questions that we will study in this paper.

1.1. Some smoothing is necessary to behave like a sieve weight

In order to gain a first understanding of the importance of smoothing, let us consider the sharp cut-off function

$$f_0 := \mathbf{1}_{(-\infty,1]}.$$

If $n=2m$ with m odd, then we have that

$$M_{f_0}(n;R)=\sum_{\substack{d\mid n\\ d\le R}}\mu(d)=\sum_{\substack{d\mid m\\ d\le R}}\mu(d)+\sum_{\substack{d\mid m\\ 2d\le R}}\mu(2d)=\sum_{\substack{d\mid m\\ R/2<d\le R}}\mu(d)=M_{\widetilde{f_0}}(m;R), \tag{1.2}$$

where, with a slight abuse of notation, we have put

$$\widetilde{f_0} := \mathbf{1}_{(1-\log 2/\log R,1]}. \tag{1.3}$$

In particular, if m is square-free and has exactly one divisor $d\in(R/2,R]$, then $M_{f_0}(n;R)=\pm 1$. An easy generalization of a deep result of Ford [5, Theorem 4] implies that (3) the proportion

(1) For example, Lemma 3.5 in Pollack's paper [17] is an example of a case where a fourth moment occurs because of the use of Cauchy's inequality, and a similar issue is encountered in Friedlander's work [7] for the combinatorial sieve instead of the Selberg sieve.

(2) We are typically interested in how large sieve weights get. If we took odd powers there might be an irrelevant cancelation, so we focus on even moments.

(3) The key estimates in the proof of the lower bound of Theorem 4 in [5] are the second part of Lemma 4.1, Lemma 4.3 (the parameters are $z=R\ll R/2=y$), Lemma 4.5, Lemma 4.8 and Lemma 4.9. A key observation is that only square-free integers are considered in Lemma 4.8, so that a stronger version of the lower bound of Theorem 4 of [5] can be immediately deduced by the same proof, that counts square-free integers with exactly one divisor in $(R/2,R]$.

of such $m \le x/2$ is $\gg (\log R)^{-\delta}(\log\log R)^{-3/2}$ with

$$\delta = 1 - \frac{1+\log\log 2}{\log 2} = 0.086071332\ldots,$$

whence we conclude that

$$\#\{n \le x :\ M_{f_0}(n;R) \neq 0\} \gg \frac{x}{(\log R)^{\delta}(\log\log R)^{3/2}} \quad (x \ge R^{1+\epsilon}).$$

In particular, we find that $M_{f_0}(n;R)$ is non-zero too often to behave like a sieve weight. This indicates that part of the importance of smoothing is to reduce the contribution of isolated divisors of n to $M_f(n;R)$.

We will prove in Section 5 that

$$\#\{n \le x :\ M_{f_0}(n;R) \neq 0\} \asymp \frac{x}{(\log R)^{\delta}(\log\log R)^{3/2}} \quad (x \ge R^{5/2}). \tag{1.4}$$

This sharpens a result by Hall and Tenenbaum [11], who used a very similar argument and the best results about divisors of integers available at that time.

1.2. A heuristic argument

Going back to the study of $M_f(n;R)$ for a smooth function f, it is reasonable to believe that the smoother f is, the larger the k are for which $M_f(n;R)^{2k}$ behaves like a sieve weight. One way to explain this phenomenon is by noticing that various integral transformations have faster decay for smooth weights, which can help to tame the arithmetic issues at play. (See, for example, Section 6.) Nevertheless, we prefer to give a number theoretic explanation in terms of the underlying sieve questions rather than an analytic one focused more on the technical issues. Assume that $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}m$, where $p_1 < \cdots < p_r$, $\alpha_i \ge 1$ and all of the prime divisors of m are $> p_r$. Then

$$\begin{aligned} M_f(n;R) &= \sum_{d\mid p_2\cdots p_r m} \mu(d) f\left(\frac{\log d}{\log R}\right) + \sum_{d\mid p_2\cdots p_r m} \mu(p_1 d) f\left(\frac{\log(p_1 d)}{\log R}\right) \\ &= \sum_{d\mid p_2\cdots p_r m} \mu(d)\left\{ f\left(\frac{\log d}{\log R}\right) - f\left(\frac{\log p_1}{\log R} + \frac{\log d}{\log R}\right)\right\}. \end{aligned} \tag{1.5}$$

Continuing as above, we find that

$$M_f(n;R) = (-1)^r \sum_{d\mid m} \mu(d) \Delta^{(r)} f\left(\frac{\log d}{\log R}; \frac{\log p_1}{\log R}, \ldots, \frac{\log p_r}{\log R}\right), \tag{1.6}$$

where $\Delta^{(r)} f(x;h_1,\ldots,h_r)$ denotes the multi-difference operator defined by

$$\Delta^{(1)} f(x;h) = f(x+h) - f(x)$$

and

$$\Delta^{(r)} f(x;h_1,\ldots,h_r) = \Delta^{(r-1)}(x+h_r;h_1,\ldots,h_{r-1}) - \Delta^{(r-1)}(x;h_1,\ldots,h_{r-1}).$$

In particular, if $f \in C^r(\mathbb{R})$, then

$$\Delta^{(r)} f(x;h_1,\ldots,h_r) = \int_0^{h_r}\int_0^{h_{r-1}}\cdots\int_0^{h_1} f^{(r)}(x+t_1+t_2+\cdots+t_r)dt_1\cdots dt_r. \tag{1.7}$$

Returning to (1.6), we see that if $f \in C^A(\mathbb{R})$ and $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} m$, $r \leq A$ is as above, then $M_f(n;R)$ should heuristically be $\ll M_{f^{(r)}}(m;R)\prod_{j=1}^r(\log p_j/\log R)$. Loosely, this indicates each additional degree of smoothness of the weight function f cuts the average size of $M_f(n;R)$ by about a factor of $1/\log R$.

The above discussion leads us to conjecture that if $f \in C^A(\mathbb{R})$ with $f(0) \neq 0$, then

$$\sum_{n\leq x} M_f(n;R)^{2k} \ll \max\left\{\frac{x}{\log R}, \frac{1}{(\log R)^{2kA}}\sum_{n\leq x} M_{f_0}(n;R)^{2k}\right\}. \tag{1.8}$$

Notice that the factor $x/\log R$ is necessary because $M_f(n;R) = f(0)$ for all integers n that are free of prime factors $\leq R$.

Naturally, for this relation to be useful, we need to understand the asymptotics of $\sum_{n\leq x} M_{f_0}(n;R)^{2k}$. Recall the relation (1.9). Expanding the k-th power and swapping the order of summation, we find that

$$\sum_{n\leq x} M_f(n;R)^k = x\cdot \mathcal{M}_{f,k}(R) + O((\|f\|_\infty R)^k)$$

for any $f:\mathbb{R}\to\mathbb{R}$ supported on $(-\infty,1]$, where

(1.9)

$$\mathcal{M}_{f,k}(R) := \sum_{d_1,\ldots,d_k\geq 1} \frac{\prod_{j=1}^k \mu(d_j) f(\log d_j/\log R)}{[d_1,\ldots,d_k]} = \prod_{p\leq R}\left(1-\frac{1}{p}\right)\sum_{p|n \Rightarrow p\leq R}\frac{M_f(n;R)^k}{n}.$$

We are generally interested in the situation when R is bounded by a small power of x, so that the error term $O((\|f\|_\infty R)^k)$ is negligible. Thus our focus is on the main term $\mathcal{M}_{f,k}(R)$, which no longer depends on x. When $k = 1$, Dress, Iwaniec and Tenenbaum [3] showed that

$$\mathcal{M}_{f_0,2}(R) \sim c_1 \quad (R\to\infty) \tag{1.10}$$

for some constant $c_1 > 0$, and when $k = 2$, Motohashi [16] showed that

$$\mathcal{M}_{f_0,4}(R) \sim c_2(\log R)^2 \quad (R\to\infty) \tag{1.11}$$

for some constant $c_2 > 0$. In general, Balazard, Naimi, and Pétermann [1] proved that

$$\mathcal{M}_{f_0,2k}(R) = P_k(\log R) + O(e^{-c(\log R)^{3/5}(\log\log R)^{-1/5}}),$$

for some polynomial P_k and some constant $c = c(k) > 0$. This is built on the work of de la Bretèche [2], who showed how a wide class of related sums can be evaluated asymptotically. However, when applying his technique to this question, one would need some strong understanding of the growth of $\zeta(s)$ near to $s = 1$ to recover the result of [1] (which, for example, follows from the Riemann Hypothesis).

Notice that if $\mathcal{E}_k = \deg(P_k)$, so that $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = 2$, then (1.8) becomes

$$\sum_{n\leq x} M_f(n;R)^{2k} \ll \max\left\{\frac{x}{\log R}, x(\log R)^{\mathcal{E}_k-2kA}\right\}. \tag{1.12}$$

This suggests that $M_f(n;R)^{2k}$ acts like a sieve weight as long as $A > \mathcal{E}_k/2k$. The big issue with the result of Balazard, Naimi and Pétermann is that the degree $\mathcal{E}_k$ is not determined for general k, and that it is essential if one wishes to gain a better understanding of how the Selberg sieve weights work. Our attention thus turns to calculating $\mathcal{E}_k$.

But first, we study seemingly analogous questions (in different settings), that one might guess would be easier and indicate what kind of estimate we should be looking for.

1.3. Analogous settings

It is well-known that many of the analytic properties of integers are shared by both polynomials of finite fields (c.f. [18]), and by permutations (c.f. [10]). Moreover, polynomials and permutations are usually easier objects to understand, so in order to gain an understanding of the exponent $\mathcal{E}_k$, it would be natural to consider what happens in these analogous settings first.

Permutations. – The easiest analogy is to analyze concerns permutations. Every $\sigma \in S_N$ (the permutations on N letters) can be decomposed in a unique way into a product of disjoint cycles. Those cycles cannot be decomposed any further and play the role of irreducibles. Divisors of σ are precisely the set of possible products of cycles. If those cycles act on the subset T of $[N]$, then σ fixes T. Moreover, if σ fixes T, then σ is a product of cycles, a subproduct of which fixes T. Hence "divisors" correspond to sets $T \subset [N]$ for which $\sigma(T) = T$.

To "calibrate" our understandings of the properties of integers and permutations, we note that for a typical integer n, its j-th largest prime factor is about e^{e^j}, whereas for a typical permutation $\sigma \in S_N$, its j-th largest cycle has length about e^j. Thus, the inequality $R/2 < d \leq R$ for a divisor d of n corresponds to having a set T that is fixed by σ of size $\#T = \log R + O(1)$. Hence we will study

$$\mathrm{Perm}(N, m; k) := \frac{1}{N!} \sum_{\sigma \in S_N} \Big| \sum_{\substack{T \subset [N],\ \#T = m, \\ \sigma(T) = T}} \mu(\sigma|_T) \Big|^{2k}, \tag{1.13}$$

where

$$[N] := \{1, \dots, N\},$$

and if $\sigma|_T = C_1 C_2 \cdots C_\ell$ is the product of ℓ disjoint cycles, then we have set $\mu(\sigma|_T) = (-1)^\ell$. We claim that $\mathrm{Perm}(N, m; k)$ is more natural than it appears at first sight. A usual function of permutations is the signature $\epsilon(\sigma)$ which counts the number of transpositions (i.e., the number of interchanges of two elements) needed to create σ. For a cycle C, one knows that $\epsilon(C) = (-1)^{\#C - 1}$ and hence $\epsilon(\sigma|_T) = \epsilon(C_1)\epsilon(C_2)\dots\epsilon(C_\ell) = (-1)^{\#T - \ell} = (-1)^m \mu(\sigma|_T)$, since $\#T = m$ here. Therefore

$$\sum_{\substack{\sigma \in S_N,\ T \subset [N], \\ \#T = m,\ \sigma(T) = T}} \mu(\sigma|_T) = (-1)^m \sum_{\substack{\sigma \in S_N,\ T \subset [N], \\ \#T = m,\ \sigma(T) = T}} \epsilon(\sigma|_T),$$

whence

$$\mathrm{Perm}(N, m; k) = \frac{1}{N!} \sum_{\sigma \in S_N} \Big| \sum_{\substack{T \subset [N],\ \#T = m, \\ \sigma(T) = T}} \epsilon(\sigma|_T) \Big|^{2k}.$$

Arguing as in the work of Eberhard, Ford and Green [4] that establishes the analogue for permutations of Ford's results [5] for integers, it is possible to show that the summands on the right hand of (1.13) (and, hence, of the above formula) are non-zero for a proportion $\ll 1/m^{\delta}(\log m)^{3/2}$ of the permutations in S_N. The following theorem provides a formula and an asymptotic estimate for $\mathrm{Perm}(N, m; k)$.

THEOREM 1.1. – *For each integer $k \geq 1$ and each integer $m \geq 1$, if $N \geq 2mk$ then*

$$\mathrm{Perm}(N, m; k) = c(m, k),$$

where $c(m, k)$ is the number of $(2^{2k} - 1)$-tuples $(r_I)_{\emptyset \neq I \subset \{1,\dots,2k\}}$ of non-negative integers such that

— *$r_I \in \{0, 1\}$ for $\#I$ odd;*

— *$\sum_{I:\, i \in I} r_I = m$, for each $i \in \{1, \dots, 2k\}$.*

Moreover, for fixed $k \in \mathbb{Z}_{\geq 1}$, the function $c(m, k)$ is increasing in m and satisfies the estimate

$$c(m, k) \asymp_k m^{2^{2k-1} - 2k - 1} + 1.$$

Proof of the formula for $\mathrm{Perm}(N, m; k)$. – Given sets $T_1, \dots, T_{2k}$, the sets

$$R_I := \Big(\bigcap_{i \in I} T_i\Big) \setminus \Big(\bigcup_{i \in [2k] \setminus I} T_i\Big) \quad (I \subset [2k])$$

form a partition of $[N]$, with the convention that $\bigcap_{i \in \emptyset} T_i = [N]$; that is to say $[N]$ equals $\bigsqcup_I R_I$, the disjoint union of the sets R_I. Using this with $T_1, \dots T_{2k}$ fixed sets of σ (i.e., $\sigma(T_i) = T_i$, so the R_I are all fixed by σ as well), we find

$$\frac{1}{N!} \sum_{\sigma \in S_N} \Big| \sum_{\substack{T \subset [N],\, \#T = m, \\ \sigma(T) = T}} \epsilon(\sigma|_T) \Big|^{2k} = \sum_{\substack{r_I \geq 0\ \forall I \\ \sum_{I:\, i \in I} r_I = m}} \sum_{\substack{[N] = \bigsqcup_I R_I \\ \#R_I = r_I\ \forall I}} \frac{1}{N!} \prod_{I \subset [2k]} \Big(\sum_{\rho_I \in S_{r_I}} \epsilon(\rho_I)^{\#I} \Big).$$

The inner sums are each $r_I!$ unless $\#I$ is odd and $r_I > 1$, in which case we get 0. Additionally, we get that the number of choices of sets of the given sizes is $N!/\prod_I r_I!$, and hence the above equals $c(m, k)$.

The bounds for $c(m, k)$ will be proven in Section 3. □

Evidently, the above results suggest that $\mathcal{E}_k = \max\{0, 2^{2k-1} - 2k - 1\}$. Relation (1.10) implies that $\mathcal{E}_1 = 0$, but relation (1.11) implies that $\mathcal{E}_2 = 2 \neq 2^3 - 5$. This suggests that there is a discrepancy between the integer and the permutation setting, a very rare difference.

Polynomials over finite fields. Positive integers are uniquely identifiable by their factorization into primes (the Fundamental theorem of Arithmetic). Note though that every non-zero integer equals a unit (that is 1 or -1) times one of those positive integers. We will work with polynomials in $\mathbb{F}_q[t]$. Monic polynomials in $\mathbb{F}_q[t]$ are uniquely identifiable by their factorization into monic irreducible polynomials of degree ≥ 1. Again, note that every non-zero polynomial in $\mathbb{F}_q[t]$ equals a unit (that is, any element $a \in \mathbb{F}_q \setminus \{0\}$) times a monic polynomial. We will work only with monic polynomials, for example when considering divisors of a given polynomial (rather like we only consider positive integer divisors of a given integer). The Möbius function of a given polynomial is a multiplicative function, where $\mu(P) = -1$, and $\mu(P^k) = 0$ if $k \geq 2$, whenever P is irreducible.

To "calibrate" our understandings of the arithmetic properties of integers and polynomials, we note that $\sim 1/\log x$ of integers around x are prime, whereas $\sim 1/m$ of monic polynomials of degree m are irreducible in $\mathbb{F}_q[t]$. Here the "$\sim$" symbol means as $q \to \infty$ running through prime powers. Thus, wherever we see $\log x$ in an estimate about the integers, we try to replace it with m in an estimate about degree m polynomials. Similarly a divisor d of n that is close to R is analogous to a polynomial divisor of $F(t)$ of degree m, where m replaces $\log R$ in estimates. Hence we will study

$$\mathrm{Poly}_q(n,m;k) := \frac{1}{q^n} \sum_{\substack{\text{monic } N \in \mathbb{F}_q[t] \\ \deg N = n}} \Big| \sum_{\substack{\text{monic } M \mid N \\ \deg M = m}} \mu(M) \Big|^{2k},$$

Here we have divided by q^n because this is how many monic polynomials N of degree n are contained in $\mathbb{F}_q[t]$, which is the analogue of

$$\frac{1}{x} \sum_{n \le x} \Big(\sum_{\substack{d \mid n \\ R/2 < d \le R}} \mu(d) \Big)^{2k},$$

a quantity directly related to $\frac{1}{x} \sum_{n \le x} M_{f_0}(n;R)^{2k}$ via (1.2). We will prove below the following estimate:

THEOREM 1.2. – *For integers $k, m \ge 1$ and $n \ge 2mk$, we have that*

$$\mathrm{Poly}_q(n,m;k) = c(m,k)(1 + O_k(1/q)) \asymp_k 1 + m^{2^{2k-1}-2k-1}.$$

We thus see that polynomials behave similarly to permutations (and thus differently than integers).

1.4. Two worlds apart and a bridge between them

Our discussion of the permutation and polynomial analogues, rather than shedding more light on the value of the exponent $\mathcal{E}_k$, gave rise to even more questions. It turns out that the integer setting is substantially more complicated than the permutation and polynomial settings. We now state our main results about integers. First, given $A \in \mathbb{Z}_{\ge 1}$, we let

$$f_A(t) := \begin{cases} (1-t)^A & \text{for } t \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

an extension of the definition of f_0. Note that $f_A \in C^{A-1}(\mathbb{R}) \setminus C^A(\mathbb{R})$ for all $A \ge 1$. We then have the following result that determines the value of $\mathcal{E}_k$:

THEOREM 1.3. – *For fixed integers $k \ge 1$ and $A \ge 0$, there is a constant $c_{k,A} > 0$ such that*

$$\mathcal{M}_{f_A,2k}(R) = c_{k,A} (\log R)^{\mathcal{E}_{k,A}} + O((\log R)^{\mathcal{E}_{k,A}-1}), \tag{1.14}$$

where

$$\mathcal{E}_{k,A} := \max\left\{ \binom{2k}{k} - 2k(A+1), -1 \right\}.$$

In particular, $\mathcal{E}_k = \mathcal{E}_{k,0} = \binom{2k}{k} - 2k$. *Additionally, we find that there is a constant* $c'_k > 0$ *such that for* $R^{2k} \le x$ *we have*

$$\frac{1}{x}\sum_{n\le x}\Big(\sum_{\substack{d\mid n\\ R/2<d\le R}}\mu(d)\Big)^{2k} = c'_k(\log R)^{\binom{2k}{k}-2k} + O\left((\log R)^{\binom{2k}{k}-2k-1}\right). \tag{1.15}$$

All implied constants depend at most on k and A.

REMARK 1.1. – We have no nice formula for the constants $c_{k,A}$ and c'_k appearing in Theorem 1.3; we only know how to write them as an enormous rational linear combination of complicated integrals, and leave it as a challenge to come up with an easy explicit description.

REMARK 1.2. – If the moments of a distribution grow slowly, then the distribution can be determined via its Laplace transform. However, in our case the moments are of rapidly increasing magnitude, indeed with different powers of $\log R$, so one cannot immediately deduce from them the distribution of the weights $M_{f_A}(n; R)$ as n varies over the integers.

REMARK 1.3. – In this paper we only consider integral A, but we would expect analogous results to hold for all real $A > 0$.

REMARK 1.4. – In this paper we only consider Selberg-style sieve weights. We would expect something somewhat analogous to hold for combinatorial-style sieve weights (such as those used in the β-sieve) but we do not consider such situations here.

For general functions f, we prove that $M_f(n; R)^{2k}$ behaves like a sieve weight as long as $f \in C^A(\mathbb{R})$ with $A > \binom{2k}{k}/2k = \mathcal{E}_{2k}/2k + 1$. Notice that this confirms a weak version of the heuristic estimate (1.12).

THEOREM 1.4. – *Let* $k \in \mathbb{Z}_{\ge 1}$, $\epsilon \in (0, 1)$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be supported in* $(-\infty, 1]$. *Assume further that for some integer* $A \ge 2$, $f \in C^A(\mathbb{R})$ *and that all functions* $f, f', \dots, f^{(A)}$ *are bounded.*

(a) *If* $A > \frac{1}{2k}\binom{2k}{k}$, *then for* $x \ge R \ge 2$ *and* $1 \ge \eta \ge \log 2/\log R$, *we have that*

$$\sum_{\substack{n\le x\\ \exists p\mid n,\ p\le R^\eta}} M_f(n; R)^{2k} \ll \frac{\eta x}{\log R}.$$

If, in addition, $f(0) \ne 0$, *then there is a constant* $c_{k,f} > 0$ *such that for* $x \ge R^{2k}\log^2 R$ *we have that*

$$\frac{1}{x}\sum_{n<x} M_f(n; R)^{2k} = \frac{c_{k,f}}{\log R} + O\left(\frac{1}{(\log R)^{2-\epsilon}}\right).$$

(b) *If* $A \le \frac{1}{2k}\binom{2k}{k}$, *then for* $x \ge R \ge 2$ *we have that*

$$\sum_{n\le x} M_f(n; R)^{2k} \ll x(\log R)^{\binom{2k}{k}-2kA}.$$

All implied constants depend at most on f, k and ϵ.

The value of $\mathcal{E}_k = \binom{2k}{k} - 2k$ given by Theorem 1.3 is significantly smaller than the exponent $2^{2k-1} - 2k - 1$ in the polynomial/permutation setting. So we see the usual analogy breaking down in quite a severe way, something surprising. We devote Section 2 to the analysis of this discrepancy. In particularly, we will see that the underlying reason is the relation

$$\sum_{\substack{\sigma\in S_N,\ T\subset[N],\\ \#T=m,\ \sigma(T)=T}} \mu(\sigma|_T) = (-1)^m \sum_{\substack{\sigma\in S_N,\ T\subset[N],\\ \#T=m,\ \sigma(T)=T}} \epsilon(\sigma|_T) \tag{1.16}$$

that we saw before. Notice here that while $\mu(\rho) = -1$ for all cycles ρ, we have that $\epsilon(\rho)$ takes the values ±1 with equal probability as ρ ranges over cycles of all possible lengths. The simplest example of a multiplicative function over $\mathbb{Z}$ demonstrating this kind of behavior is that of a real Dirichlet character. To this end, we consider

$$\mathcal{X}_{2k}(R) = \prod_{p\le R}\Big(1-\frac{1}{p}\Big)\sum_{P^+(n)\le R}\frac{1}{n}\Big(\sum_{\substack{d|n\\ R/2<d\le R}}\chi(d)\Big)^{2k},$$

which, as in (1.9), is the main term of

$$\frac{1}{x}\sum_{n<x}\Big|\sum_{\substack{d|n\\ R/2<d\le R}}\chi(d)\Big|^{2k}.$$

We then have the following theorem, which shows that it is possible to bridge the gap between the two worlds of integers and of permutations/polynomials. All implied constants below depend at most on k, and we have set

$$\mathcal{S}^+(2k) := \{I\subset\{1,2,\dots,2k\} : \#I \text{ even}\}\setminus\{\emptyset\}.$$

Theorem 1.5. – *Let $\chi \pmod q$ be a real non-principal character and $k\in\mathbb{Z}_{\ge1}$.*

(a) *If $k=1$, then*

$$\mathcal{X}_2(R) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P(t,\chi)|L(1+it,\chi)|^2\sin^2(t(\log 2)/2)}{t^2}\mathrm{d}t + O\Big(\frac{1}{(\log R)^{100}}\Big),$$

where $P(\cdot,\chi)$ is a real-valued Euler product whose factors are $1+O(1/p^2)$. In particular, $P(t,\chi)\asymp 1$ for all t, uniformly in χ.

(b) *Assume that $k\ge2$. Let $V_k(m)$ be the Lebesgue volume in $\mathbb{R}^{2^{2k-1}-1}$ given by*

$$V_k(m) = \mathrm{vol}\{(x_I)_{I\in\mathcal{S}^+(2k)} : x_I\ge 0,\ m-\log 2\le \sum_{I\ni i}x_I\le m\},$$

and let $\mathfrak{S}_k(\chi)$ be the singular series

$$\mathfrak{S}_k(\chi) = \prod_p\Big(1-\frac{1}{p}\Big)^{2^{2k-1}} f_p,$$

where

$$f_p = \begin{cases} \sum_{j\ge0}(j+1)^{2k}/p^j, & \textit{if } \chi(p)=1,\\ (1-1/p^2)^{-1}, & \textit{if } \chi(p)=-1,\\ (1-1/p)^{-1}, & \textit{if } p|q.\end{cases}$$

Then $V_k(m) \asymp_k m^{2^{2k-1}-2k-1}$, $\mathfrak{S}_k(\chi) \asymp_k L(1,\chi)^{2^{2k-1}}$, *and*

$$\mathcal{X}_{2k}(R) = \mathfrak{S}_k(\chi) \cdot V_k(\log R) + O\left((\log R)^{2^{2k-1}-2k-2}(\log\log R)^{O(1)} \log q\right).$$

(c) *Assume that* $k \geq 2$ *and that* $L(\beta,\chi) = 0$ *for some* $\beta > 1 - 1/(100\log q)$. *If* $Q = e^{1/(1-\beta)}$ *and* $e^{(\log q)^C} \leq R \leq Q$ *for some large enough* $C = C(k)$, *then there is a constant* $c_k(\chi) = (\log q)^{O(1)}$ *such that*

$$\mathcal{X}_{2k}(R) = c_k(\chi)(\log R)^{\binom{2k}{k}-2k}\left(1 + O\left(\frac{(\log(q\log R))^{O(1)}}{\log R}\right)\right).$$

In the case of our polynomial and permutation models, we have an exponent of $2^{2k-1} - 2k - 1$ for the $2k$-th moment with $k \geq 2$, whilst over the integers we have an exponent $\binom{2k}{k} - 2k$. We see that our Dirichlet character model interpolates between these two settings. If the Dirichlet L-function associated with the character has a zero very close to 1, then $\chi(p) = -1$ for many small primes p, and so by multiplicativity χ behaves similarly to μ (at least in appropriate ranges). This is represented by our exponent of $\binom{2k}{k} - 2k$ in this case. On the other hand, χ is a periodic character, and if the L function does not have a zero very close to 1, we see that we have an exponent $2^{2k-1} - 2k - 1$, matching the exponent of our polynomial and permutation models. Notice that if $L(s,\chi)$ does have an exceptional zero, then the asymptotic of case (c) for $\mathcal{X}_{2k}(R)$ holds for small R, and transitions into the asymptotic of case (b) as R grows.

REMARK 1.5. – Relation (1.16) has a polynomial analogue whose consequences are worth exploring further. Given $I \subset \{F \in \mathbb{F}_q[t] : \deg(F) = n\}$, we consider the sum

$$\sum_{F \in I} \mu(F).$$

For example, we could take $I = \{F \in \mathbb{F}_q[t] : \deg(F) = n\}$, or $I = \{F \in \mathbb{F}_q[t] : \deg(F - F_0) \leq h\}$ for some $F_0 \in \mathbb{F}_q[t]$ of degree n and for some integer $h \in [1, n-1]$, which can be seen as the polynomial analogue of a short interval. Then

$$\sum_{F \in I} \mu(F) = (-1)^n \sum_{F \in I} \chi(F),$$

where $\chi(F) = (-1)^{\deg(F)}\mu(F)$, which is also a multiplicative function. However, we note that, even though $\mu(P) = -1$ for all irreducibles, we have that $\chi(P) = 1$ for about half of the irreducibles P, and $\chi(P) = -1$ for the other half, that is to say χ behaves on average much more like a real Dirichlet character rather than the Möbius function.

This phenomenon is striking and sharply different from what happens over $\mathbb{Z}$, where there is a dichotomy between multiplicative functions that look like the Möbius functions and other ones whose average prime value is 0, as is exemplified by Theorem 1.5 (see, also, [13]).

1.5. Further analysis of truncated Möbius divisor sums

As we saw in Theorem 1.4, if $f \in C^A(\mathbb{R})$ with $A > \frac{1}{2k}\binom{2k}{k} = \mathcal{E}_k/2k+1$, then $M_f(n;R)^{2k}$ behaves like a sieve weight. When $f = f_A$, we can be more precise:

If $A > \frac{1}{2k}\binom{2k}{k} - 1 = \mathcal{E}_k/2k$, then $\mathcal{E}_{k,A} = -1$ in Theorem 1.3, and so $\mathcal{M}_{f_A,2k}(R) \asymp (\log R)^{-1}$. Since integers $n \le x$ with no prime factors less than R contribute a total $\gg x(\log R)^{-1}$ to $\sum_{n\le x} M_{f_A}(n;R)^{2k}$, we see that $M_{f_A}(n;R)^{2k}$ is behaving like a sieve weight in this case.

If $A \le \frac{1}{2k}\binom{2k}{k} - 1 = \mathcal{E}_k/2k$, then $\mathcal{E}_{k,A} \ge 0$, and so $\mathcal{M}_{f_A,2k}(R) \gg 1$. In particular, $M_{f_A}(n;R)^{2k}$ no longer behaves like a sieve weight, and the main contribution is from numbers with several prime factors in $[1,R]$.

The following theorem illustrates further this distinction.

THEOREM 1.6. – *Let $x \ge R \ge 2$, $k \in \mathbb{Z}_{\ge 1}$ and $A \in \mathbb{Z}_{\ge 0}$. Moreover, let $\Omega(n;R)$ denote the number of prime factors of n in $[1,R]$, counted with multiplicity.*

(a) *If $A > \frac{1}{2k}\binom{2k}{k} - 1$, then*

$$\sum_{\substack{n<x \\ \Omega(n;R)\ge C}} M_{f_A}(n;R)^{2k} \ll_{k,A} \frac{x}{C\log R}.$$

(b) *If $A \le \frac{1}{2k}\binom{2k}{k} - 1$ and $\epsilon > 0$ is fixed, then there is a $\delta = \delta(\epsilon,k) > 0$ such that*

$$\sum_{\substack{n<x \\ |\Omega(n;R)/\log\log R - \binom{2k}{k}|\ge\epsilon}} M_{f_A}(n;R)^{2k} \ll_{k,A} x(\log R)^{\binom{2k}{k}-2k(A+1)-\delta}.$$

In other words, if $A > \frac{1}{2k}\binom{2k}{k} - 1$, then the main contribution to the sum defining $\mathcal{M}_{f_A,2k}(R)$ comes from integers with a bounded number of prime factors $\le R$; whereas if $A \le \frac{1}{2k}\binom{2k}{k} - 1$, then the main contribution to the sum comes from integers with $\left(\binom{2k}{k} + o(1)\right)\log R$ prime factors $\le R$.

Analogous results hold with $\Omega(n;R)$ replaced by the function $\#\{p|n : p \le R\}$. We note that typically one requires $x > R^{2k}$, as in Theorem 1.3, to estimate a $2k$-th moment of a sum of divisors of size at most R, but the estimates of Theorem 1.6 hold in the much wider range $x \ge R$. We can show similar (but slightly weaker) results for general weights f:

THEOREM 1.7. – *Let $k \in \mathbb{Z}_{\ge 1}$ and $f : \mathbb{R} \to \mathbb{R}$ be supported in $(-\infty,1]$. Assume further that $f \in C^A(\mathbb{R})$ and that all functions $f, f', \dots, f^{(A)}$ are uniformly bounded for some integer $A \ge 2$, and fix some $\epsilon \in (0,1)$.*

(a) *Assume that $A > \frac{1}{2k}\binom{2k}{k}$. For $x \ge R \ge 2$ and $C \ge 1$, we have that*

$$\sum_{\substack{n<x \\ \Omega(n;R)\ge C}} M_f(n;R)^{2k} \ll_{k,f} \frac{x}{C\log R}.$$

(b) *If $A \le \frac{1}{2k}\binom{2k}{k}$ and $\epsilon > 0$ is fixed, then there is a $\delta = \delta(\epsilon,k) > 0$ such that*

$$\sum_{\substack{n<x \\ |\Omega(n;R)/\log\log R - \binom{2k}{k}|\ge\epsilon}} M_f(n;R)^{2k} \ll_{k,f,\epsilon} x(\log R)^{\binom{2k}{k}-2kA-\delta} \quad (x \ge R \ge 2).$$

1.6. Outline of the paper

We start the paper in Section 2 with a discussion on the discrepancy between the exponent of $\log R$ in (1.15) and the exponent of m in Theorem 1.1, which is surprising at first sight.

Sections 3 and 4 study the analogies for permutations and polynomials over finite fields, respectively. These analogies are considerably easier to analyze than the integer case.

The main body of the paper is then dedicated to the study of moments of $M_{f_A}(n; R)$ over Sections 5 – 10. Specifically, in Section 5 we establish relation (1.4) for the size of the support of $M_{f_0}(n; R)$, and in Section 6 we study inversion formulas for our divisor sums $M_f(n; R)$ that will be essential when dealing with their moments. The proof of Theorem 1.3 is separated over three sections: in Section 7, we establish certain combinatorial inequalities that will be instrumental in understanding the leading term in the asymptotics for $\mathcal{M}_{f_A,2k}(R)$. Then, in Section 8 we establish Theorem 1.3 by a multidimensional contour shifting argument, except for showing the positivity of the constants $c_{k,A}$ and c'_k. The latter will be accomplished with a different argument in Section 9. Section 10 contains an analysis of the anatomy of the integers that give the main contribution to moments of $M_f(n; R)$. Specifically, we prove Theorems 1.4, 1.6 and 1.7 there.

Finally, in Sections 11 and 12 we study the moments of the sum weighted by Dirichlet characters, and establish Theorem 1.5, first for non-exceptional Dirichlet characters (where the proof is similar to Theorem 1.2), and then for exceptional Dirichlet characters (where the proof is similar to Theorem 1.3).

1.7. Notation

Given an integer $N \geq 1$, we set throughout the paper

$$[N] := \{1, 2, \ldots, N\},$$

$$\mathcal{S}^+(N) := \{\emptyset \neq I \subset [N] : \#I \text{ even}\}, \qquad \mathcal{S}^-(N) := \{I \subset [N] : \#I \text{ odd}\},$$
$$\mathcal{S}(N) := \{I \subset [N]\} \qquad \text{and} \qquad \mathcal{S}^*(N) := \mathcal{S}^+(N) \cup \mathcal{S}^-(N).$$

Also, we recall that, given an integer $n \geq 1$, we write $P^+(n)$ and $P^-(n)$ for its largest and smallest prime divisors, respectively, with the convention that $P^+(1) = 1$ and $P^-(1) = \infty$.

Finally, given $2k$ variables $s_1, \ldots, s_{2k}$ and $I \subset [2k]$, we will use the notation $s_I = \sum_{i \in I} s_i$.

2. The discrepancy between integers and polynomials

The goal of this section is to analyze in detail why we have such a different behavior when considering integers vs. polynomials or permutations.

2.1. Integer setting

Assume that $k \geq 2$. We mimic the proof of Theorem 1.1. Recall the definition of $\widetilde{f_0}$ in (1.3). Given square-free integers $d_1, \dots, d_{2k}$ and $I \subset \mathcal{S}^*(2k)$, we let D_I be the product of those primes p that divide each of the d_i's with $i \in I$ but do not divide $\prod_{i \in [2k]\setminus I} d_i$. Then the integers D_I for $I \in \mathcal{S}^*(2k)$ are pairwise coprime and $d_i = \prod_{I: i \in I} D_I$ for each i, so that

$$\begin{aligned}\mathcal{M}_{\widetilde{f_0},2k}(R) &= \sum_{R/2<d_1,\dots,d_{2k}<R} \frac{\mu(d_1)\dots\mu(d_{2k})}{[d_1,\dots,d_{2k}]} \\ &= \sideset{}{^\flat}\sum_{\substack{D_I \ (I\in\mathcal{S}^*(2k)) \\ R/2<\prod_{I\ni i} D_I \leq R \ (1\leq i\leq 2k)}} \Bigg(\prod_{I\in\mathcal{S}^-(2k)} \frac{\mu(D_I)}{D_I}\Bigg)\Bigg(\prod_{I\in\mathcal{S}^+(2k)} \frac{1}{D_I}\Bigg),\end{aligned}$$

where the notation $\sum^\flat$ means that the summation is running over squarefree and pairwise coprime integers D_I. Set $L = e^{(\log\log R)^3}$. The contribution of those tuples with $D_I > L$ for some I odd to $\mathcal{M}_{\widetilde{f_0},2k}(R)$ can be seen to be $\ll 1/e^{c(\log\log R)^{3/2}}$ for some $c = c(k) > 0$, by the Prime Number Theorem. So assume that $D_I \leq L$ for all I odd. Then it is natural to write

$$\mathcal{M}_{\widetilde{f_0},2k}(R) \sim \sideset{}{^\flat}\sum_{\substack{D_I\leq L \ (I\in\mathcal{S}^*(2k)) \\ D=\prod_{I\in\mathcal{S}^-(2k)} D_I}} \frac{\mu(D)}{D} \cdot T_{2k}(R_1,\dots,R_{2k};D),$$

where $R_i = R/\prod_{I\ni i,\ I\in\mathcal{S}^+(2k)} D_I$ and

$$T_{2k}(\boldsymbol{R};a) = \sideset{}{^\flat}\sum_{\substack{(D_I,a)=1 \ (I\in\mathcal{S}^+(2k)) \\ R_i/2<\prod_{I\in\mathcal{S}^+(2k): i\in I} D_I\leq R_i \ (1\leq i\leq 2k)}} \prod_{I\in\mathcal{S}^+(2k)} \frac{1}{D_I}.$$

When $\log R_i = \log R + O(\log L) = \log R + O((\log\log R)^{3/2})$, as it is here, we should be expecting that $T_{2k}(\boldsymbol{R};a)$ has an asymptotic formula of the form

$$T_{2k}(\boldsymbol{R};a) = g(a)(\log R)^{2^{2k-1}-2k-1} + O\left((\log R)^{2^{2k-1}-2k-2}(\log\log R)^{O(1)}\right),$$

since we have $2^{2k-1}-1$ variables on a logarithmic scale and $2k$ multiplicative constraints in dyadic intervals, where $g(a)$ is a multiplicative function with $g(p) = 1 + O(1/p)$. Since $\sum_{n=1}^{\infty} \mu(n)/n = 0$, we then find that the total contribution of the main terms to $\mathcal{M}_{\widetilde{f_0},2k}(R)$ is

$$(\log R)^{2^{2k-1}-2k-1} \sideset{}{^\flat}\sum_{\substack{D_I\leq L \ (I\in\mathcal{S}^-(2k)) \\ D=\prod_{I\in\mathcal{S}^-(2k)} D_I}} \frac{\mu(D)g(D)}{D} \ll e^{-c'(\log\log R)^{3/2}},$$

for some $c' = c'(k) > 0$, which is negligible. Consequently,

$$\mathcal{M}_{\widetilde{f_0},2k}(R) \ll (\log\log R)^{O(1)}(\log R)^{2^{2k-1}-2k-2},$$

whereas the power of m in Theorem 1.1 is $2^{2k-1}-2k-1$. So this heuristic indicates that we should get more cancelation in the integer setting than we will obtain in the analogous permutation question, as established in Theorem 1.1.

2.2. Polynomial analogue

The reader might be skeptical of the argument presented above, because a direct analogue exists for polynomials over finite fields too. Specifically, expanding the $2k$-th power in $\mathrm{Poly}_q(n,m;k)$, we find that

$$\mathrm{Poly}_q(n,m;k) = \sum_{G_1,\dots,G_{2k}} \frac{\mu(G_1)\cdots\mu(G_{2k})}{q^{\deg([G_1,\dots,G_{2k}])}}$$

for $n \geq 2mk$. Given square-free, monic polynomials $G_1,\dots,G_{2k}$ over $\mathbb{F}_q[t]$ and $I \subset \mathcal{S}^*(2k)$, we let G_I be the product of those monic irreducibles P that divide each of the G_i's with $i \in I$ but do not divide $\prod_{i\in[2k]\setminus I} G_i$. Then the polynomials G_I for $I \in \mathcal{S}^*(2k)$ are pairwise coprime and $G_i = \prod_{I: i\in I} G_I$ for each i, so that

$$\mathrm{Poly}_q(n,m;k) = \sideset{}{^\flat}\sum_{\substack{G_I\ (I\in\mathcal{S}^*(2k)) \\ \sum_{I\ni i}\deg(G_I)=m\ (1\leq i\leq 2k)}} \left(\prod_{I\in\mathcal{S}^-(2k)} \frac{\mu(G_I)}{q^{\deg(G_I)}}\right)\left(\prod_{I\in\mathcal{S}^+(2k)} \frac{1}{q^{\deg(G_I)}}\right),$$

where the notation $\sum^\flat$ means that the summation is running over squarefree and pairwise monic polynomials G_I. As in the integer case, the contribution to $\mathrm{Poly}_q(n,m;k)$ of those tuples $(G_I)_{I\in\mathcal{S}^*(2k)}$ such that $\deg(G_I)$ is large for some $I \in \mathcal{S}^-(2k)$ is negligible, by the Prime Number Theorem over $\mathbb{F}_q[t]$. Hence, we may assume that $\deg(G_I) \leq \log m$ for all $I \in \mathcal{S}^-(2k)$. Then it is natural to write

$$\mathrm{Poly}_q(n,m;k) = \sideset{}{^\flat}\sum_{\substack{\deg(G_I)\leq\log m\ (I\in\mathcal{S}^-(2k)) \\ G=\prod_{I\in\mathcal{S}^-(2k)} G_I}} \frac{\mu(G)}{q^{\deg(G)}}\cdot \widetilde{T}_{q,2k}(m_1,\dots,m_{2k};G),$$

where $m_i = m - \sum_{I\ni i,\, I\in\mathcal{S}^+(2k)} \deg(G_I)$ and

$$\widetilde{T}_{q,2k}(\boldsymbol{m};A) := \sideset{}{^\flat}\sum_{\substack{(G_I,A)=1\ (I\in\mathcal{S}^+(2k)) \\ \sum_{I\in\mathcal{S}^+(2k):\, i\in I}\deg(G_I)=m_i\ (1\leq i\leq 2k)}} \prod_{I\in\mathcal{S}^+(2k)} \frac{1}{q^{\deg(G_I)}}.$$

As before, when $\ell_i = m + O(\log m)$, as above, we should be expecting that $\widetilde{T}_{q,2k}(\boldsymbol{\ell};A)$ has an asymptotic formula of the form

$$\widetilde{T}_{q,2k}(\boldsymbol{\ell};A) = \widetilde{g}(A) m^{2^{2k-1}-2k-1} + O\left(m^{2^{2k-1}-2k-2}(\log m)^{O(1)}\right), \tag{2.1}$$

where $\widetilde{g}(A)$ is a multiplicative function with $\widetilde{g}(P) = 1 + O(1/q^{\deg(P)})$ for irreducibles P.

The above argument suggests that we should have an asymptotic behavior of $\mathrm{Poly}_q(n,m;k)$ that is smaller than what Theorem 1.2 states, which is absurd. The problem is that if $\sum_{I\ni i}\deg(G_I) = m$ for all i, then we also have that

$$2km = \sum_{i=1}^{2k}\sum_{I: i\in I}\deg(G_I) = \sum_I \#I\,\deg(G_I).$$

Reducing this formula mod 2, we find that

$$\sum_{I\in\mathcal{S}^-(2k)} \deg(G_I) \equiv 0 \pmod{2}, \tag{2.2}$$

a local constraint that is not present in the integer analogue. In particular, we see that (2.1) is true only when (2.2) is satisfied. We thus find that the main term for $\mathrm{Poly}_q(n,m;k)$ equals

$$\begin{aligned} & m^{2^{2k-1}-2k-1}\cdot \sideset{}{^\flat}\sum_{\substack{\deg(G_I)\le \log m\ (I\in\mathcal{S}^-(2k))\\ G=\prod_{I\in\mathcal{S}^-(2k)} G_I,\ 2\mid\deg(G)}} \frac{\mu(G)\widetilde{g}(G)}{q^{\deg(G)}} \\ &= \frac{m^{2^{2k-1}-2k-1}}{2}\cdot \sideset{}{^\flat}\sum_{\substack{\deg(G_I)\le \log m\ (I\in\mathcal{S}^-(2k))\\ G=\prod_{I\in\mathcal{S}^-(2k)} G_I}} \frac{\mu(G)(1+(-1)^{\deg(G)})}{q^{\deg(G)}} + O\Big(\frac{1}{q}\Big) \\ &\asymp m^{2^{2k-1}-2k-1}, \end{aligned}$$

because $\sum_F \mu(F)/q^{\deg(F)} = 0$ and $\sum_F \mu(F)(-1/q)^{\deg(F)} = \prod_P (1-(-1/q)^{\deg(P)}) > 0$. Thus we see the local constraint associated to the discreteness of degrees in the polynomial setting means we have a genuinely different asymptotic behavior.

2.3. Further analysis

The above arguments suggest a possible route to proving Theorem 1.3, by working out the full asymptotic expansion of $T_{2k}(\boldsymbol{x};a)$. Controlling the coefficients in this expansion is a highly non-trivial problem. Instead, we take another route, using a high-dimensional contour shifting argument. Our starting point is Perron's inversion formula which, ignoring convergence issues, yields

$$T_{2k}(\boldsymbol{x};a) \sim \frac{1}{(2\pi i)^{2k}} \int\cdots\int_{\substack{\mathrm{Re}(s_j)=1/\log R\\ 1\le j\le 2k}} \sideset{}{^\flat}\sum_{\substack{(D_I,a)=1\\ I\in\mathcal{S}^+(2k)}} \prod_{I\in\mathcal{S}^+(2k)} \frac{1}{D_I^{1+s_I}} \prod_{j=1}^{2k} \frac{x_j^{s_j}(1-2^{-s_j})}{s_j} \mathrm{d}s_1\cdots\mathrm{d}s_{2k},$$

with the notational convention that $s_I = \sum_{j\in I} s_j$. Therefore

$$\mathcal{M}_{\widetilde{f}_0,2k}(R) \sim \frac{1}{(2\pi i)^{2k}} \int\cdots\int_{\substack{\mathrm{Re}(s_j)=1/\log R\\ 1\le j\le 2k}} F(\boldsymbol{s}) \frac{\prod_{I\in\mathcal{S}^+(2k)}\zeta(1+s_I)}{\prod_{I\in\mathcal{S}^-(2k)}\zeta(1+s_I)} \prod_{j=1}^{2k} \frac{1-2^{-s_j}}{s_j} \mathrm{d}s_1\cdots\mathrm{d}s_{2k},$$

where $F(\boldsymbol{s})$ is analytic and non-zero when $\mathrm{Re}(s_j) > -1/4k$ for all j. As we will see in Section 8, shifting contours, we pick up poles any time $s_I = 0$ for some $I \in \mathcal{S}^+(2k)$. What is the difficulty in proving Theorem 1.3 is that some of these poles can get annihilated by poles of the zeta factors in the denominator, which is an analytic way of saying that the higher order terms in the asymptotic expansions of $T_{2k}(\boldsymbol{x};a)$ are canceled out.

It is clear from the above discussion that the underlying reason why we got a genuinely smaller main term for $\mathcal{M}_{\widetilde{f}_0,2k}(R)$ is the identity $\sum_{n=1}^\infty \mu(n)/n = 0$, that is to say the fact that $1/\zeta$ has a zero at 1. This also explains the phenomenon we see in Theorem 1.5. If we replace μ by a real valued multiplicative function f whose Dirichlet series $F(s) = \sum_{n=1}^\infty f(n)/n^s$ which is not very small at $s=1$, then the behavior of the respective divisor sums should be similar to the permutation analogue, whilst if $F(s)$ is close to 0 at $s=1$ (which occurs if F has a zero very close to 1) the behavior is the same as in the original integer setting.

2.4. Further obstructions to the analogy

Is it possible that the local constraints at the prime 2 described above are the only thing separating integers and polynomials? In order to study this question, we consider the variations

$$\mathrm{Poly}_q(n,m,h;k) := \frac{1}{q^n} \sum_{\substack{N\in\mathbb{F}_q[t] \\ \deg N=n}} \Bigg| \sum_{\substack{M|N \\ m-h<\deg M\le m}} \mu(M) \Bigg|^{2k},$$

where $h \in \mathbb{Z}\cap[1,m+1]$. If $h \geq 2$, then the local problems at the prime 2 should be resolved. However, we will see that this is not sufficient and that the discrepancy between the integer and the polynomial analogues goes even deeper.

First, let us consider the case $h = m+1$ in order to convince the reader that resolving the constraints at the prime 2 is not sufficient. It is known that a positive proportion of polynomials $N \in \mathbb{F}_q[t]$ of degree at most r have a simple zero over $\mathbb{F}_q$, and that the number of zeroes of such a polynomial over $\mathbb{F}_q$ is, on average, bounded. So we should expect

$$\mathrm{Poly}_q(n,m,m+1;k) \asymp \frac{1}{q^n} \sum_{\substack{N\in\mathbb{F}_q[t] \\ \deg N=n}} \sum_{\substack{\alpha\in\mathbb{F}_q \\ N(\alpha)=0 \\ N'(\alpha)\neq 0}} \Bigg| \sum_{\substack{M|N \\ \deg M\le m}} \mu(M) \Bigg|^{2k}.$$

If N has a simple zero at α, then we can factor $N(x) = (x-\alpha)\widetilde{N}(x)$, where $\widetilde{N}(\alpha) \neq 0$, that is to say $x-\alpha$ and $\widetilde{N}$ are co-prime. Then

$$\sum_{\substack{M|N \\ \deg M\le m}} \mu(M) = \sum_{\substack{M|\widetilde{N} \\ \deg M\le m}} \mu(M) + \sum_{\substack{M|\widetilde{N} \\ 1+\deg M\le m}} \mu((x-\alpha)\cdot M) = \sum_{\substack{M|\widetilde{N} \\ \deg M=m}} \mu(M),$$

so that

$$\mathrm{Poly}_q(n,m,m+1;k) \asymp \frac{1}{q^n} \sum_{\alpha\in\mathbb{F}_q} \sum_{\substack{\widetilde{N}\in\mathbb{F}_q[t] \\ \deg \widetilde{N}=n-1 \\ \widetilde{N}(\alpha)\neq 0}} \Bigg| \sum_{\substack{M|\widetilde{N} \\ \deg M=m}} \mu(M) \Bigg|^{2k} \asymp m^{2^{2k-1}-2k-1} + 1$$

for $n \geq 2mk$, by an easy variation of Theorem 1.2. This argument can be made rigorous; we leave this task to the interested reader.

Let us now study $\mathrm{Poly}_q(n,m,h;k)$ more generally. For any $h \in \mathbb{Z}_{\geq 1}$ and $n \geq 2mk$, we note that

$$\begin{aligned}\mathrm{Poly}_q(n,m,h;k) &= \sum_{\substack{G_1,\ldots,G_{2k} \\ m-h<\deg(G_i)\le m \\ 1\le i\le 2k}} \frac{\mu(G_1)\cdots\mu(G_{2k})}{q^{\deg([G_1,\ldots,G_{2k}])}} \\ &= \sum_{\substack{G_I\ (I\in\mathcal{S}^*(2k)) \\ m-h<\sum_{I\ni i}\deg(G_I)\le m\ (1\le i\le 2k)}}^{\flat} \Bigg(\prod_{I\in\mathcal{S}^-(2k)} \frac{\mu(G_I)}{q^{\deg(G_I)}} \Bigg) \Bigg(\prod_{I\in\mathcal{S}^+(2k)} \frac{\mu^2(G_I)}{q^{\deg(G_I)}} \Bigg),\end{aligned}$$

as before. Applying Fourier inversion $2k$ times, we find that, for any $r \in (0,1)$,

$$\mathrm{Poly}_q(n,m,h;k) = \sum_{\substack{m-h<\ell_j\le m\\ 1\le j\le 2k}} \sideset{}{^\flat}\sum_{G_I\ (I\in\mathcal{S}^*(2k))} \Bigg(\prod_{I\in\mathcal{S}^-(2k)} \frac{\mu(G_I)}{q^{\deg(G_I)}}\Bigg)\Bigg(\prod_{I\in\mathcal{S}^+(2k)} \frac{\mu^2(G_I)}{q^{\deg(G_I)}}\Bigg) \times \prod_{j=1}^{2k}\int_0^1 (re(\theta_j))^{-\ell_j+\sum_{I\ni j}\deg(G_I)}\mathrm{d}\theta_j.$$

So, if we set

$$\mathcal{Z}_q(w) = \sum_{G\in\mathbb{F}_q[t]}\left(\frac{w}{q}\right)^{\deg(G)} = \prod_P (1-(w/q)^{\deg(P)})^{-1},$$

the $\mathbb{F}_q[t]$ analogue of the Riemann zeta function, then

$$\begin{aligned}\mathrm{Poly}_q(n,m,h;k) &= \sum_{\substack{m-h<\ell_j\le m\\ 1\le j\le 2k}} \int_{[0,1]^{2k}} \widetilde{F}_q((re(\theta_j))_j)\frac{\prod_{I\in\mathcal{S}^+(2k)}\mathcal{Z}_q(r^{\#I}e(\theta_I))}{\prod_{I\in\mathcal{S}^-(2k)}\mathcal{Z}_q(r^{\#I}e(\theta_I))}\prod_{j=1}^{2k}\frac{e(-\ell_j\theta_j)}{r^{\ell_j}}\mathrm{d}\boldsymbol{\theta}\\ &= \int_{[0,1]^{2k}} \widetilde{F}_q\left((re(\theta_j))_j\right)\frac{\prod_{I\in\mathcal{S}^+(2k)}\mathcal{Z}_q(r^{\#I}e(\theta_I))}{\prod_{I\in\mathcal{S}^-(2k)}\mathcal{Z}_q(r^{\#I}e(\theta_I))}\prod_{j=1}^{2k}\sum_{\ell=m-h+1}^{m}\frac{e(-\ell\theta_j)}{r^{\ell}}\mathrm{d}\boldsymbol{\theta},\end{aligned}$$

where $\theta_I = \sum_{j\in I}\theta_j$ and $\widetilde{F}_q(\boldsymbol{w})$ is a certain function that is analytic and non-zero when $|w_j| < \sqrt{q}/2k$ for all j.

We take $r = 1-1/m$ and note that the main contribution to $\mathrm{Poly}_q(n,m,h;k)$ should come from those values of $\boldsymbol{\theta}$ for which there are many $I \in \mathcal{S}^+(2k)$ such that $\theta_I \equiv 0 \,(\mathrm{mod}\ )1$. This is the key difference with the integer case: before, we needed many $I \in \mathcal{S}^+(2k)$ with $s_I = 0$. So we see two different linear algebra problems: one over the group $\mathbb{R}/\mathbb{Z}$, which has torsion, and one over $\mathbb{R}$, which does not. The presence of torsion in $\mathbb{R}/\mathbb{Z}$ is a reflection of the discreteness of the polynomial setting (of the degree of the polynomials, more precisely), and the fact that $\mathbb{R}$ is a field reflects the continuous nature of the integer problem (of the logarithms of integers, more precisely).

When $h = 1$, then the integrand is $\asymp 1/m$ when $\theta_j = O(1/m)$ mod 1 for all j, much like the integer analogue. However, if we take $\theta_j = 1/2 + O(1/m)$ for all j, then we see that $\theta_I = O(1/m)$ mod 1 for $I \in \mathcal{S}^+(2k)$, whereas $\theta_I = 1/2 + O(1/m)$ mod 1 for $I \in \mathcal{S}^-(2k)$, so the integrand has size $m^{2^{2k-1}-1}$ for such $\boldsymbol{\theta}$. The volume of this region is $\asymp 1/m^{2k}$, leading to a contribution of size $m^{2^{2k-1}-2k-1}$ to $\mathrm{Poly}_q(n,m,1;k)$, which is precisely its order of magnitude for $k \ge 2$. Note that the fact the main contribution comes from when $\theta_j \approx 1/2$ and not when $\theta_j \approx 0$ is a reflection of the local constraint at the prime 2 we noticed above.

Similarly to the above case, if $h = 2$ and $\theta_j = 1/2 + O(1/m)$, then the integrand becomes

$$\widetilde{F}_q\,(1/2,\dots,1/2)\,\frac{m^{2^{2k-1}-1}(1+O(1/m))}{\mathcal{Z}_q(1/2)^{4^k}}\prod_{j=1}^{2k}(1+e(\theta_j)).$$

By Taylor expansion, we have that

$$1+e(\theta_j) = 1-e(\theta_j-1/2) = -(\theta_j-1/2) - \frac{(\theta_j-1/2)^2}{2} - \cdots$$

By symmetry, we should then have that

$$\begin{aligned}&\int\cdots\int_{\substack{|\theta_j-1/2|\le 1/m\\ 1\le j\le 2k}} \widetilde{F}_q\,(1/2,\ldots,1/2)\,\frac{m^{2^{2k-1}-1}(1+O(1/m))}{\mathscr{Z}_q(1/2)^{4^k}}\prod_{j=1}^{2k}(1+e(\theta_j))\mathrm{d}\boldsymbol{\theta}\\ &\quad=(1+O(1/m))\int\cdots\int_{\substack{|\theta_j-1/2|\le 1/m\\ 1\le j\le 2k}} \widetilde{F}_q\,(1/2,\ldots,1/2)\,\frac{m^{2^{2k-1}-1}}{\mathscr{Z}_q(1/2)^{4^k}}\prod_{j=1}^{2k}\frac{(\theta_j-1/2)^2}{2}\mathrm{d}\boldsymbol{\theta},\end{aligned}$$

which leads to a contribution of size $m^{2^{2k-1}-6k-1}$ to $\mathrm{Poly}_q(n,m,2;k)$. This should be the dominant contribution for large k, even though for small k other regions can dominate. For example, if $k=2$ and we take $\theta_1,\theta_2\in[0.33,0.34]$, $\theta_3,\theta_4\in[0.66,0.67]$, and $\theta_1+\theta_2,\theta_3+\theta_4$, $\theta_1+\theta_4=O(1/m)$, then the integrand becomes $\asymp m^5$, and we are integrating over a region of volume $\asymp 1/m^3$, so we see that $\mathrm{Poly}_q(n,m,1;2)\gg m^2$. In fact, this is the exact order of magnitude of $\mathrm{Poly}(n,m,1;2)$.

We conclude our discussion with another peculiar fact: if $h=3$, then

$$\sum_{\ell=m-h+1}^{m} e(\ell\theta)=e(m\theta)(1+e(\theta)+e(2\theta))=e(m\theta)+O(1/m)$$

when $\theta=1/2+O(1/m)$. So $\mathrm{Poly}_q(n,m,3;k)$ should have the same size as $\mathrm{Poly}_q(n,m,1;k)$, whereas $\mathrm{Poly}_q(n,m,h;k)$ is a bit smaller, by a factor of size $m^{O(k)}$. In general, no matter how we choose h, we cannot make the sum $\sum_{\ell=m-h+1}^{m} e(\ell\theta)$ small enough to cancel the contribution of the factors $\mathscr{Z}_q(r^{\#I}e(\theta_I))$ for even I in the region $\theta_j\sim 1/2$, so the quantities $\mathrm{Poly}_q(n,m,h;k)$ do not behave in the same way as $\mathscr{M}_{\widetilde{f}_0,2k}(R)$ for k large.

3. The analogy for permutations

Completion of the proof of Theorem 1.1. – It remains to prove the two claims for the quantity $c(m,k)$, which we recall is defined as the number of $(2^{2k}-1)$-tuples $(r_I)_{\emptyset\neq I\subset[2k]}$ of non-negative integers such that $r_I\in\{0,1\}$ for $\#I$ odd and such that $\sum_{I:\,i\in I} r_I=m$, for each $i\in[2k]$.

Given any vector $\{r_I\,:\,\emptyset\neq I\subset[2k]\}$ counted by $c(m,k)$, the vector $\{r'_I\,:\,\emptyset\neq I\subset[2k]\}$ is counted by $c(m+1,k)$ where $r'_{\{1,2\}}=r_{\{1,2\}}+1$ and $r'_{\{3,4,\ldots,2k\}}=r_{\{3,4,\ldots,2k\}}+1$, and $r'_I=r_I$ otherwise. Since $r_I\mapsto r'_I$ is injective, we see $c(m,k)\le c(m+1,k)$ for all $m\ge 0$, as claimed.

We now estimate $c(m,k)$. When $k=1$, we find immediately that $c(m,1)=2$, so there is nothing to prove. Assume now that $k\ge 2$. Since $c(m,k)$ is increasing in m and $c(0,k)=1$, we may assume that m is even and large enough. We note that there are $\asymp_k 1$ possibilities for the r_I for the odd-sized I. Otherwise we have to satisfy $2k$ equations with $2^{2k-1}-1$ variables. Hence the number of solutions should be

$$\asymp_k m^{2^{2k-1}-2k-1}+1,$$

as claimed. Certainly, this argument yields an appropriate upper bound. To prove the lower bound for $k \geq 2$ we will construct this number of solutions. Let

$$\mathcal{J} = \{\{i, j\} : 1 \leq i < j \leq 4\} \cup \{\{1, j\} : 5 \leq j \leq 2k\},$$

so that $\#\mathcal{J} = 2k + 2$. Set $r_I = 0$ if $I \in \mathcal{S}^-(2k)$ and, given $\delta > 0$ to be chosen later, let r_I be any even integer from the range $[0, \delta m/4^k]$ if $I \in \mathcal{S}^+(2k) \setminus (\mathcal{J} \cup \{\{5, 6, \ldots, 2k\}\})$. Finally, if $k > 2$, let $r_{\{5,6,\ldots,2k\}}$ be an even integer from the range $[m - 2\delta m, m - \delta m]$. There are $\asymp_{k,\delta} m^{2^{2k-1}-3-2k}$ such choices of r_I, $I \in \mathcal{S}^+(2k) \setminus \mathcal{J}$. Then select

$$r_{\{1,j\}} := m - \sum_{I \in \mathcal{S}^+(2k) \setminus \mathcal{J},\ j \in I} r_I \quad (5 \leq j \leq 2k),$$

which is an even integer lying in the interval $[0, 2\delta m]$, so that $\sum_{I \in \mathcal{S}^+(2k),\, j \in I} r_I = m$ for $5 \leq j \leq 2k$. Now set

$$\mathcal{J}_2 = \{\{i, j\} : 1 \leq i < j \leq 4\},$$
$$m_j = m - \sum_{I \in \mathcal{S}^+(2k) \setminus \mathcal{J}_2,\ j \in I} r_I \quad (1 \leq j \leq 4).$$

We note that the m_j are even integers lying in the interval $[m - 2k\delta m, m]$. It remains to choose r_I, $I \in \mathcal{J}_2$, such that $\sum_{I \in \mathcal{J}_2,\ j \in I} r_I = m_j$ for $1 \leq j \leq 4$. Then, we select any even integers $r_{\{2,4\}}, r_{\{3,4\}}$ from $[\sqrt{\delta} m - \delta m, \sqrt{\delta} m + \delta m]$, and we set

$$r_{\{1,4\}} = m_4 - r_{\{2,4\}} - r_{\{3,4\}}.$$

Finally, we define $r_{\{1,2\}}, r_{\{1,3\}}, r_{\{2,3\}}$ such that

$$r_{\{1,2\}} + r_{\{1,3\}} = m_1 - r_{\{1,4\}} = m_1 - m_4 + r_{\{2,4\}} + r_{\{3,4\}};$$
$$r_{\{1,2\}} + r_{\{2,3\}} = m_2 - r_{\{2,4\}}; \text{ and}$$
$$r_{\{1,3\}} + r_{\{2,3\}} = m_3 - r_{\{3,4\}}.$$

Note that the right-hand sides are all even so there is no parity problem, and the solutions we obtain are non-negative integers for δ small enough. We have thus constructed $\gg_k m^{2^{2k-1}-2k-1}$ solutions counted by $c(m, k)$. This completes the proof of the lemma. $\square$

REMARK 3.1. – It should not be too difficult to determine $c(m, k)$ exactly in some special cases. For example, we have that $c(m, 1) = 2$ and $c(m, 2) = \frac{1}{3}(64m^3 - 135m^2 + 182m - 66)$ for all $m \geq 1$.

Finally, we prove a probabilistic interpretation for $c(m, k)$. In its statement, we have set with a slight abuse of notation

$$M(\boldsymbol{c}; r) := \sum_{\substack{0 \leq b_j \leq c_j \\ 1 \leq j \leq m \\ \sum_j j b_j = r}} (-1)^{b_1 + \cdots + b_m} \tag{3.1}$$

for an m-tuple of non-negative integers $\boldsymbol{c} = (c_1, \ldots, c_m)$.

PROPOSITION 3.1. – *Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$ be a vector of pairwise independent Poisson random variables, where X_j has parameter $1/j$. For every $k \in \mathbb{Z}_{\geq 1}$, we have that*

$$c(m, k) = \mathbb{E}[M(\boldsymbol{X}; m)^{2k}].$$

In passing, we note that Proposition 3.1 is purely a statement about Poisson random variables and not immediately related to permutations or polynomials over finite fields, but our proof makes use of this connection.

Before we prove Proposition 3.1, we need a lemma.

LEMMA 3.2. – *Let $N \geq m \geq 1$. The proportion of permutations $\sigma \in S_N$ that have no cycles of length $\leq m$ is*

$$\prod_{j=1}^{m} e^{-1/j} + O\left(\frac{m^2}{N}\right).$$

Proof. – Note that this lemma was proven by the first author in [10] for large m, but here we are mainly interested in the case when m is very small compared to N. We apply inclusion-inclusion. If $\mathcal{C}_j$ denotes the j-cycles in S_N, we write $|\pi| = j$ for an element π of $\mathcal{C}_j$, and we let $\mathcal{C}$ be the union of $\mathcal{C}_1, \ldots, \mathcal{C}_m$. Then

$$\begin{aligned}
&\#\{\sigma \in S_N : \sigma \text{ has no cycles of length} \leq m\} \\
&\quad = N! - \sum_{\pi \in \mathcal{C}} (N - |\pi|)! + \sum_{\substack{\pi_1, \pi_2 \in \mathcal{C} \\ \pi_1, \pi_2 \text{ disjoint}}} (N - |\pi_1| - |\pi_2|)! \mp \cdots \\
&\quad = \sum_{\substack{c_1, \ldots, c_m \geq 0 \\ c_1 + 2c_2 + \cdot + mc_m \leq n}} (-1)^{c_1 + \cdots + c_m} (N - c_1 - 2c_2 \cdots - mc_m)! \sum_{\substack{\pi_1, \pi_2, \cdots \in \mathcal{C} \text{ disjoint} \\ \#\{i : |\pi_i| = j\} = c_j \ \forall j}} 1.
\end{aligned}$$

In order to count the inner quantity, we note that if $r = c_1 + \cdots + c_m$ is the total number of disjoint cycles we are choosing, and we have fixed our choice for $\pi_1, \pi_2, \ldots, \pi_{r-1}$, then there are

$$\frac{(N - |\pi_1| - \cdots - |\pi_{r-1}|)!}{|\pi_r|!(N - |\pi_1| - \cdots - |\pi_{r-1}| - |\pi_r|)!}$$

choices for the set of size $|\pi_r|$ fixed by π_r, and then $(|\pi_r| - 1)!$ possibilities for a cycle on $|\pi_r|$ given elements. Inductively, we then find that the total number of possibilities for $\pi_1, \ldots, \pi_r$ should be

$$\frac{N!}{(N - |\pi| - \cdots - |\pi_r|)!} \cdot \frac{1}{|\pi_1| \cdots |\pi_r|} = \frac{N!}{(N - c_1 - 2c_2 - \cdots - mc_m)!} \prod_{j=1}^{m} \frac{1}{j^{c_j}}.$$

Note though we have overcounted: each possibility of j-cycles occurs $c_j!$ times, depending on the order they are picked, so we must divide the above expression by $c_1! \cdots c_m!$. We then find that

$$\begin{aligned}
\frac{\#\{\sigma \in S_N : \sigma \text{ has no cycles of length} \leq m\}}{N!} &= \sum_{\substack{c_1, \ldots, c_m \geq 0 \\ c_1 + 2c_2 + \cdot + mc_m \leq n}} \prod_{j=1}^{m} \frac{(-1/j)^{c_j}}{c_j!} \\
&= \prod_{j=1}^{m} e^{-1/j} + O\left(\frac{m^2}{N}\right),
\end{aligned}$$

where the error term is obtained by noting that

$$\mathbf{1}_{c_1 + 2c_2 + \cdots + mc_m \leq N} \leq (c_1 + 2c_2 + \cdots + mc_m)/N. \qquad \square$$

Proof of Proposition 3.1. – We recall that we have already proved that

$$c(m,k) = \frac{1}{N!}\sum_{\sigma\in S_N}\Bigg(\sum_{\substack{T\subset[n]\\ \sigma(T)=T\\ \#T=m}}\mu(\sigma|_T)\Bigg)^{2k}$$

for any $N \geq 2mk$. We will now rewrite the right hand side for n much larger than m and k. Note that if σ has c_j cycles of length j for each $j \in \{1,\dots,m\}$, then

$$\sum_{\substack{T\subset[n]\\ \sigma(T)=T\\ \#T=m}}\mu(\sigma|_T) = M(\boldsymbol{c};m) = \sum_{\substack{0\leq b_j\leq c_j\\ 1\leq j\leq m\\ \sum_j jb_j=m}}(-1)^{b_1+\cdots+b_m},$$

where $\boldsymbol{c} = (c_1,\dots,c_m)$. Moreover, a generalization of Cauchy's formula (see Lemma 2.2 in [4]) implies that if $t := c_1 + 2c_2 + \cdots + mc_m \leq N$, then

$$\begin{aligned}&\frac{\#\{\sigma\in S_N : \sigma \text{ has } c_j\ j\text{-cycles of length } j\ (1\leq j\leq m)\}}{N!}\\ &= \Bigg(\prod_{j=1}^m \frac{1}{j^{c_j}c_j!}\Bigg)\cdot\frac{\#\{\sigma\in S_{N-t} : \sigma \text{ has no cycles of length} \leq m\}}{(N-t)!}.\end{aligned}$$

Applying Lemma 3.2, it is then easy to conclude that

$$\begin{aligned}c(m,k) &= \lim_{N\to\infty}\frac{1}{N!}\sum_{\sigma\in S_N}\Bigg(\sum_{\substack{T\subset[n]\\ \sigma(T)=T\\ \#T=m}}\mu(\sigma|_T)\Bigg)^{2k}\\ &= \sum_{c_1,\dots,c_m\geq 0} M(\boldsymbol{c};m)^{2k}\prod_{j=1}^m\frac{e^{-1/j}}{j^{c_j}c_j!}.\end{aligned}$$

Since $\mathbb{P}(X_j = c_j) = e^{-1/j}/(j^{c_j}c_j!)$, this is $\mathbb{E}[M(\boldsymbol{X};m)^{2k}]$, and so completes the proof. □

4. The analogy for polynomials over finite fields

Proof of Theorem 1.2. – Throughout this proof all polynomials we consider are monic, and P denotes a generic monic irreducible polynomial over $\mathbb{F}_q$. Note that

$$\begin{aligned}\mathrm{Poly}_q(n,m;k) &= \frac{1}{q^n}\sum_{\deg(F)=n}\Bigg(\sum_{\substack{G|F\\ \deg(G)=m}}\mu(G)\Bigg)^{2k}\\ &= \prod_{\deg(P)\leq m}\Big(1-q^{-\deg(P)}\Big)\sum_{P|F\ \Rightarrow\ \deg(P)\leq m}\frac{1}{q^{\deg(F)}}\Bigg(\sum_{\substack{G|F\\ \deg(G)=m}}\mu(G)\Bigg)^{2k}\end{aligned}$$

for $n \geq 2km$, as can be proven by expanding the $2k$-th power in both sides, and noticing that if $G_j|F$ for each $j \leq 2k$, then we may write $F = [G_1,\dots,G_{2k}]H$ for some monic polynomial H.

Next, note that if $F = P_1^{n_1} \cdots P_r^{n_r}$ is the factorisation of F into monic irreducible factors, and we write $c_j = \#\{i : \deg(P_i) = j\}$ for $1 \le j \le m$, then

$$\sum_{\substack{G|F \\ \deg(G)=m}} \mu(G) = M(\boldsymbol{c}; m)$$

with $M(\boldsymbol{c}; m)$ defined by (3.1). In particular, we see that $\sum_{G|F,\ \deg(G)=m} \mu(G)$ is a function of the vector $c(F) := (c_1, \dots, c_m)$. Moreover, given a fixed vector $\boldsymbol{c}$, we see that

$$\begin{aligned}\prod_{\deg(P)\le m}\left(1-q^{-\deg(P)}\right)\sum_{\substack{F:c(F)=\boldsymbol{c}\\ P|F\ \Rightarrow\ \deg(P)\le m}}\frac{1}{q^{\deg(F)}} &= \prod_{\deg(P)\le m}\left(1-q^{-\deg(P)}\right)\prod_{j=1}^{m}\frac{\binom{N_j}{c_j}}{(q^j-1)^{c_j}}\\ &= \prod_{j=1}^{m}\frac{\binom{N_j}{c_j}(1-q^{-j})^{N_j}}{(q^j-1)^{c_j}},\end{aligned}$$

where

$$N_j := \#\{P \in \mathbb{F}_q[t] : P \text{ irreducible},\ \deg(P) = j\}.$$

(Note that we have $(q^j-1)^{c_j}$ and not q^{jc_j} in the denominator because we have to sum over powers of P_j too.) Galois theory implies that $q^j = \sum_{j'|j} j' N_{j'}$, whence

$$N_j = \frac{1}{j}\sum_{j'|j}\mu(j')q^{j/j'} = \frac{q^j}{j}\left(1 + O\left(\frac{\mathbf{1}_{j\ge 2}}{(q^j/j)^{1/2}}\right)\right) \tag{4.1}$$

and

$$q + jN_j \le q^j \quad (j \ge 2). \tag{4.2}$$

Our next task is to control the quantity

$$\prod_{j=1}^{m}\frac{\binom{N_j}{c_j}(1-q^{-j})^{N_j}}{(q^j-1)^{c_j}}$$

and remove the dependence on q. First, note that

$$\prod_{j=1}^{m}(1-q^{-j})^{N_j} = (1+O(1/q))\prod_{j=1}^{m}e^{-1/j}.$$

Furthermore,

$$\frac{\binom{N_j}{c_j}}{(q^j-1)^{c_j}} = \frac{N_j^{c_j}(1+O(c_j/N_j))^{c_j}}{c_j!(q^j-1)^{c_j}} = \frac{1}{c_j!j^{c_j}}\left(1+O\left(\frac{\mathbf{1}_{j\ge2}\cdot c_j}{(q^j/j)^{1/2}}+\frac{\mathbf{1}_{j=1}c_j}{q^j}\right)\right),$$

provided that $c_1 \le q$ and that $c_j \le \sqrt{q^j/j}$ if $j \ge 2$.

Therefore, if $c_1 \le q$ and $\sum_{2\le j\le m} c_j \cdot (j/q^j)^{1/2} \le 1$, then

$$\prod_{j=1}^{m}\frac{\binom{N_j}{c_j}(1-q^{-j})^{N_j}}{(q^j-1)^{c_j}} = \left(1+O\left(\frac{c_1+1}{q}+\sum_{j=2}^{m}\frac{c_j j^{1/2}}{q^{j/2}}\right)\right)\prod_{j=1}^{m}\frac{e^{-1/j}}{c_j!j^{c_j}}.$$

Together with Proposition 3.1, this implies that

$$\mathrm{Poly}_q(n,m;k) = c(m,k) + O(R_1+R_2+R_3) \quad (n \ge 2mk),$$

where

$$R_1 = \sum_{c_1,\dots,c_m\geq 0} M(\boldsymbol{c};m)^{2k}\left(\frac{c_1+1}{q}+\sum_{j=2}^m \frac{c_j\, j^{1/2}}{q^{j/2}}\right)\prod_{j=1}^m \frac{e^{-1/j}}{c_j!\, j^{c_j}},$$

$$R_2 = \sum_{\substack{c_1,\dots,c_m\geq 0\\ c_1>q \text{ or } \sum_{j>1} c_j j^{1/2}/q^{j/2}>1}} M(\boldsymbol{c};m)^{2k}\prod_{j=1}^m \frac{e^{-1/j}}{c_j!\, j^{c_j}}$$

$$\leq \sum_{c_1,\dots,c_m\geq 0} M(\boldsymbol{c};m)^{2k}\left(\frac{c_1}{q}+\sum_{j=2}^m \frac{c_j\, j^{1/2}}{q^{j/2}}\right)\prod_{j=1}^m \frac{e^{-1/j}}{c_j!\, j^{c_j}} \leq R_1,$$

and

$$R_3 = \sum_{\substack{c_1,\dots,c_m\geq 0\\ c_1>q \text{ or } \sum_{j>1} c_j j^{1/2}/q^{j/2}>1}} M(\boldsymbol{c};m)^{2k}\prod_{j=1}^m \frac{e^{-1/j}\binom{N_j}{c_j}}{(q^j-1)^{c_j}}.$$

For R_3, we note that $c_j \leq N_j$ in its range; otherwise, $\binom{N_j}{c_j}=0$. In particular, $c_1\leq N_1=q$. Moreover, (4.2) implies that

$$\binom{N_j}{c_j}\leq \frac{N_j^{c_j}}{c_j!}\leq \begin{cases} \dfrac{(q^j-1)^{c_j}}{c_j!\, j^{c_j}} & \text{if } j\geq 2,\\[2ex] \dfrac{q^{c_1}}{c_1}\leq (1-1/q)^{-q}\cdot\dfrac{(q-1)^{c_1}}{c_1!} & \text{if } j=1.\end{cases}$$

Therefore

$$R_3 \ll \sum_{c_1,\dots,c_m\geq 0} M(\boldsymbol{c};m)^{2k}\left(\sum_{j=2}^m \frac{c_j\, j^{1/2}}{q^{j/2}}\right)\prod_{j=1}^m \frac{e^{-1/j}}{c_j!\, j^{c_j}} \leq R_1.$$

We thus see that Theorem 1.2 is reduced to proving that $R_1 \ll_k c(m,k)/q$. It suffices to show that

$$T_i := \sum_{c_1,\dots,c_m\geq 0} c_i M(\boldsymbol{c};m)^{2k}\prod_{j=1}^m \frac{e^{-1/j}}{j^{c_j}\, c_j!} \ll c(m,k)\quad (1\leq i\leq m).$$

Indeed, we note that the term with $c_i=0$ does not contribute, and we replace c_i by c_i+1 to find that

$$T_i = \frac{1}{i}\sum_{c_1,\dots,c_m\geq 0} M(\boldsymbol{e}_i+\boldsymbol{c};m)^{2k}\prod_{j=1}^m \frac{e^{-1/j}}{j^{c_j}\, c_j!},$$

where $\boldsymbol{e}_i$ denotes the m-th dimensional vector that has the i-th coordinate equal to 1 and all other coordinates equal to 0. Note that

$$M(\boldsymbol{e}_i+\boldsymbol{c};m) = \sum_{\substack{0\leq b_j\leq c_j\ \forall j\neq i\\ 0\leq b_i\leq c_i+1\\ \sum_j jb_j=m}} (-1)^{b_1+\cdots+b_m} = M(\boldsymbol{c};m)+(-1)^{c_i+1}M(\boldsymbol{c}_i;m-i(c_i+1)),$$

where $\boldsymbol{c}_i=(c_1,\dots,c_{i-1},0,c_{i+1},\dots,c_m)$, so that

$$M(\boldsymbol{e}_i+\boldsymbol{c};m)^{2k}\leq 2^{2k-1}\left(M(\boldsymbol{c};m)+M(\boldsymbol{c}_i;m-i(c_i+1))\right)^{2k},$$

by Hölder's inequality. We thus conclude that

$$\begin{aligned} T_i &\le \frac{2^{2k-1}}{i} c(m,k) + \frac{2^{2k-1}}{i} \sum_{c_i=0}^{\infty} \frac{e^{-1/i}}{c_i! i^{c_i}} \sum_{(c_j)_{j\le m,\, j\neq i}} M(\boldsymbol{c}_i; m - i(c_i+1))^{2k} \prod_{j\neq i} \frac{e^{-1/j}}{c_j! j^{c_j}} \\ &\le \frac{2^{2k-1}}{i} c(m,k) + \frac{2^{2k-1}}{i} \sum_{c_i=0}^{\infty} \frac{e^{-1/i}}{c_i! i^{c_i}} c(m - i(c_i+1), k), \end{aligned}$$

since the $M(\boldsymbol{c}_i; m-i(c_i+1))^{2k}$ is independent of the value of the c_j's with $j > m-i(c_i+1)$. Recalling that $c(\ell,k)$ is an increasing function of ℓ by Theorem 1.1, we arrive to the claimed bound $T_i \ll_k c(m,k)$, whence Theorem 1.2 follows. □

5. The support of $M_{f_0}(n;R)$

We prove here (1.4), which we recall is the statement that

$$\#\{n \le x : \ M_{f_0}(n;R) \neq 0\} \asymp \frac{x}{(\log R)^{\delta} (\log\log R)^{3/2}} \quad (x \ge R^4).$$

The lower bound was proven in the introduction, so we are left to show the upper bound. We recall the relation (1.5)

$$M_f(n;R) = \sum_{d | p_2 \cdots p_r m} \mu(d) \left\{ f\left(\frac{\log d}{\log R}\right) - f\left(\frac{\log p_1}{\log R} + \frac{\log d}{\log R}\right) \right\},$$

where $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} m$, where $p_1 < \cdots < p_r$, $\alpha_i \ge 1$ and all of the prime divisors of m are $> p_r$. Taking $r = 2$, letting q be the smallest prime dividing n and writing $n = q^j m$ with $q \nmid m$, we see that

$$M_{f_0}(n;R) = \sum_{\substack{d|m \\ R/q<d\le R}} \mu(d).$$

Therefore,

$$\#\{n \le x : M_{f_0}(n;R) \neq 0\} \le \sum_{q^j \le y} H(x/q^j, q; R/q, R) + O\left(\frac{x}{\log y}\right), \tag{5.1}$$

for any parameter $y \le R^{1/3}$ to be chosen later, where

$$H(X,Y;Z,W) := \#\{n \le X : P^-(n) > Y,\ \exists d | n \text{ with } Z < d \le W\}.$$

We have the following estimate, that is useful in its own right.

Proposition 5.1. – *Uniformly for $1 \le Y \le Z \le W \le X/(2Z)$ and $2Z \le W \le Z^2$, we have*

$$H(X,Y;Z,W) \ll \frac{X}{\log Y} \cdot \frac{1}{\lambda^{\delta}(1+\log\lambda)^{3/2}},$$

where λ is defined by the relation $W = Z^{1+1/\lambda}$ and $\delta = 1 - \frac{1+\log\log 2}{\log 2} = 0.086071\ldots$

Remark. – In the special case when $W = 2Z$, Ford [6] used a more refined argument and determined the exact order of magnitude of $H(X,Y;Z,W)$. The exact statement is a bit complicated, so we refer the interested reader to Ford's paper.

Proof. – We adapt the proof of Lemma 6.1 in Ford's paper [5]. By a dyadic decomposition argument, it suffices to upper bound the difference $H(X,Y;Z,W)-H(X/2,Y;Z,W)$. Let n be counted by this difference, so that it can be written as $n=n_1n_2$ with $n_1\in(Z,W]$. We thus have that $n_2\in(X/2W,X/Z]$. If $p=\min\{P^+(n_1),P^+(n_2)\}\in(Y,W]$, then we may write $n=apb$, where:

(i) all prime factors of a are in (Y,p);

(ii) all prime factors of b are $\geq p$ (and there is at least one such prime factor);

(iii) there is a divisor $d|a$ such that $pd\in(Z,W]\cup(X/2W,X/Z]$.

If we set $\mathscr{L}(a;\sigma):=\bigcup_{d|a}[\log d-\sigma,\log d)$ and $\eta=\log(W/Z)$, the last condition can be also written as:

(iii′) either $\log(Z/p)\in\mathscr{L}(a;\eta)$, or $\log(X/(2Wp))\in\mathscr{L}(a;\eta+\log 2)$.

Let $\eta'=\eta+\log 2$, and note that $\eta'\asymp\eta$ by our assumption that $W\geq 2Z$. Moreover, let $Z_1=Z$ and $Z_2=X/2W$, so that condition (iii′) yields condition

(iii″) $\log(Z_j/p)\in\mathscr{L}(a;\eta')$ for some $j\in\{1,2\}$.

Finally, note that since there is $d|a$ with $dp>Z_j$, we must have that $p>Z_j/d\geq Z_j/a$. We thus conclude that we must have the condition

(iv) $p>Q_j(a):=\max\{P^+(a),Z_j/a\}$.

Given a and p satisfying conditions (i), (iii″) and (iv), the number of $b\in(1,X/ap]$ such that $P^-(b)>p$ is $\ll X/(ap\log p)$. Indeed, notice that if there is one such b, then $X/ap\geq b\geq p$, so that the claimed estimate follows by a standard sieve bound, such as Theorem 4.3 of [8]. We thus conclude that

$$H(X,Y;Z,W)-H(X/2,Y;Z,W)\ll X\sum_{j=1}^{2}\sum_{a\in\mathscr{P}(Y,W)}\frac{1}{a}\sum_{\substack{p>Q_j(a)\\ \log(Z_j/p)\in\mathscr{L}(a;\eta')}}\frac{1}{p\log p},$$

where $\mathscr{P}(Y,W)$ denotes the set of integers all of whose prime factors are in $(Y,W]$. As in the proof of Lemma 6.1 in [5], we have that the sum over p is $\ll L(a;\eta')/\log^2 Q_j(a)$, where $L(a;\sigma)$ denotes the Lebesgue measure of $\mathscr{L}(a;\sigma)$. We conclude that

$$\begin{aligned} H(X,Y;Z,W)-H(X/2,Y;Z,W) &\ll X\sum_{j=1}^{2}\sum_{a\in\mathscr{P}(Y,W)}\frac{L(a;\eta')}{a\log^2 Q_j(a)}\\ &\ll\frac{X}{\log Y}\sum_{j=1}^{2}\sum_{P^+(a')\leq Y}\sum_{a\in\mathscr{P}(Y,W)}\frac{L(a;\eta')}{aa'\log^2 Q_j(a)}. \end{aligned}$$

Since $Q_j(a)\geq Q_j(aa')$ and $L(a;\eta')\leq L(aa';\eta')$, we have the estimate

$$H(X,Y;Z,W)-H(X/2,Y;Z,W)\ll\frac{X}{\log Y}\sum_{j=1}^{2}\sum_{P^+(m)\leq W}\frac{L(m;\eta')}{m\log^2 Q_j(m)}.$$

Since $Z_2=X/2W\geq Z=Z_1$, the contribution for $j=2$ is bounded by the contribution from $j=1$, and so it suffices to just consider $Z_j=Z$. In this case the contribution is $\ll 1/(\lambda^\delta(1+\log\lambda)^{3/2}$ by Lemma 3.3, equation (3.8) and Lemma 3.7 of [5]. This completes the proof. □

Proposition 5.1 implies that

$$H(x/q^j, q; R/q, R) \ll \frac{x}{q^j \log q} \cdot \left(\frac{\log q}{\log R}\right)^{\delta} \left(\log \frac{\log R}{\log q}\right)^{-3/2},$$

uniformly in $2 \le q^j \le y \le R^{1/2}$ and $x \ge R^{5/2}$. Inserting this bound to (5.1), we deduce that

$$\#\{n \le x : M_{f_0}(n; R) \neq 0\} \ll \frac{x}{(\log R)^{\delta}(\log\log R)^{3/2}} + \frac{x}{\log y}.$$

Selecting $y = \exp((\log R)^{\delta}(\log\log R)^{3/2})$ completes the proof of (1.4).

REMARK 5.1. – It is possible to construct integers n for which $M_{f_0}(n; R)$ is quite large. Indeed, let $y \ge 3$ and $k \in \mathbb{Z}_{\ge 1}$ be two parameters such that the interval $(y, 2^{1/k}y)$ contains at least $2k$ primes, and let $q_1 < \cdots < q_{2k}$ be the smallest such primes. Then we set $n = 2q_1 \cdots q_{2k}$ and $R = 2y^k$. By (1.5),

$$M_{f_0}(n; R) = \sum_{\substack{d | q_1 \cdots q_{2k} \\ R/2 < d \le R}} \mu(d).$$

The choice of R implies that the above sum runs over all divisors d of $q_1, \ldots, q_{2k}$ with precisely k prime factors, so that

$$M_{f_0}(n; R) = (-1)^k \binom{2k}{k}.$$

Optimizing the choice of k and y, and using the fact that there are infinitely many y such that $\pi(y + \sqrt{y \log y}) - \pi(y) \gg \sqrt{y/\log y}$ (see, for example, [12, Exercice 5, p. 266]), we find that there exist arbitrarily large integers n such that $|M_f(n; R)| \gg n^{c/\log\log n}$, for any fixed $c < \frac{\log 2}{2}$ with $R \approx n^{1/2}$.

On the other hand, such extreme values of $M_{f_0}(n; R)$ are very rare, as Theorem 1.3 indicates.

6. Inversion formulas

Given $f : \mathbb{R} \to \mathbb{R}$, $R \ge 2$ and $s \in \mathbb{C}$, we set

$$\widehat{f}_R(s) = \int_0^{\infty} f\left(\frac{\log x}{\log R}\right) x^{s-1} \mathrm{d}x = (\log R) \int_{-\infty}^{\infty} f(u) R^{su} \mathrm{d}u,$$

provided that the above integral converges. If f is Lebesgue measurable, supported in $(-\infty, 1]$ and bounded, which will always be the case for us, then $\widehat{f}_R$ defines an analytic function for $\mathrm{Re}(s) > 0$. If, in addition, $f \in C^j(\mathbb{R})$ for some $j \ge 1$ and the derivatives $f', f'', \ldots, f^{(j)}$ are all bounded, then integrating by parts j times yields the formula

$$\widehat{f}_R(s) = \frac{(-1)^j}{s^j (\log R)^{j-1}} \int_{-\infty}^{\infty} f^{(j)}(u) R^{su} \mathrm{d}u. \tag{6.1}$$

In particular, we see that

$$\left|\widehat{f}_R(s)\right| \le \|f^{(j)}\|_{\infty} \cdot \frac{R^{\mathrm{Re}(s)}}{\mathrm{Re}(s)(|s| \log R)^j} \tag{6.2}$$

for $\mathrm{Re}(s) > 0$, where we used our assumption that $\mathrm{supp}(f) \subset (-\infty, 1]$.

Now, for $m \in \mathbb{Z}_{\geq 1}$, the Mellin inversion formula implies that for $c > 0$

$$f\left(\frac{\log m}{\log R}\right) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} \widehat{f}_R(s) m^{-s}\mathrm{d}s. \tag{6.3}$$

In the proof of Theorem 1.3 with $A \geq 1$ and of Theorem 1.6, our assumption that f is a few times differentiable in $\mathbb{R}$ allows us to apply (6.2) and write $M_f(a; R)$ in terms of an absolutely convergent integral, which can easily be truncated at some appropriate height. However, when $A = 0$ in Theorem 1.3, we have $f_0 = \chi_{(-\infty,1]}$, so that $\widehat{(f_0)}_R(s) = R^s/s$. Truncating Perron's Formula is still feasible but rather technical. Instead, we perform a technical maneuver and smoothen f_0 a bit. We consider a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} h(x) = 1 & \text{if } x \leq 1-\eta, \\ 0 \leq h(x) \leq 1 & \text{if } 1-\eta \leq x \leq 1, \\ h(x) = 0 & \text{if } x \geq 1, \end{cases}$$

where $\eta = 1/(\log R)^C$ for some constant $C > 0$ that will be chosen appropriately later. We choose h so that $h^{(j)}(x) \ll_j \eta^{-j}$, for all $j \in \mathbb{Z}_{\geq 0}$. We claim that, for any fixed $L > 0$ and $k \geq 1$, there is $C = C(k, L)$ such that

$$\mathcal{M}_{f_0,2k}(R) = \mathcal{M}_{h,2k}(R) + O\left(\frac{1}{(\log R)^L}\right). \tag{6.4}$$

Indeed, we have that

$$|\mathcal{M}_{f_0,2k}(R) - \mathcal{M}_{h,2k}(R)| \leq 2k \sum_{\substack{d_1,\dots,d_{2k-1}\leq R \\ R^{1-\eta}<d_{2k}\leq R}} \frac{\prod_{j=1}^{2k}\mu^2(d_j)}{[d_1,\dots,d_{2k}]} \leq 2k \sum_{m\leq R^{2k}} \frac{\tau(m)^{2k-1}}{m} \sum_{\substack{d|m \\ R^{1-\eta}<d\leq R}} 1,$$

by setting $m = [d_1,\dots,d_{2k}]$ and $d = d_{2k}$. We split the above sum according to whether $\tau(m) \leq (\log R)^B$ or not, where B is some parameter. We then find that

$$\begin{aligned} |\mathcal{M}_{f_0,2k}(R) - \mathcal{M}_{h,2k}(R)| &\leq 2k \sum_{\substack{m\leq R^{2k} \\ \tau(m)\leq(\log R)^B}} \frac{(\log R)^{(2k-1)B}}{m} \sum_{\substack{d|m \\ R^{1-\eta}<d\leq R}} 1 \\ &\quad + 2k \sum_{\substack{m\leq R^{2k} \\ \tau(m)>(\log R)^B}} \frac{\tau(m)^{2k+1}(\log R)^{-B}}{m} \\ &\ll_k (\log R)^{(2k-1)B+2-C} + (\log R)^{2^{2k+1}-B}. \end{aligned}$$

We choose $B = L + 2^{2k+1}$ and $C \geq (2k-1)2^{2k+1} + 2kL + 2$ to complete the proof of our claim. For the purposes of Theorem 1.3, we may take $L = 1$, so that having $C \geq (2k-1)2^{2k+1} + 2k + 2$ suffices. We also note that

$$\widehat{h}_R(s) = -\frac{1}{s}\int_{1-\eta}^{1} h'(u)R^{su}\mathrm{d}u = -\frac{R^s}{s}\int_0^{\eta} h'(1-u)R^{-su}\mathrm{d}u,$$

by (6.1) and the fact that h is constant outside $[1-\eta, 1]$. In particular, this relation implies that $\widehat{h}_R$ has a meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s = 0$ of residue

$-\int_0^\eta h'(1-u)\mathrm{d}u = 1$. We further note that

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\frac{s\widehat{h}_R(s)}{R^s}\right) = (-1)^{j-1}\int_0^\eta h'(1-u)(u\log R)^j R^{-su}\mathrm{d}u \ll \eta\cdot\eta^{-1}\cdot(\eta\log R)^j = (\eta\log R)^j \quad (s\in\mathbb{C},\ \mathrm{Re}(s)\geq -1). \tag{6.5}$$

Moreover, we have that

$$\begin{aligned}\widehat{h}_R(s) &= (\log R)\int_{-\infty}^1 R^{us}du + O(\eta(\log R)R^{\mathrm{Re}(s)}) \\ &= \frac{R^s}{s} + O((\log R)^{-C+1}R^{\mathrm{Re}(s)}) \quad (\mathrm{Re}(s)\geq 0),\end{aligned} \tag{6.6}$$

a relation that we will use at the very end of the proof of Theorem 1.3.

7. A combinatorial problem in linear algebra

Recall the notations from Section 1.7. Consider the $2k$-dimensional vector space (over $\mathbb{Q}$) of linear forms in the free variables $s_1,\dots,s_{2k}$, which we denote by W_k. Given a subspace V of W_k, we define

$$\mathscr{A}(V) = \sum_{\substack{J\in\mathcal{S}^*(2k)\\ s_J\in V}} (-1)^{\#J}.$$

(We recall that in our notation $s_J = \sum_{j\in J} s_j$.) We will prove the following result.

Proposition 7.1. – *Let $k\geq 1$ and V be a subspace of W_k containing the form $s_{[2k]} = \sum_{i=1}^{2k} s_i$.*

(a) *If $s_j\in V$ for some $j\in[2k]$, then $\mathscr{A}(V) = -1$.*

(b) *If $\dim(V) = 2k-1$, then*

$$\mathscr{A}(V) - \dim(V) \leq \binom{2k}{k} - 2k,$$

with equality if, and only if, there is a set $J\subset[2k]$ such that $\#J = k$, $1\in J$, and $V = \mathrm{Span}_{\mathbb{Q}}(\{s_j - s_1\}_{j\in J}, \{s_j + s_1\}_{j\in[2k]\setminus J})$.

(c) *If $\dim(V)\leq 2k-2$, then*

$$\mathscr{A}(V) - \dim(V) \leq \binom{2k}{k} - 2k - 2.$$

Proof. – (a) If $s_j\in V$, then we immediately see that $\mathscr{A}(V) = -1$ by pairing s_J with $s_{J\cup\{j\}}$ for each $J\subset[2k]\setminus\{j\}$.

(b) We may assume that $s_1,\dots,s_{2k}\notin V$, by part (a). Since $\dim(V) = 2k-1$ and $s_1\notin V$, for each $j = 1,\dots,2k$ we have that $s_j\equiv r_j s_1 \pmod V$, for some $r_j\in\mathbb{Q}\setminus\{0\}$. We may write $r_j = b_j/q$ for some $b_j\in\mathbb{Z}\setminus\{0\}$ and $q\in\mathbb{Z}_{\geq 1}$, so that $s_J\in V$ if, and only if, $b_J = \sum_{j\in J} b_j = 0$. Therefore

$$\mathscr{A}(V) = -1 + \int_0^1 \prod_{j=1}^{2k}(1 - e(b_j\theta))\mathrm{d}\theta \leq -1 + \int_0^1 \prod_{j=1}^{2k}|1 - e(b_j\theta)|\mathrm{d}\theta.$$

Hölder's inequality then implies that

$$\mathscr{A}(V) \leq -1 + \prod_{j=1}^{2k} \left(\int_0^1 |1 - e(b_j\theta)|^{2k} \mathrm{d}\theta \right)^{\frac{1}{2k}} = -1 + \binom{2k}{k},$$

whence

$$\mathscr{A}(V) \leq \binom{2k}{k} - 1. \tag{7.1}$$

Finally, we claim that (7.1) is an equality if, and only if, the multiset $\{b_1, \ldots, b_{2k}\}$ is of the form $\{b, -b, \ldots, b, -b\}$ with $b = b_1$ (which must equal q). This claim immediately implies (b) of the proposition.

If the multiset $\{b_1, \ldots, b_{2k}\}$ is of the form $\{b, -b, \ldots, b, -b\}$, then the integral formula for $\mathscr{A}(V)$ becomes $\mathscr{A}(V) = -1 + \int_0^1 |1 - e(b\theta)|^{2k} d\theta = \binom{2k}{k} - 1$. Conversely, we know that Hölder's inequality above is an equality if, and only if, there exist real numbers $\lambda_1, \ldots, \lambda_{2k}$ such that $|1 - e(b_j\theta)| = \lambda_j|1 - e(b_1\theta)|$ for $\theta \in [0, 1]$ and $j \in \{1, \ldots, 2k\}$. Since $\int_0^1 |1 - e(b\theta)| d\theta = 4/\pi$ for $b \neq 0$, we must have that $\lambda_j = 1$ for all j. Moreover, taking θ close enough to 0, we find that the condition $|1 - e(b_j\theta)| = |1 - e(b_1\theta)|$ implies that $|b_j| = |b_1|$ for all j. So $\{b_1, \ldots, b_{2k}\}$ has ℓ copies of b_1 and $2k - \ell$ copies of $-b_1$, for some $\ell \in \{1, \ldots, 2k\}$. Since $b_{[2k]} = 0$ by our assumption that $s_{[2k]} \in V$, we must have that $\ell = k$, which completes the proof of our claim.

(c) Write $\dim(V) = 2k - n$, where $n \geq 2$. By part (a), we may assume that $s_1, \ldots, s_{2k} \notin V$.

We first deal with the case $n = 2, k = 2$ by direct computation. In this case, we have $s_1 + s_2 + s_3 + s_4 \in V$ and $s_1, \ldots, s_4 \notin V$, by assumption. It is thus easy to see that either $V \cap \{s_I : I \in \mathcal{S}^*(2k)\} = \{s_1 + \cdots + s_4\}$ or $V \cap \{s_I : I \in \mathcal{S}^*(2k)\} = \{s_1 + \cdots + s_4, s_J\}$, for some J containing two elements. (Here we recall that $\mathcal{S}^*(2k) = \{I \subseteq [2k] : I \neq 0\}$.) In any case, $\mathscr{A}(V) \leq 2$, as required. This completes the proof of part (c) when $n = 2$ and $k = 2$.

We now assume that either $n > 2$ or $k > 2$. Choose a maximal subset of linear forms $\{s_{j_1}, \ldots, s_{j_{n'}}\}$ that are linearly independent when reduced mod V. Clearly, $n' = n$. Moreover, a permutation of the variables $s_1, \ldots, s_{2k}$ allows us to assume without loss of generality that $j_i = i$ for each i. Then

$$s_j \equiv \sum_{i=1}^{n} r_{i,j} s_i \pmod V \qquad (1 \leq j \leq 2k),$$

for certain $r_{i,j} \in \mathbb{Q}$. We write $r_{i,j} = b_{i,j}/q$, where $b_{i,j} \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geq 1}$, so that $s_J \in V$ if, and only if, $b_{i,J} := \sum_{j \in J} b_{i,j} = 0$ for each $i \in [n]$. Thus

$$\mathscr{A}(V) + 1 = \int_{[0,1]^n} \prod_{j=1}^{2k} (1 - e(b_{1,j}\theta_1 + \cdots + b_{n,j}\theta_n)) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.$$

We set

$$J_m = \{1 \leq j \leq 2k : b_{n-m+1,j} = \cdots = b_{n,j} = 0\} \quad (0 \leq m \leq n)$$

to be the set of j such that s_j is in the span of $\{s_1, \ldots, s_{n-m}\}$ (mod V). In particular, $J_0 = [2k]$ and $J_n = \emptyset$. By construction, s_i is a basis vector of W_k/V for $1 \leq i \leq n$, so for $0 \leq m \leq n-1$

we have $n-m \in J_m$ but $n-m \notin J_{m+1}$. In particular, $\#(J_m \setminus J_{m+1}) \geq 1$ for $0 \leq m \leq n-1$. Then

$$\mathscr{A}(V)+1 \leq \int_{[0,1]^{n-1}} \prod_{j \in J_1} |1-e(b_{1,j}\theta_1+\cdots+b_{n-1,j}\theta_{n-1}))| \times \left(\int_0^1 \prod_{j\in[2k]\setminus J_1} |1-e(b_{1,j}\theta_1+\cdots+b_{n,j}\theta_n)|\mathrm{d}\theta_n\right)\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{n-1}.$$

By Hölder's inequality, the innermost integral is bounded by

$$\prod_{j\in[2k]\setminus J_1}\left(\int_0^1 |1-e(b_{1,j}\theta_1+\cdots+b_{n,j}\theta_n)|^{2k-\#J_1}\mathrm{d}\theta_n\right)^{\frac{1}{2k-\#J_1}}.$$

Since $b_{n,j}\neq 0$ for $j \notin J_1$, we make the change of variables $\theta_n \to b_{1,j}\theta_1+\cdots+b_{n,j}\theta_n$ and use periodicity to find that

$$\int_0^1 |1-e(b_{1,j}\theta_1+\cdots+b_{n,j}\theta_n)|^{2k-\#J_1}\mathrm{d}\theta_n = \frac{1}{|b_{n,j}|}\int_0^{|b_{n,j}|}|1-e(\theta)|^{2k-\#J_1}d\theta = \int_0^1 |1-e(\theta)|^{2k-\#J_1}d\theta.$$

We set

$$M(\lambda)=\int_0^1 |1-e(\theta)|^{\lambda}d\theta = 2^{\lambda}\int_0^1 |\sin(\pi\theta)|^{\lambda}\mathrm{d}\theta.$$

Note that $M(2k)=\binom{2k}{k}$. Thus we find that

$$\mathscr{A}(V)+1 \leq M(2k-\#J_1)\int_{[0,1]^{n-1}} \prod_{j\in J_1}|1-e(b_{1,j}\theta_1+\cdots+b_{n-1,j}\theta_{n-1}))|\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{n-1}.$$

We repeat the process to obtain

$$\begin{aligned}\mathscr{A}(V)+1 &\leq M(2k-\#J_1)M(\#J_1-\#J_2)\\ &\quad\times\int_{[0,1]^{n-1}}\prod_{j\in J_2}|1-e(b_{1,j}\theta_1+\cdots+b_{n-1,j}\theta_{n-1}))|\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{n-2}\\ &\leq M(2k-\#J_1)M(\#J_1-\#J_2)\cdots M(\#J_{n-2}-\#J_{n-1})M(\#J_{n-1}).\end{aligned}$$

Thus

$$\mathscr{A}(V)+1 \leq \sup\{M(\lambda_1)\cdots M(\lambda_n) : \lambda_1+\cdots+\lambda_n = 2k,\ \lambda_j \geq 1\ (1\leq j\leq n)\}. \tag{7.2}$$

By Cauchy-Schwarz, for any positive reals x, y we have

$$2(xy)^{(A+B)/2} = 2(xy)^B(xy)^{(A-B)/2} \leq (xy)^B(x^{A-B}+y^{A-B}) = x^Ay^B+x^By^A.$$

Thus, applying this with $x=|\sin\theta_1|$, $y=|\sin\theta_2|$ we find

$$\begin{aligned}M(\lambda_1)M(\lambda_2) &= \frac{2^{\lambda_1+\lambda_2}}{2}\int_0^1\int_0^1\Big(|\sin(\pi\theta_1)^{\lambda_1}\sin(\pi\theta_2)^{\lambda_2}|+|\sin(\pi\theta_1)^{\lambda_1}\sin(\pi\theta_2)^{\lambda_2}|\Big)\mathrm{d}\theta_1\mathrm{d}\theta_2\\ &\geq 2^{\lambda_1+\lambda_2}\Big(\int_0^1|\sin(\pi\theta)|^{(\lambda_1+\lambda_2)/2}\mathrm{d}\theta\Big)^2 = M\Big(\frac{\lambda_1+\lambda_2}{2}\Big)^2.\end{aligned}$$

In particular, $\log M(\lambda)$ is a convex function. It is then easy to see that supremum in (7.2) is attained when $\lambda_j = 1$ for $n-1$ of the indices $j \in [n]$, and with the remaining λ_j being equal to $2k-n+1$. Indeed, without loss of generality $\lambda_1, \dots, \lambda_{n-1} \le \lambda_n$, and if $\lambda_j \neq 1$ for some $j < n$, then we can increase the size of $M(\lambda_1)\dots M(\lambda_n)$ by replacing λ_j with $\lambda_j - 1$ and λ_n with $\lambda_n + 1$. So

$$\mathscr{A}(V) \le M(1)^{n-1} M(2k-n+1) - 1.$$

Thus, it suffices to show that

$$M(1)^{n-1} M(2k-n+1) < \binom{2k}{k} - n = M(2k) - n \tag{7.3}$$

for $2 \le n \le 2k-1$ and $k \ge 2$.

Firstly, consider $n = 2$ and $k \ge 3$. The function $k \mapsto M(1)M(2k-1)/M(2k)$ is decreasing in k by the convexity of $\log M(\lambda)$. Thus

$$M(1)M(2k-1) \le \frac{M(1)M(3)}{M(4)} M(2k) = \frac{64}{9\pi^2}\binom{2k}{k} < \binom{2k}{k} - 2.$$

Here we have used the fact that $M(1) = 4/\pi$, $M(3) = 32/3\pi$ and performed a quick computation to verify $64\binom{2k}{k}/(9\pi^2) < \binom{2k}{k} - 2$ for all $k \ge 3$.

Now consider $3 \le n \le 2k-1$. The function $n \mapsto M(1)^{n-1}M(2k-n+1)$ is decreasing in m since

$$\frac{M(1)^{n-1}M(2k-n+1)}{M(1)^{n-2}M(2k-n+2)} = \frac{M(1)M(2k-n+1)}{M(2k-n+2)} \le \frac{M(1)^2}{M(2)} < 1.$$

Similarly $k \mapsto M(2k-2)/M(2k)$ is decreasing in k respectively by the convexity of $\log M(\lambda)$. Thus we have

$$\begin{aligned} M(1)^{n-1}M(2k-n+1) &\le M(1)^2 M(2k-2) \\ &\le \frac{M(1)^2 M(2)}{M(4)} M(2k) \\ &= \frac{16}{3\pi^2}\binom{2k}{k} \\ &< \binom{2k}{k} - 2k + 1 = M(2k) - 2k + 1. \end{aligned}$$

Here we have performed a short computation to verify the final inequality. This completes the proof of the proposition. □

8. Contour integration

In this section we begin our attack on Theorem 1.3. All implied constants might depend on k and on A. We will actually prove a result that is a little weaker than Theorem 1.3; we will show that there exist constants $c_{k,A}$ and c'_k for which

$$\mathcal{M}_{f_A,2k}(R) = c_{k,A}(\log R)^{\mathcal{E}_{k,A}} + O((\log R)^{\mathcal{E}_{k,A}-1}), \tag{8.1}$$

and

$$\mathcal{M}_{\widetilde{f_0},2k}(R) = c_k'(\log R)^{\binom{2k}{k}-2k} + O((\log R)^{\binom{2k}{k}-2k-1}). \tag{8.2}$$

Here we recall that $\mathcal{E}_{k,A} = \max(\binom{2k}{k} - 2k(A+1), -1)$, and we have defined $\widetilde{f_0}(x) = f_0(x) - f_0(x + \frac{\log 2}{\log R})$, so that

$$M_{\widetilde{f_0}}(n; R) = \sum_{\substack{d|n \\ R/2<d\le R}} \mu(d).$$

Notice that we do not claim here that $c_{k,A} \neq 0$ and $c_k \neq 0$, as is required in order to prove Theorem 1.3. We do obtain a very complicated expression for these constants, but we are unable to prove they are non-zero (or evaluate them at all) with the approach of this section. Showing that $c_{k,A} > 0$ and $c_k' > 0$ is the objective of Section 9.

8.1. Initial preparations

We will first prove relation (8.1). The proof of relation (8.2) is very similar, and we indicate the necessary changes in the end of Section 8.

We note that

$$\widehat{(f_A)}_R(s) = \frac{A!R^s}{(\log R)^A s^{A+1}}. \tag{8.3}$$

This function is absolutely integrable over vertical lines $\mathrm{Re}(s) = c \neq 0$ when $A \ge 1$, but this is not the case when $A = 0$. However, recall from relation (6.4) that

$$\mathcal{M}_{f_0,2k}(R) = \mathcal{M}_{h,2k}(R) + O\left(\frac{1}{(\log R)^2}\right),$$

where h is a smooth function such that $h(x) = 1$ for $x \le 1 - 1/(\log R)^C$ and $h(x) = 0$ for $x \ge 1$, for some constant $C \ge (2k-1)2^{2k+1} + 2k + 2$ to be chosen later. Therefore relation (8.1) is reduced to showing that

$$\mathcal{M}_{g,2k}(R) = c_{k,A}(\log R)^{\mathcal{E}_{k,A}} + O\left((\log R)^{\mathcal{E}_{k,A}-1}\right), \tag{8.4}$$

where $g = h$ when $A = 0$, and $g = f_A$ when $A \ge 1$.

For any $\lambda > 1$, which will be chosen to be sufficiently large in terms of k, relation (6.3) implies that

$$\mathcal{M}_{g,2k}(R) = \sum_{\substack{m_j\in\mathbb{Z}_{\ge 1} \\ 1\le j\le 2k}} \frac{\prod_{j=1}^{2k}\mu(m_j)}{[m_1,\ldots,m_{2k}]} \cdot \frac{1}{(2i\pi)^{2k}} \int\cdots\int_{\substack{\mathrm{Re}(s_j)=\lambda^j/\log R \\ 1\le j\le 2k}} \prod_{j=1}^{2k} m_j^{-s_j} \left(\prod_{j=1}^{2k} \widehat{g}_R(s_j)\right) ds_{2k}\cdots ds_1.$$

To this end, we introduce the multiple Dirichlet series

$$D(\boldsymbol{s}) := \sum_{\substack{m_j\in\mathbb{Z}_{\ge 1} \\ 1\le j\le 2k}} \frac{\prod_{j=1}^{2k} m_j^{-s_j}\mu(m_j)}{[m_1,\ldots,m_{2k}]},$$

which converges absolutely when $\mathrm{Re}(s_j) > 0$ for all j as can be seen, for example, by the Euler product expansion

$$D(\boldsymbol{s}) = \prod_p \left(\sum_{\nu_1,\ldots,\nu_k \in \{0,1\}} \frac{(-1)^{\nu_1+\cdots+\nu_k}}{p^{\nu_1 s_1+\cdots+\nu_k s_k}} \cdot \frac{1}{[p^{\nu_1},\ldots,p^{\nu_k}]} \right)$$

$$= \prod_p \left(1 + \frac{1}{p} \sum_{\emptyset \neq I \subset [2k]} \frac{(-1)^{\#I}}{p^{s_I}} \right) \tag{8.5}$$

$$= \prod_p \left(1 - \frac{1}{p} + \frac{1}{p} \prod_{j=1}^{2k} \left(1 - \frac{1}{p^{s_j}} \right) \right), \tag{8.6}$$

where we have used the notation $s_I = \sum_{i \in I} s_i$. (Similar computations are performed in [1].) We thus see that

$$\mathscr{M}_{g,2k}(R) = \frac{1}{(2i\pi)^{2k}} \int \cdots \int_{\substack{\mathrm{Re}(s_j) = \lambda^j / \log R \\ 1 \le j \le 2k}} D(\boldsymbol{s}) \left(\prod_{j=1}^{2k} \widehat{g}_R(s_j) \right) ds_{2k} \cdots ds_1,$$

for any $\lambda > 1$.

We shall truncate all variables of integration at height

$$T := \exp\{(\log\log R)^2\}.$$

To do so, we notice that $\widehat{g}_R(s) \ll (\log R)^{O(1)}/|s|^2$ for $\mathrm{Re}(s) = \lambda^j / \log R$, a consequence of (6.2) when $A = 0$ and of (8.3) when $A \ge 1$, as well as that $D(\boldsymbol{s}) \ll (\log R)^{O(1)}$, an estimate that follows by (8.5) and the Prime Number Theorem. We conclude that

$$\mathscr{M}_{g,2k}(R) = I_{g,2k}(R) + O\left(\frac{1}{(\log R)^2} \right),$$

where

$$I_{g,2k}(R) := \frac{1}{(2i\pi)^{2k}} \int \cdots \int_{\substack{\mathrm{Re}(s_j) = \lambda^j / \log R \\ |\mathrm{Im}(s_j)| \le T \\ 1 \le j \le 2k}} D(\boldsymbol{s}) \left(\prod_{j=1}^{2k} \widehat{g}_R(s_j) \right) ds_{2k} \cdots ds_1.$$

Motivated by (8.5) and (8.6), we set

$$P(\boldsymbol{s}) := D(\boldsymbol{s}) \prod_{I \in \mathscr{S}^*(2k)} \zeta(1+s_I)^{(-1)^{1+\#I}}$$

$$= \prod_p \left\{ \left(1 - \frac{1}{p} + \frac{1}{p} \prod_{j=1}^{2k} \left(1 - \frac{1}{p^{s_j}} \right) \right) \prod_{I \in \mathscr{S}^*(2k)} \left(1 - \frac{1}{p^{1+s_I}} \right)^{(-1)^{\#I}} \right\},$$

which is analytic when $\mathrm{Re}(s_j) > -1/(4k)$ for all j, as well as

$$F(\boldsymbol{s}) := P(\boldsymbol{s}) \prod_{j=1}^{2k} \frac{(\log R)^A \widehat{g}_R(s_j)}{R^{s_j} \zeta(1+s_j)^{A+1}}$$

and

$$e_I := \begin{cases} A & \text{if } \#I = 1, \\ (-1)^{\#I} & \text{if } \#I \geq 2, \end{cases}$$

so that

$$I_{g,2k}(R) = \frac{1}{(2i\pi)^{2k}} \int \cdots \int_{\substack{\mathrm{Re}(s_j)=\lambda^j/\log R \\ |\mathrm{Im}(s_j)|\leq T \\ 1\leq j\leq 2k}} \frac{F(\boldsymbol{s})R^{s_1+\cdots+s_{2k}}}{(\log R)^{2kA}} \prod_{I\in\mathcal{S}^*(2k)} \zeta(1+s_I)^{e_I}\, ds_{2k}\cdots ds_1.$$

Given $\ell \in \mathbb{N}$, we now let

$$\Omega_\ell := \{\boldsymbol{s} \in \mathbb{C}^\ell : |\mathrm{Re}(s_j)| < 2/(\log T)^{4/3},\ |\mathrm{Im}(s_j)| < T+1\ (1 \leq j \leq \ell)\}$$

and define $\mathcal{C}_\ell$ to be the class of complex-valued functions f such that: (a) f is defined over a complex domain containing Ω_ℓ; (b) f is analytic in Ω_ℓ; (c) the derivatives of f satisfy the bound

$$\frac{\partial^{j_1+\cdots+j_\ell} f}{\partial s_1^{j_1}\cdots\partial s_\ell^{j_\ell}}(\boldsymbol{s}) \ll_{j_1,\dots,j_\ell} \frac{(\log\log R)^{O(j_1+\cdots+j_\ell)}}{(|s_1|+1)\cdots(|s_\ell|+1)} \tag{8.7}$$

for all $j_1,\dots,j_\ell \geq 0$ and all $\boldsymbol{s} = (s_1,\dots,s_\ell) \in \Omega_\ell$.

We claim that $F \in \mathcal{C}_{2k}$. Indeed, there are absolute constants $\delta, c_0 > 0$ such that $\zeta(s)(s-1)$ is analytic and non-vanishing for $|s-1| \leq \delta$ and

$$\zeta^{(j)}(s), \left(\frac{1}{\zeta}\right)^{(j)}(s) \ll_j \log^{j+1}(|t|+2) \qquad \text{when} \quad \sigma \geq 1 - \frac{c_0}{\log(|t|+2)},\ |s-1| \geq \delta, \tag{8.8}$$

with (8.8) being a consequence [(4)] of the classical zero-free region for ζ. Moreover,

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\frac{s^{A+1}\widehat{g}_R(s)}{R^s}\right) \ll \frac{1}{(\log R)^A} \quad (\mathrm{Re}(s) \geq -1,\ j \in \mathbb{Z}_{\geq 0}), \tag{8.9}$$

an estimate that follows from (6.5) when $A = 0$ and from the Formula (8.3) for $\widehat{(f_A)}_R$ otherwise. Our claim that $F \in \mathcal{C}_{2k}$ then follows.

8.2. Contour shifting

We will simplify $I_{g,2k}(R)$ and prove (8.1) by a $2k$-dimensional contour shifting argument that we will perform in an iterative fashion. The general idea is to move the variables s_j to the left in a certain order. When we move the contour corresponding to the variable s_j, we will pick up contributions from poles of the integrand (coming from solutions to linear equations of the form $s_I = 0$, $I \in \mathcal{S}^*(2k)$ with $e_I > 0$), and be left with a residual contour (which will be negligible in size). Thus we only need to consider the contributions from the poles, and these contributions will all be multi-integrals similar to $I_{g,2k}(R)$ but involving one fewer variable. By iterating this, we show that $I_{g,2k}(R)$ is (up to a small error term) given by the total contribution of all the successive poles we have encountered having shifted all $2k$ variables. We will show that provided one moves the contours in a suitable order, all

(4) The claimed bound follows by [23, Theorems 3.8 and 3.11] and the fact that if f is analytic in a neighborhood of the circle $|z| \leq r$, then $f^{(j)}(z_0) = \frac{1}{2\pi i}\oint_{|z|=r} f(z)dz/z$ for any z_0 with $|z_0| < r$.

the contributions from all the multi-poles and all the residual integrals give a contribution $c_{k,A}(\log R)^{\mathcal{E}_{k,A}} + O\left((\log R)^{\mathcal{E}_{k,A}-1}\right)$.

When we consider poles we encounter equations of the form $s_I = 0$, where we think of $s_I = \sum_{i \in I} s_i$ as a linear form in the variables $s_1, \dots, s_{2k}$. To avoid any ambiguity when we consider such multiple equations, we will let $L_{0,I} \in \mathbb{Q}[x_1, \dots, x_{2k}]$ be the linear form corresponding to s_I, that is to say

$$L_{0,I}(\boldsymbol{x}) := \sum_{i \in I} x_i.$$

Before we setup the necessary notation to keep track of all the terms we encounter when performing the multiple contour shifting, we first describe the first two contour shifting steps to help motivate the basic idea.

The first variable we move to the left is s_{2k}. When doing so, we pick up the contribution from some poles in the integrand. Such a pole must occur when $L_{0,I_1}(\boldsymbol{s}) = 0$ for some $I_1 \subset [2k]$ with $e_{I_1} > 0$ and $2k \in I_1$ (a possible pole from $\prod_{I \in \mathcal{S}^*(2k)} \zeta^{e_I}(1 + L_{0,I}(\boldsymbol{s}))$.) Having fixed such a pole and the corresponding set I_1, we use this equation $L_{0,I_1}(\boldsymbol{s}) = 0$ to rewrite s_{2k} in terms of s_j, $j \in [2k] \setminus \{2k\}$. Imposing the same condition on the x_j's, we find that for each $I \subset [2k]$, the linear form $L_{0,I}(x_1, \dots, x_{2k})$ becomes a linear form $L_{1,I}$ in the variables x_j for $j \in [2k] \setminus \{2k\}$. Trivially, $L_{1,I} = 0$ if, and only if, $I \in \mathcal{J}_1 := \{\emptyset, I_1\}$. This pole contribution can be written as an integral over $s_1, \dots, s_{2k-1}$, with an integrand that has poles only when $L_{1,I}(\boldsymbol{s}) = 0$.

Next, for this integral over $s_1, \dots, s_{2k-1}$, we choose some other variable s_{j_2} (precisely how we choose s_{j_2} will be specified later), and move the s_{j_2} contour. This produces a residual contour (which will be negligible) and contributions from further poles in the integrand which occur only when s_{j_2} satisfies a linear equation $L_{1,I_2}(\boldsymbol{s}) = 0$ for some $I_2 \in \mathcal{S}^*(2k) \setminus \mathcal{J}_1$ with $e_{I_2} > 0$ and with $L_{1,I_2}(\boldsymbol{x})$ having a non-zero x_{j_2} coefficient. We use this to write s_{j_2} in terms of s_j, $j \in [2k] \setminus \{2k, j_2\}$. Imposing the corresponding condition on the variables x_j makes $L_{1,I}$ a linear form $L_{2,I}$ in the variables x_j, $j \in [2k] \setminus \{2k, j_2\}$. Some of these new linear forms will vanish identically, and the total number will determine the order of this pole.

Continuing in this manner, we eventually write our original integral $I_{g,2k}(R)$ in terms of $O(1)$ contributions from repeatedly encountered poles (all of which will be of the form $c(\log R)^m$ for some c, m) or from terms which correspond to encountering a residual integral (which will always be small). In order to control this process, we need to keep track of which poles we encounter, the order of the poles, and the integrands of the new multi-integrals corresponding to these poles. To do this we introduce some notation and terminology.

— Let us be given an integer $N \in \{0, 1, \dots, 2k\}$, which we shall often refer to as the **level**. It describes how many iterations we have performed (i.e., how many variables s_j we have shifted). The case $N = 0$ corresponds to the initial integral $I_{g,2k}(R)$.

— Let us be given sets $I_1, \dots, I_N \subset [2k]$ and indices $j_1, \dots, j_N$ such that:

(i) $j_n \in I_n$ for each $1 \le n \le N$.

(ii) $j_1, \dots, j_N$ are distinct.

(iii) $e_{I_n} > 0$ for each $1 \le n \le N$.

Here the sets $I_1, \dots I_N$ correspond to the sequence of poles which we have encountered from performing N contour shifts, and the index j_n corresponds to the variable we have chosen to use to shift the n-th contour.

Since the j_i are distinct, the linear forms $L_{0,I_1}, \dots, L_{0,I_N}$ are linearly independent over $\mathbb{Q}$. We let V_N be their $\mathbb{Q}$-span and $\mathcal{J}_N$ to be those forms that vanish identically subject to the conditions $L_{0,I_1} = \cdots = L_{0,I_N} = 0$. More generally, for $0 \le n \le N$ let

(8.10) $$V_n = \mathrm{Span}_{\mathbb{Q}}(L_{0,I_1}(\boldsymbol{x}), \dots, L_{0,I_n}(\boldsymbol{x})) \quad \text{and} \quad \mathcal{J}_n = \{I \in \mathcal{S}(2k) : L_{0,I}(\boldsymbol{x}) \in V_n\},$$

with the conventions that $V_0 = \{0\}$ and $\mathcal{J}_0 = \{\emptyset\}$. Since $j_r \in I_r$ for all r, and $j_1, \dots, j_n$ are distinct integers, if we impose the conditions $\sum_{i \in I_n} x_i = 0$ on the variables x_i, then we may write $x_{j_1}, \dots, x_{j_n}$ as $\mathbb{Q}$-linear combinations of the other variables. Hence the linear form $L_{0,I}$ becomes a linear form $L_{n,I}$ in the variables x_j, $j \in [2k] \setminus \{j_1, \dots, j_n\}$. Clearly, $L_{0,I} \in V_n$ if and only if $L_{0,I} = 0$ after we have "quotiented" the space of linear forms in the variables x_j with the relations $L_{0,I_1} = \cdots = L_{0,I_n} = 0$, if and only if $L_{n,I} = 0$.

Remark. – We will show later on that the variables $s_1, \dots, s_{2k}$ can be permuted in a way that allows us to assume that $j_n = 2k - n + 1$ for all n.

Definition 8.1. – Let N be a level. The triplet $(\boldsymbol{I}, \boldsymbol{h}, d)$ is called a *type of level* N if:

(a) $\boldsymbol{I} = (I_1, \dots, I_N)$ is an N-tuple of sets such that $2k - n + 1 \in I_n$ and $e_{I_n} > 0$ for all $n = 1, 2, \dots, N$.

(b) $\boldsymbol{h} = (h_{n,I})_{0 \le n \le N,\, I \in \mathcal{S}^*(2k)}$ is a tuple of non-negative integers such that:

  (i) $h_{n,I} = 0$ for $0 \le n \le N$ if $e_I = 0$ (i.e., if $A = 0$ and $\#I = 1$);

  (ii) $0 = h_{0,I} \le h_{1,I} \le \cdots \le h_{N,I}$ for $I \in \mathcal{S}^*(2k)$;

  (iii) If $I \in \mathcal{J}_n \setminus \mathcal{J}_{n-1}$ for some $n \in [N]$, then $h_{m,I} = h_{n,I}$ for all $m \ge n$.

The integers $h_{n,I}$ will describe the different terms coming up in poles of high order, corresponding to taking many derivatives of different parts of the integrand. (The $h_{N,I}$-th derivative of $\zeta^{e_I}(1 + L_{N,I}(\boldsymbol{s}))$ will occur in the integrand of the term we are considering.)

(c) d is a non-negative integer.

We will further say that the triplet $(\boldsymbol{I}, \boldsymbol{h}, d)$ is an *admissible type of level* N if

$$H_N \ge N + d,$$

where

$$H_N = H_N(\boldsymbol{h}, \mathcal{J}_N, A) := \sum_{I \in \mathcal{J}_N \setminus \{\emptyset\}} (-1)^{\#I} - \sum_{I \in \mathcal{S}(2k) \setminus \mathcal{J}_N} h_{N,I} + (A+1) \sum_{j \in [2k],\, \{j\} \in \mathcal{J}_N} 1.$$

Remark 8.1. – We must have that $H_N \ge N$ if $(\boldsymbol{I}, \boldsymbol{h}, d)$ is an admissible type of level N. We will see that the quantity H_N is related to the total order of the poles we have picked up from the first N contour shiftings. We further note that, in the notation of Section 7, it can be written as

$$H_N = \mathscr{A}(V_N) - \sum_{I \in \mathcal{S}(2k) \setminus \mathcal{J}_N} h_{N,I} + (A+1) \sum_{j \in [2k],\, \{j\} \in \mathcal{J}_N} 1.$$

The data in a type of level N will keep track of all the relevant information on terms we encounter from poles having shifted N contours. Given a type, we can now define the key objects we wish to consider:

Definition 8.2. – Let $(\boldsymbol{I},\boldsymbol{h},d)$ be a type of level N. A function $J:\mathbb{R}_{\geq 2}\to\mathbb{C}$ is called a *fundamental component of level N and of type $(\boldsymbol{I},\boldsymbol{h},d)$* if:

— the type $(\boldsymbol{I},\boldsymbol{h},d)$ is admissible, that is to say, we have $H_N\geq N+d$;

— when $N=2k$, we have $J(R)=(\log R)^{H_N-N-d-2kA}$;

— when $N<2k$, we have

$$J(R)=\frac{(\log R)^{H_N-N-d-2kA}}{(2i\pi)^{2k-N}}\int\cdots\int_{\substack{\mathrm{Re}(s_j)=\lambda_j/\log R\\|\mathrm{Im}(s_j)|\leq T\\1\leq j\leq 2k-N}}G(\boldsymbol{s})R^{E_N(\boldsymbol{s})}$$
$$\times\prod_{I\in\mathcal{S}(2k)\setminus\mathcal{J}_N}(\zeta^{e_I})^{(h_{N,I})}(1+L_{N,I}(\boldsymbol{s}))\mathrm{d}s_{2k-N}\cdots\mathrm{d}s_1$$

where $\lambda_j/\lambda_{j-1}\geq\lambda$,

$$E_N(s_1,\dots,s_{2k-N}):=L_{N,[2k]}(s_1,\dots,s_{2k-N}),$$

and G is a function in the variables $s_1,\dots,s_{2k-N}$ that belongs to the class $\mathcal{C}_{2k-N}$. Moreover, if we have additionally that $d=0$, then G is given by

$$G(\boldsymbol{s})=F(L_{N,\{1\}}(\boldsymbol{s}),\dots,L_{N,\{2k\}}(\boldsymbol{s})).$$

We note that when $d=0$, we have that G is non-vanishing in Ω_{2k-N} by (8.8) and the preceding discussion.

Definition 8.3. – A fundamental component of level N and type $(\boldsymbol{I},\boldsymbol{h},d)$ is called *irreducible* if either $N=2k$ or $E_N=0$. Otherwise, it is called *reducible*.

With the above notation, the integral $I_{g,2k}(R)$ is a reducible fundamental component of level 0 and of type $(\emptyset,\emptyset,0)$.

If we say that $J(R)$ is a fundamental component of level N, we mean that there exists an admissible type $(\boldsymbol{I},\boldsymbol{h},d)$ of level N such that $J(R)$ is a fundamental component of level N and type $(\boldsymbol{I},\boldsymbol{h},d)$.

We begin with a lemma that justifies the terms *irreducible vs. reducible*, showing how reducible components are a linear combination of irreducible ones (up to a very small error term). First, we need to introduce a last piece of notation. Notice that if $E_N\neq 0$, then we may write uniquely

$$E_N(\boldsymbol{x})=\gamma_1x_1+\gamma_2x_2+\cdots+\gamma_{j_{N+1}}x_{j_{N+1}},$$

for some $\gamma_j\in\mathbb{Q}$ with $\gamma_{j_{N+1}}\neq 0$. If λ is big enough, then the sign of $\mathrm{Re}(E_N(\boldsymbol{s}))$ throughout the region of integration is constant and equal to the sign of $\gamma_{j_{N+1}}$. The behavior of reducible fundamental components differs according to this sign:

LEMMA 8.4. – *Assume the above setup. Let $J(R)$ be a reducible fundamental component of level $N < 2k$ and type $(\boldsymbol{I}, \boldsymbol{h}, d)$, and let $\gamma_1, \dots, \gamma_{j_{N+1}}$ be as above. Assume that λ is large enough in terms of $(\boldsymbol{I}, \boldsymbol{h}, d)$.*

(a) *If $\gamma_{j_{N+1}} > 0$, then $J(R)$ is a linear combination of $O(1)$ fundamental components of level $N + 1$ with coefficients of size $O(1)$, up to an error term of size $\ll T^{-1+o(1)}$. Each of these fundamental components has (admissible) type $(\boldsymbol{I}', \boldsymbol{h}', d')$ that depends only on $N, \boldsymbol{I}, \boldsymbol{h}, d$.*

(b) *If $\gamma_{j_{N+1}} < 0$, then $J(R) \ll T^{-1+o(1)}$.*

The implied constants depend at most on $(\boldsymbol{I}, \boldsymbol{h}, d)$, A and the function G in the definition of J, and are independent of R.

We iterate the above lemma until all the fundamental components we are dealing with are irreducible. For such components, we have the following asymptotic formula.

LEMMA 8.5. – *Assume the above setup. If $J(R)$ is an irreducible fundamental component, then there is some $c \in \mathbb{C}$ such that*

$$J(R) = c(\log R)^{\mathcal{E}_{k,A}} + O((\log R)^{\mathcal{E}_{k,A}-1}),$$

where we recall that $\mathcal{E}_{k,A} = \max(\binom{2k}{k} - 2k(A+1), -1)$. The implied constant and the constant c are independent of R.

Since $I_{g,2k}(R)$ is a reducible fundamental component of level 0, we apply Lemma 8.4 repeatedly to write it as a linear combination of $O(1)$ irreducible fundamental components, and then estimate these components by Lemma 8.5. This establishes (8.4). We now prove the above two key lemmas.

8.3. Proof of the auxiliary Lemmas 8.4 and 8.5

Proof of Lemma 8.4. – Note that if $E_N \neq 0$, then we must have that either $N = 0$, or $k \geq 2$ or $A \geq 1$: when $N = k = 1$ and $A = 0$, the only $I \subset \{1, 2\}$ with $e_I > 0$ is $I = \{1, 2\}$. But if $x_{\{1,2\}} = 0$, we must have that $E_1 = 0$, a contradiction.

(a) Here $\gamma_{j_{N+1}} > 0$. For notational simplicity, we make the change of variables

$$s'_j = s_j \ (1 \leq j < j_{N+1}), \ s'_j = s_{j+1} \ (j_{N+1} \leq j < 2k - N), \ s'_{2k-N} = s_{j_{N+1}},$$

which corresponds to a cyclic permutation of the variables $s_{j_{N+1}}, \dots, s_{2k-N}$. We similarly define the linear forms x'_j, using the corresponding permutation of the forms x_j, as well as the parameters λ'_j. We shift the s'_{2k-N} contour to the line $\mathrm{Re}(s'_{2k-N}) = -1/(\log T)^{3/2}$. The contribution of the horizontal integrals is $\ll (\log R)^{O(1)}/T$. Moreover, when $\mathrm{Re}(s'_{2k-N}) = -1/(\log T)^{3/2}$ and $\mathrm{Re}(s'_j) = O(1/\log R)$ for $j < 2k - N$, we have that

$$\mathrm{Re}(E_N(\boldsymbol{s}')) = -\frac{\gamma_{j_{N+1}}}{(\log T)^{3/2}} + O\left(\frac{1}{\log R}\right).$$

It thus follows that the contribution of the integral with $\mathrm{Re}(s'_{2k-N}) = -1/(\log T)^{3/2}$ is $\ll e^{-\sqrt{\log R}}$, say, which is of negligible size. So we need only worry about the poles that the contour shifting introduces.

The poles occur when $L_{N,I_{N+1}}(\boldsymbol{s}') = 0$ for some $I_{N+1} \in \mathcal{S}(2k) \setminus \mathcal{J}_N$ with $e_{I_{N+1}} > 0$ such that the coefficient of s'_{2k-N} in $L_{N,I_{N+1}}$ is non-zero. As we discussed in Section 8.2, imposing the relation $L_{N,I_{N+1}}(\boldsymbol{x}') = 0$ allows us to write x'_{2k-N} as a linear combination of the forms $x'_1, \dots, x'_{2k-N-1}$, say $x'_{2k-N} = C(x'_1, \dots, x'_{2k-N-1})$. We then define the sets V_{N+1} and $\mathcal{J}_{N+1}$ as in (8.10), and similarly let $E_{N+1} = L_{N+1,[2k]}$.

We need to understand the order of the pole at $s'_{2k-N} = C(s'_1, \dots, s'_{2k-N-1})$. We only look at *generic points* $(s'_1, \dots, s'_{2k-N-1})$: it could be the case that for some measure-zero subset of points, we get a different pole order. For example, for fixed $s_1 \in \mathbb{C}$, the function $s_2 \mapsto s_1/s_2$ has generically a pole of order 1 at $s_2 = 0$, unless $s_1 = 0$, when there is no pole. This reduced pole order however would not affect an integral over s_1, because it only occurs for a measure-zero set of s_1 values.

With the above discussion in mind, we note that the generic order of the zero of the analytic function

$$\prod_{I \in \mathcal{S}(2k) \setminus \mathcal{J}_{N+1}} (\zeta^{e_I})^{(h_{N,I})} (1 + L_{N,I}(\boldsymbol{s}'))$$

at $s'_{2k-N} = C(s'_1, \dots, s'_{2k-N-1})$ is 0. Indeed, for this product to vanish we must have that $L_{N+1,I}(\boldsymbol{s}') = 0$, which happens non-generically when $I \in \mathcal{S}(2k) \setminus \mathcal{J}_{N+1}$.

Next, let ν be the generic order of the zero of the analytic function

$$G(\boldsymbol{s}') \prod_{\substack{I \in \mathcal{J}_{N+1} \setminus \mathcal{J}_N \\ e_I = -1,\, h_{N,I} \geq 2}} \left(\frac{1}{\zeta}\right)^{(h_{N,I})} (1 + L_{N,I}(\boldsymbol{s}')) \tag{8.11}$$

at $s'_{2k-N} = C(s'_1, \dots, s'_{2k-N-1})$. If $d = 0$ and $h_{N,I} = 0$ for all $I \in \mathcal{S}^-(2k) \cap (\mathcal{J}_{N+1} \setminus \mathcal{J}_N)$, then the function in (8.11) equals $F(L_{N,\{1\}}(\boldsymbol{s}'), \dots, L_{N,\{2k\}}(\boldsymbol{s}'))$, which does not vanish in Ω_{2k}, so that $\nu = 0$.

From the above discussion, we conclude that the generic order of the pole of the integrand of $J(R)$ at $s'_{2k-N} = C(s'_1, \dots, s'_{2k-N-1})$ is

$$\begin{aligned} m &= \sum_{\substack{I \in \mathcal{J}_{N+1} \setminus \mathcal{J}_N \\ \#I = \text{even}}} (h_{N,I} + 1) - \sum_{\substack{I \in \mathcal{J}_{N+1} \setminus \mathcal{J}_N \\ e_I = -1,\, h_{N,I} = 0}} 1 + \sum_{\substack{1 \leq j \leq 2k \\ \{j\} \in \mathcal{J}_{N+1} \setminus \mathcal{J}_N}} (h_{N,\{j\}} + A) - \nu \\ &= \sum_{I \in \mathcal{J}_{N+1} \setminus \mathcal{J}_N} (h_{N,I} + (-1)^{\#I}) + (A+1) \sum_{j \in [2k],\, \{j\} \in \mathcal{J}_{N+1} \setminus \mathcal{J}_N} 1 \\ &\quad - \nu - \sum_{\substack{I \in \mathcal{S}^-(2k) \cap (\mathcal{J}_{N+1} \setminus \mathcal{J}_N) \\ h_{N,I} \geq 2,\, \#I \geq 3}} (h_{N,I} - 1). \end{aligned} \tag{8.12}$$

Note that it could be the case that $m \leq 0$, in which case there is no pole contribution to $J(R)$ from the pole with $L_{N,I_{N+1}}(\boldsymbol{s}') = 0$.

Assume, now, that $m \geq 1$. Then, $m + H_N \geq 1 + N + d$ by our assumption that $H_N \geq N + d$. Moreover,

$$\begin{aligned}(8.13)\qquad m + H_N = &\sum_{I \in \mathcal{J}_{N+1} \setminus \{\emptyset\}} (-1)^{\# I} - \sum_{I \in \mathcal{S}(2k) \setminus \mathcal{J}_{N+1}} h_{N,I} + (A+1) \sum_{j \in [2k],\, \{j\} \in \mathcal{J}_{N+1}} 1 \\ &- \nu - \sum_{\substack{I \in \mathcal{S}^-(2k) \cap (\mathcal{J}_{N+1} \setminus \mathcal{J}_N) \\ h_{N,I} \geq 1,\, \# I \geq 3}} (h_{N,I} - 1).\end{aligned}$$

In order to continue, we separate two subcases depending on whether $N = 2k - 1$ or $N \leq 2k - 2$.

Case 1 of the proof of Lemma 8.4: $N = 2k - 1$. – In this case, we have that $s'_j = L_{2k-1,\{j\}}(s'_1) = a_j s'_1$ for all j, where $a_j \in \mathbb{Q}$. Thus, the only potential pole is at $s'_1 = 0$. If $m \geq 1$, so that there is a genuine pole at $s'_1 = 0$, then we obtain an expression for $J(R)$ as a finite linear combination of powers of $\log R$ (up to an error term of size $O((\log R)^{O(1)}/T)$), the highest of which has exponent

$$H_{2k-1} + m - 2k - 2kA - d = \sum_{I \in \mathcal{S}^*(2k)} (-1)^{\# I} - \nu - \sum_{\substack{I \in \mathcal{S}^-(2k) \setminus \mathcal{J}_{2k-1} \\ h_{2k-1,I} \geq 2,\, \# I \geq 3}} (h_{2k-1,I} - 1) - d \leq -1,$$

since $\mathcal{J}_{2k} = \mathcal{S}(2k)$ in this case. We have thus written $J(R)$ as a linear combination of irreducible fundamental components of level $2k$ and suitable type (taking $h_{2k,I} = h_{2k-1,I}$ and $I_{2k} = \{1\}$), up to a small error term. This proves Lemma 8.4 in this case.

As an amusing remark, we note that the above exponent equals $\mathcal{E}_{k,A}$ only when $A > \frac{1}{2k}\binom{2k}{k} - 1$, $d = 0$, $\nu = 0$, $h_{2k-1,I} \in \{0, 1\}$ for $I \in \mathcal{S}^-(2k) \setminus \mathcal{J}_{2k-1}$, and $G(s) = F(a_1 s, \ldots, a_{2k} s)$, in which case the residue is

$$G(0) = A!^{2k} F(0, \ldots, 0) = A!^{2k}.$$

Otherwise, these poles contribute towards the error term of $I_{g,2k}(R)$.

Case 2 of the proof of Lemma 8.4: $N \leq 2k - 2$. – Then the contribution of the pole $s'_{2k-N} = C(s'_1, \ldots, s'_{2k-N-1})$ to $J(R)$ equals

$$\frac{(\log R)^{H_N - N - d - 2kA}}{(2i\pi)^{2k-N-1} m!} \int \cdots \int_{\substack{\mathrm{Re}(s'_j) = \lambda'_j/\log R, \\ |\mathrm{Im}(s'_j)| \leq T \\ 1 \leq j \leq 2k-N-1}} \frac{\mathrm{d}^{m-1}}{\mathrm{d}(s'_{2k-N})^{m-1}}\bigg|_{s'_{2k-N} = C(s'_1, \ldots, s'_{2k-N-1})} (Z(s')) \mathrm{d}s'_{2k-N-1} \cdots \mathrm{d}s'_1,$$

where

$$Z(s') := G(s') R^{E_N(s')} (s'_{2k-N} - C(s'_1, \ldots, s'_{2k-N-1}))^m \prod_{I \in \mathcal{S}(2k) \setminus \mathcal{J}_N} (\zeta^{e_I})^{(h_{N,I})} (1 + L_{N,I}(s')).$$

Applying the generalized product rule and writing s_j in place of s'_j, we claim that the above integral can be expressed as a finite sum of terms of the form

$$c \cdot \frac{(\log R)^{H_N+m-h-N-1-d-2kA}}{(2i\pi)^{2k-N-1}} \int\cdots\int_{\substack{\mathrm{Re}(s'_j)=\lambda'_j/\log R,\,|\mathrm{Im}(s'_j)|\le T\\ 1\le j\le 2k-N-1}} \widetilde{G}(\boldsymbol{s}) R^{E_{N+1}(\boldsymbol{s})}$$
$$\times \prod_{I\in\mathcal{S}(2k)\setminus\mathcal{J}_{N+1}} (\zeta^{e_I})^{(h_{N+1,I})}(1+L_{N+1,I}(\boldsymbol{s}))\mathrm{d}s_{2k-N-1}\cdots\mathrm{d}s_1,$$

where:

- $c \ll 1$;
- $h \in \{0,\ldots,m-1\}$;
- $h_{N+1,I} \ge h_{N,I}$ with equality if $I \in \mathcal{J}_{N+1}\setminus\{\emptyset\}$;
- $\sum_{I\in\mathcal{S}(2k)\setminus\mathcal{J}_{N+1}}(h_{N+1,I}-h_{N,I}) \le h$;
- $\widetilde{G}$ is in the class $\mathcal{C}_{2k-N-1}$.

The first four claims are easy to verify, but our claim about $\widetilde{G}$ requires some justification. To simplify the notation, we make the change of variables $s'_{2k-N} = \tau + C(s'_1,\ldots,s'_{2k-N-1})$. Let $\delta_{N,I}$ denote the coefficient of x'_{2k-N} in the linear form $L_{N,I}$. If $I \in \mathcal{J}_{N+1}$, so that $L_{N+1,I} = 0$, we find that $L_{N,I}(s'_1,\ldots,s'_{2k-N}) = \delta_{N,I}\tau$. So, if $I \in \mathcal{J}_{N+1}\setminus\mathcal{J}_N$, then $\delta_{N,I}\neq 0$. Finally, we let $G_1(s'_1,\ldots,s'_{2k-N-1},\tau)$ be the function $G(s'_1,\ldots,s'_{2k-N})$ after our change of variables. If ν_1 denotes the generic order of the zero of $G_1(s'_1,\ldots,s'_{2k-N-1},\tau)$ at $\tau=0$, then the function $\widetilde{G}$ will simply be a linear combination of the functions

$$\frac{\partial^j}{\partial\tau^j}\bigg|_{\tau=0}(\tau^{-\nu_1}G_1(s'_1,\ldots,s'_{2k-N-1},\tau)) = \frac{j!}{(j+\nu_1)!}\cdot\frac{\partial^{j+\nu_1}G_1}{\partial\tau^{j+\nu_1}}(s'_1,\ldots,s'_{2k-N-1},0).$$

Since G is in the class $\mathcal{C}_{2k-N}$, the above functions are in the class $\mathcal{C}_{2k-N-1}$, which proves our claim about $\widetilde{G}$.

It is straightforward to verify that $\boldsymbol{h}'$ and $\boldsymbol{I}'$ satisfy the required properties. Thus it remains to show that there is a suitable d'.

Now, relation (8.13) implies that the exponent of $\log R$ is $H_{N+1}-(N+1)-d'-2kA$, with

$$d' = d+\nu+\sum_{\substack{I\in\mathcal{S}^-(2k)\cap(\mathcal{J}_{N+1}\setminus\mathcal{J}_N)\\ h_{N,I}\ge 2,\,\#I\ge 3}}(h_{N,I}-1)+h-\sum_{I\in\mathcal{S}(2k)\setminus\mathcal{J}_{N+1}}(h_{N+1,I}-h_{N,I}) \ge 0.$$

Moreover, we have that

$$H_{N+1}-d' = H_N-d+m-h \ge N+1 \quad\Longrightarrow\quad H_{N+1}\ge d'+N+1,$$

as needed. Finally, if $d'=0$, it is easy to check that $G(\boldsymbol{s}) = F(L_{N+1,\{1\}}(\boldsymbol{s}),\ldots,L_{N+1,\{2k\}}(\boldsymbol{s}))$.

(b) Here $\gamma_{j_{N+1}} < 0$. We then shift the contours of $s_{2k-N}, s_{2k-N-1},\ldots,s_{j_{N+1}}$ in this order to the lines $\mathrm{Re}(s_j) = \lambda_j/(\log T)^{3/2}$, $j_{N+1}\le j\le 2k-N$. If λ is large enough, then we do not encounter any poles and the horizontal lines contribute $\ll (\log R)^{O(1)}/T$ when we make this shift. Finally, when $\mathrm{Re}(s_j) = \lambda_j/(\log T)^{3/2}$ for $j_{N+1} \le j \le 2k-N$, and $\mathrm{Re}(s_j) = \lambda_j/\log R$ for $1 \le j < j_{N+1}$, then we have that

$$\mathrm{Re}(E(\boldsymbol{s})) = -\frac{|\gamma_{j_{N+1}}|\lambda_{j_{N+1}}(1+O(1/\lambda))}{(\log T)^{3/2}},$$

so that our integrand is $\ll e^{-\sqrt{\log R}}$ if λ is large enough. We thus find that in this case

$$J(R) \ll T^{-1+o(1)},$$

as needed. □

REMARK 8.2. – Case 1 is feasible for some choice of $I_1,\dots,I_{2k-1}$. Indeed, if $k=1$ and $A \ge 1$, so that $N=1$, then we note that at least one of the zeta factors must have survived in the numerator after shifting the s_2 contour, and there are none in the denominator, so there is a pole at $s_1'=0$. On the other hand, if $k \ge 2$, then if $I_1=\{2k-1,2k\}$, $I_2=\{2k-1,2k-2\}$ and $I_3=\{2k-2,2k\}$, then $a_{2k}=a_{2k-1}=a_{2k-2}=0$, and $a_1 \neq 0$. Taking $h_{2k-1,I}=0$ for all I, we see that

$$m = \sum_{I\in\mathcal{S}(2k)\setminus\mathcal{I}_{2k-1}} (-1)^{\#I} + (A+1)\sum_{j\in[2k],\,a_j\neq 0} 1 = (A+1)\sum_{j\in[2k],\,a_j\neq 0} 1 \ge 1,$$

where we used Proposition 7.1(a). Thus we see indeed that there is a genuine pole at $s_1'=0$.

It remains to prove the second intermediate step in the proof of (8.4).

Proof of Lemma 8.5. – We separate into three cases depending on whether $N=2k$, $N=2k-1$ or $N\le 2k-2$.

Case 1 of the proof of Lemma 8.5: $N=2k$. – Here there is no integral and we have that $J(R)=c(\log R)^{H_{2k}-2k(A+1)-d}$.

Since $\mathcal{I}_{2k}=\{I\subset[2k]\}$, we find that $H_{2k}=-1+2k(A+1)$, whence $J(R)=(\log R)^{-d-1}$. If $d=0$ and $\mathcal{E}_{k,A}=-1$, the lemma follows with $c=1$; otherwise, we take $c=0$.

Case 2 of the proof of Lemma 8.5: $N=2k-1$. – In this case $J(R)$ is given by a one-dimensional integral over s_1. Moreover, there exist coefficients $a_j\in\mathbb{Q}$ such that $x_j=L_{2k-1,\{j\}}(x_1)=a_jx_1$ for all j. Therefore the only possible pole of the integrand in $J(R)$ is when $s_1=0$. If there is such a pole, then it means that $\{1\}\notin\mathcal{I}_{2k-1}$. In this case, the order of this potential pole, say m, would be given by (8.12) with $N=2k-1$, $\mathcal{I}_{2k}=\mathcal{S}(2k)$ and ν defined analogously, so that (8.13) implies that

(8.14)
$$m+H_{2k-1}-(2k-1)-2kA = 1+\sum_{I\in\mathcal{S}^*(2k)}(-1)^{\#I} - \sum_{\substack{I\in\mathcal{S}^-(2k)\setminus\mathcal{I}_{2k-1}\\ \#I\ge 3,\,h_{2k-1,I}\ge 2}}(h_{2k-1,I}-1)-\nu \le 0,$$

First, let us assume that $m\ge 1$ (i.e., there is a genuine pole at $s_1=0$). We find that $H_{2k-1}+1-2k(A+1)\le -1$. We then move the line of integration of $s_1=\sigma_1+it_1$ to the contour $\sigma_1=1/(\log(2+|t_1|))^{3/2}$, $|t_1|\le T$. No poles are encountered and the horizontal integrals contribute $\ll(\log R)^{O(1)}/T$. Moreover, the integral converges fast enough now (even when $A = 0$) that we may remove the condition $|t_1| \le T$ at the cost of an error term of size $\ll (\log R)^{O(1)}/T$. We thus conclude that

$$J(R) = c \cdot (\log R)^{H_{2k-1}-2k(A+1)+1-d} + O(T^{-1+o(1)}),$$

where

$$c = \int_{\sigma_1 = 1/(\log(2+|t_1|))^{3/2}} G(s_1) \prod_{I \in \mathcal{S}(2k) \setminus \mathcal{J}_{sk-1}} (\zeta^{e_I})^{(h_{2k-1,I})}(1 + a_I s_1) \mathrm{d}s_1$$

is some constant. This contributes towards the error term if $m \ge 2$, $d \ge 1$ or $A \le \frac{1}{2k}\binom{2k}{k} - 1$, and towards the main term if $m = 1$, $d = 0$, $A > \frac{1}{2k}\binom{2k}{k} - 1$ and $h_{2k-1,I} \in \{0, 1\}$ for each $I \in \mathcal{S}^-(2k)$, in which case $H_{2k-1} - 2k(A+1) + 1 = -1$ by (8.14).

Alternatively, assume that $m \le 0$, so that there is no pole at $s_1 = 0$. We move s_1 to the line $\mathrm{Re}(s_1) = 0$. The horizontal lines contribute $\ll (\log R)^{O(1)}/T$. Furthermore, we note that we may extend the range of integration to all $s_1 \in \mathbb{C}$ with $\mathrm{Re}(s_1) = 0$ at the cost of an error term of size $\ll (\log R)^{O(1)}/T$. Consequently,

$$J(R) = c \cdot (\log R)^{H_{2k-1}+1-2k(A+1)-d} + O((\log R)^{O(1)}/T),$$

where

$$c = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(it) \prod_{I \in \mathcal{S}(2k) \setminus \mathcal{J}_{2k-1}} (\zeta^{e_I})^{(h_{2k-1,I})}(1 + i a_I t) \mathrm{d}t$$

with $a_I := \sum_{j \in I} a_j$. The power of $\log R$ is

$$\begin{aligned} H_{2k-1} + 1 - 2k(A+1) - d &= 1 + \sum_{I \in \mathcal{S}_{2k-1} \setminus \{0\}} (-1)^{\#I} - \sum_{I \in \mathcal{S}(2k) \setminus \mathcal{J}_{2k-1}} h_{2k-1,I} \\ &\quad - (A+1) \cdot \#\{1 \le j \le 2k : a_j \ne 0\} - d \\ &= 1 + \mathscr{A}(V_{2k-1}) - \sum_{I \in \mathcal{S}(2k) \setminus \mathcal{J}_{2k-1}} h_{2k-1,I} \\ &\quad - (A+1) \cdot \#\{1 \le j \le 2k : a_j \ne 0\} - d \end{aligned}$$

in the notation of Proposition 7.1. Clearly, this is maximized when $h_{2k-1,I} = 0$ for all $I \in \mathcal{S}(2k) \setminus \mathcal{J}_{2k-1}$ and $d = 0$, in which case

$$G(s) = F(a_1 s, \dots, a_{2k} s) = \frac{P(a_1 s, \dots, a_{2k} s)}{\prod_{j=1}^{2k} (a_j s \zeta(1 + a_j s))^{A+1}} + O\left(\frac{(\log(2 + |s|))^{O(1)}}{(1 + |s|)^{(2k-1)(A+1)}} \right) \tag{8.15}$$

when $\mathrm{Re}(s) = 0$, by (6.6).

Now, if $a_j = 0$ for some $j \in [2k]$, then $\mathscr{A}(V_{2k-1}) = -1$, by Proposition 7.1(a). Note that there is at least one j such that $a_j \ne 0$; otherwise, the dimension of V_{2k-1} would be $2k$, as it would contain the independent forms $s_1, \dots, s_{2k}$, which is a contradiction. We conclude that the power of $\log R$ is $\le -(A+1) \cdot \#\{1 \le j \le 2k : a_j \ne 0\} \le -A - 1$. Consequently,

$$J(R) \ll (\log R)^{-A-1}$$

in this case, which contributes towards the error term (i.e., $c = 0$ in this case).

Finally, assume that $a_j \neq 0$ for all $j \in [2k]$. Since $\dim(V_{2k-1}) = 2k-1$, Proposition 7.1(b) implies that the power of $\log R$ is

$$H_{2k-1} + 1 - 2k(A+1) - d \leq \mathscr{A}(V_{2k-1}) - \dim(V_{2k-1}) - 2kA \leq \binom{2k}{k} - 2kA,$$

with the second inequality being an equality when half of the a_j's equal $+1$ and the other half -1.

Even though this is not needed for the proof, we remark that when $a_j \neq 0$ for all j, we can give an asymptotic formula for $J(R)$. For simplicity, let us assume that $a_j = 1$ for $j \leq k$ and $a_j = -1$ for $j > k$. Then $a_I = \#(I \cap [1,k]) - \#(I \cap (k,2k])$, which has the same parity as $\#I$. In particular, $\mathscr{J}^-(2k) \cap \mathscr{J}_{2k-1} = \emptyset$, so that $h_{2k-1,I} = 0$ for all $I \in \mathscr{S}^-(2k)$. Moreover, given $\ell \in \mathbb{Z}$, we have that $a_I = \ell$ for exactly $\binom{2k}{k+|\ell|}$ sets $I \subset [2k]$. Therefore

$$J(R) = \frac{(\log R)^{\binom{2k}{k}-2kA}}{2\pi} \int_{-\infty}^{\infty} \frac{P(it,-it,\dots,it,-it)}{t^{2k(A+1)}} \cdot \frac{\prod_{\ell \text{ even},\geq 2} |\zeta(1+i\ell t)|^{2\binom{2k}{k+\ell}}}{\prod_{\ell \text{ odd},\geq 1} |\zeta(1+i\ell t)|^{2\binom{2k}{k+\ell}}} \mathrm{d}t + O(T^{-1+o(1)}).$$

This completes the study of Case 2.

Case 3 of the proof of Lemma 8.5: $N \leq 2k-2$. – We shift the contours of $s_{2k-N}, s_{2k-N-1}, \dots, s_1$ in this order to the lines $\mathrm{Re}(s_j) = \lambda^j/(\log T)^{3/2}$, $1 \leq j \leq 2k-N$. If λ is large enough in terms of k (but independently of R), then the functions $\mathrm{Re}(L_{N,I}(\boldsymbol{s}))$ with $I \in \mathscr{S}(2k) \setminus \mathscr{J}_N$ have constant sign in the entire domain where the contour shifting is performed, so that no poles are encountered. The horizontal lines contribute $\ll (\log R)^{O(1)}/T$. Finally, we note that the integrand on the new lines of integration is

$$\ll (\log\log R)^{O(1)}/[(1+|s_1|)\cdots(1+|s_{2k-N}|)],$$

by (8.8) and our assumption that G is in the class $\mathcal{C}_{2k-N}$. We thus find that

$$J(R) \ll (\log R)^{H_N - N - 2kA - d} (\log\log R)^{O(1)}$$

in this case. We need to understand the power of $\log R$. Firstly, note that

$$H_N - N - 2kA - d \leq 2k - N + \sum_{I \in \mathscr{J}_N \setminus \{\emptyset\}} (-1)^{\#I} - (A+1)\cdot \#\{1 \leq j \leq 2k : \{j\} \notin \mathscr{J}_N\}.$$

To continue, we separate two cases.

If there is $\{j\} \in \mathscr{J}_N$, then $\sum_{I \in \mathscr{J}_N \setminus \{\emptyset\}} (-1)^{\#I} = -1$ by Proposition 7.1(a). Since $\dim(V_N) = N$ by construction, there are $\leq N$ integers j with $\{j\} \in \mathscr{J}_N$. The power of $\log R$ is thus

$$\leq 2k - N - 1 - (A+1)(2k-N) = -1 - A(2k-N) \leq -1 - 2\cdot \mathbf{1}_{A \geq 1},$$

and we can see that $J(R)$ satisfies the conclusion of the lemma with no main term (i.e., $c = 0$). So assume that there is no $\{j\} \in \mathscr{J}_N$. Then we find that

$$H_N - N - 2kA \leq 2k - N + \sum_{I \in \mathscr{J}_N \setminus \{0\}} (-1)^{\#I} - 2k(A+1) \leq \binom{2k}{k} - 2k(A+1) - 2,$$

by Proposition 7.1(c), which again means that the lemma holds with $c = 0$. $\square$

8.4. Dyadic intervals

We conclude this section with a brief explanation of the proof of (8.2). We have that

$$\mathcal{M}_{\widetilde{f_0},2k}(R) = \mathcal{M}_{\widetilde{h},2k}(R) + O\left(\frac{1}{\log R}\right),$$

where $\widetilde{h}(x) = h(x) - h(x + \frac{\log 2}{\log R})$, by the argument leading to (6.4). We then note that

$$\widehat{(\widetilde{h})}_R(s) = \widehat{h}_R(s)(1 - 2^{-s}),$$

so that Perron's inversion formula and relation (6.2) imply that

$$\mathcal{M}_{\widetilde{f_0},2k}(R) = \frac{1}{(2i\pi)^{2k}} \underset{\substack{\mathrm{Re}(s_j)=\lambda^j/\log R \\ 1\le j\le 2k}}{\int\cdots\int} D(\boldsymbol{s}) \left(\prod_{j=1}^{2k} \widehat{h}_R(s_j)(1-2^{-s_j})\right) ds_{2k}\cdots ds_1 + O\left(\frac{1}{\log R}\right),$$

where λ and T are as before. We thus find that

(8.16)
$$\mathcal{M}_{\widetilde{f_0},2k}(R) = \frac{1}{(2i\pi)^{2k}} \underset{\substack{\mathrm{Re}(s_j)=\lambda^j/\log R \\ 1\le j\le 2k}}{\int\cdots\int} \widetilde{F}(\boldsymbol{s}) R^{s_1+\cdots+s_{2k}} \prod_{I\in\mathcal{S}^*(2k)} \zeta(1+s_I)^{(-1)^{\#I}} \mathrm{d}s_1\cdots\mathrm{d}s_{2k} + O\left(\frac{1}{\log R}\right),$$

where

$$\widetilde{F}(\boldsymbol{s}) := P(\boldsymbol{s}) \prod_{j=1}^{2k} \frac{\widehat{h}_R(s_j)(1-2^{-s_j})}{R^{s_j}}$$

and P is defined as above. The function $\widetilde{F}$ is in the class $\mathcal{C}_{2k}$, since the factor $1 - 2^{-s_j}$ annihilates the pole of $\widehat{h}_R(s_j)$ at $s_j = 0$. We thus see that the above integral has the same shape as the integral $I_{g,2k}(R)$ with $A = 0$, with the difference that $e_I = -1$ when $\#I = 1$. We thus follow the argument leading to (8.1) when $A = 0$ with the obvious modifications. The only difference is that in the analogue of (8.12) we have instead

$$\begin{aligned} m &= -\nu + \sum_{\substack{I\in\mathcal{I}_{N+1}\setminus\mathcal{I}_N \\ \#I=\text{even}}} (h_{N,I}+1) - \sum_{\substack{I\in\mathcal{I}_{N+1}\setminus\mathcal{I}_N \\ \#I=\text{odd},\, h_{N,I}=0}} 1 \\ &= -\nu + \sum_{I\in\mathcal{I}_{N+1}\setminus\mathcal{I}_N} (h_{N,I}+(-1)^{\#I}) - \sum_{\substack{I\in\mathcal{S}^-(2k)\cap(\mathcal{I}_{N+1}\setminus\mathcal{I}_N) \\ h_{N,I}\ge 2,\, \#I\ge 3}} (h_{N,I}-1), \end{aligned}$$

with ν defined as in the proof of Lemma 8.4. We thus find that m has the same expression as when $A = -1$, and relation (8.2) follows by the proof of (8.4) when $A = -1$. An important remark is that when $A = -1$ there is a power of $(\log R)^{-2k}$ in the denominator of the integrand of $I_{g,2k}(R)$ that is not present in the denominator of the right hand side of (8.16).

9. Lower bounds

In this section we complete our proof of Theorem 1.3 by showing that, for fixed $k \in \mathbb{Z}_{\geq 1}$, $A \in \mathbb{Z}_{\geq 0}$ and $\epsilon > 0$, there are positive constants $c'_{k,A} > 0$ and $c''_k > 0$ such that

$$\mathcal{M}_{f_A,2k}(R) \geq c'_{k,A}(\log R)^{\mathcal{E}_{k,A}-\epsilon} + O_\epsilon((\log R)^{\mathcal{E}_{k,A}-1}), \tag{9.1}$$

and

$$\mathcal{M}_{\widetilde{f}_0,2k}(R) \geq c''_k(\log R)^{D_{k,0}-\epsilon} + O_\epsilon((\log R)^{D_{k,0}-1}), \tag{9.2}$$

where $\widetilde{f}_0(x) = f_0(x) - f_0(x + \frac{\log 2}{\log R})$, as before. Evidently, this completes the proof of Theorem 1.3, since if the constants in the leading terms in (8.1) or (8.2) were 0, we would obtain a contradiction to the above lower bounds by letting $R \to \infty$.

As in the previous section, there is some smooth function h such that $h(x) = 1$ for $x \leq 1 - 1/(\log R)^C$ and $h(x) = 0$ for $x \geq 1$, with $C = (2k-1)2^{2k+1} + 2k + 2$, so that

$$\mathcal{M}_{f_0,2k}(R) = \mathcal{M}_{h,2k}(R) + O\left(\frac{1}{\log R}\right) \quad \text{and} \quad \mathcal{M}_{\widetilde{f}_0,2k}(R) = \mathcal{M}_{\widetilde{h},2k}(R) + O\left(\frac{1}{\log R}\right),$$

where $\widetilde{h}(x) = h(x) - h(x + \frac{\log 2}{\log R})$. So, it suffices to prove that

$$\mathcal{M}_{g,2k}(R) \geq c'_{k,A}(\log R)^{\mathcal{E}_{k,A}-\epsilon} + O((\log R)^{\mathcal{E}_{k,A}-1}), \tag{9.3}$$

where $g \in \{h, \widetilde{h}\}$ when $A = 0$, and $g = f_A$ when $A \geq 1$.

Positivity is a key to this proof: we will consider the sum restricted to those integers with a convenient prime factorization, which clearly provides a lower bound. The fact that these integers have a convenient prime factorization means the corresponding sum is technically easier to analyze.

9.1. First manipulations

To ease notation, let

$$\Pi_R := \prod_{p \leq R}\left(1 - \frac{1}{p}\right).$$

We start by observing that

$$\mathcal{M}_{g,2k}(R) = \Pi_R \sum_{P^+(n) \leq R} \frac{1}{n}\left(\sum_{d|n}\mu(d)g\left(\frac{\log d}{\log R}\right)\right)^{2k},$$

since $\operatorname{supp}(g) \subset (-\infty, 1]$. So (9.1) follows immediately when $A > -1 + \frac{1}{2k}\binom{2k}{k}$ by noticing that $\mathcal{E}_{k,A} = -1$ in this case and that we always have that $\mathcal{M}_{g,2k}(R) \geq \Pi_R \gg 1/\log R$.

For the rest of the proof, we assume that $A \leq -1 + \frac{1}{2k}\binom{2k}{k}$, so that

$$\mathcal{E}_{k,A} = \binom{2k}{k} - 2k(A+1) \geq 0.$$

Let $q_1 < q_2 < \cdots$ be the sequence of all prime numbers and set

$$Q = \prod_{j=1}^{A+1} q_j.$$

If $A \geq 1$, or if $A = 0$ and $g = h$, then we restrict our attention to integers of the form Qn with $P^-(n) > Q$, so that

$$\mathcal{M}_{g,2k}(R) \geq \frac{\Pi_R}{Q} \sum_{p|n \Rightarrow q_{A+1}<p\leq R} \frac{1}{n} \left(\sum_{J\subset[A+1]} (-1)^{\#J} \sum_{d|n} \mu(d) g\left(\frac{\log(q_J d)}{\log R}\right)\right)^{2k},$$

where $q_J := \prod_{j\in J} q_j$. We define

$$w(x) := \sum_{J\subset[A+1]} (-1)^{\#J} g\left(x + \frac{\log q_J}{\log R}\right), \tag{9.4}$$

so that

$$\mathcal{M}_{g,2k}(R) \geq \frac{\Pi_R}{Q} \sum_{p|n \Rightarrow q_{A+1}<p\leq R} \frac{1}{n} \left(\sum_{d|n} \mu(d) w\left(\frac{\log d}{\log R}\right)\right)^{2k}. \tag{9.5}$$

We further note that $w = \widetilde{h}$ when $A = 0$, so that the right hand side of (9.5) is a trivial lower bound for $\mathcal{M}_{\widetilde{h},2k}(R)$. Therefore, relation (9.3) will follow in all cases if we can show that

$$\begin{aligned} W := \Pi_R \sum_{p|n \Rightarrow q_{A+1}<p\leq R} \frac{1}{n} \left(\sum_{d|n} \mu(d) w\left(\frac{\log d}{\log R}\right)\right)^{2k} \\ \geq c''_{k,A} (\log R)^{\mathcal{E}_{k,A}-\epsilon} + O_\epsilon((\log R)^{\mathcal{E}_{k,A}-1}) \end{aligned} \tag{9.6}$$

for some $c''_{k,A} > 0$, where w is defined by (9.4) with $g = h$ if $A = 0$ and $g = f_A$ if $A \geq 1$.

Next, observe that

$$\widehat{w}_R(s) = (\log R) \sum_{J\subset[A+1]} (-1)^{\#J} \int_{-\infty}^{\infty} g\left(u + \frac{\log q_J}{\log R}\right) R^{su} \mathrm{d}u = \widehat{g}_R(s) \prod_{j=1}^{A+1} (1 - q_j^{-s}). \tag{9.7}$$

In particular, we see that $\widehat{w}_R$ has an analytic continuation to $\mathbb{C}$ and it satisfies the bound

$$\widehat{w}_R(s) \ll \frac{R^{\mathrm{Re}(s)}}{(|s|+1)(\log R)^A} \quad (\mathrm{Re}(s) \geq -1), \tag{9.8}$$

which follows by the definition of f_A when $A \geq 1$ and by (6.5) otherwise. Finally, we note that we also have the bound

$$\widehat{w}_R(s) \ll \frac{R^{\mathrm{Re}(s)} (\log R)^{O(1)}}{1 + |s|^2} \quad (\mathrm{Re}(s) \geq 1/\log R), \tag{9.9}$$

which follows from (6.2) when $A = 0$. (This bound can be shown to hold in a larger range, but the above range is good enough for our purposes.)

Before we apply Perron's inversion formula to write the right hand side of (9.5) in terms of $\widehat{w}_R$, we use positivity again to focus on integers n of a certain convenient form. We set

$$y = \exp\{(\log R)^{1-\epsilon'}\} \quad \text{and} \quad Y = \exp\{(\log R)^{1-\epsilon'/2}\},$$

where $\epsilon' > 0$ will be taken to be small enough in terms of ϵ, and write

$$\mathcal{N} = \{n \in \mathbb{Z}_{\geq 1} : p|n \Rightarrow q_{A+1} < p \leq y\}.$$

We then focus on integers of the form $n = mp_1 \cdots p_k$, where $m \in \mathcal{N}$ with $m \le Y$, and $p_1, \dots, p_k$ are distinct primes from the interval $(R^{1/2}, R]$. For such an integer n, if $d|n$, then either $d = d'$ or $d = d' p_\ell$, for some $d'|m$ and some $\ell \in \{1, \dots, k\}$. So, we conclude that

$$W \ge \frac{\Pi_R}{k!} \sum_{\substack{m \in \mathcal{N} \\ m \le Y}} \sum_{\substack{\sqrt{R} < p_1, \dots, p_k \le R \\ \text{distinct primes}}} \frac{1}{mp_1 \cdots p_k} \times \left(\sum_{\ell=1}^{k} \sum_{d|m} \mu(d) w\left(\frac{\log(p_\ell d)}{\log R}\right) - \sum_{d|m} \mu(d) w\left(\frac{\log d}{\log R}\right) \right)^{2k}.$$

Note that the condition that $m \le Y = R^{o(1)}$ certainly implies that $d < R/(2q_{A+1})$ for all divisors d of m (since $q_{A+1} \ll_A 1$), so that

$$\sum_{d|m} \mu(d) w\left(\frac{\log d}{\log R}\right) = \sum_{J \subset [A+1]} (-1)^{\#J} \sum_{d|m} \mu(d) \left(1 - \frac{\log(q_J d)}{\log R}\right)^{A} = \sum_{d|Qm} \mu(d) \left(1 - \frac{\log d}{\log R}\right)^{A} = 0$$

by observing that $\Delta^{(A+1)}(1-x)^A = 0$ by (1.7), and by applying (1.6) with $r = A+1$, since $\omega(Qm) \ge \omega(Q) \ge A+1$ here. We thus conclude that

$$W \ge \frac{\Pi_R}{k!} \sum_{\substack{m \in \mathcal{N} \\ m \le Y}} \sum_{\substack{\sqrt{R} < p_1, \dots, p_k \le R \\ \text{distinct primes}}} \frac{1}{mp_1 \cdots p_k} \left(\sum_{\ell=1}^{k} \sum_{d|m} \mu(d) w\left(\frac{\log(p_\ell d)}{\log R}\right) \right)^{2k}$$
$$\ge \frac{\Pi_R}{k!(\log R)^k} \sum_{\substack{m \in \mathcal{N} \\ m \le Y}} \sum_{\substack{\sqrt{R} < p_1, \dots, p_k \le R \\ \text{distinct primes}}} \frac{\prod_{j=1}^{k} \log p_j}{mp_1 \cdots p_k} \left(\sum_{\ell=1}^{k} \sum_{d|m} \mu(d) w\left(\frac{\log(p_\ell d)}{\log R}\right) \right)^{2k}.$$

We note that

$$\sum_{n \in \mathcal{N}, n > Y} \frac{\tau_r(n)}{n} \le Y^{-1/\log y} \sum_{n \in \mathcal{N}} \frac{\tau_r(n)}{n^{1-1/\log y}} \ll \frac{1}{(\log R)^B},$$

for any fixed $B \ge 1$ and $r \in \mathbb{Z}_{\ge 1}$. Therefore, we may drop the conditions that $m \le Y$ and that the p_ℓ's are distinct at the cost of an admissible error term, finding that

$$W \ge \frac{\Pi_R}{k!(\log R)^k} \sum_{m \in \mathcal{N}} \sum_{\sqrt{R} < p_1, \dots, p_k \le R} \frac{\prod_{j=1}^{k} \log p_j}{mp_1 \cdots p_k} \left(\sum_{\ell=1}^{k} \sum_{d|m} \mu(d) w\left(\frac{\log(p_\ell d)}{\log R}\right) \right)^{2k} + O\left(\frac{1}{(\log R)^{100}}\right).$$

Next, we expand the $2k$-th power as follows:

$$\left(\sum_{\ell=1}^{k} \sum_{d|m} \mu(d) w\left(\frac{\log(p_\ell d)}{\log R}\right) \right)^{2k} = \sum_{\substack{J_1 \cup \cdots \cup J_k = [2k] \\ J_i \cap J_j = \emptyset \text{ for } i \ne j}} \prod_{\ell=1}^{k} \prod_{j \in J_\ell} \sum_{d_j|m} \mu(d_j) w\left(\frac{\log(p_\ell d_j)}{\log R}\right),$$

with the convention that if $J_\ell = \emptyset$, then the corresponding factor equals 1. Clearly, if $J_i\ell$ is even for all ℓ, then the corresponding summand yields a non-negative contribution to the above sum. We write $\mathscr{J}$ for the set of k-tuples $\boldsymbol{J}$ such that $(J_1, \dots, J_k)$ is a partition of $[2k]$ and either $\#J_\ell = 2$ for all ℓ, or there is some ℓ such that $\#J_\ell$ is an odd number. Then

$$\left(\sum_{\ell=1}^{k}\sum_{d|m}\mu(d)w\left(\frac{\log(p_\ell d)}{\log R}\right)\right)^{2k} \geq \sum_{\boldsymbol{J}\in\mathscr{J}}\prod_{\ell=1}^{k}\prod_{j\in J_\ell}\sum_{d_j|m}\mu(d_j)w\left(\frac{\log(p_\ell d_j)}{\log R}\right).$$

Moreover, if $D \in \mathscr{N}$, then

$$\sum_{\substack{m\in\mathscr{N}\\ D|m}}\frac{1}{m} = \frac{1}{D}\prod_{q_{A+1}<p\leq y}\left(1-\frac{1}{p}\right)^{-1} = \frac{c_1\prod_{p\leq y}(1-1/p)^{-1}}{D}$$

with $c_1 = \prod_{p\leq q_{A+1}}(1-1/p) \asymp 1$. So, we conclude that

$$\begin{aligned}W \geq{}& \frac{c_1\prod_{y<p\leq R}(1-1/p)}{k!(\log R)^k}\sum_{\boldsymbol{J}\in\mathscr{J}}\sum_{d_1,\dots,d_{2k}\in\mathscr{N}}\frac{\mu(d_1)\cdots\mu(d_{2k})}{[d_1,\dots,d_{2k}]}\\ &\times\prod_{\ell=1}^{k}\sum_{\sqrt{R}<p_\ell\leq R}\frac{\log p_\ell}{p_\ell}\prod_{j\in J_\ell}w\left(\frac{\log(p_\ell d_j)}{\log R}\right) - O\left(\frac{1}{(\log R)^{100}}\right).\end{aligned}\tag{9.10}$$

For the convenience of notation, set

$$W(\boldsymbol{J}) = \frac{1}{(\log R)^k}\sum_{d_1,\dots,d_{2k}\in\mathscr{N}}\frac{\mu(d_1)\cdots\mu(d_{2k})}{[d_1,\dots,d_{2k}]}\prod_{\ell=1}^{k}\sum_{\sqrt{R}<p_\ell\leq R}\frac{\log p_\ell}{p_\ell}\prod_{j\in J_\ell}w\left(\frac{\log(p_\ell d_j)}{\log R}\right),$$

so that

$$W \geq \frac{c_1\prod_{y<p\leq R}(1-1/p)}{k!}\sum_{\boldsymbol{J}\in\mathscr{J}}W(\boldsymbol{J}) - O\left(\frac{1}{(\log R)^{100}}\right).$$

We will show that the dominant contribution comes from the terms $\boldsymbol{J}$ with $\#J_\ell = 2$ for all ℓ.

9.2. Mellin transformation

Fix a choice of sets $\boldsymbol{J} = (J_1, \dots, J_k) \in \mathscr{J}$ and let

$$\mathscr{L} = \{1 \leq \ell \leq k : J_\ell \neq \emptyset\}.$$

For $\ell \notin \mathscr{L}$, the sum over p_ℓ is

$$\sum_{\sqrt{R}<p_\ell\leq R}\frac{\log p_\ell}{p_\ell} = \frac{\log R}{2} + O\left(e^{-c\sqrt{\log R}}\right).$$

So

$$\begin{aligned}W(\boldsymbol{J}) = \frac{2^{\#\mathscr{L}-k}}{(\log R)^{\#\mathscr{L}}}\sum_{d_1,\dots,d_{2k}\in\mathscr{N}}\frac{\mu(d_1)\cdots\mu(d_{2k})}{[d_1,\dots,d_{2k}]}\prod_{\ell\in\mathscr{L}}\sum_{p_\ell>\sqrt{R}}\frac{\log p_\ell}{p_\ell}\prod_{j\in J_\ell}w\left(\frac{\log(p_\ell d_j)}{\log R}\right)&\\ +O\left(\frac{1}{(\log R)^{100}}\right),&\end{aligned}$$

where the condition that $p_\ell \leq R$ was dropped because it is implied by the fact that $\mathrm{supp}(w) \subset (-\infty, 1]$.

Next, we use Perron's formula $2k$ times to write each appearance of w as an integral of $\widehat{w}_R$. We thus find that

$$W(\boldsymbol{J}) = \frac{2^{\#\mathcal{L}-k}(\log R)^{-\#\mathcal{L}}}{(2\pi i)^{2k}} \int\cdots\int_{\substack{\mathrm{Re}(s_j)=1/\log R\\ 1\le j\le 2k}} \sum_{d_1,\dots,d_{2k}\in\mathcal{N}} \frac{\prod_{j=1}^{2k}\mu(d_j)d_j^{-s_j}}{[d_1,\dots,d_{2k}]}\left(\prod_{\ell\in\mathcal{L}}\sum_{p_\ell>\sqrt{R}}\frac{\log p_\ell}{p_\ell^{1+s_{J_\ell}}}\right)$$

$$\times\left(\prod_{j=1}^{2k}\widehat{w}_R(s_j)\right)\mathrm{d}s_1\cdots\mathrm{d}s_{2k} + O\left(\frac{1}{\log R}\right), \tag{9.11}$$

with the notational convention that $s_J := \sum_{j\in J}s_j$. By possibly re-indexing the variables $s_1,\dots,s_{2k}$, we may assume that $\mathcal{L} = \{1,\dots,L\}$, where $L = \#\mathcal{L}$, and that $\max J_\ell = 2k-L+\ell$ for all $\ell\in\{1,\dots,L\}$. We want to move the variables $s_{2k-L+1},\dots,s_{2k}$ to the left. First, we need some bounds on the sum over p_ℓ. We note that

$$\sum_{p>\sqrt{R}}\frac{\log p}{p^{1+s}} = -\frac{\zeta'}{\zeta}(1+s)+O(1)-\sum_{p\le R^{1/2}}\frac{\log p}{p^{1+s}}$$

for $\mathrm{Re}(s)\ge -1/3$. Using standard bounds on the Riemann zeta function (see, for example, Titchmarsch [23, Theorem 3.11]), we find that

$$\begin{aligned}\sum_{p>\sqrt{R}}\frac{\log p}{p^{1+s}} &\ll \frac{1}{|s|}+\log(2+|t|)+R^{\max\{0,-\sigma\}/2}\sum_{p\le\sqrt{R}}\frac{\log p}{p}\\ &\ll R^{\max\{0,-\sigma\}/2}\log(R+|t|),\end{aligned}$$

where $s=\sigma+it$, as usual. Moreover, note that if $\sigma_j\ge -1/\log y$ for all $j\in\{1,\dots,2k\}$, then

$$\begin{aligned}\left|\sum_{d_1,\dots,d_{2k}\in\mathcal{N}}\frac{\prod_{j=1}^{2k}\mu(d_j)d_j^{-s_j}}{[d_1,\dots,d_{2k}]}\right| &\le \sum_{d_1,\dots,d_{2k}\in\mathcal{N}}\frac{\prod_{j=1}^{2k}\mu^2(d_j)d_j^{1/\log y}}{[d_1,\dots,d_{2k}]}\\ &\le\prod_{p\le y}\left(1+\frac{p^{1/\log y}(4^k-1)}{p}\right)\ll(\log y)^{4^k-1},\end{aligned}$$

using the estimate $p^{1/\log y}=1+O(\log p/\log y)$ for $p\le y$.

We are now ready to move the variables $s_{2k-L+1},\dots,s_{2k}$ in (9.11) to the left. First, we move the variable s_{2k} to the line $\mathrm{Re}(s_{2k})=-1/\log y$. (Here we can use (9.9) to justify the convergence required for this maneuver.) We pick up a simple pole from the sum over p_{2k} when $s_{J_L}=0$. The integrand when $\mathrm{Re}(s_{2k})=-1/\log y$ is

$$\begin{aligned}&\ll(\log y)^{4^k-1}(\log R)^{L-1}\log(R+|t|)R^{1/(2\log y)}\prod_{j=1}^{2k}\left|\widehat{w}_R(s_j)\right|\\ &\ll\frac{(\log R)^{O(1)}\log(2+|t|)R^{-1/(2\log y)}}{(|s_1|^2+1)\cdots(|s_{2k}|^2+1)}\end{aligned}$$

by (9.9). So we find that

$$W(\boldsymbol{J}) = \frac{2^{L-k}(\log R)^{-L}}{(2\pi i)^{2k-1}} \int\cdots\int_{\substack{\mathrm{Re}(s_j)=1/\log R\\ 1\le j\le 2k-1\\ s_{J_L}=0}} \sum_{d_1,\dots,d_{2k}\in\mathcal{N}} \frac{\prod_{j=1}^{2k}\mu(d_j)d_j^{-s_j}}{[d_1,\dots,d_{2k}]}\left(\prod_{\ell=1}^{L-1}\sum_{p_\ell>R^{1/2}}\frac{\log p_\ell}{p_\ell^{1+s_{J_\ell}}}\right)$$
$$\times\left(\prod_{j=1}^{2k}\widehat{w}_R(s_j)\right)\mathrm{d}s_1\cdots\mathrm{d}s_{2k-1} + O\left(\frac{1}{(\log R)^{100}}\right).$$

Now the product $\prod_{\ell=1}^{L-1}\sum_{p_\ell>R^{1/2}}(\log p_\ell)p_\ell^{-1-s_{J_\ell}}$ doesn't depend on the variables $s_j\in J_L\setminus\{2k\}$, and so we encounter no poles if we move all of these variables to the lines $\mathrm{Re}(s_j)=0$. Having done this, by (9.8) the growth of $\widehat{w}_R(s_j)$ only depends weakly on R.

Then we repeat the same argument by moving s_{2k-1} to the left, then s_{2k-2}, and so on and so forth, until all the sums over primes have been removed and replaced by contributions coming from poles. Writing $s_j = it_j$, we conclude that

$$W(\boldsymbol{J}) = \frac{2^{L-k}(\log R)^{-L}}{(2\pi)^{2k-L}} \int\cdots\int_{\substack{t_1,\dots,t_{2k}\in\mathbb{R}\\ t_{J_\ell}=0\,(1\le\ell\le L)}} \sum_{d_1,\dots,d_{2k}\in\mathcal{N}} \frac{\prod_{j=1}^{2k}\mu(d_j)d_j^{-it_j}}{[d_1,\dots,d_{2k}]}\left(\prod_{j=1}^{2k}\widehat{w}_R(it_j)\right)\mathrm{d}t_1\cdots\mathrm{d}t_{2k-L}$$
$$+ O\left(\frac{1}{(\log R)^{100}}\right). \tag{9.12}$$

We note that, for any $t_1,\dots,t_{2k}$, we have

$$\sum_{d_1,\dots,d_{2k}\in\mathcal{N}} \frac{\prod_{j=1}^{2k}\mu(d_j)d_j^{-it_j}}{[d_1,\dots,d_{2k}]} = \prod_{q_{A+1}<p\le y}\left(1+\frac{\lambda_p(\boldsymbol{t})}{p}\right),$$

where

$$\lambda_p(\boldsymbol{t}) := \sum_{I\in\mathcal{S}^*(2k)}(-1)^{\#I}p^{-it_I} = -1 + \prod_{j=1}^{2k}(1-p^{-it_j}). \tag{9.13}$$

9.3. An auxiliary result

Before we continue with the estimation of $W(\boldsymbol{J})$, we establish a preliminary (and fairly standard) result.

LEMMA 9.1. – *For $z\ge y\ge 3$ and $t\in\mathbb{R}$, we have that*

$$\sum_{y<p\le z}\frac{1}{p^{1+it}} = \begin{cases} \log\left(\dfrac{\log z}{\log y}\right)+O(1) & \text{if } |t|\le 1/\log z,\\ \log\left(\dfrac{1}{|t|\log y}\right)+O(1) & \text{if } 1/\log z<|t|\le 1/\log y,\\ O(1) & \text{if } y\ge|t|\ge 1/\log y.\end{cases}$$

Finally, if $|t| \geq y$, *then*

$$\Bigg|\sum_{y<p\leq z}\frac{1}{p^{1+it}}\Bigg| \leq \log\left(\frac{\log(\min\{|t|,z\})}{\log y}\right)+O(1).$$

Proof. – If $|t| \leq 1/\log z$, then we note that $p^{-it} = 1 + O(\log p/\log z)$ for $p \leq z$, so that

$$\sum_{y<p\leq z}\frac{1}{p^{1+it}} = \sum_{y<p\leq z}\frac{1}{p}+O(1) = \log\left(\frac{\log z}{\log y}\right)+O(1),$$

as claimed. For $|t| \geq 1/\log y$ and $y \geq |t|$, then we note that

$$\sum_{y<p\leq z}\frac{1}{p^{1+it}} \ll 1$$

by relation (4.4) in [14]. Next, if $1/\log z \leq |t| \leq 1/\log y$, then we apply the results we just proved to deduce that

$$\sum_{y<p\leq e^{1/|t|}}\frac{1}{p^{1+it}} = \log\left(\frac{\log e^{1/|t|}}{\log y}\right)+O(1) = \log\left(\frac{1}{|t|\log y}\right)+O(1)$$

and that

$$\sum_{e^{1/|t|}<p\leq z}\frac{1}{p^{1+it}} \ll 1.$$

Finally, if $|t| \geq y$, then we note that

$$\sum_{|t|<p\leq z}\frac{1}{p^{1+it}} \ll 1,$$

so that

$$\begin{aligned}\Bigg|\sum_{y<p\leq z}\frac{1}{p^{1+it}}\Bigg| = \Bigg|\sum_{y<p\leq\min\{|t|,z\}}\frac{1}{p^{1+it}}\Bigg| + O(1) &\leq \sum_{y<p\leq\min\{|t|,z\}}\frac{1}{p}+O(1)\\ &\leq \log\left(\frac{\log(\min\{|t|,z\})}{\log y}\right)+O(1). \qquad \square\end{aligned}$$

9.4. Lower bound for the main term

We now return to the study of the quantity $W(\boldsymbol{J})$, defined by (9.12). First, we show that the term when $\#J_\ell = 2$ for all ℓ contributes what we claim to be our main term. In this case $L = \#\mathscr{L} = k$ and, by possibly permuting the t_j's, we may assume that $J_\ell = \{\ell, k+\ell\}$ for all $\ell \in \{1,\dots,k\}$. Thus for these terms we have $t_{k+\ell} = -t_\ell$. We want to show that the integrand in (9.12) is non-negative for all choices of $t_1,\dots,t_k$. We have that

$$\prod_{j=1}^{2k}\widehat{w}_R(it_j) = \prod_{j=1}^{k}\widehat{w}_R(it_j)\widehat{w}_R(-it_j) = \prod_{j=1}^{k}\left|\widehat{w}_R(it_j)\right|^2 = \prod_{j=1}^{k}\left|t_j^{A+1}\widehat{g}_R(it_j)\right|^2\prod_{a=1}^{A+1}\frac{|1-q_a^{-it_j}|^2}{t_j^2},$$

which is clearly non-negative for all $t_1,\dots,t_k$. Moreover, if $t_j \ll 1$ for all j, then relations (9.7) and (6.6) imply that

$$\prod_{j=1}^{2k}\widehat{w}_R(it_j) \geq \frac{c_2}{(\log R)^{2kA}}$$

for some $c_2 > 0$. Furthermore, the Definition (9.13) implies that in this case we have

$$\lambda_p(\boldsymbol{t}) = -1 + \prod_{j=1}^{k} |1 - p^{-it_j}|^2 \geq -1,$$

so

$$\sum_{d_1,\ldots,d_{2k} \in \mathcal{N}} \frac{\prod_{j=1}^{2k} \mu(d_j) d_j^{-it_j}}{[d_1, \ldots, d_{2k}]} = \prod_{q_{A+1} < p \leq y} \left(1 + \frac{\lambda_p(\boldsymbol{t})}{p}\right) \geq 0$$

for such $\boldsymbol{t}$.

Since the integrand in (9.12) is non-negative, we may obtain a lower bound by restricting the range of integration to any region we wish. We restrict to $t_1 \in [1,2]$ and $t_j \in [t_1, t_1 + 1/\log y]$ for $2 \leq j \leq k$. The volume of this region is $\asymp 1/(\log y)^{k-1}$. Moreover, in this region we find that

$$\lambda_p(\boldsymbol{t}) = -1 + |1 - p^{it_1}|^{2k} + O\left(\frac{\log p}{\log y}\right).$$

Therefore

$$\begin{aligned}\sum_{d_1,\ldots,d_{2k} \in \mathcal{N}} \frac{\prod_{j=1}^{2k} \mu(d_j) d_j^{-it_j}}{[d_1, \ldots, d_{2k}]} &= \exp\left\{\sum_{p \leq y} \frac{-1 + |1 - p^{it_1}|^{2k}}{p} + O\Big(\frac{1}{p^2}\Big) + O\Big(\frac{\log p}{p \log y}\Big)\right\} \\ &\gg \frac{1}{\log y} \exp\left\{\sum_{p \leq y} \frac{|1 - p^{it_1}|^{2k}}{p}\right\}.\end{aligned}$$

By binomial expansion

$$|1 - p^{it_1}|^{2k} = \sum_{j + j' = 2k} \binom{2k}{k} (-1)^{j'} p^{i(j-j')t_1}.$$

The terms with $j = j'$ contribute a factor

$$\exp\Big(\sum_{p \leq y} \binom{2k}{k} \frac{1}{p}\Big) \asymp (\log y)^{\binom{2k}{k}}.$$

By the final part of Lemma 9.1, since $t_1 \in [1,2]$ the terms with $j \neq j'$ contribute a factor

$$\exp\Big(O\Big(\sup_{|j| \leq 2k} \Big|\sum_{p \leq y} \frac{p^{ijt_1}}{p}\Big|\Big)\Big) \asymp 1.$$

Thus we obtain a lower bound of $\gg (\log y)^{\binom{2k}{k}-1}$ for our sum over $d_1, \ldots d_{2k}$. Since the region of integration has volume $\gg (\log y)^{-(k-1)}$, this gives

$$W(\boldsymbol{J}) \gg \frac{(\log y)^{\binom{2k}{k}-k}}{(\log R)^{k+2kA}}$$

in this case. So we find that the total contribution to the right hand side of (9.10) from such $\boldsymbol{J}$ is $\geq c_2 (\log y)^{\binom{2k}{k}-k+1}/(\log R)^{(2A+1)k}$ for some $c_2 > 0$, which is greater than the claimed main term in (9.6) if ϵ' is small enough in terms of ϵ and k.

9.5. Upper bound for the error term

It remains to show that the contribution of the $\boldsymbol{J}$'s for which at least one of the $\#J_a$'s is odd, is smaller than what we have above. Before we get started, we note that

$$\lambda_p(\boldsymbol{t}) = -1 + \prod_{j=1}^{2k}(1 - p^{-it_j}) = -1 + \prod_{j=1}^{2k}(p^{it_j/2} - p^{-it_j/2}) = -1 + (-4)^k \prod_{j=1}^{2k} \sin\left(\frac{t_j \log p}{2}\right)$$

whenever $t_1 + \cdots + t_{2k} = 0$, which is the case here. In particular, $-4^k \leq \lambda_p(\boldsymbol{t}) + 1 \leq 4^k$, whence

$$\left|\sum_{d_1,\ldots,d_{2k}\in\mathcal{N}} \frac{\prod_{j=1}^{2k}\mu(d_j)d_j^{-it_j}}{[d_1,\ldots,d_{2k}]}\right| = \left|\prod_{q_{A+1}<p\leq y}\left(1+\frac{\lambda_p(\boldsymbol{t})}{p}\right)\right| \ll \prod_{4^k<p\leq y}\left(1+\frac{\lambda_p(\boldsymbol{t})}{p}\right).$$

Next, we split the region of integration in (9.12) into various subsets. First, we note that, by a dyadic decomposition argument and (9.9), we have that

$$W(\boldsymbol{J}) \ll \frac{(\log T)^{2k}}{(\log R)^{L+2kA}} \max_{1\leq T_1,\ldots,T_{2k}\leq T} \frac{\Lambda(\boldsymbol{T})}{T_1\cdots T_{2k-L}} + \frac{(\log R)^{O(1)}}{T},$$

where

$$\Lambda(\boldsymbol{T}) := \int\cdots\int_{\substack{T_j\leq|t_j|+1\leq 2T_j\\ 1\leq j\leq 2k-L\\ t_{J_\ell}=0\ (\ell\in\mathcal{L})}} \prod_{4^k<p\leq y}\left(1+\frac{\lambda_p(\boldsymbol{t})}{p}\right)dt_1\cdots dt_{2k-L}.$$

We take

$$T = \exp\{(\log\log R)^2\}$$

and fix a choice of $T_1,\ldots,T_{2k-L}$ as above. We will further break the region of integration according to which sums $t_J = \sum_{j\in J} t_j$ are small. Indeed, by Lemma 9.1 the product $\prod_{q_{A+1}<p\leq y}(1+\lambda_p(\boldsymbol{t})/p)$ can become large only if there are such configurations. Note that if t_{J_1} and t_{J_2} are both small, so is any linear combination of t_{J_1} and t_{J_2}. Thus, we are naturally led to the following definition: given free variables $x_1,\ldots,x_{2k}$ and $J\subset[2k]$, we define the linear form

$$L_J(x_1,\ldots,x_{2k}) := \sum_{j\in J} x_j,$$

(thinking of the linear form as acting on the space $\mathbb{Q}^{2k}$) and, given $\mathcal{J}\subset\mathcal{S}(2k)$, we set

$$V_B(\mathcal{J}) := \left\{\sum_{I\in\mathcal{J}}\frac{a_I}{q_I}\cdot L_I \ :\ a_I, q_I\in\mathbb{Z}\cap[-B,B]\right\}$$

(thought of as a subspace in the dual of $\mathbb{Q}^{2k}$). For the simplicity of notation, we also set

$$V(\mathcal{J}) := V_\infty(\mathcal{J}) = \mathrm{Span}_{\mathbb{Q}}(\{L_I : I\in\mathcal{J}\}).$$

Since there are only finitely many linear forms L_I, $I\subset[2k]$, there is some finite $B_0 = B_0(k)$ such that, for any $\mathcal{J}$, if $L_J\in V(\mathcal{J})$, then $L_J\in V_B(\mathcal{J})$ for all $B\geq B_0$. We assume that $B\geq B_0$ from now on (we will eventually choose B to be large enough in terms of k).

We set $m = \lfloor\log\log y\rfloor$ and, to each $\boldsymbol{t}$, we associate the sets

$$\mathcal{J}_j = \mathcal{J}_j(\boldsymbol{t}) := \{I\subset[2k] : e^m|t_I| + 1\leq e^{j+1}\}\quad(0\leq j\leq m).$$

Note that if $L_I \notin V_B(\mathscr{J}_m)$, then $|t_I| > 1$. Conversely, if $L_I \in V_B(\mathscr{J}_m)$, then $|t_I| \leq eB\#\mathscr{J}_m \ll 1$. Since $t_I \ll T$ for all I, Lemma 9.1 implies that

$$\begin{aligned}\prod_{4^k<p\leq y}\Big(1+\frac{\lambda_p(\boldsymbol{t})}{p}\Big) &= \exp\Big(\sum_{I\in\mathcal{S}^*(2k)}(-1)^{\#I}\sum_{4^k<p\leq y}\Big(p^{-1-it_I}+O\Big(\frac{1}{p^2}\Big)\Big)\Big)\\ &= \exp\Big(\sum_{\substack{I\in\mathcal{S}^*(2k)\\ L_I\in V_B(\mathscr{J}_m)}}(-1)^{\#I}\min\Big\{\log\log y,\log\Big(\frac{1}{|t_I|}\Big)\Big\}\Big)(\log\log R)^{O(1)}\\ &\ll (\log\log R)^{O(1)}\prod_{\substack{I\in\mathcal{S}^*(2k)\\ L_I\in V_B(\mathscr{J}_m)}}\min\Big\{\log y,\frac{1}{|t_I|}\Big\}^{(-1)^{\#I}}.\end{aligned}$$

If $L_I \in V_B(\mathscr{J}_j)\setminus V_B(\mathscr{J}_{j-1})$, where $\mathscr{J}_{-1}=\{\emptyset\}$ so that $V_B(\mathscr{J}_{-1})=\{0\}$, then $I\notin\mathscr{J}_{j-1}$, which means that $e^m|t_I|+1>e^j$. On the other hand, since $L_I\in V_B(\mathscr{J}_j)$, then we find that $e^m|t_I|\leq B\cdot\#\mathscr{J}_j\cdot e^{j-m}$. We thus conclude that

$$\min\Big\{\log y,\frac{1}{|t_I|}\Big\}\asymp e^{m-j}\quad\text{for}\quad L_I\in V_B(\mathscr{J}_j)\setminus V_B(\mathscr{J}_{j-1}),\ 0\leq j\leq m.$$

Therefore

$$\prod_{4^k<p\leq y}\Big(1+\frac{\lambda_p(\boldsymbol{t})}{p}\Big)\ll(\log\log R)^{4^k}\prod_{j=0}^{m}e^{(m-j)F_j},$$

where

$$F_j:=\sum_{\substack{I\subset[2k]\\ L_I\in V_B(\mathscr{J}_j)\setminus V_B(\mathscr{J}_{j-1})}}(-1)^{\#I}.$$

(Here we use the fact that there are only finitely many indices j for which $\mathscr{J}_j\neq\mathscr{J}_{j-1}$.) Summing over the $\ll(\log\log R)^{O(1)}$ choices for $\mathscr{J}_0,\mathscr{J}_1,\dots,\mathscr{J}_m$, we conclude that

$$\Lambda(\boldsymbol{T})\ll(\log\log R)^{O(1)}\max_{\mathscr{J}_m\supset\cdots\supset\mathscr{J}_0\supset\{J_1,\dots,J_L\}}\Bigg(\nu(\boldsymbol{T},\mathscr{J})\cdot\prod_{j=0}^{m}e^{(m-j)F_j}\Bigg),$$

where $\nu(\boldsymbol{T},\mathscr{J})$ denotes the $(2k-L)$-dimensional Lebesgue measure of $(t_1,\dots,t_{2k-L})\in\mathbb{R}$ such that $T_j\leq|t_j|+1\leq 2T_j$ for $j\leq 2k-L$ and

$$\{I\subset[2k]:e^m|t_I|+1<e^{j+1}\}=\mathscr{J}_j\quad(0\leq j\leq m),$$

where the variables $t_{2k-L+1},\dots,t_{2k}$ are defined via the equations $t_{J_\ell}=0$ for $\ell\in\{1,\dots,2k\}$. (In particular, $\mathscr{J}_0\supset\{J_1,\dots,J_L\}$.)

Next, we use linear algebra to understand $\nu(\boldsymbol{T},\mathscr{J})$. If

$$D_j=\dim V(\mathscr{J}_j),$$

then we may find sets $I_1,\dots,I_{D_m}$ such that, for each $j\leq m$, $\{L_{I_1},\dots,L_{I_{D_j}}\}$ is a basis of $V(\mathscr{J}_j)$. We may also assume that $\{J_\ell:1\leq\ell\leq L\}$ is contained in $\{I_1,\dots,I_{D_1}\}$.

Eliminating variables from linear combinations of the asymptotic formulas

$$t_I=O(e^{j-m})\quad(I\in\{I_1,\dots,I_{D_j}\})$$

(for example, as in Gaussian elimination), yields D_j of the variables t_i, say the variables $\{t_i:i\in\mathscr{D}_j\}$ (where $\#\mathscr{D}_j=D_j$), in terms of linear combinations of the other variables, up to an error of $O(e^{j-m})$. We can also arrange the sets $\mathscr{D}_0, \dots, \mathscr{D}_L$ to satisfy $\mathscr{D}_0 \subset \cdots \subset \mathscr{D}_L$. (Recall that $\mathscr{J}_j \neq \mathscr{J}_{j-1}$ for only finitely many j's.) We may therefore prove that

$$\nu(\boldsymbol{T}, \boldsymbol{I}) \ll (\log y)^L \left(\prod_{j=0}^{m} e^{(D_j - D_{j-1})(j-m)}\right)\Bigg(\prod_{\substack{1\le j\le 2k \\ j\notin \mathscr{D}_0\cup\cdots\cup\mathscr{D}_L}} T_j\Bigg),$$

where the extra factor $(\log y)^L$ is included because we are not integrating over the variables $t_{2k-L+1}, \dots, t_{2k}$, which are fixed via the conditions $t_{J_\ell} = 0$, $1 \le \ell \le L$. By the above discussion, we find that

$$\Lambda(\boldsymbol{T}) \ll T_1 \cdots T_{2k-L} (\log y)^L (\log\log R)^{O(1)} \max_{\mathscr{J}_m \supset \cdots \supset \mathscr{J}_0 \supset \{J_1,\dots,J_L\}} \prod_{j=0}^{m} e^{(m-j)(F_j - (D_j - D_{j-1}))}.$$

We note that

$$\begin{aligned}
\sum_{j=0}^{m} (m-j)(F_j - (D_j - D_{j-1})) &= \sum_{j=0}^{m-1}\sum_{i=1}^{m-j} (F_j - (D_j - D_{j-1})) \\
&= \sum_{i=1}^{m}\sum_{j=0}^{m-i} (F_j - (D_j - D_{j-1})) \\
&= \sum_{i=1}^{m} (\mathscr{A}(V(\mathscr{J}_{m-i})) - \dim(V(\mathscr{J}_{m-i})))
\end{aligned}$$

in the notation of Section 7, provided that B is large enough. Since exp is a convex function, we have that

$$e^{\sum_{j=0}^{m}(m-j)(F_j-(D_j-D_{j-1}))} \le \frac{1}{m}\sum_{j=0}^{m-1} e^{m(\mathscr{A}(V(\mathscr{J}_j)) - \dim(V(\mathscr{J}_j)))} \asymp \frac{1}{m}\sum_{j=0}^{m-1} (\log y)^{\mathscr{A}(V(\mathscr{J}_j)) - \dim(V(\mathscr{J}_j))}.$$

We then conclude that

$$W(\boldsymbol{J}) \ll \frac{(\log\log R)^{O(1)} (\log y)^L}{(\log R)^{L+2kA}} \max_{\mathscr{J} \supset \{J_1,\dots,J_L\}} (\log y)^{\mathscr{A}(V(\mathscr{J})) - \dim(V(\mathscr{J}))}.$$

The above discussion reduces (9.6) to proving that

$$\mathscr{A}(V(\mathscr{J})) - \dim(V(\mathscr{J})) \le \binom{2k}{k} - 2k - 1 \tag{9.14}$$

whenever $\mathscr{J}$ contains the sets $J_1, \dots, J_L$ and at least one of the J_i's has an odd number of elements. This follows directly by Proposition 7.1, thus completing the proof of (9.1) and, hence, of Theorem 1.3.

10. On the anatomy of integers contributing to $\mathscr{M}_{f,2k}(R)$

This section is devoted to establishing Theorems 1.6 and 1.4. Throughout this section, given $n \in \mathbb{Z}_{\ge 1}$ and $R \ge 1$, we adopt the notations

$$\Omega(n; R) := \sum_{p^\alpha \| n,\, p \le R} \alpha \quad \text{and} \quad \Omega(n; r, R) := \sum_{p^\alpha \| n,\, r < p \le R} \alpha.$$

A key observation, that we will use several times, is that if $(a,b)=1$, then

$$\begin{aligned} M_{f_A}(ab;R) &= \sum_{d|a}\sum_{d'|b}\mu(d)\mu(d')f_A\left(\frac{\log d}{\log R}+\frac{\log d'}{\log R}\right) \\ &= \sum_{\substack{d'|b\\ d\le R}}\mu(d')\left(\frac{\log(R/d')}{\log R}\right)^A\sum_{d|a}\mu(d)f_A\left(\frac{\log d}{\log(R/d')}\right) \end{aligned} \tag{10.1}$$

by the definition of f_A.

10.1. An estimate for the logarithmic means

We start by proving a preliminary result, where the integer n is weighted by $1/n$. The transition to the uniform weights will be accomplished in the subsequent section.

THEOREM 10.1. – *Let $R\ge 2$, $k\in\mathbb{Z}_{\ge 1}$ and $A\in\mathbb{Z}_{\ge 0}$.*

(a) *If $A>\frac{1}{2k}\binom{2k}{k}-1$ and $\eta\in[\log 2/\log R,1]$, then*

$$\prod_{p\le R}\left(1-\frac{1}{p}\right)\sum_{P^+(n)\le R}\frac{\Omega(n;R^\eta)M_{f_A}(n;R)^{2k}}{n}\ll_{k,A}\frac{\eta^{2k}}{\log R}+(\log R)^{\binom{2k}{k}-2k(A+1)}\log\log R.$$

(b) *If $A\le\frac{1}{2k}\binom{2k}{k}-1$ and $1-1/\binom{2k}{k}\le v\le 2-\epsilon$, then*

$$\prod_{p\le R}\left(1-\frac{1}{p}\right)\sum_{P^+(n)\le R}\frac{v^{\Omega(n)}M_{f_A}(n;R)^{2k}}{n}\ll_{\epsilon,k,A}(\log R)^{v\binom{2k}{k}-2k(A+1)}(\log\log R)^{O_k(1)}.$$

Proof. – To ease notation, for this proof we let all implied constants depend on k, A and ϵ without explicitly stating this.

(a) First of all, we claim we may restrict our attention to small enough η. To see this, it suffices to show that

$$\sum_{P^+(n)\le R}\frac{\Omega(n;R^\delta,R)M_{f_A}(n;R)^{2k}}{n}\ll_\delta 1, \tag{10.2}$$

for any fixed $\delta>0$. Now, Hölder's inequality applied to (10.1) yields

$$M_{f_A}(ab;R)^{2k}\le\tau(b)^{2k-1}\sum_{\substack{d'|b\\ d'\le R}}\mu^2(d')\left(\frac{\log(R/d')}{\log R}\right)^{2kA}M_{f_A}(a;R/d')^{2k}. \tag{10.3}$$

Therefore, writing $n = ab$, where $b = \prod_{p^v \| n,\, R^\delta < p \le R} p^v$, we find that

$$\begin{aligned}
&\sum_{P^+(n)\le R} \frac{\Omega(n; R^\delta, R) M_{f_A}(n; R)^{2k}}{n} \\
&\le \sum_{p|b \Rightarrow R^\delta < p \le R} \frac{\tau(b)^{2k-1}\Omega(b)}{b} \sum_{\substack{d'|b\\ d'\le R}} \mu^2(d') \left(\frac{\log(R/d')}{\log R}\right)^{2kA} \sum_{P^+(a)\le R} \frac{M_{f_A}(a; R/d')^{2k}}{a} \\
&\ll \sum_{p|b \Rightarrow R^\delta < p \le R} \frac{\tau(b)^{2k-1}\Omega(b)}{b} \sum_{\substack{d'|b\\ d'\le R}} \mu^2(d') \left(\frac{\log(R/d')}{\log R}\right)^{2kA} \frac{\log R}{\log(R/d')} \ll_\delta 1
\end{aligned}$$

by Theorem 1.3, since $A > \frac{1}{2k}\binom{2k}{k} - 1$ and so $\mathcal{E}_{k,A} = -1$ and $A \ge 1$. This proves (10.2), so for the rest of the proof, we may assume that η is sufficiently small.

Expanding $\Omega(n, R^\eta)$, we have that

$$\begin{aligned}
S &:= \prod_{p\le R}\left(1 - \frac{1}{p}\right) \sum_{P^+(n)\le R} \frac{\Omega(n; R^\eta) M_{f_A}(n; R)^{2k}}{n} \\
&= \prod_{p\le R}\left(1 - \frac{1}{p}\right) \sum_{\substack{q\le R^\eta\\ j\ge 1}} \frac{j}{q^j} \sum_{\substack{P^+(m)\le R\\ q\nmid m}} \frac{M_{f_A}(mq^j; R)^{2k}}{m} \\
&\le \prod_{p\le R}\left(1 - \frac{1}{p}\right) \sum_{q\le R^\eta} \frac{q}{(q-1)^2} \sum_{P^+(m)\le R} \frac{\left(\sum_{d|m} \mu(d)\left\{ f_A\left(\frac{\log d}{\log R}\right) - f_A\left(\frac{\log(qd)}{\log R}\right)\right\}\right)^{2k}}{m},
\end{aligned}$$

where we used (10.1) with $a = q^j$ and $b = m$. Expanding the $2k$-th power, we find that

$$S \le \sum_{q\le R^\eta} \frac{q}{(q-1)^2} \sum_{\substack{P^+(d_j)\le R\\ 1\le j\le 2k}} \frac{\prod_{j=1}^{2k} \mu(d_j)\left\{ f_A\left(\frac{\log d_j}{\log R}\right) - f_A\left(\frac{\log(qd_j)}{\log R}\right)\right\}}{[d_1, \dots, d_{2k}]}.$$

Here $A \ge 1$, so that $\widehat{(f_A)}(s) = A! R^s/(s^{A+1}(\log R)^A)$ decays fast, and Perron inversion implies that

(10.4)

$$S \le \sum_{q\le R^\eta} \frac{q}{(q-1)^2} \int\!\!\cdots\!\!\int_{\substack{\mathrm{Re}(s_j)=1/\log R\\ 1\le j\le 2k}} \sum_{d_1,\dots,d_{2k}\ge 1} \frac{\prod_{j=1}^{2k} \mu(d_j) d_j^{-s_j}}{[d_1, \dots, d_{2k}]} \prod_{j=1}^{2k} \frac{A! R^{s_j}(1 - q^{-s_j})}{(\log R)^A s_j^{A+1}} \mathrm{d}s_1 \cdots \mathrm{d}s_{2k}.$$

The above integral is amenable to the methods of Section 8. Precisely, we note that the integrand is of the form

$$A!^{2k} \left(\frac{\log q}{\log R}\right)^{2k} \frac{P(\boldsymbol{s}) R^{s_{[2k]}}}{(\log R)^{2k(A-1)}} \prod_{I\in \mathcal{S}^*(2k)} \zeta^{e_I}(1 + s_I) \prod_{j=1}^{2k} \frac{(1 - q^{-s_j})/(s_j \log q)}{(s_j \zeta(1 + s_j))^A},$$

where $P(\boldsymbol{s})$ is as in Section 8, $e_I = +1$ for $I \in \mathcal{S}^+(2k)$, $e_I = -1$ for $I \in \mathcal{S}^-(2k)$ with $\#I > 1$, and $e_I = A - 1 \ge 0$ for $\#I = 1$. If $q \le R^\delta$ with δ small enough, then the argument leading to (8.1) yields that

$$\begin{aligned}(10.5)\quad &\int_{\substack{\mathrm{Re}(s_j)=1/\log R\\ 1\le j\le 2k}}\cdots\int \sum_{d_1,\dots,d_{2k}\ge 1} \frac{\prod_{j=1}^{2k}\mu(d_j)d_j^{-s_j}}{[d_1,\dots,d_{2k}]}\prod_{j=1}^{2k}\frac{A!R^{s_j}(1-q^{-s_j})}{(\log R)^A s_j^{A+1}}\mathrm{d}s_1\cdots\mathrm{d}s_{2k}\\ &\ll \frac{(\log q)^{2k}}{(\log R)^{2k+1}}+(\log R)^{\binom{2k}{k}-2k(A+1)},\end{aligned}$$

with the first term coming from Case 1a and the second one from Case 2. Inserting the above inequality into (10.4) completes the proof of part (a).

(b) We will use a variation of the argument of Section 9. The fact that Proposition 7.1 requires $s_{[2k]} = 0$ complicates the proof; otherwise, a direct application of the method of Section 9.5 would be possible.

Call S' the sum in question. As usual, we may replace f_0 by a sufficiently smooth function h. So write $g = h$ if $A = 0$, and $g = f_A$ otherwise. Note that, since $M_g(n; R)$ depends only on the square-free kernel of n, we have that

$$S' = \prod_{p\le R}\left(1-\frac{1}{p}\right)\sum_{P^+(n)\le R}\frac{\mu^2(n)v^{\omega(n)}}{\varphi_v(n)}M_{f_A}(n;R)^{2k}$$

with $\varphi_v(n) = \prod_{p|n}(p-v)$. Set $y = R^{c/\log\log R}$ for a small enough but fixed c. Given an integer n, we decompose it as $n = ab$ with $P^+(a) \le y < P^-(b)$. If $\omega(a) \le A + 1$, then $M_{f_A}(n; R) \ll 4^{k\omega(b)}$, and we immediately find that such n's contribute at most $\ll (\log\log R)^{O(1)}/\log R$. Otherwise, we sum over all possible choices $a = qa'$ with $\omega(q) = A+1$ to deduce that

$$(10.6)\qquad S' \le v^{A+1}\sum_{\substack{P^+(q)\le y\\ \omega(q)=A+1}}\frac{\mu^2(q)}{\varphi_v(q)}S'(q) + O\left(\frac{(\log\log R)^{O(1)}}{\log R}\right),$$

where

$$S'(q) := \prod_{p\le R}\left(1-\frac{1}{p}\right)\sum_{\substack{P^+(a)\le y\\ p|b\,\Rightarrow\, y<p\le R\\ (a',q)=1}}\frac{\mu^2(a'b)v^{\omega(a'b)}}{\varphi_v(a'b)}M_{f_A}(qa'b;R)^{2k}.$$

Next, we apply (10.3) with $a = a'q$ to find that

$$S'(q) \le \prod_{p\le R}\left(1-\frac{1}{p}\right)\sum_{\substack{P^+(a')\le y\\ p|b\,\Rightarrow\, y<p\le R\\ (a',q)=1}}\frac{\mu^2(a'b)v^{\omega(a'b)}\tau(b)^{2k-1}}{\varphi_v(a'b)}\sum_{\substack{d|b\\ d\le R}}\left(\frac{\log(R/d)}{\log R}\right)^{2kA}M_{f_A}(a'q;R/d)^{2k}.$$

Note that $\varphi_v(n) \gg n/(\log\log n)^v$. Hence, fixing a' and d, and then summing over b yields that

$$S'(q) \ll (\log\log R)^{O(1)}\sum_{\substack{d\le R\\ P^-(d)>y}}\frac{4^{k\Omega(d;y,R)}}{d}S''(d,q) \le (\log\log R)^{O(1)}\sum_{\substack{d\le R\\ P^-(d)>y}}\frac{S''(d,q)}{d^{1-2k/\log y}},$$

where

$$S''(d,q) := \prod_{p\le R}\left(1-\frac{1}{p}\right)\sum_{\substack{P^+(a')\le y\\(a',q)=1}}\frac{\mu^2(a')v^{\omega(a')}}{\varphi_v(a')}\left(\frac{\log(R/d)}{\log R}\right)^{2kA} M_{f_A}(a'q;R/d)^{2k}$$

$$\le \prod_{p\le R}\left(1-\frac{1}{p}\right)\sum_{\substack{P^+(a)\le y\\(a,q)=1}}\frac{v^{\Omega(a)}}{a}\left(\frac{\log(R/d)}{\log R}\right)^{2kA} M_{f_A}(aq;R/d)^{2k}.$$

Before we proceed to the estimation of $S''(d,q)$, we note that

$$S'(q) \ll (\log\log R)^{O(1)}\sum_{d\le R}\frac{(\lambda^+ * 1)(d)}{d^{1-2k/\log y}}S''(d,q), \tag{10.7}$$

where $(\lambda^+(m))_{m\le R^\delta}$ is an upper bound sieve with δ small enough, constructed using the fundamental lemma of sieve methods, taking $\kappa = 1$ in [12, Lemma 6.3, p. 159]. Then the Dirichlet series $\sum_d (1*\lambda^+)(d)/d^s$ has a simple pole at $s = 1$ of residue $\sum_m \lambda^+(m)/m \asymp (\log\log R)^{O(1)}/\log R$.

Next, if $q = p_1^{r_1}\cdots p_{A+1}^{r_{A+1}}$ is the prime factorisation of q, then (10.1) implies that

$$\left(\frac{\log(R/d)}{\log R}\right)^{A} M_{f_A}(aq;R/d) = \sum_{d'|aq}\mu(d')f_A\left(\frac{\log(dd')}{\log R}\right) = \sum_{d''|a}\mu(d'')w_q\left(\frac{\log(dd'')}{\log R}\right)$$

with

$$w_q(x) := \sum_{J\subset[A+1]}(-1)^{\#J} f_A\left(x+\frac{\sum_{j\in J}\log p_j}{\log R}\right).$$

As usual, if $A = 0$, we may replace f_0 by a sufficiently smooth function h at the cost of a small error. Letting $g = h$ when $A = 0$ and $g = f_A$ otherwise, and letting W_q have the same definition as w_q with g in place of f_A, we find that

$$S''(d,q) \le \prod_{p\le R}\left(1-\frac{1}{p}\right)\sum_{\substack{P^+(a)\le y\\(a,q)=1}}\frac{v^{\Omega(a)}}{a}\left(\sum_{f|a}\mu(f)W_q\left(\frac{\log(df)}{\log R}\right)\right)^{2k} + O\left(\frac{1}{\log R}\right)$$

$$= \frac{\prod_{p\le R}(1-1/p)}{\prod_{p\le y,\,p\nmid q}(1-v/p)}\sum_{\substack{(f_j,q)=1\\1\le j\le 2k}}\frac{v^{\Omega([f_1,\dots,f_{2k}])}}{[f_1,\dots,f_{2k}]}\prod_{j=1}^{2k}\mu(f_j)W_q\left(\frac{\log(df_j)}{\log R}\right) + O\left(\frac{1}{\log R}\right).$$

We apply Mellin inversion $2k$ times to find that

$$S''(d,q) \le \frac{\prod_{p\le R}(1-1/p)}{(2\pi i)^{2k}\prod_{p\le y,\,p\nmid q}(1-v/p)}\int\cdots\int_{\substack{\mathrm{Re}(s_j)=4k/\log y\\|\mathrm{Im}(s_j)|\le T\\1\le j\le 2k}}\sum_{\substack{P^+(f_j)\le y,\,(f_j,q)=1\\1\le j\le 2k}}\frac{v^{\Omega([f_1,\dots,f_{2k}])}\prod_{j=1}^{2k}\mu(f_j)f_j^{-s_j}}{[f_1,\dots,f_{2k}]}$$

$$\times\prod_{j=1}^{2k}\frac{\widehat{g}_R(s_j)}{d^{s_j}}\prod_{a=1}^{A+1}(1-p_a^{-s_j})\mathrm{d}s_j + O\left(\frac{1}{\log R}\right), \text{ where } T := \exp\{(\log\log R)^2\}.$$

Together with (10.7), this implies that

$$S'(q) \le \frac{(\log\log R)^{O(1)} \prod_{p\le R}(1-1/p)}{(2\pi i)^{2k} \prod_{p\le y,\, p\nmid q}(1-v/p)} \int\cdots\int_{\substack{\mathrm{Re}(s_j)=4k/\log y\\ |\mathrm{Im}(s_j)|\le T\\ 1\le j\le 2k}} \sum_{\substack{P^+(f_j)\le y\\ (f_j,q)=1\\ 1\le j\le 2k}} \frac{v^{\Omega([f_1,\dots,f_{2k}])}\prod_{j=1}^{2k}\mu(f_j)f_j^{-s_j}}{[f_1,\dots,f_{2k}]}$$
$$\times P(1+s_{[2k]}-2k/\log y)\prod_{j=1}^{2k}\widehat{g}_R(s_j)\prod_{a=1}^{A+1}(1-p_a^{-s_j})\mathrm{d}s_j + O\left(\frac{1}{\log R}\right),$$

where

$$P(s) := \sum_{d=1}^{\infty}\frac{(\lambda^+ * 1)(d)}{d^s} = \zeta(s)\sum_{m\le R^\delta}\frac{\lambda^+(m)}{m^s}.$$

We fix $s_1,\dots,s_{2k-1}$ and move s_{2k} to the line $\mathrm{Re}(s_{2k}) = -8k^2/\log y$ to pick up the pole at $s_{[2k]} = 2k/\log y$. If c is small enough in the definition of y, and the same is true for δ, the complementary contours contribute $\ll 1/\log R$. There are no other poles, since the factors $1-p_a^{-s_j}$ are annihilating the poles of $\widehat{g}_R(s_j)$. We conclude that

$$S'(q) \ll \frac{(\log\log R)^{O(1)}}{(\log R)^{2kA+2-v}} \int\cdots\int_{\substack{\mathrm{Re}(s_j)=4k/\log y\\ |\mathrm{Im}(s_j)|\le T\\ 1\le j<2k\\ s_{[2k]}=2k/\log y}} \left| \sum_{\substack{P^+(f_j)\le y\\ 1\le j\le 2k}} \frac{v^{\Omega([f_1,\dots,f_{2k}])}\prod_{j=1}^{2k}\mu(f_j)f_j^{-s_j}}{[f_1,\dots,f_{2k}]}\right|$$
$$\times\left(\prod_{j=1}^{2k}\frac{1}{|s_j|^{A+1}}\right)|\mathrm{d}s_1\cdots\mathrm{d}s_{2k-1}| + \frac{1}{\log R}.$$

Recall the notation $\lambda_p(\boldsymbol{t})$ defined in (9.13), and combine the above with (10.6) to find that

$$S' \ll \frac{(\log\log R)^{O(1)}}{(\log R)^{2kA+2-v}} \int\cdots\int_{\substack{-T\le t_j\le T\ (1\le j<2k)\\ t_{[2k]}=0}} \prod_{p\le y}\left(1+\frac{v\cdot\lambda_p(\boldsymbol{t})+(A+1)\sum_{j=1}^{2k}\cos(t_j\log p)}{p}\right)$$
$$\times\left(\prod_{j=1}^{2k}\frac{(\log(2+|t_j|))^{2k(A+1)}}{(1+|t_j|)^{A+1}}\right)\mathrm{d}t_1\cdots\mathrm{d}t_{2k-1} + \frac{(\log\log R)^{O(1)}}{\log R},$$

where we used Lemma 9.1, which implies that $\exp\{\sum_{p\le y}\cos(t\log p)/p\} \gg \frac{1}{\log(2+|t|)}\cdot\frac{|t|+1}{|t|+1/\log y}$ for $t\in[-T,T]$. Then, following the arguments of Section 9.5 (with $L=1$), and recalling the notations $V(\mathscr{J})$ and $\mathscr{A}(V(\mathscr{J}))$, we find that

$$S' \ll \max_{\mathscr{J}\subset\mathscr{J}(2k)}(\log R)^{v(1+\mathscr{A}(V(\mathscr{J})))-\dim(V(\mathscr{J}))+(A+1)U(\mathscr{J})-2kA-1}(\log\log R)^{O(1)},$$

where

$$U(\mathscr{J}) := \#\{1\le j\le 2k : L_{\{j\}}\in V(\mathscr{J})\}.$$

If $U(\mathscr{J})\ge 1$, then Proposition 7.1(a) implies that $\mathscr{A}(V(\mathscr{J})) = -1$, and the exponent of $\log R$ is then

$$(A+1)\cdot U(\mathscr{J}) - \dim(V(\mathscr{J})) - 2kA - 1 \le A\cdot U(\mathscr{J}) - 2kA - 1 \le -1,$$

since we clearly have that $2k \geq \dim(V(\mathscr{J})) \geq U(\mathscr{J})$. Assume now that $U(\mathscr{J}) = 0$. If $\mathscr{A}(V(\mathscr{J})) = \binom{2k}{k} - 1$ and $\dim(V(\mathscr{J})) = 2k-1$, then the exponent of $\log R$ is $v\binom{2k}{k} - 2k(A+1)$. Finally, if this is not the case, then Proposition 7.1 (together with the argument in the end of Section 9.5) implies that

$$\begin{aligned} v(1 + \mathscr{A}(V(\mathscr{J}))) - \dim(V(\mathscr{J})) - 1 &\leq \max\{0, v-1\}\binom{2k}{k} + \mathscr{A}(V(\mathscr{J})) - \dim(V(\mathscr{J})) \\ &\leq \max\{0, v-1\}\binom{2k}{k} + \binom{2k}{k} - 2k - 1 \\ &\leq v\binom{2k}{k} - 2k, \end{aligned}$$

by our assumption that $v \geq 1 - 1/\binom{2k}{k}$. This completes the proof of the theorem. □

10.2. From logarithmic weights to uniform weights

In this section, we show how to go from Theorem 10.1 to the analogous result for the regular mean value and then prove Theorem 1.6.

THEOREM 10.2. – *Let $x \geq R \geq 2$, $k \in \mathbb{Z}_{\geq 1}$, $A \in \mathbb{Z}_{\geq 0}$ and $\epsilon \in (0, 1/2)$.*

(a) *Assume that $A > \frac{1}{2k}\binom{2k}{k} - 1$. Then uniformly for $\eta \in [\log 2/\log R, 1]$, we have*

$$\frac{1}{x}\sum_{n \leq x} \Omega(n; R^{\eta}) M_{f_A}(n; R)^{2k} \ll_{k,A} \frac{\eta}{\log R}.$$

(b) *If $A \leq \frac{1}{2k}\binom{2k}{k} - 1$ and $1 - 1/\binom{2k}{k} + \epsilon \cdot \mathbf{1}_{k=1} \leq v \leq 2 - \epsilon$, then*

$$\frac{1}{x}\sum_{n \leq x} v^{\Omega(n;R)} M_{f_A}(n; R)^{2k} \ll_{k,A,\epsilon} (\log R)^{v\binom{2k}{k} - 2k(A+1)} (\log\log R)^{O(1)}.$$

Proof. – We start by proving a preparatory estimate. Our goal is to show that there is some constant $C = C(k)$ such that

$$\sum_{n \leq x} M_{f_A}(n; R)^{2k} \leq \frac{Cx}{\log R} \quad (x \geq R > 1). \tag{10.8}$$

When $x \leq 2^H$ for some fixed $H \in \mathbb{Z}_{\geq 1}$ that will be taken large enough in terms of k and A, relation (10.8) trivially holds by taking C to be large enough in terms of H. Assume now that (10.8) holds for all $x \leq 2^h$, where $h \geq H$. We wish to prove that (10.8) is also true when $x \in (2^h, 2^{h+1}]$.

We follow a variation of the argument leading to Theorem III.3.5 in [22, p.308]: note that

$$\sum_{n \leq x} M_{f_A}(n; R)^{2k} = \sum_{n \leq x} M_{f_A}(n; R)^{2k} \frac{\log(x/n)}{\log x} + \sum_{n \leq x} M_{f_A}(n; R)^{2k} \frac{\log n}{\log x}. \tag{10.9}$$

For the first sum, we bound $\log(x/n)$ by x/n to give

$$\begin{aligned}\sum_{n\le x} M_{f_A}(n;R)^{2k}\frac{\log(x/n)}{\log x} &\le \sum_{n\le x} M_{f_A}(n;R)^{2k}\frac{x}{n\log x}\\ &\ll \frac{x}{\log R}\sum_{P^+(n)\le R}\frac{M_{f_A}(n;R)^{2k}}{n}\\ &\ll \frac{x}{\log R},\end{aligned}\tag{10.10}$$

by Theorem 1.3. In the second sum, we write $\log n = \sum_{p^j\|n} j\log p$ to find that

$$\sum_{n\le x} M_{f_A}(n;R)^{2k}\frac{\log n}{\log x} = \sum_{p^j\le x}\frac{j\log p}{\log x}\sum_{\substack{m\le x/p^j\\ p\nmid m}} M_{f_A}(mp^j;R)^{2k}.$$

Since

$$M_{f_A}(mp^j;R) = M_{f_A}(m;R) - \mathbf{1}_{p<R}\left(\frac{\log(R/p)}{\log R}\right)^A M_{f_A}(m;R/p)$$

by (10.1), Minkowski's inequality implies that

$$\sum_{n\le x} M_{f_A}(n;R)^{2k}\frac{\log n}{\log x} \le \left(S_1^{\frac{1}{2k}} + S_2^{\frac{1}{2k}}\right)^{2k},$$

where

$$S_1 := \sum_{p^j\le x}\frac{j\log p}{\log x}\sum_{m\le x/p^j} M_{f_A}(m;R)^{2k}.$$

and

$$S_2 := \sum_{\substack{p^j\le x\\ p<R}}\frac{j\log p}{\log x}\left(\frac{\log(R/p)}{\log R}\right)^{2kA}\sum_{m\le x/p^j} M_{f_A}(m;R/p)^{2k}.$$

For S_1, we note that

$$S_1 = \sum_{m\le x} M_{f_A}(m;R)^{2k}\sum_{p^j\le x/m}\frac{j\log p}{\log x} \ll \frac{x}{\log x}\sum_{m\le x}\frac{M_{f_A}(m;R)^{2k}}{m} \ll \frac{x}{\log R},$$

by (10.10). Finally, we need to bound S_2. First, we bound its subsum with $j\ge 2$. We have that

$$\begin{aligned}&\sum_{\substack{p^j\le x\\ j\ge 2\\ p<R}}\frac{j\log p}{\log x}\left(\frac{\log(R/p)}{\log R}\right)^{2kA}\sum_{m\le x/p^j} M_{f_A}(m;R/p)^{2k}\\ &\quad\le \sum_{\substack{p^j\le x\\ j\ge 2\\ p<R}}\frac{j\log p}{\log x}\left(\frac{\log(R/p)}{\log R}\right)^{2kA}\sum_{m\le x/p^j} M_{f_A}(m;R/p)^{2k}\frac{x}{p^j m}\end{aligned}$$

$$\ll x \sum_{\substack{p^j \le x \\ j \ge 2 \\ p < R}} \frac{j \log p}{p^j \log x} \left(\frac{\log(R/p)}{\log R} \right)^{2kA} \cdot \frac{\log x}{\log(R/p)},$$

by (10.10) with R replaced by R/p. Since $A \ge 1$ here, we find that the above is

$$\ll \sum_{\substack{p^j \le x \\ j \ge 2}} \frac{j \log p}{p^j} \cdot \frac{x}{\log R} \ll \frac{x}{\log R},$$

where the implied constant is independent of C.

Finally, we need to bound the part of S_2 with $j = 1$. We note that $x/p \le 2^h$ and that $R/p \le x/p$, so we may apply the induction hypothesis. This gives a bound

$$\le \sum_{p < R} \frac{\log p}{\log x} \left(\frac{\log(R/p)}{\log R} \right)^{2kA} \cdot \frac{Cx}{p \log(R/p)} \le \frac{Cx}{(\log x)(\log R)^2} \sum_{p < R} \frac{(\log p)(\log(R/p))}{p}$$
$$\le \frac{2C}{3} \cdot \frac{x}{\log R},$$

provided that H (and hence x) is big enough, where we used our assumptions that $A \ge 1$ and $x \ge R$. Combining the above, and assuming that C is big enough in terms of k and A completes the inductive step and hence the proof of (10.8).

We now turn to proving part (a), that is bounding the sum

$$S(x, R, Q) := \sum_{n \le x} \Omega(n; Q) M_{f_A}(n; R)^{2k}.$$

The proof is similar to the proof of (10.8), so we only give the main technical twists: we use induction on the dyadic interval in which x lies to prove that there is some constant $C' = C'(k, \epsilon)$ such that

$$S(x, R, Q) \le \frac{C' x \log Q}{(\log R)^2} \quad (x \ge R \ge Q \ge 2). \tag{10.11}$$

When $x \le 2^H$, for some fixed $H \in \mathbb{Z}_{\ge 1}$ that will be taken large enough in terms of k, A and ϵ, this trivially holds by taking C' to be large enough in terms of H. Assume now that (10.8) holds for all $x \le 2^h$, where $h \ge H$. We will prove that it also holds for $x \in (2^h, 2^{h+1}]$. Note that the analogues of (10.9) and (10.10) hold here as well, so let us focus on understanding the sum

$$T := \sum_{n \le x} \Omega(n; Q) M_{f_A}(n; R)^{2k} \frac{\log n}{\log x}.$$

Fix a large integer N and call T_1 the portion of this sum with $\Omega(n; Q) > 2N$ and T_2 the remaining sum. Writing $\log n = \sum_{p^j \| n} j \log p$, we find that

$$T_1 = \sum_{p^j \le x} \frac{j \log p}{\log x} \sum_{\substack{m \le x/p^j,\ p \nmid m \\ \Omega(mp^j; Q) > 2N}} \Omega(mp^j; Q) M_{f_A}(mp^j; R)^{2k}.$$

If T_1' is the part with $\Omega(mp^j;Q)\le 2j$ and T_1'' is the rest, then

$$\begin{aligned}
T_1' &\le \sum_{\substack{p^j\le x\\ j>N}} \frac{2j^2\log p}{\log x} \sum_{\substack{m\le x/p^j\\ p\nmid m}} M_{f_A}(mp^j;R)^{2k}\frac{x}{p^j m}\\
&\ll x \sum_{\substack{p^j\le x,\, p\le e^{\sqrt{\log R}}\\ j>N}} \frac{j^2\log p}{p^j\log R} \sum_{\substack{P^+(m)\le R\\ p\nmid m}} \frac{M_{f_A}(mp;R)^{2k}}{m} + \frac{x(\log R)^{O(1)}}{e^{\sqrt{\log R}}}\\
&\ll_N x \sum_{\substack{p^j\le x,\, p\le e^{\sqrt{\log R}}\\ j>N}} \frac{j^2\log p}{p^j\log R}\cdot\left\{\left(\frac{\log p}{\log R}\right)^{2k} + (\log R)^{1+\binom{2k}{k}-2k(A+1)}\right\}\\
&\ll \frac{x}{(\log R)^{2k+1}} + x(\log R)^{\binom{2k}{k}-2k(A+1)},
\end{aligned}$$

where the second to last bound follows from (10.5). Finally, in the range of T_1'', we note that $\Omega(m;Q)\ge(\Omega(m;Q)+j)/2>N$, so that $\Omega(mp^j;Q)\le j(1+\Omega(m;Q))\le(1+1/N)\cdot j\cdot\Omega(m;Q)$. Therefore,

$$\begin{aligned}
T_1 &\le \frac{N+1}{N}\sum_{p^j\le x}\frac{j^2\log p}{\log x}\sum_{\substack{m\le x/p^j,\, p\nmid m\\ \Omega(mp^j;Q)>2N}} \Omega(m;Q)M_{f_A}(mp^j;R)^{2k}\\
&\quad + O\left(\frac{x}{(\log R)^{2k+1}} + x(\log R)^{\binom{2k}{k}-2k(A+1)}\right).
\end{aligned}\tag{10.12}$$

Next, we need to bound

$$T_2 = \sum_{\substack{n\le x\\ \Omega(n;Q)\le 2N}} \Omega(n;Q)M_{f_A}(mp^j;R)^{2k}\frac{\log n}{\log x}.$$

If $Q>R^{1/(2N^2)}$, then we simply note that

$$T_2\le 2N\sum_{n\le x}M_{f_A}(n;R)^{2k}\ll_N \frac{x}{\log R}\ll_N\frac{x\log Q}{(\log R)^2}$$

by (10.8). Here the implied constant depends on N but does not depend on C'.

On the other hand, if $Q\le R^{1/2N^2}$, then we have that $\sum_{p^j\| n,\, p\le Q} j\log p\le(\log x)/N$, so that

$$\begin{aligned}
T_2 &\le \frac{S(x,R,Q)}{N} + \sum_{\substack{n\le x\\ \Omega(n;Q)\le 2N}} \Omega(n;Q)M_{f_A}(n;R)^{2k}\sum_{p^j\| n,\, p>Q}\frac{j\log p}{\log x}\\
&\le \frac{S(x,R,Q)}{N} + \sum_{\substack{p^j\le x\\ p>Q}}\frac{j\log p}{\log x}\sum_{\substack{m\le x/p^j,\, p\nmid m\\ \Omega(mp^j;Q)\le 2N}}\Omega(m;Q)M_{f_A}(mp^j;R)^{2k}.
\end{aligned}$$

Combining the above inequality and (10.12), we deduce that

$$T\le\frac{S(x,R,Q)}{N}+\frac{N+1}{N}\sum_{p^j\le x}\frac{j^2\log p}{\log x}\sum_{m\le x/p^j,\, p\nmid m}\Omega(m;Q)M_{f_A}(mp^j;R)^{2k}+O_N\left(\frac{x\log Q}{(\log R)^2}\right).$$

We thus conclude that

$$S(x,R,Q) \le \frac{N+1}{N-1} \sum_{p^j \le x} \frac{j^2 \log p}{\log x} \sum_{m \le x/p^j,\, p \nmid m} \Omega(m;Q) M_{f_A}(mp^j;R)^{2k} + O_N\left(\frac{x \log Q}{(\log R)^2}\right).$$

We can bound the sum on the right hand side in an entirely analogous way to the proof of (10.8). Choosing N sufficiently large, and then C' large enough in terms of N completes the inductive step and thus the proof of (10.11). This proves part (a) of the theorem.

(b) The proof of part (b) is, for the most part, similar to the proof of (10.8). An important detail is that, after applying the induction hypothesis, we use the fact that

$$\begin{aligned} &\sum_{p<R} \frac{v \log p}{\log x} \left(\frac{\log(R/p)}{\log R}\right)^{2kA} \cdot \frac{(\log(R/p))^{v\binom{2k}{k}-2k(A+1)} (\log\log(R/p))^D}{p} \\ &\sim \frac{v}{v\binom{2k}{k} - 2k + 1} \cdot \frac{\log R}{\log x} \cdot (\log R)^{\binom{2k}{k}-2k(A+1)} (\log\log R)^D \end{aligned} \tag{10.13}$$

for any fixed $D \ge 0$, as $R \to \infty$. This is sufficient when $k \ge 2$, because

$$\frac{v}{v\binom{2k}{k} - 2k + 1} \le \frac{1 - 1/\binom{2k}{k}}{\binom{2k}{k} - 2k} \le \frac{1}{2}$$

in this case.

However, when $k = 1$, the situation is more tricky. First of all, we note that the above argument allows to establish the theorem for $x \ge R^{2v/(2v-1)}$. (Note that if $p < R$ and $x \ge R^{2v/(2v-1)}$, then we also have that $x/p \ge (R/p)^{2v/(2v-1)}$, so that the inductive hypothesis can be applied.) Finally, it remains to treat the case when $x \le R^{2v/(2v-1)}$. We then observe that $\Omega(n; R^\delta, R) \ll_{\epsilon,\delta} 1$, for any fixed δ. It would thus suffice to prove that

$$\sum_{n \le x} v^{\Omega(n;R^\delta)} \delta^{\Omega(n;R^\delta,R)} M_{f_0}(n;R)^2 \le C'' x (\log R)^{2v-2} (\log\log R)^D \quad (x \ge R \ge 2), \tag{10.14}$$

for some appropriate constants $C'', D > 0$. Then, in place of (10.13), we note that

$$\begin{aligned} &\sum_{p<R} \frac{(v \cdot \mathbf{1}_{p \le R^\delta} + \delta \cdot \mathbf{1}_{R^\delta < p \le R}) \log p}{\log x} \left(\frac{\log(R/p)}{\log R}\right)^{2kA} \cdot \frac{(\log(R/p))^{v\binom{2k}{k}-2k(A+1)} (\log\log(R/p))^D}{p} \\ &\quad < \frac{1}{2} \cdot (\log R)^{\binom{2k}{k}-2k(A+1)} (\log\log R)^D \end{aligned}$$

as $R \to \infty$, as long as δ is small enough. This allows us to complete the inductive step and establish (10.14), thus completing the proof of the theorem. □

Given the above result, proving Theorem 1.6 is quite easy:

Proof of Theorem 1.6. – (a) This an immediate consequence of Theorem 10.2(a).

(b) We use Rankin's trick: Given a small $\alpha > 0$ and $1 \le v \le 3/2$, we have that

$$\sum_{\substack{n \le x \\ \Omega(n;R) > \binom{2k}{k}(1+\alpha)\log\log R}} M_{f_A}(n;R)^{2k} \le \sum_{n \le x} v^{\Omega(n;R) - \binom{2k}{k}(1+\alpha)\log\log R} M_{f_A}(n;R)^{2k}.$$

Theorem 10.2(b) then implies that

$$\sum_{\substack{n\le x\\ \Omega(n;R)>\binom{2k}{k}(1+\alpha)\log\log R}} M_f(n;R)^{2k} \ll (\log R)^{\binom{2k}{k}-2k(A+1)+\binom{2k}{k}(v-1-(1+\alpha)\log v)},$$

provided that v is close enough to 1. We optimize this by choosing $v = 1+\alpha$, so that $1-v+(1+\alpha)\log v = \int_1^{1+\alpha}(\log t)\mathrm{d}t > 0$.

Similarly, we have that

$$\sum_{\substack{n\le x\\ \Omega(n;R)<\binom{2k}{k}(1-\alpha)\log\log R}} M_{f_A}(n;R)^{2k} \le \sum_{n\le x} v^{\Omega(n;R)-\binom{2k}{k}(1-\alpha)\log\log R} M_{f_A}(n;R)^{2k},$$

for any $1-1/\binom{2k}{k}+\epsilon\cdot\mathbf{1}_{k=1} v \le 1$. Applying Theorem 10.2(b) and choosing $v = 1-\alpha$ for small enough α completes the proof of the theorem. □

10.3. Estimates for general weight functions

It is not so hard to go from estimates for the moments of $M_{f_A}(n;R)$ to the moments of $M_f(n;R)$ for a general weight function f. The following lemma provides the key link.

LEMMA 10.3. – *Let $f:\mathbb{R}\to\mathbb{R}$ be supported in $(-\infty,1]$. Assume further that $f\in C^A(\mathbb{R})$ and that all functions $f, f',\dots,f^{(A)}$ are uniformly bounded for some integer $A\ge 1$. Then*

$$M_f(n;R)^{2k} \ll_{A,k,f} \int_{\frac{\log 2}{\log R}}^1 u^{2k(A-1)} M_{f_{A-1}}(n;R^u)^{2k}\mathrm{d}u + \frac{1}{(\log R)^{2kA}}.$$

Proof. – Since $f(x)=0$ for $x>0$ and $f\in C^A(\mathbb{R})$, we must have that $f^{(j)}(1)=0$ for $j\le A$. Taylor's theorem with the integral form of the remainder term implies that

$$f(x) = \int_1^x \frac{f^{(A)}(u)}{(A-1)!}(x-u)^{A-1}\mathrm{d}u = \frac{(-1)^A}{(A-1)!}\int_{[x,1]} f^{(A)}(u)(u-x)^{A-1}\mathrm{d}u,$$

for all x, since both sides vanish if $x>1$. Therefore,

$$\begin{aligned} M_f(n;R) &= \frac{(-1)^A}{(A-1)!}\sum_{d|n}\mu(d)\int_{\frac{\log d}{\log R}\le u\le 1} f^{(A)}(u)\left(u-\frac{\log d}{\log R}\right)^{A-1}\mathrm{d}u \\ &= \frac{(-1)^A}{(A-1)!}\int_0^1 f^{(A)}(u)\sum_{\substack{d|n\\ d\le R^u}}\mu(d)\left(u-\frac{\log d}{\log R}\right)^{A-1}\mathrm{d}u \\ &= \frac{(-1)^A}{(A-1)!}\int_{\frac{\log 2}{\log R}}^1 f^{(A)}(u)u^{A-1}M_{f_{A-1}}(n;R^u)\mathrm{d}u + O\left(\frac{1}{(\log R)^A}\right), \end{aligned}$$

by noting that $d=1$ if $d\le R^u<2$. Hölder's inequality then completes the proof. □

We use the above lemma to show the analogue of Theorem 10.2 for general weight functions f.

THEOREM 10.4. – *Let $k\in\mathbb{Z}_{\ge 1}$, $x\ge R\ge 2$ and $f:\mathbb{R}\to\mathbb{R}$ be supported in $(-\infty,1]$. Assume further that $f\in C^A(\mathbb{R})$ and that all functions $f,f',\dots,f^{(A)}$ are uniformly bounded for some integer $A\ge 1$, and fix some $\epsilon\in(0,1)$.*

(a) *Let* $A > \frac{1}{2k}\binom{2k}{k}$. *Uniformly for* $\eta \in [\log 2/\log R, 1]$, *we have*

$$\frac{1}{x}\sum_{n\le x}\Omega(n;R^\eta)M_f(n;R)^{2k} \ll \frac{\eta}{\log R}.$$

(b) *If* $A \le \frac{1}{2k}\binom{2k}{k}$ *and* $1-1/\binom{2k}{k}+\epsilon\cdot\mathbf{1}_{k=1} \le v \le 2-\epsilon$, *then*

$$\frac{1}{x}\sum_{n\le x} v^{\Omega(n;R)}M_f(n;R)^{2k} \ll (\log R)^{v\binom{2k}{k}-2kA}(\log\log R)^{O(1)}.$$

All implied constants depend at most on k, f *and* ϵ.

Proof. – (a) Lemma 10.3 implies that

$$\sum_{n\le x}\Omega(n;R^\eta)M_f(n;R)^{2k} \ll \int_{\frac{\log 2}{\log R}}^{1} u^{2k(A-1)}\sum_{n\le x}\Omega(n;R^\eta)M_{f_{A-1}}(n;R^u)^{2k}\mathrm{d}u + O\left(\frac{x\log\log R}{(\log R)^{2kA}}\right). \tag{10.15}$$

When $u \ge \eta$, Theorem 10.2(a) implies that

$$\int_\eta^1 u^{2k(A-1)}\sum_{n\le x}\Omega(n;R^\eta)M_{f_{A-1}}(n;R^u)^{2k}\mathrm{d}u \ll \frac{\eta x}{\log R}. \tag{10.16}$$

Finally, we consider the integral over $u \in [\log 2/\log R, \eta]$. Observe that

$$\sum_{n\le x}\Omega(n;R^u)M_{f_{A-1}}(n;R^u)^{2k} \ll \frac{x}{u\log R},$$

by Theorem 10.2(a). If $x \le R^{100}$, then we also have that $\Omega(n;R^u,R^\eta) \le 100/u$ for each $n \le x$, so that

$$\sum_{n\le x}\Omega(n;R^u,R^\eta)M_{f_{A-1}}(n;R^u)^{2k} \ll \frac{x}{u^2\log R}$$

by Theorem 10.2. We thus find that

$$\int_{\frac{\log 2}{\log R}}^{\eta} u^{2k(A-1)}\sum_{n\le x}\Omega(n;R^\eta)M_{f_{A-1}}(n;R^u)^{2k}\mathrm{d}u \ll \int_{\frac{\log 2}{\log R}}^{\eta}\frac{u^{2k(A-1)-2}x}{\log R}\mathrm{d}u \ll \frac{\eta x}{\log R}.$$

Together with (10.15) and (10.16), this proves part (a) in the case where $x \le R^{100}$.

Finally, let us consider the case where $x > R^{100}$. Then

$$\sum_{n\le x}\Omega(n;R^u,R^\eta)M_{f_{A-1}}(n;R^u)^{2k} \le \sum_{\substack{p^j\le x\\ R^u<p\le R^\eta}} j\sum_{m\le x/p^j} M_{f_{A-1}}(m;R^u)^{2k}.$$

When $p^{j-1} \ge \sqrt{x}$, then we bound the inner sum trivially by $\ll (x/p^j)(\log R)^{O(1)}$, so that the total contribution of such summands is $\ll \sqrt{x}(\log R)^{O(1)}$. Finally, when $p^{j-1} \le \sqrt{x}$, then we see that $x/p^j \ge \sqrt{x}/R \ge x^{0.499} \ge R^u$, so that the sum over m is $\ll (x/p^j)/\log(R^u)$, by Theorem 10.2(a). We thus conclude that

$$\sum_{n\le x}\Omega(n;R^u,R^\eta)M_{f_{A-1}}(n;R^u)^{2k} \ll \frac{x\log(\eta/u)}{u\log R} + \frac{x}{(u\log R)^2} \ll \frac{x}{u\log R}.$$

Therefore,

$$\int_{\frac{\log 2}{\log R}}^{\eta} u^{2k(A-1)} \sum_{n\le x} \Omega(n;R^\eta) M_{f_{A-1}}(n;R^u)^{2k} \mathrm{d}u \ll \frac{\eta x}{\log R},$$

in this case as well, thus completing the proof of part (a) of the theorem.

(b) The proof of this part is similar. We start again with Lemma 10.3 to find that Lemma 10.3 implies that

$$\begin{aligned}\sum_{n\le x} v^{\Omega(n;R)} M_f(n;R)^{2k} \ll \int_{\frac{\log 2}{\log R}}^{1} u^{2k(A-1)} \sum_{n\le x} v^{\Omega(n;R)} M_{f_{A-1}}(n;R^u)^{2k} \mathrm{d}u \\ + O\left(x(\log R)^{v-1-2kA}\right),\end{aligned} \tag{10.17}$$

where we also used Theorem III.3.5 in [22, p.308].

Next, if $w = \max\{v,1\}$ and $\log 2/\log R \le u \le 1$, then note that

$$\sum_{n\le x} v^{\Omega(n;R)} M_{f_{A-1}}(n;R^u)^{2k} \le \sum_{\substack{a\le x\\ P^+(a)\le R^u}} v^{\Omega(a;R^u)} M_{f_{A-1}}(a;R^u)^{2k} \sum_{\substack{b\le x/a\\ P^-(b)>R^u}} w^{\Omega(b;R^u,R)}.$$

When $x/a \ge R^u$, the sum over b is $\ll u^{1-w} x/(a\log(R^u))$ by Theorem III.3.5 in [22, p.308]. Hence,

$$\begin{aligned}\sum_{n\le x} v^{\Omega(n;R)} M_{f_{A-1}}(n;R^u)^{2k} \ll \frac{u^{1-w}x}{\log(R^u)} \sum_{\substack{a\le x\\ P^+(a)\le R^u}} \frac{v^{\Omega(a;R^u)} M_{f_{A-1}}(a;R^u)^{2k}}{a} \\ + \sum_{\substack{a\le x\\ P^+(b)\le R^u}} v^{\Omega(a;R^u)} M_{f_{A-1}}(a;R^u)^{2k}.\end{aligned}$$

We bound the first sum by Theorem 10.1(b) and the second one by Theorem 10.2(b) to find that

$$\sum_{n\le x} v^{\Omega(n;R)} M_{f_{A-1}}(n;R^u)^{2k} \ll x u^{1-w+v\binom{2k}{k}-2kA} (\log R)^{v\binom{2k}{k}-2kA}.$$

Inserting the above bound into (10.17) completes the proof of the theorem. □

We conclude this section with the proof of Theorem 1.4. We need a preliminary lemma.

LEMMA 10.5. – *If $m \in \mathbb{Z}_{\ge 1}$, $g \in C^1(\mathbb{R}^m)$ and $z \ge y \ge 3$, then there is a positive constant $c > 0$ such that*

$$\begin{aligned}\sum_{y<p_1<\cdots<p_m\le z} \frac{g(\log p_1,\ldots,\log p_m)}{(p_1-1)\cdots(p_m-1)} = \frac{1}{m!}\int_{[\log y,\log z]^m} \frac{g(t_1,\ldots,t_m)}{t_1\cdots t_m}\mathrm{d}t_1\cdots \mathrm{d}t_m \\ + O\left(\frac{\|g\|_\infty + \|\nabla g\|_\infty}{e^{c\sqrt{\log y}}}\cdot \frac{(\sum_{y<p\le z} 1/p)^{m-1} + (\int_y^z \mathrm{d}t/t\log t)^{m-1}}{m!/m^2}\right).\end{aligned}$$

Proof. – First of all, note that

$$\sum_{y<p_1<\cdots<p_m\le z}\frac{g(\log p_1,\dots,\log p_m)}{(p_1-1)\cdots(p_m-1)}=\frac{1}{m!}\sum_{\substack{y<p_1,\dots,p_m\le z\\ \text{distinct}}}\frac{g(\log p_1,\dots,\log p_m)}{(p_1-1)\cdots(p_m-1)}$$
$$=\frac{1}{m!}\sum_{y<p_1,\dots,p_m\le z}\frac{g(\log p_1,\dots,\log p_m)}{p_1\cdots p_m}+O\left(\frac{m^2\|g\|_\infty(\sum_{y<p\le z}1/p)^{m-1}}{m!y}\right).$$

So, if we can show that

$$\sum_{y<p_1,\dots,p_m\le z}\frac{g(\log p_1,\dots,\log p_m)}{p_1\cdots p_m}=\int_{[\log y,\log z]^m}\frac{g(t_1,\dots,t_m)}{t_1\cdots t_m}\mathrm{d}t_1\cdots\mathrm{d}t_m$$
$$+O\left(\sum_{j=1}^m\frac{\|g\|_\infty+\|\partial g/\partial x_j\|_\infty}{e^{c\sqrt{\log y}}}\left(\int_y^z\frac{\mathrm{d}t}{t\log t}\right)^{j-1}\left(\sum_{y<p\le z}\frac{1}{p}\right)^{m-j}\right),$$

then the lemma will follow. But this estimate can be easily proved by induction on m and the Prime Number Theorem, and the proof is completed. □

Let us now see how we can deduce Theorem 1.4 from the above results:

Proof of Theorem 1.4. – The first estimate of part (a) follows by 10.4(a) and part (b) follows by Theorem 10.2(b). It remains to prove the second estimate of part (a). Note that it suffices to prove that

$$\prod_{p\le R}\left(1-\frac{1}{p}\right)\sum_{P^+(n)\le R}\frac{M_f(n;R)^{2k}}{n}=\frac{c_{k,f}}{\log R}+O\left(\frac{1}{(\log R)^{2-\epsilon}}\right),$$

since $x\ge R^{2k}\log^2 R$ here. Fix $\eta\in[\log 2/\log R,1]$ to be chosen later. Then Theorem 10.4(a) implies that

$$\sum_{\substack{P^+(n)\le R\\ P^-(n)\le R^\eta}}\frac{M_f(n;R)^{2k}}{n}\le\sum_{P^+(n)\le R}\frac{\Omega(n;R^\eta)M_f(n;R)^{2k}}{n}\ll\eta.$$

Moreover, for each positive integer m, we have that

$$\sum_{\substack{p|n\Rightarrow R^\eta<p\le R\\ \omega(n)=m}}\frac{M_f(n;R)^{2k}}{n}=\sum_{R^\eta<p_1<\cdots<p_m\le R}\frac{M_f(p_1\cdots p_m;R)}{(p_1-1)\cdots(p_m-1)}.$$

We note that

$$M_f(p_1\cdots p_m;R)=g_m\left(\frac{\log p_1}{\log R},\dots,\frac{\log p_m}{\log R}\right),$$

where

$$g_m(t_1,\dots,t_m):=\sum_{J\subset\{1,\dots,m\}}(-1)^{\#J}f\left(\sum_{j\in J}t_j\right),$$

which is a smooth function satisfying the estimates $\|g_m^{2k}\|_\infty \le 4^{km}\|f\|_\infty^{2k}$ and $\|\nabla g_m^{2k}\|_\infty \le 2k4^{km}\|f'\|_\infty\|f\|_\infty^{2k-1}$. Therefore Lemma 10.5 implies that

$$\sum_{\substack{p|n \Rightarrow R^\eta < p \le R \\ \omega(n)=m}} \frac{M_f(n;R)^{2k}}{n} = \frac{1}{m!}\int_{[\eta,1]^m} \frac{g_m(t_1,\dots,t_m)^{2k}}{t_1\cdots t_m} dt_1\cdots dt_m + O\left(\frac{(4^k\log(1/\eta)+O(1))^m}{e^{c\sqrt{\log(R^\eta)}}m!/m^2}\right)$$

for all $m \in \mathbb{Z}_{\ge 1}$. We thus conclude that

$$\sum_{P^+(n)\le R} \frac{M(n;R)^{2k}}{n} = F(\eta) + O\left(\eta + \frac{\log^2(1/\eta)}{\eta^{4^k} e^{c\sqrt{\log(R^\eta)}}}\right) \quad (0<\eta\le 1,\ R^\eta \ge 2), \tag{10.18}$$

where

$$F(\eta) := 1 + \sum_{m=1}^\infty \frac{1}{m!}\int_{[\eta,1]^m} \frac{g_m(t_1,\dots,t_m)^{2k}}{t_1\cdots t_m}\mathrm{d}t_1\cdots\mathrm{d}t_m.$$

Completing the proof is now an exercice in real analysis. We start by proving that $\lim_{\eta\to 0^+} F(\eta)$ exists. Indeed, applying (10.18) twice, we deduce that

$$F(\eta_1) - F(\eta_2) \ll \max_{j\in\{1,2\}} \left\{\eta_j + \frac{\log^2(1/\eta_j)}{\eta_j^{4^k} e^{c\sqrt{\log(R^{\eta_j})}}}\right\}$$

whenever $0 < \eta_1 \le \eta_2 \le 1$ and $R^{\eta_1} \ge 2$. Letting $R \to \infty$, we find that

$$F(\eta_1) - F(\eta_2) \ll \eta_2 \quad (0<\eta_1\le\eta_2\le 1). \tag{10.19}$$

In particularly, Cauchy's convergence criterion implies that $\lim_{\eta\to 0^+} F(\eta)$ exists. Call F this limit, which clearly equals $F(0)$, and note that letting $\eta_1 \to 0^+$ in (10.19) implies that

$$F(\eta) = F + O(\eta) \quad (0<\eta\le 1).$$

Together with (10.18), this yields the estimate

$$\sum_{P^+(n)\le R} \frac{M(n;R)^{2k}}{n} = F + O\left(\eta + \frac{\log^2(1/\eta)}{\eta^{4^k} e^{c\sqrt{\log(R^\eta)}}}\right) \quad (0<\eta\le 1,\ R^\eta \ge 2).$$

Selecting η such that

$$R^\eta = e^{(\log\log R)^3}$$

completes the proof of Theorem 1.4. □

We conclude this section with the proof of Theorem 1.7.

Proof of Theorem 1.7. – The first estimate of part (a) can be proven following *mutatis mutandis* the proof of Theorem 1.6(a) above, using Theorem 10.4 in place of Theorem 10.2. Similarly, part (b) follows from the proof of Theorem 1.6(b). □

11. The analogy for non-exceptional Dirichlet characters

In the section, we study the sum

$$\mathcal{X}_{2k}(R) = \prod_{p\le R}\left(1-\frac{1}{p}\right)\sum_{P^+(n)\le R}\frac{1}{n}\left(\sum_{\substack{d|n\\ R/2<d\le R}}\chi(d)\right)^{2k}$$

when $k \ge 2$ and $L(1,\chi)$ is not very small and prove Theorem 1.5(b). We may assume throughout that $q \le R^{1/\log\log R}$; otherwise, the needed estimate holds trivially.

We have that

$$\mathcal{X}_{2k}(R) = \sum_{R/2<d_1,\dots,d_{2k}\le R}\frac{\chi(d_1)\chi(d_2)\cdots\chi(d_{2k})}{[d_1,\dots,d_{2k}]}.$$

We want to introduce new variables D_I, $I \in \mathcal{S}^*(2k)$, as in Section 2, but first we perform a technical maneuver to simplify the situation. We write $d_i = d_i'd_i''$, where $P^+(d_i') \le y < P^-(d_i'')$, where

$$y := (\log R)^{4^k+1}.$$

The contribution to $\mathcal{X}_{2k}(R)$ of $\boldsymbol{d}$'s for which d_i'' is not square-free for some i is $\ll (\log R)^{4^k-1}/y$ by a crude upper bound, and so is the contribution of those $\boldsymbol{d}$'s with $\max_i d_i' > B$, where

$$B := e^{(\log\log R)^3},$$

by Rankin's trick. Then we let D_I, $I \in \mathcal{S}^*(2k)$, be the product of those primes that divide d_i'' when $i \in I$, and are coprime to the other d_i'''s. The numbers D_I are pairwise coprime and square-free, and $d_i'' = \prod_{I\in\mathcal{S}^*(2k),\, I\ni i} D_I$, so that

$$\frac{\chi(d_1'')\chi(d_2'')\cdots\chi(d_{2k}'')}{[d_1'',\dots,d_{2k}'']} = \frac{\prod_{I\in\mathcal{S}^+(2k)}\chi_0(D_I)\prod_{I\in\mathcal{S}^-(2k)}\chi(D_I)}{\prod_{I\in\mathcal{S}^*(2k)}D_I}.$$

We may now drop the condition that the D_I's are square-free and coprime, since the contribution to $\mathcal{J}_{2k}(R)$ of the D_I's not satisfying these conditions is $\ll (\log R)^{4^k-1}/y$. Finally, we may drop the condition that $(D_I,q) = 1$ for $I \in \mathcal{S}^+(2k)$, encoded in the notation $\chi_0(D_I)$, since the contribution of D_I's with $P^-(D_I) > y$ and $(D_I,q) > 1$ is $\ll (\log R)^{4^k-1}\sum_{p|q,\, p>y}1/p \ll (\log R)^{4^k}/y$. The above discussion implies that

$$\mathcal{X}_{2k}(R) = \sum_{\substack{P^+(d_i')\le y,\, d_i'\le B\\ 1\le i\le 2k}}\frac{\chi(d_1')\cdots\chi(d_{2k}')}{[d_1',\dots,d_{2k}']}\sum_{\substack{P^-(D_I)>y\ (I\in\mathcal{S}^*(2k))\\ R/(2d_i')<\prod_{I\in\mathcal{S}^*(2k),\, I\ni i}D_I\le R/d_i'\\ 1\le i\le 2k}}\frac{\prod_{I\in\mathcal{S}^-(2k)}\chi(D_I)}{\prod_{I\in\mathcal{S}^*(2k)}D_I}$$
$$+\,O\left(\frac{1}{\log R}\right).$$

Next, we note that

$$\sum_{\substack{n\le x\\ P^-(n)>y}}\chi(n) \ll \frac{x^{1-1/(30\log y)}}{\log y}\quad (x\ge\max\{q^4,y\}),$$

by Lemma 2.4 in [14]. Therefore,

$$\sum_{\substack{n>q^4B \\ P^-(n)>y}} \frac{\chi(n)}{n} \ll \frac{1}{y}. \tag{11.1}$$

This implies that the contribution to $\mathscr{X}_{2k}(R)$ with $D_I > q^4B$ for some $I \in \mathcal{S}^-(2k)$ is $\ll (\log R)^{4^k-1}/y$. To conclude, we have shown that

(11.2)

$$\mathscr{X}_{2k}(R) = \sum_{\substack{P^+(d_i')\le y,\, d_i'\le B \\ 1\le i\le 2k}} \sum_{\substack{D_I\le q^4B \\ P^-(D_I)>y \\ I\in\mathcal{S}^-(2k)}} \frac{\chi(d_1')\cdots\chi(d_{2k}')}{[d_1',\dots,d_{2k}']} \cdot \frac{\prod_{I\in\mathcal{S}^-(2k)}\chi(D_I)}{\prod_{I\in\mathcal{S}^-(2k)} D_I} \cdot T(R_1,\dots,R_{2k}) + O\left(\frac{1}{\log R}\right).$$

where $R_i = R/(d_i'\prod_{I\in\mathcal{S}^-(2k),\, I\ni i} D_I)$ and

$$T(\boldsymbol{R}) := \sum_{\substack{P^-(D_I)>y\ (I\in\mathcal{S}^+(2k)) \\ R_i/2<\prod_{I\in\mathcal{S}^+(2k),\, I\ni i} D_I\le R_i \\ 1\le i\le 2k}} \frac{1}{\prod_{I\in\mathcal{S}^*(2k)} D_I}.$$

Our task now becomes estimating $T(\boldsymbol{R})$. Let $d = 2^{2k-1} - 2k$ and recall the definition of $V_d(\cdot)$ from the statement of Theorem 1.5(b). The proof of Theorem 1.1 shows that $V_k(m) \asymp m^d$. We claim that

$$T(\boldsymbol{R}) = V_k(\log R)\left(1 + O\left(\frac{\log(qB)}{\log R}\right)\right)\prod_{p\le y}\left(1-\frac{1}{p}\right)^{2^{2k-1}-1} \tag{11.3}$$

whenever $R/(q^4D)^{4^k} \le R_i \le R$ for all i, as is the case here. Proving (11.3) can be accomplished easily using a lattice point count and the fundamental lemma of sieve methods. First of all, note that the part of $T(\boldsymbol{R})$ where $D_I \le B$ for some $I \in \mathcal{S}^+(2k)$ is trivially $\ll (\varphi(P)/P)^{2^{2k-1}-1}(\log R)^{d-1}\log B$ by an upper bound sieve, where we have set $P := \prod_{p\le y} p$ for simplicity. In the rest of the range, we set $\rho = 1 + 1/\log R$ and divide the variables D_I into boxes of the form $(\rho^{m_I}, \rho^{m_I+1}]$, $m_I \ge 0$. Replacing D_I by ρ^{m_I} in the conditions $R_i/2 < \prod_{J\in\mathcal{S}^+(2k),\, J\ni i} D_J \le R_i$ creates a total error of size $\ll (\varphi(P)/P)^{2^{2k-1}-1}(\log R)^{d-1}$. In addition, if $\rho^{m_I} \ge B = e^{(\log\log R)^3}$, then we have that

$$\sum_{\substack{\rho^{m_I}<D_I\le\rho^{m_I+1} \\ P^-(D_I)>y}} \frac{1}{D_I} = (\log\rho)\frac{\varphi(P)}{P} + O_C\left(\frac{1}{(\log R)^C}\right)$$

for any fixed C, by the fundamental lemma of sieve methods (see, for example, [22, Theorem I.4.3]). We thus conclude that

$$T(\boldsymbol{R}) = ((\log\rho)\varphi(P)/P)^{2^{2k-1}-1} \sum_{\substack{m_I \ge \log B/\log\rho\ (I\in\mathcal{S}^+(2k)) \\ \frac{\log(R_i/2)}{\log\rho} < \sum_{I\in\mathcal{S}^+(2k),\, I\ni i} m_I \le \frac{\log R_i}{\log\rho}}} 1 \\ + O\left((\varphi(P)/P)^{2^{2k-1}-1}(\log R)^{d-1}\log B\right).$$

A straightforward lattice point counting argument implies that the sum on the right hand side equals

$$\frac{W_k(\log R_1,\dots,\log R_{2k})}{(\log\rho)^{2^{2k-1}-1}}\left(1+O\left(\frac{\log B}{\log R}\right)\right),$$

where $W_k(\boldsymbol{m})$ is the volume of the polytope $\{(x_I)_{I\in\mathcal{S}^+(2k)} : x_I \ge 0\ \forall I,\ m_i - \log 2 \le \sum_{I\ni i} x_I \le m_i\ \forall i\}$. Since $m_i = \log R_i = \log R + O(\log(qB))$ here, we may show using the Mean Value Theorem that $W_k(\boldsymbol{m}) = V_k(m) + O((\log R)^{d-1}\log(qB))$. Relation (11.3) then follows.

We are now ready to complete the proof of Theorem 1.5(a): inserting the estimate (11.3) into (11.2), we conclude that

$$\mathcal{X}_{2k}(R) = \frac{V_k(\log R)}{(P/\varphi(P))^{2^{2k-1}-1}} \sum_{\substack{P^+(d_i')\le y,\, d_i\le B \\ 1\le i\le 2k}} \sum_{\substack{D_I\le q^4 B \\ P^-(D_I)>y \\ I\in\mathcal{S}^-(2k)}} \frac{\chi(d_1')\cdots\chi(d_{2k}')}{[d_1',\dots,d_{2k}']}\cdot\frac{\prod_{I\in\mathcal{S}^-(2k)}\chi(D_I)}{\prod_{I\in\mathcal{S}^-(2k)} D_I} \\ + O((\log R)^{d-1}(\log\log R)^{O(1)}\log q).$$

We now remove the conditions $D_I \le q^4 B$ and $d_i' \le B$ via (11.1) and Rankin's trick, respectively. We conclude that

$$\mathcal{X}_{2k}(R) = \frac{V_k(\log R)}{(P/\varphi(P))^{2^{2k-1}-1}} \sum_{\substack{P^+(d_i')\le y \\ 1\le i\le 2k}} \sum_{\substack{P^-(D_I)>y \\ I\in\mathcal{S}^-(2k)}} \frac{\chi(d_1')\cdots\chi(d_{2k}')}{[d_1',\dots,d_{2k}']}\cdot\frac{\prod_{I\in\mathcal{S}^-(2k)}\chi(D_I)}{\prod_{I\in\mathcal{S}^-(2k)} D_I} \\ + O((\log R)^{d-1}((\log\log R)^{O(1)}\log q).$$

Finally, we note that

$$\sum_{\substack{P^+(d_i')\le y \\ 1\le i\le 2k}} \frac{\chi(d_1')\cdots\chi(d_{2k}')}{[d_1',\dots,d_{2k}']} = \prod_{p\le y}\left(1+\sum_{j=1}^{\infty}\sum_{\substack{j_1,\dots,j_{2k}\ge 0 \\ \max\{j_1,\dots,j_{2k}\}=j}} \frac{\chi(p)^{j_1+\dots+j_{2k}}}{p^j}\right).$$

The coefficient of $1/p$ is $(2^{2k-1}-1)\chi_0(p)+2^{2k-1}\chi(p)$. We thus conclude that

$$(\varphi(P)/P)^{2^{2k-1}-1}\sum_{\substack{P^+(d_i')\le y\\ 1\le i\le 2k}}\ \sum_{\substack{P^-(D_I)>y\\ I\in\mathcal{S}^-(2k)}}\frac{\chi(d_1')\cdots\chi(d_{2k}')}{[d_1',\ldots,d_{2k}']}\cdot\frac{\prod_{I\in\mathcal{S}^-(2k)}\chi(D_I)}{\prod_{I\in\mathcal{S}^-(2k)}D_I}$$
$$=\prod_{p\le y}\left(1+\sum_{j=1}^{\infty}\sum_{\substack{j_1,\ldots,j_{2k}\ge 0\\ \max\{j_1,\ldots,j_{2k}\}=j}}\frac{\chi(p)^{j_1+\cdots+j_{2k}}}{p^j}\right)\left(1-\frac{1}{p}\right)^{2^{2k-1}-1}\prod_{p>y}\left(1-\frac{\chi(p)}{p}\right)^{-2^{2k-1}}$$
$$=\prod_{p}\left(1+\sum_{j=1}^{\infty}\sum_{\substack{j_1,\ldots,j_{2k}\ge 0\\ \max\{j_1,\ldots,j_{2k}\}=j}}\frac{\chi(p)^{j_1+\cdots+j_{2k}}}{p^j}\right)\left(1-\frac{1}{p}\right)^{2^{2k-1}-1}+O\left(\frac{1}{y}\right).$$

An easy calculation then completes the proof of Theorem 1.5(b).

12. The analogy for exceptional Dirichlet characters

In this section, we consider the quantity $\mathcal{X}_{2k}(R)$ when $k=1$, or when $\chi(p)=-1$ for most primes $p\le R$, and complete the proof of Theorem 1.5. Our arguments here resemble closely the ones of Section 8, so we only highlight the key points here. Throughout the proof, we assume that $R\ge q^{2c_1}$, the complimentary case being trivial.

12.1. Initial preparations

Arguing as in Section 8.4, we find that

$$\mathcal{X}_{2k}(R)=\frac{1}{(2i\pi)^{2k}}\int\cdots\int_{\substack{\mathrm{Re}(s_j)=\lambda^j/\log R\\ |\mathrm{Im}(s_j)|\le T\\ 1\le j\le 2k}}\sum_{d_1,\ldots,d_{2k}\ge 1}\frac{\prod_{j=1}^{2k}\chi(d_j)d_j^{-s_j}}{[d_1,\ldots,d_{2k}]}\prod_{j=1}^{2k}\widehat{h}_R(s_j)(1-2^{-s_j})\mathrm{d}s_1\cdots\mathrm{d}s_{2k}$$
$$+O\left(\frac{1}{(\log R)^{100}}\right),$$

where h is a smooth function with $h(x)=1$ for $x\le 1-1/(\log R)^{(2k-1)2^{2k+1}+200k+2}$ and $h(x)=0$ for $x>1$, $T=\exp\{(\log\log R)^2\}$, and λ is some large parameter >1 to be chosen later.

By expanding as an Euler product, we find that for $\mathrm{Re}(s_1),\ldots\mathrm{Re}(s_{2k})\ge -1/4k$ we have

$$\sum_{d_1,\ldots,d_{2k}\ge 1}\frac{\prod_{j=1}^{2k}\chi(d_j)d_j^{-s_j}}{[d_1,\ldots,d_{2k}]}=\prod_p\left(1+\sum_{I\in\mathcal{S}^*(2k)}\frac{\chi(p)^{\#I}}{p^{1+s_I}}+O\left(\frac{1}{p^{2-2k/4^k}}\right)\right)$$
$$=P(\boldsymbol{s})\prod_{I\in\mathcal{S}^*(2k)}L(1+s_I,\chi^{\#I}),$$

where $P(\boldsymbol{s})$ is given by an Euler product which converges absolutely in the region $\mathrm{Re}(s_j) > -1/4k$ for all j. Next, we set

$$F(\boldsymbol{s}) = P(\boldsymbol{s}) \prod_{j=1}^{2k} \frac{\widehat{h}_R(s_j)(1-2^{-s_j})}{R^{s_j}}$$

and

$$\zeta_q(s) := L(s,\chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right),$$

so that

$$\begin{aligned} \mathcal{X}_{2k}(R) = \frac{1}{(2i\pi)^{2k}} & \underset{\substack{\mathrm{Re}(s_j)=\lambda^j/\log R \\ |\mathrm{Im}(s_j)|\le T \\ 1\le j\le 2k}}{\int \cdots \int} F(\boldsymbol{s}) R^{s[2k]} \prod_{I \in \mathcal{S}^-(2k)} L(1+s_I,\chi) \\ & \times \prod_{I\in\mathcal{S}^+(2k)} \zeta_q(1+s_I) \mathrm{d}s_1 \cdots \mathrm{d}s_{2k} + O\left(\frac{1}{(\log R)^{100}}\right). \end{aligned} \tag{12.1}$$

Similarly to Section 8, we let $\widetilde{\mathcal{C}}_\ell$ denote the class of complex-valued functions f defined in a domain containing

$$\widetilde{\Omega}_\ell := \{\boldsymbol{s} \in \mathbb{C}^\ell : |\mathrm{Re}(s_j)| < 1/5k, |\mathrm{Im}(s_j)| < T+1 \ (1 \le j \le \ell)\},$$

it is analytic in $\widetilde{\Omega}_\ell$, and its derivatives satisfy the bound

$$\frac{\partial^{j_1+\cdots+j_\ell} f}{\partial s_1^{j_1} \cdots \partial s_\ell^{j_\ell}}(\boldsymbol{s}) \ll_{j_1,\ldots,j_\ell} \prod_{m=1}^{\ell} \frac{[(1+(qT)^{-\mathrm{Re}(s_m)})(\log\log R)^{j_m}]^{O(1)}}{|s_m|+1} \tag{12.2}$$

for all $j_1,\ldots,j_\ell \ge 0$ and all $\boldsymbol{s} = (s_1,\ldots,s_\ell) \in \widetilde{\Omega}_\ell$.

Since there is an absolute constant $c_1 > 0$ such that

$$L^{(j)}(s,\psi) \ll_m (1+(q+|t|)^{c_1(1-\sigma)}) \log^{j+1}(q+|t|) + \frac{\mathbf{1}_{\psi=\chi_0}}{|s-1|^{j+1}} \tag{12.3}$$

for $j \in \mathbb{Z}_{\ge 0}$, $\psi \in \{\chi,\chi_0\}$ and $j \in \{0,1\}$, a standard consequence of bounds on the exponential sum $\sum_{n\le N,\ n\equiv a \,(\mathrm{mod}\, q)} n^{it}$ (see, for example, Lemma 4.1 in [14]), we have that F is in the class $\widetilde{\mathcal{C}}_{2k}$.

12.2. The case $k = 1$

We first deal with the case $k=1$ that is easy and will help us clarify some of the technical details of the argument. When $k=1$, we move the variable s_2 to the line $\mathrm{Re}(s_2) = -\epsilon$, for a sufficiently small ϵ. The contribution of the horizontal contours is $\ll (\log R)^{O(1)}/T$, and the contribution of the contour $\mathrm{Re}(s_2) = -\delta$ is $\ll R^{-\delta}$, for some positive $\delta = \delta(\epsilon)$ by (12.3), and by our assumptions that $F \in \mathcal{C}_2$ and that $R \ge q^{2c_1}$. In conclusion,

$$\mathcal{X}_2(R) = \frac{1}{2i\pi} \int\limits_{\substack{\mathrm{Re}(s_1)=\lambda/\log R \\ |\mathrm{Im}(s_1)|\le T \\ 1\le j\le 2k}} F(s_1,-s_1) L(1+s_1,\chi) L(1-s_1,\chi) \mathrm{d}s_1 + O\left(\frac{1}{(\log R)^{100}}\right).$$

Finally, we move s_1 to the line $\mathrm{Re}(s_1) = 0$. No poles are encountered, and the contribution of the horizontal lines is easily seen to be $\ll (\log R)^{O(1)}/T$, so that

$$\begin{aligned}\mathcal{X}_2(R) &= \frac{1}{2\pi}\int_{-T}^{T} F(it,-it)|L(1+it,\chi)|^2 \mathrm{d}t + O\left(\frac{1}{(\log R)^{100}}\right)\\ &= \frac{1}{2\pi}\int_{-(\log R)^{102}}^{(\log R)^{102}} P(it,-it)\left|L(1+it,\chi)\widehat{h}_R(it)(1-2^{it})\right|^2 \mathrm{d}t + O\left(\frac{1}{(\log R)^{100}}\right),\end{aligned}$$

since $\widehat{h}_R(it)(1-2^{it}) \ll 1/(1+|t|)$. Finally, using 6.6 to replace $\widehat{h}_R(s)$ by R^s/s, choosing C to be large enough, and then extending the range of integration to $\mathbb{R}$ yields the estimate

$$\mathcal{X}_2(R) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(it,-it)\left|L(1+it,\chi)\cdot\frac{\sin(t(\log 2)/2)}{t}\right|^2 \mathrm{d}t + O\left(\frac{1}{(\log R)^{100}}\right),$$

which proves Theorem 1.5(a). (Obviously, we can obtain a much stronger error term, but we have chosen to content ourselves with a more qualitative result.)

12.3. Contour shifting

Next, we focus on the case $k \geq 2$ and prove Theorem 1.5(c). From now on, we will always be working under the assumptions and notations

$$L(\beta,\chi) = 0, \quad \beta > 1 - 1/(100\log q), \quad Q = e^{1/(1-\beta)}.$$

As it is well-known, we have that $\sum_{q<p\leq Q}(1+\chi(p))/p \ll 1$ and $\sum_{p>Q}\chi(p)/p \ll 1$ (see, for example, Theorems 2.1 and 2.4 in [13]). As a consequence, we note once and for all that

$$L(1,\chi) \asymp \frac{1}{\log Q}\prod_{p\leq q}\left(1+\frac{1+\chi(p)}{p}\right). \tag{12.4}$$

As in Section 8, we shift the contours of the variables $s_1,\dots,s_{2k}$ in a certain order. As in that section, to describe the general contour shifting argument after N steps, $0 \leq N \leq 2k$, we fix sets $I_1,\dots,I_N$, and distinct integers $j_n \in I_n$ for each n. Recall, also, that s_j denotes a variable and x_j denotes a linear form. We then define

$$V_n = \mathrm{Span}_{\mathbb{Q}}(x_{I_1},\dots,x_{I_n}) \quad\text{and}\quad \mathcal{J}_N = \{I \in \mathcal{S}(2k) : x_I \in V_n\} \quad (0 \leq n \leq N).$$

Imposing the conditions $x_{I_1} = \cdots = x_{I_n} = 0$, we may write $x_{j_1},\dots,x_{j_n}$ in terms of the variables s_j with $j \in [2k]\setminus\{j_1,\dots,j_N\}$. Hence x_I becomes a linear form $L_{N,I}$ in the variables s_j with $j \in [2k]\setminus\{j_1,\dots,j_N\}$. Moreover, $x_I \in V_N$ if and only if $L_{N,I} = 0$.

As we will see, we will always be able to assume that $j_n = 2k-n+1$. Let $d \in \mathbb{Z}_{\geq 0}$ and given the above set-up with $j_n = 2k-n+1$, an integer $d \in \mathbb{Z}_{\geq 0}$ and $\boldsymbol{h} = (h_{n,I})_{0\leq n\leq N,\, I\in\mathcal{S}^*(2k)}$ be a vector of non-negative integers such that:

— $0 = h_{0,I} \leq h_{1,I} \leq \cdots \leq h_{N,I}$ for $I \in \mathcal{S}^*(2k)$;

— if $I \in \mathcal{J}_n \setminus \mathcal{J}_{n-1}$ for some $n \in \{1,\dots,N\}$, then $h_{m,I} = h_{n,I}$ for all $m \geq n$;

— $X_N \geq N + d$, where

$$X_N := \#(\mathcal{J}_N \cap \mathcal{S}^+(2k)) - \sum_{I\in\mathcal{S}^-(2k)\cup(\mathcal{S}^+(2k)\setminus\mathcal{J}_N)} h_{N,I}.$$

A function $J : \mathbb{R}_{\geq 2} \to \mathbb{C}$ is a called a *fundamental component of level N and of type $(\boldsymbol{I},\boldsymbol{h},d)$* if:

— when $N = 2k$, it equals

$$J(R) = (\log R)^{X_N - N - d} \prod_{I \in \mathcal{S}^-(2k)} L^{(h_{N,I})}(1, \chi);$$

— when $N < 2k$, it is of the form

$$J(R) = \left(\frac{\varphi(q)}{q}\right)^{M_N} \frac{(\log R)^{X_N - N - d}}{(2i\pi)^{2k-N}} \prod_{I \in \mathcal{S}^-(2k) \cap \mathcal{I}_N} L^{(h_{N,I})}(1, \chi)$$
$$\times \int \cdots \int_{\substack{\mathrm{Re}(s_j) = \lambda_j / \log R \\ |\mathrm{Im}(s_j)| \le T \\ 1 \le j \le 2k-N}} G(\boldsymbol{s}) R^{E_N(\boldsymbol{s})} \prod_{I \in \mathcal{S}^-(2k) \setminus \mathcal{I}_N} L^{(h_{N,I})}(1 + L_{N,I}(\boldsymbol{s}), \chi)$$
$$\times \prod_{I \in \mathcal{S}^+(2k) \setminus \mathcal{I}_N} \zeta_q^{(h_{N,I})}(1 + L_{N,I}(\boldsymbol{s})) \mathrm{d}s_{2k-N} \cdots \mathrm{d}s_1,$$

where $\lambda_j / \lambda_{j-1} \ge \lambda$, $E_N(\boldsymbol{s}) := L_{N,[2k]}(\boldsymbol{s})$,

$$M_N := \sum_{n=1}^{N} \sum_{I \in (\mathcal{I}_n \setminus \mathcal{I}_{n-1}) \cap \mathcal{S}^+(2k)} (h_{n-1,I} + 1),$$

and G is a function in the variables $s_1, \ldots, s_{2k-N}$ that belongs to the class $\mathcal{C}_{2k-N}$, given by

$$G(\boldsymbol{s}) = F(L_{N,\{1\}}(\boldsymbol{s}), \ldots, L_{N,\{2k\}}(\boldsymbol{s}))$$

when $d = 0$. In particular, G is non-vanishing in Ω_{2k-N} when $d = 0$ by (12.3).

As in Section 8.2, a fundamental component of level N is called reducible when $N < 2k$ and $E_N \ne 0$. Otherwise, it is called irreducible. With this above terminology, the integral on the right hand side of (12.1) is a reducible fundamental component of level 0 and of type $(\emptyset, \emptyset, 0)$.

Again as in Section 8.2, when $E_N \ne 0$ there are some $\gamma_j \in \mathbb{Q}$ with $\gamma_{j_{N+1}} \ne 0$ such that

$$E_N(\boldsymbol{x}) = \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_{j_{N+1}} x_{j_{N+1}}$$

If λ is big enough, then the sign of $\mathrm{Re}(E_N(\boldsymbol{s}))$ throughout the region of integration is constant and equal to the sign as $\gamma_{j_{N+1}}$.

The analogies of Lemmas 8.4 and 8.5 can be proven in this setting:

LEMMA 12.1. – *Assume the above setup, let $J(R)$ be a reducible fundament component of level $N < 2k$, and let $\gamma_{j_{N+1}}$ be as above. Suppose, further, that $k \ge 2$ and that $e^{(\log q)^2} \le R \le Q$. All implied constants below depend at most on k.*

(a) *If $\gamma_{j_{N+1}} > 0$, then $J(R)$ is a linear combination of fundamental components of level $N+1$, up to an error term of size $\ll 1/\log R$. Moreover, the coefficients of this linear combination are $\ll (\log q)^{O(1)}$.*

(b) *If $\gamma_{j_{N+1}} < 0$, then $J(R) \ll T^{-1+o(1)}$.*

We iterate the above lemma till all the fundamental components we are dealing with are irreducible. For such components, we have the following asymptotic formula.

LEMMA 12.2. – *Assume the above setup. Suppose, further, that $k \geq 2$ and that $e^{(\log q)^2} \leq R \leq Q$. If $J(R)$ is an irreducible fundamental component, then there is some complex number $c \ll (\log q)^{O(1)}$ such that*

$$J(R) = c(\log R)^{\binom{2k}{k}-2k} + O((\log(q \log R))^{O(1)}(\log R)^{\binom{2k}{k}-2k-1}).$$

All implied constants depend at most on k.

Since the integral on the right hand side of (12.1) is a reducible fundamental component of level 0, we apply Lemma 8.4 repeatedly to write it as a linear combination of irreducible fundamental components, and then estimate these components by Lemma 8.5. This proves that there is a constant $c_k(\chi) \ll (\log q)^{O(1)}$ such that

$$\mathscr{X}_{2k}(R) = c_k(\chi) \cdot (\log R)^{\binom{2k}{k}-2k} + O\left((\log(q \log R))^{O(1)}(\log R)^{\binom{2k}{k}-2k-1}\right) \tag{12.5}$$

when $k \geq 2$ and $e^{(\log q)^2} \leq R \leq Q$. We will show that $c_k(\chi) \gg (\log q)^{-O(1)}$ in Section 12.5 and complete the proof of Theorem 1.5. The key intermediate Lemmas 12.1 and 12.2 are proven in the next section.

12.4. Proof of the auxiliary Lemmas 12.1 and 12.2

For easy reference, we record the following bound that we will repeatedly use: for $R \leq Q$, we have

$$(\log R)^{X_N - N - d} \prod_{I \in \mathcal{S}^-(2k) \cap \mathcal{J}_N} L^{(h_{N,I})}(1, \chi) \ll (\log q)^{O(1)}(\log R)^{\mathscr{A}(V_N)-N-D}, \tag{12.6}$$

where

$$D = d + \sum_{I \in \mathcal{S}(2k) \setminus \mathcal{J}_N} h_{N,I} + \sum_{\substack{I \in \mathcal{S}^-(2k) \cap \mathcal{J}_N \\ h_{N,I} \geq 1}} (h_{N,I} - 1)$$

and $\mathscr{A}(V_N)$ is defined in Section 7. Indeed, when $h_{N,I} = 0$ with $I \in \mathcal{S}^-(2k) \cap \mathcal{J}_N$, we use (12.4) to find that

$$L(1, \chi) \ll \frac{(\log q)^2}{\log Q} \leq \frac{(\log q)^2}{\log R}.$$

Otherwise, we use the bound $L^{(h_{N,I})}(1, \chi) \ll (\log q)^{h_{N,I}+1}$. Putting these estimates together yields (12.6).

Proof of Lemma 12.1. – (a) Here $\gamma_{j_{N+1}} > 0$. We make the change of variables

$$s'_j = s_j \ (1 \leq j < j_{N+1}), \ s'_j = s_{j+1} \ (j_{N+1} \leq j \leq 2k - N), \ s'_{2k-N} = s_{j_{N+1}},$$

and similarly for the forms x_j and the parameters λ_j.

Next, we shift the s'_{2k-N} contour to the line $\mathrm{Re}(s'_{2k-N}) = -\epsilon$ for a small enough $\epsilon > 0$. The integral on the new contour is $\ll (\log R)^{O(1)}/T$, which is negligible, and we are left with having to analyze the pole contributions. The poles occur when $L_{N,I_{N+1}}(\boldsymbol{s}') = 0$ for some $I_{N+1} \in \mathcal{S}^+(2k) \setminus \mathcal{J}_N$ such that the coefficient of s'_{2k-N} in $L_{N,I_{N+1}}$ is non-zero. As we discussed in the previous section, imposing the relation $L_{N,I_{N+1}}(\boldsymbol{x}') = 0$ to write the form x'_{2k-N} as a linear combination of $x'_1, \ldots, x'_{2k-N-1}$, say $x'_{2k-N} = C(x'_1, \ldots, x'_{2k-N-1})$. In particular, $L_{N,I}(\boldsymbol{x}')$ becomes a linear form $L_{N+1,I}$ in the variables $x'_1, \ldots, x'_{2k-N-1}$. We also set $E_{N+1} = L_{N+1,[2k]}$ and let $\mathcal{J}_{N+1}$ be the set of $I \subset [2k]$ such that $L_{N+1,I} = 0$.

The generic order of the pole at $s'_{2k-N} = C(s'_1, \dots, s'_{2k-N-1})$ is

$$m = \sum_{I \in (\mathcal{J}_{N+1} \setminus \mathcal{J}_N) \cap \mathcal{S}^+(2k)} (h_{N,I} + 1) - \nu, \tag{12.7}$$

where ν is the generic order of the zero of

$$G(\boldsymbol{s}) \prod_{I \in \mathcal{S}^-(2k) \cap (I_{N+1} \setminus I_N)} L^{(h_{N,I})}(1 + L_{N,I}(\boldsymbol{s}), \chi)$$

at the same point. In particular, $\nu = 0$ if $d = 0$ (so that $G(\boldsymbol{s}) = F(L_{N,\{1\}}(\boldsymbol{s}), \dots, L_{N,\{2k\}}(\boldsymbol{s})))$ and $h_{N,I} = 0$ for all $I \in \mathcal{S}^-(2k) \cap (\mathcal{J}_{N+1} \setminus \mathcal{J}_N)$. By a direct computation, we then find that

$$X_{2k-1} + m = \#(\mathcal{J}_{N+1} \cap \mathcal{S}^+(2k)) - \sum_{I \in \mathcal{S}^-(2k) \cup (\mathcal{S}^+(2k) \setminus \mathcal{J}_{N+1})} h_{N,I} - \nu. \tag{12.8}$$

We note that $m \geq 1$ for all $N \leq 2k-1$ when $k \geq 2$ and $\nu = 0$; otherwise, we would have that $\mathcal{S}^+(2k) \subset \mathcal{J}_N$, which is impossible because the dimension of V_N is N, whereas the dimension of the span of the linear forms s_I, $I \in \mathcal{S}^+(2k)$, is $2k$.

We want to understand the pole contribution when $m \geq 1$. We separate two subcases:

Case 1 of the proof of Lemma 12.1: $N = 2k-1$. – In this case, we have that $s'_j = L_{2k-1,\{j\}}(s'_1) = a_j s'_1$ for all j, where $a_j \in \mathbb{Q}$. Then the pole occurs necessarily when $s'_1 = 0$. Thus $\mathcal{J}_{2k} = \mathcal{S}(2k)$, and we obtain an evaluation of $J(R)$ as finite linear combination of powers of $\log R$, the highest of which has exponent

$$X_{2k-1} + m - 2k = 2^{2k-1} - 2k - 1 - \sum_{I \in \mathcal{S}^-(2k)} h_{2k-1,I} - \nu,$$

up to an admissible error. The coefficients of the polynomial in $\log R$ are given in terms of the derivatives $L^{(j)}(1,\chi)$. Specifically, the coefficient of $(\log R)^{X_{2k-1}+m-2k-h}$, $0 \leq h \leq m-1$, is a linear combination of products of the form

$$\left(\frac{\varphi(q)}{q}\right)^{M_{2k}} \prod_{I \in \mathcal{S}^-(2k)} L^{(h_{2k,I})}(1,\chi),$$

with the coefficients of this linear combination being $\ll 1$, and with the parameters $h_{2k,I}$ satisfying $h_{2k,I} \geq h_{2k-1,I}$ with equality if $I \in \mathcal{J}_{2k-1} \setminus \{0\}$, and $\sum_{I \in \mathcal{S}^-(2k)} (h_{2k,I} - h_{2k-1,I}) \leq h$. Arguing as in the proof of (12.6), we find that

$$J(R) \ll \frac{(\log q)^{O(1)} (\log R)^{X_{2k-1}+m-2k-h}}{(\log Q)^{\#\{I \in \mathcal{S}^-(2k) : h_{2k,I} = 0\}}} \leq \frac{(\log q)^{O(1)} (\log R)^{2^{2k-1}-2k-1-\sum_{I \in \mathcal{S}^-(2k)} h_{2k,I}}}{(\log R)^{\#\{I \in \mathcal{S}^-(2k) : h_{2k,I} = 0\}}},$$

where we used that $Q \leq R$, $\nu \geq 0$ and $h \geq \sum_{I \in \mathcal{S}^-(2k)} (h_{2k,I} - h_{2k-1,I})$. We thus conclude that

$$J(R) \ll \frac{(\log q)^{O(1)}}{(\log R)^{2k+1}} \ll \frac{1}{\log R}.$$

This completes the proof of the lemma in this case (the linear combination is empty).

Case 2 of the proof of Lemma 12.1: $N \le 2k-2$. – Arguing as in the proof of part (b) of Lemma 8.4, the contribution of the pole $s'_{2k-N} = C(s'_1, \dots, s'_{2k-N-1})$ to $J(R)$ can be seen to be a linear combination of terms of the form

$$\left(\frac{\varphi(q)}{q}\right)^{M_{N+1}} \frac{(\log R)^{X_N+m-h-N-1-d}}{(2i\pi)^{2k-N-1}} \int\cdots\int_{\substack{\mathrm{Re}(s_j)=\lambda'_j/\log R,\,|\mathrm{Im}(s_j)|\le T\\ 1\le j\le 2k-N-1}} \widetilde{G}(\boldsymbol{s}) R^{E_{N+1}(\boldsymbol{s})} \times \prod_{I\in\mathcal{S}^-(2k)} L^{(h_{N+1,I})}(1+L_{N+1,I}(\boldsymbol{s}),\chi)$$
$$\times \prod_{I\in\mathcal{S}^+(2k)\setminus\mathcal{J}_{N+1}} \zeta_q^{(h_{N+1,I})}(1+L_{N+1,I}(\boldsymbol{s}))\mathrm{d}s_{2k-N-1}\cdots\mathrm{d}s_1$$

plus an error term of size $O(1/\log R)$, where $h \in \{0,\dots,m-1\}$, $h_{N+1,I} \ge h_{N,I}$ with equality if $I \in \mathcal{J}_{N+1}\setminus\{0\}$, and $\sum_{I\in\mathcal{S}^-(2k)\cup(\mathcal{S}^+(2k)\setminus\mathcal{J}_{N+1})}(h_{N+1,I}-h_{N,I}) \le h$. Relation (12.8) then implies that the power of $\log R$ is then $X_{N+1}-N-1-d'$ with

$$d' = d + \nu + h - \sum_{I\in\mathcal{S}^-(2k)\cup(\mathcal{S}^+(2k)\setminus\mathcal{J}_{N+1})} (h_{N+1,I}-h_{N,I}) \ge 0.$$

Moreover, $X_{N+1}-d' = X_N + m - h \ge N+1$. This completes the proof of part (a).

(b) Here $\gamma_{j_{N+1}} < 0$. We treat this case using the same argument as in part (b) of Lemma 8.4, with the difference that the contours of $s_{2k-N}, s_{2k-N-1}, \dots, s_{j_{N+1}}$ are shifted to the lines $\mathrm{Re}(s_j) = \lambda^j/((\log q)+(\log T)^{3/2})$, $j_{N+1} \le j \le 2k-N$. Since $q \le e^{\sqrt{\log R}}$ by assumption, we find that

$$J(R) \ll T^{-1+o(1)},$$

as needed. □

Proof of Lemma 12.2. – We distinguish three cases.

Case 1 of the proof of Lemma 12.2: $N = 2k$. – Here $\mathcal{J}_{2k} = \mathcal{S}(2k)$ and thus $J(R) \ll (\log q)^{O(1)}/(\log R)^{2k+1}$ by (12.6), which proves Lemma 12.2 in this case.

Case 2 of the proof of Lemma 12.2: $N = 2k-1$. – As in Case 1 of the proof of Lemma 12.1, we have that $s_j = L_{2k-1,\{j\}}(s_1) = a_j s_1$ for all j, where $a_j \in \mathbb{Q}$. Then

$$J(R) = \left(\frac{\varphi(q)}{q}\right)^{M_{2k-1}} \frac{(\log R)^{X_{2k-1}-2k+1-d}}{2\pi i} \prod_{I\in\mathcal{S}^-(2k)\cap\mathcal{J}_{2k-1}} L^{(h_{2k-1,I})}(1,\chi)$$
$$\times \int_{\substack{\mathrm{Re}(s_1)=\lambda_1/\log R\\ |\mathrm{Im}(s_1)|\le T}} G_{2k-1}(s_1) \prod_{I\in\mathcal{S}^-(2k)\setminus\mathcal{J}_{2k-1}} L^{(h_{2k-1,I})}(1+a_I s_1,\chi)$$
$$\times \prod_{I\in\mathcal{S}^+(2k)\setminus\mathcal{J}_{2k-1}} \zeta_q^{(h_{2k-1,I})}(1+a_I s_1)\mathrm{d}s_1.$$

We first show a crude bound on $J(R)$, that will allow us to focus on a more convenient subcase. We move the line of integration to $\mathrm{Re}(s_1) = \lambda/\log(qT)$ and use (12.3) to find that the integral over s_1 is $\ll (\log(q\log R))^{O(1)}$. Together with (12.6) and Proposition 7.1, this implies that

$$J(R) \ll (\log(q\log R))^{O(1)} \cdot (\log R)^{\binom{2k}{k}-2k-1},$$

unless $d = 0$, $h_{2k-1,I} \in \{0,1\}$ for all $I \in \mathcal{S}^-(2k) \cap \mathcal{J}_{2k-1}$, $h_{2k-1,I} = 0$ for all $I \in \mathcal{S}(2k) \setminus \mathcal{J}_{2k-1}$, half of the a_j's are $+b$ and the other half are $-b$, for some $b \neq 0$.

We have thus reduced proving the lemma to the case when $d = 0$, $h_{2k-1,I} \in \{0,1\}$ for all $I \in \mathcal{S}^-(2k) \cap \mathcal{J}_{2k-1}$, $h_{2k-1,I} = 0$ for all $I \in \mathcal{S}(2k) \setminus \mathcal{J}_{2k-1}$, half of the a_j's are $+b$ and the other half are $-b$, where $b \neq 0$. In particular, we find that $\mathcal{J}_{2k-1} \cap \mathcal{S}^-(2k) = \emptyset$ and that $\#(\mathcal{J}_{2k-1} \cap \mathcal{S}^+(2k)) = \binom{2k}{k} - 1$, so that

$$J(R) = \left(\frac{\varphi(q)}{q}\right)^{M_{2k-1}} \frac{(\log R)^{\binom{2k}{k}-2k}}{2\pi i} \int\limits_{\substack{\mathrm{Re}(s_1)=\lambda/\log R \\ |\mathrm{Im}(s_1)|\le T}} G(s_1) \prod_{I\in\mathcal{S}^-(2k)} L(1+a_I s_1,\chi) \times \prod_{I\in\mathcal{S}^+(2k)\setminus\mathcal{J}_{2k-1}} \zeta_q(1+a_I s_1)\mathrm{d}s_1.$$

Since $\nu = 0$ here, we saw before that the integrand has a genuine pole of order $m \ge 1$ at $s_1 = 0$ by a dimension argument. In fact, we have that $m \ge 2$: indeed, we know that $[2k] \in \mathcal{J}_{2k-1}$ by our assumption that $E_{2k-1} = 0$, so that $I \in \mathcal{J}_{2k-1}$ if and only if $[2k] \setminus I \in \mathcal{J}_{2k-1}$. In particular, since we know that there is at least one $I \in \mathcal{S}^+(2k) \setminus \mathcal{J}_{2k-1}$, there must be at least two such I's.

The presence of this pole makes the estimation of $J(R)$ tricky. In particular, we cannot shift the contour to the line $\mathrm{Re}(s_1) = 0$ as in Case 2 of Section 8.2. Instead, we write

$$L(1+a_I s_1,\chi) = L(\beta + a_I s_1,\chi) + \Delta(a_I s_1),$$

where

$$\Delta(s) := L(1+s,\chi) - L(\beta+s,\chi).$$

We thus find that

$$\Delta(s) = \int_\beta^1 L'(u+s,\chi)\mathrm{d}u \ll \frac{\log(q+|t|)}{\log Q} \quad (\sigma \ge -1/\log(q+|t|)),$$

by (12.3) and the assumption that $\beta > 1 - 1/(100\log q)$. With this notation,

$$J(R) = M + E,$$

where

$$M = \left(\frac{\varphi(q)}{q}\right)^{M_{2k-1}} \frac{(\log R)^{\binom{2k}{k}-2k}}{2\pi i} \int\limits_{\substack{\mathrm{Re}(s_1)=\lambda/\log R \\ |\mathrm{Im}(s_1)|\le T}} G_{2k-1}(s_1) \prod_{I\in\mathcal{S}^-(2k)} L(\beta+a_I s_1,\chi) \times \prod_{I\in\mathcal{S}^+(2k)\setminus\mathcal{J}_{2k-1}} \zeta_q(1+a_I s_1)\mathrm{d}s_1$$

and E is a sum of similar expressions where at least one of the $L(\beta + a_I s_1,\chi)$ factors is replaced by $\Delta(a_I s_1)$.

First, we bound E. Moving s_1 to the line $\mathrm{Re}(s_1) = 1/\log(qT)$, we find that

$$E \ll \frac{(\log(q\log R))^{O(1)}(\log R)^{\binom{2k}{k}-2k}}{\log Q} \le (\log(q\log R))^{O(1)}(\log R)^{\binom{2k}{k}-2k-1}.$$

Finally, we estimate M by moving the line of integration of s_1 to $\mathrm{Re}(s_1) = 0$, and use the fact that $G(s_1) \ll 1/(1+|s_1|)^{2k}$ (note that the integrand is now analytic, since the pole of the ζ_q's is annihilated by the zeroes of the $L(\cdot,\chi)$'s) to find that

$$J(R) = \left(\frac{\varphi(q)}{q}\right)^{M_{2k-1}} \frac{(\log R)^{\binom{2k}{k}-2k}}{2\pi} \int_{-T}^{T} G(it) \prod_{I\in\mathcal{S}^-(2k)} L(\beta + ia_I t,\chi)$$
$$\times \prod_{I\in\mathcal{S}^+(2k)\setminus\mathcal{I}_{2k-1}} \zeta_q(1+ia_I t)\mathrm{d}t + O((\log(q\log R))^{O(1)}(\log R)^{\binom{2k}{k}-2k-1}).$$

Note that $G(s) = F(a_1 s,\dots,a_{2k}s)$ here, because $d=0$. When $|t| \le \log R$, we use (6.6) to replace $\widehat{h}_R$ by R^s/s, and when $|t| > \log R$ we use the bound $\widehat{h}_R(s) \ll R^\sigma/|s|$ by (6.5). Taking C to be large enough, we thus conclude that

$$\frac{J(R)}{(\log R)^{\binom{2k}{k}-2k}} = \frac{(\varphi(q)/q)^{M_{2k-1}}}{2\pi} \int_{-\infty}^{\infty} F(ia_1 t,\dots,ia_{2k}t) \prod_{I\in\mathcal{S}^-(2k)} L(\beta+ia_I t,\chi)$$
$$\times \prod_{I\in\mathcal{S}^+(2k)\setminus\mathcal{I}_{2k-1}} \zeta_q(1+ia_I t) \prod_{j=1}^{2k} \frac{1-2^{-ia_j t}}{ia_j t}\mathrm{d}t + O\left(\frac{(\log(q\log R))^{O(1)}}{\log R}\right).$$

This completes the proof of Lemma 12.2 in this case.

Case 3 of the proof of Lemma 12.2: $N \le 2k-2$. – As in the corresponding case of the proof of Lemma 8.5, and using (12.6), we find that

$$J(R) \ll (\log(q\log R))^{O(1)}(\log R)^{X_N-N-d} \prod_{I\in\mathcal{S}^-(2k)\cap\mathcal{I}_N} |L^{(h_{N,I})}(1,\chi)|$$
$$\ll (\log(q\log R))^{O(1)}(\log R)^{\mathcal{A}(V_N)-N}.$$

Proposition 7.1(c) then implies that $\mathcal{A}(V_N) - N \le \binom{2k}{k} - 2k - 2$, thus completing the proof of Lemma 12.2. □

12.5. Lower bounds

In order to complete the proof of Theorem 1.5, we show that constant $c_k(\chi)$ in (12.5) is $\gg (\log q)^{-O(1)}$. In order to do so, we follow the argument of Section 9 and prove that, for any $\epsilon > 0$, there is a constant $c_k > 0$ such that

$$\mathcal{X}_{2k}(R) \ge c_k \frac{(\log R)^{\binom{2k}{k}-2k-\epsilon}}{(\log q)^{O(1)}} - O_\epsilon\left((\log(q\log R))^{O(1)}(\log R)^{\binom{2k}{k}-2k-1}\right), \tag{12.9}$$

provided that $\sqrt{\log Q} \ge \log R \ge 2c_1 \log q$, where c_1 is the constant appearing in (12.3). (For this section, all constants will be independent of ϵ, unless specified by a subscript, as above.)

We set

$$y = \exp\{(\log R)^{1-\epsilon'}\} \quad \text{and} \quad Y = \exp\{(\log R)^{1-\epsilon'/2}\},$$

where ϵ' will be taken to be small enough in terms of ϵ, and focus our attention on integers of the form $n = ap_1 \cdots p_k$ with $P^+(a) \le y$, $a \le Y$, and $p_1, \dots, p_k$ are distinct primes such that $p_\ell > \sqrt{R}$ and $\chi(p_\ell) = -1$ for all $\ell \in \{1, \dots, k\}$. Then

$$\mathcal{X}_{2k}(R) \ge \frac{\prod_{p \le R}(1 - 1/p)}{k!} \sum_{\substack{P^+(a) \le y \\ a \le Y}} \sum_{\substack{p_j > \sqrt{R} \\ \chi(p_j) = -1 \\ 1 \le j \le k}} \frac{\mu^2(p_1 \cdots p_k)}{ap_1 \cdots p_k} \left(\sum_{\ell=1}^{k} \sum_{\substack{R/(2p_\ell) < d \le R/p_\ell \\ d | a}} \chi(d) \right)^{2k}.$$

The next step is to drop the condition that $a \le Y$ by an application of Rankin's trick and to remove the condition that the p_j's are distinct. We further replace the sharp cut-off $R/(2p_\ell) < d \le R/p_\ell$ by the smooth cut-off $h(\frac{\log(dp_\ell)}{\log R}) - h(\frac{\log(2dp_\ell)}{\log R})$, where $h(x) = 1$ for $x \le 1 - 1/(\log R)^B$ and $h(x) = 0$ for $x \ge 1$, with B sufficiently large. To conclude, we have that

$$\mathcal{X}_{2k}(R) \ge \frac{\prod_{p \le R}(1 - 1/p)}{k!(\log R)^k} \sum_{P^+(a) \le y} \sum_{\substack{\sqrt{R} < p_j \le R \\ \chi(p_j) = -1 \\ 1 \le j \le k}} \frac{\prod_{j=1}^{k} \log p_j}{ap_1 \cdots p_k} \left(\sum_{\ell=1}^{k} \sum_{d | a} \chi(d) w \left(\frac{\log(dp_\ell)}{\log R} \right) \right)^{2k} + O\left(\frac{1}{\log R} \right),$$

where $w(x) = h(x) - h(x + \log 2/\log R)$.

The rest of the proof follows the argument of Section 9, with a small twist, as we will explain in the end. We expand the $2k$-th power and focus on a convenient subset of summands. We then conclude that

$$\mathcal{X}_{2k}(R) \ge \frac{\prod_{y < p \le R}(1 - 1/p)}{k!} \sum_{\boldsymbol{J} \in \mathcal{J}} X(\boldsymbol{J}) + O\left(\frac{1}{\log R} \right)$$

with

$$X(\boldsymbol{J}) = \frac{1}{(\log R)^k} \sum_{\substack{P^+(d_j) \le y \\ 1 \le j \le 2k}} \frac{\chi(d_1) \cdots \chi(d_{2k})}{[d_1, \dots, d_{2k}]}$$

$$\times \prod_{\ell=1}^{k} \sum_{\sqrt{R} < p_\ell \le R} \frac{(1 - \chi(p_\ell)) \log p_\ell}{2p_\ell} \prod_{j \in J_\ell} w\left(\frac{\log(p_\ell d_j)}{\log R} \right),$$

where $\boldsymbol{J}$ is as in Section 9. We set

$$S := \sum_{\sqrt{R} < p \le R} \frac{(1 - \chi(p)) \log p}{2p} \asymp \log R$$

and write $\mathcal{L}$ for the set of $\ell \in \{1, \dots, k\}$ such that $J_\ell \neq \emptyset$. Then, using Perron's formula $2k$ times to write each appearance of w as an integral of $\widehat{w}_R$, we find that

$$X(\boldsymbol{J}) = \frac{S^{k-\#\mathcal{L}}(\log R)^{-k}}{(2\pi i)^{2k}} \int \cdots \int_{\substack{\mathrm{Re}(s_j)=1/\log R\\ 1\le j\le 2k}} \sum_{\substack{P^+(d_j)\le y\\ 1\le j\le 2k}} \frac{\prod_{j=1}^{2k}\chi(d_j)d_j^{-s_j}}{[d_1,\dots,d_{2k}]}$$
$$\times \left(\prod_{\ell\in\mathcal{L}} \sum_{p_\ell>\sqrt{R}} \frac{(1-\chi(p_\ell))\log p_\ell}{2p_\ell^{1+s_{J_\ell}}}\right)\left(\prod_{j=1}^{2k}\widehat{w}_R(s_j)\right)\mathrm{d}s_1\cdots\mathrm{d}s_{2k} + O\left(\frac{1}{\log R}\right),$$

where $s_J = \sum_{j\in J} s_j$, as usual, and the condition that $p_\ell \le R$ was dropped because it is encoded in the support of w. By possibly re-indexing the variables $s_1, \dots, s_{2k}$, we may assume that $\mathcal{L} = \{1, \dots, L\}$, where $L = \#\mathcal{L}$, and that $\max J_\ell = 2k - L + \ell$ for all $\ell \in \{1, \dots, L\}$ with $L = \#\mathcal{L}$. As in Section 9, we will move the variables $s_{2k-L+1}, \dots, s_{2k}$ to the left. We note that

$$\sum_{p>\sqrt{R}} \frac{(1-\chi(p))\log p}{p^{1+s}} = -\frac{\zeta'}{\zeta}(1+s) + \frac{L'}{L}(1+s,\chi) + O(1) - \sum_{p\le R^{1/2}} \frac{\log p}{p^{1+s}}$$

for $\mathrm{Re}(s) \ge -1/3$. The above has simple poles at $s = 0$ and $s = \beta - 1$, each of residue 1. Therefore, using the argument leading to (9.12), we find that

(12.10)
$$X(\boldsymbol{J}) = \sum_{\substack{\epsilon_\ell\in\{0,\beta-1\}\\ 1\le\ell\le L}} \frac{S^{k-L}(\log R)^{-k}}{2^L(2\pi i)^{2k-L}} \int \cdots \int_{\substack{\mathrm{Re}(s_j)=1/\log R\ (1\le j\le 2k-L)\\ s_{J_\ell}=\epsilon_\ell\ (1\le\ell\le L)}} \sum_{\substack{P^+(d_j)\le y\\ 1\le j\le 2k}} \frac{\prod_{j=1}^{2k}\mu(d_j)d_j^{-s_j}}{[d_1,\dots,d_{2k}]}$$
$$\times \left(\prod_{j=1}^{2k}\widehat{w}_R(s_j)\right)\mathrm{d}s_1\cdots\mathrm{d}s_{2k-L} + O\left(\frac{1}{\log R}\right).$$

The above expression is sufficient for handling the terms $\boldsymbol{J} \in \mathcal{J}$ with at least one J_ℓ of odd cardinality: following the argument of Section 9.5 with the obvious modifications implies that

$$X(\boldsymbol{J}) \ll (\log(q\log R))^{O(1)} \frac{(\log y)^{\binom{2k}{k}-2k-1+L}}{(\log R)^L}, \tag{12.11}$$

where we used the fact that $\sup_{q<p\le Q}(1+\chi(p))/p \ll 1$.

However, we need to be more careful on our lower bound for the main term, that is to say for $X(\boldsymbol{J})$ with $\#J_\ell = 2$ for all ℓ. First of all, by relabeling our variables, we may assume that $J_\ell = \{\ell, \ell+k\}$ for all ℓ. In our expression (12.10) for $X(\boldsymbol{J})$, we see that $s_{J_\ell} = \epsilon_\ell \in \{0, \beta-1\}$ implies that $s_{\ell+k} = -s_\ell + O(1/\log Q)$. We want to replace $s_{\ell+k}$ by $-s_\ell$. This introduces an error that we will control by an application of the mean value theorem. In particular, we need to understand the derivative of the integrand. If

$$G(\boldsymbol{s}) = \sum_{\substack{P^+(d_j)\le y\\ 1\le j\le k}} \frac{\prod_{j=1}^{2k}\chi(d_j)d_j^{-s_j}}{[d_1,\dots,d_{2k}]},$$

then $-G'(\boldsymbol{s})/G(\boldsymbol{s})$ equals $\sum_{I\in\mathcal{S}^*(2k)}\sum_{p\le y}\chi^{\#I}(p)\log p/p^{1+s_I}$, plus lower order terms, so that $G'(\boldsymbol{s}) \ll (\log y)G(\boldsymbol{s})$ for the vectors $\boldsymbol{s}$ we are considering. Similarly, $\widehat{w}_R(s+\epsilon)/R^{s+\epsilon} = \widehat{w}_R(s)/R^s + O(\epsilon/(|s|+1))$ by (6.5). Since we also have that

$$\int\!\!\cdots\!\!\int_{\substack{\mathrm{Re}(s_j)=1/\log R\\ |s_{j+k}+s_j|\le 1-\beta\\ 1\le j\le k}} |G(\boldsymbol{s})|\prod_{j=1}^{k}\frac{|\mathrm{d}s_j|}{1+|s_j|^2} \ll (\log(q\log R))^{O(1)}\frac{(\log y)^{\binom{2k}{k}-k}}{(\log R)^k},$$

by the argument leading to (12.11), we conclude that

$$X(\boldsymbol{J}) = \frac{(1+R^{\beta-1})^k}{(2\pi i\log R)^k}\int\!\!\cdots\!\!\int_{\substack{\mathrm{Re}(s_j)=1/\log R\\ s_{j+k}=-s_j\\ 1\le j\le k}} G(\boldsymbol{s})\left(\prod_{j=1}^{2k}\widehat{w}_R(s_j)\right)\mathrm{d}s_1\cdots\mathrm{d}s_k$$
$$+\,O\left(\frac{\log y}{\log Q}\cdot\frac{(\log(q\log R))^{O(1)}(\log y)^{\binom{2k}{k}-k}}{(\log R)^k}\right).$$

The main term can now be bounded from below as in Section 9.4. We thus arrive to the lower bound

$$X(\boldsymbol{J}) \ge c_k\frac{(\log y)^{\binom{2k}{k}-k}(\log R)^{-k}}{(\log q)^{O(1)}} - O\left(\frac{(\log(q\log R))^{O(1)}(\log y)^{\binom{2k}{k}-k+1}}{(\log R)^{k+2}}\right),$$

using that $\sum_{q<p\le Q}(1+\chi(p))/p \ll 1$ and $y \le R \le e^{\sqrt{\log Q}}$ here. This completes the proof of (12.9), and thus of Theorem 1.5(c).

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TAUTOLOGICAL CLASSES WITH TWISTED COEFFICIENTS

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ABSTRACT. – Let M_g be the moduli space of smooth genus g curves. We define a notion of Chow groups of M_g with coefficients in a representation of $\mathrm{Sp}(2g)$, and we define a subgroup of tautological classes in these Chow groups with twisted coefficients. Studying the tautological groups of M_g with twisted coefficients is equivalent to studying the tautological rings of all fibered powers C_g^n of the universal curve $C_g \to M_g$ simultaneously. By taking the direct sum over all irreducible representations of the symplectic group in fixed genus, one obtains the structure of a twisted commutative algebra on the tautological classes. We obtain some structural results for this twisted commutative algebra, and we are able to calculate it explicitly when $g \leq 4$. Thus we completely determine the tautological rings of all fibered powers of the universal curve over M_g in these genera. We also give some applications to the Faber conjecture.

RÉSUMÉ. – Notons par M_g l'espace de modules des courbes lisses de genre g. Nous définissons une notion de groupes de Chow de M_g à coefficients dans une représentation de $\mathrm{Sp}(2g)$, et nous définissons en outre un sous-groupe de classes tautologiques dans ces groupes de Chow à coefficients tordus. L'étude des groupes tautologiques de M_g à coefficients tordus est équivalente à l'étude simultanée des anneaux tautologiques de toutes les puissances fibrées C_g^n de la courbe universelle $C_g \to M_g$. En prenant la somme directe de toutes les représentations irréductibles du groupe symplectique en genre fixe, on obtient sur les classes tautologiques la structure d'une algèbre tordue commutative. Nous obtenons des résultats structurels pour cette algèbre tordue commutative, et nous la calculons explicitement lorsque $g \leq 4$. Ainsi, nous déterminons complètement les anneaux tautologiques de toutes les puissances fibrées de la courbe universelle sur M_g pour $g \leq 4$. Quelques applications à la conjecture de Faber sont données.

1. Introduction

Say that $g \geq 2$, and let C_g^n be the moduli space of smooth genus g curves with n ordered not necessarily distinct marked points. Equivalently, C_g^n is the n-fold fibered power of the universal curve over M_g with itself. Suppose that we want to study the cohomology of the spaces C_g^n. A natural approach is to apply the Leray-Serre spectral sequence for the fibration

$f\colon C_g^n \to M_g$ that forgets the n markings. Since f is smooth and proper, the spectral sequence degenerates by Deligne's decomposition theorem [8], and

$$H^k(C_g^n, \mathbf{Q}) \cong \bigoplus_{p+q=k} H^p(M_g, R^q f_*\mathbf{Q}).$$

To each dominant weight λ of $\mathrm{Sp}(2g)$ there is associated a local system $\mathbb{V}_{\langle\lambda\rangle}$ on A_g, the moduli space of principally polarized abelian varieties of dimension g. The sheaves $R^q f_*\mathbf{Q}$ decompose into direct sums of local systems $\mathbb{V}_{\langle\lambda\rangle}$, where we use the notation $\mathbb{V}_{\langle\lambda\rangle}$ also for their pullback along the Torelli map. The local systems $\mathbb{V}_{\langle\lambda\rangle}$ occuring as summands of $Rf_*\mathbf{Q}$ are precisely those with $|\lambda| \leq n$.

It follows that the two collections of cohomology groups

$$H^\bullet(C_g^n, \mathbf{Q}) \text{ for } n \leq N \qquad \text{and} \qquad H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \text{ for } |\lambda| \leq N$$

contain more or less the same information. However, this information is "packaged" in a much more efficient way in the local systems. The cohomology groups of C_g^n are generally very large, but when expressed in terms of local systems we see that most of the cohomology just encodes how the complex $Rf_*\mathbf{Q}$ decomposes into summands—that is, it encodes the Künneth formula for the n-fold self-product of a genus g curve, and some representation theory of $\mathrm{Sp}(2g)$. By studying the local systems we may focus our attention on the "interesting" part of the cohomology in a systematic way.

Our first goal of this paper is to do the same thing for the *tautological rings* of C_g^n. We remind the reader that the tautological ring $R^\bullet(C_g^n)$ is the subalgebra of $\mathrm{CH}^\bullet(C_g^n)$ generated by the classes of the diagonal loci Δ_{ij} where two markings coincide, the classes $\psi_1, \ldots, \psi_n$ which are the Chern classes of the n cotangent line bundles at the marked points, and the Morita-Mumford-Miller classes κ_d. The image of $R^\bullet(C_g^n)$ in cohomology under the cycle class map is denoted $RH^\bullet(C_g^n)$.

We will be able to define tautological cohomology groups $RH^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \subseteq H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle})$, with the property that the collections of tautological groups

$$RH^\bullet(C_g^n) \text{ for } n \leq N \qquad \text{and} \qquad RH^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \text{ for } |\lambda| \leq N$$

bear exactly the same relation to each other as the collections of cohomology groups $H^\bullet(C_g^n, \mathbf{Q})$ and $H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle})$. Thus we are able to decompose the tautological groups of C_g^n into pieces indexed by local systems; the tautological groups of the local systems package all the information about the tautological groups of C_g^n in a much more efficient way, and working with twisted coefficients allows us to "zoom in" on particularly interesting parts of the tautological groups. Moreover, the groups $RH^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle})$ turn out to be more computable than the groups $RH^\bullet(C_g^n)$.

In fact, we will actually not only do this on the level of cohomology groups, but for Chow groups. (The results are new already on the level of cohomology, though.) For this we should not work with local systems on M_g, but with relative Chow motives over the base M_g. Instead of decomposing the complex $Rf_*\mathbf{Q}$ into local systems $\mathbb{V}_{\langle\lambda\rangle}$, we will decompose the Chow motive $h(C_g^n/M_g)$ into Chow motives $\mathbf{V}_{\langle\lambda\rangle}$ which are motivic lifts of the local systems $\mathbb{V}_{\langle\lambda\rangle}$. Once the correct framework is in place, working with motives rather than local systems provides no extra difficulties.

The utility of working with the local systems is illustrated by our Theorem 10.1, in which we completely determine all tautological groups with twisted coefficients when $g = 2, 3, 4$. It is an easy matter to compute from Theorem 10.1 the ranks of all the groups $R^k(C_g^n)$ when $g \leq 4$, the decompositions of these tautological groups into $\mathfrak{S}_n$-representations, and the socle pairing. Thus a lot of useful information about the tautological rings is encoded in a few lines of information about the local systems.

Since the tautological rings are defined in terms of explicit generators, understanding the tautological rings is equivalent to finding the complete list of relations between these generators. A conjectural complete description of the tautological rings was formulated by Faber [12]. Namely, a theorem of Looijenga [43] asserts that $R^{g-2+n}(C_g^n) \cong \mathbf{Q}$, and that the tautological ring vanishes above this degree. Thus any two monomials of degree $g-2+n$ in the generators of the tautological ring are proportional to each other, and the proof of the $\lambda_g\lambda_{g-1}$-conjecture [19, 21] gives explicit proportionalities. (In fact, both Looijenga's theorem and the proportionalities were part of Faber's original conjecture.) What Faber then conjectured was that any possible relation which is consistent with the pairing into the top degree is a true relation; that is, the ring $R^\bullet(C_g^n)$ should satisfy Poincaré duality. The general belief now is that this conjecture should fail. One reason is that the original conjecture was later extended to a "trinity" of conjectures for the spaces $M_{g,n}^{\mathrm{rt}}$, $M_{g,n}^{\mathrm{ct}}$ and $\overline{M}_{g,n}$ [56, 13], and the conjectures for $\overline{M}_{2,n}$ and $M_{2,n}^{\mathrm{ct}}$ are known to fail when $n \geq 20$ and $n \geq 8$, respectively [62, 60]. The Faber conjecture for the spaces $M_{g,n}^{\mathrm{rt}}$ is equivalent to the Faber conjecture for C_g^n [59], and is still open. It has more recently been conjectured that Pixton's extension of the FZ relations (see Section 9) gives rise to all relations between tautological classes, and this conjecture is known to contradict the Faber conjecture [63].

An interesting aspect of our work is that even though the decomposition of the tautological groups $R^\bullet(C_g^n)$ into pieces indexed by representations of $\mathrm{Sp}(2g)$ is not compatible with the ring structure, the multiplication into the top degree behaves very well: the matrix describing the top degree pairing is block diagonal with respect to our decomposition of the tautological groups. This has the consequence that the Faber conjecture can be fruitfully studied from the perspective of the motives $\mathbf{V}_{\langle\lambda\rangle}$—Poincaré duality can be checked for each $\mathbf{V}_{\langle\lambda\rangle}$ separately. Using this we show that the Faber conjecture is true for the moduli space C_g^n (hence also the space $M_{g,n}^{\mathrm{rt}}$) when $g \leq 4$ and n is arbitrary, and we make some progress in trying to understand likely failures of the Faber conjectures in higher genera.

A completely different perspective on our results is provided by work of Kawazumi-Morita and Hain. For a fixed genus $g \geq 2$, one can define a structure of commutative ring on the direct sum

$$\mathsf{T}_g = \bigoplus_\lambda H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes \mathbb{V}_{\langle\lambda\rangle}^*,$$

where the direct sum is taken over all dominant weights λ of $\mathrm{Sp}(2g)$. Let $\mathsf{A}_g = \bigwedge \mathbb{V}_{\langle 1,1,1\rangle}^*/(\mathbb{V}_{\langle 2,2\rangle}^*)$ denote the exterior algebra on the representation $\mathbb{V}_{\langle 1,1,1\rangle}^*$, modulo the ideal generated by the subrepresentation $\mathbb{V}_{\langle 2,2\rangle}^* \subset \bigwedge^2 \mathbb{V}_{\langle 1,1,1\rangle}^*$. If $\mathbb{V}_{\langle 1,1,1\rangle}^*$ is placed in degree 1, then one can define a natural $\mathrm{Sp}(2g)$-equivariant homomorphism of graded commutative rings $\varphi\colon \mathsf{A}_g \to \mathsf{T}_g$. In particular, we get a morphism between the subalgebras of symplectic invariants,

$$\varphi^{\mathrm{Sp}(2g)}\colon \mathsf{A}_g^{\mathrm{Sp}(2g)} \to \mathsf{T}_g^{\mathrm{Sp}(2g)} = H^\bullet(M_g, \mathbf{Q}).$$

According to a theorem of Kawazumi-Morita [38] the image of $\varphi^{\mathrm{Sp}(2g)}$ coincides with the tautological cohomology ring of M_g. For this reason it is natural, following Hain [25], to define the image of φ as the "tautological subalgebra" $\mathsf{R}_g \subset \mathsf{T}_g$. By considering the individual summands of R_g one obtains an a priori completely different definition of the tautological subgroup of $H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle})$. We prove in Theorem 12.8 that the two definitions coincide, which in particular re-proves the theorem of Kawazumi-Morita. Our low genus results can be seen as calculations of the ring R_g for $g \leq 4$; these are the first nontrivial cases where this ring is completely known. A consequence of our general results is that the morphism $\varphi\colon \mathsf{A}_g \to \mathsf{T}_g$ can be lifted to take values in Chow groups rather than cohomology groups, answering a question of Hain. A final remark is that Morita has conjectured [51] that the morphism $\varphi^{\mathrm{Sp}(2g)}$ is injective, so that $\mathsf{A}_g^{\mathrm{Sp}(2g)}$ is *isomorphic* to the tautological ring of M_g for any g. By extension it is natural to ask also whether φ is injective. Our results in genus four show that this is not the case, however: $\mathsf{A}_4 \to \mathsf{R}_4$ is not an isomorphism.

Let us now state in some more detail what we do in this paper:

1. For any partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0$, we construct a relative Chow motive $\mathbf{V}_{\langle\lambda\rangle}$ over the moduli space M_g, which is a motivic version of the local system over M_g associated to a representation of $\mathrm{Sp}(2g)$ of highest weight λ.
2. For any $n \geq 0$, we let $h(C_g^n/M_g)$ be the relative Chow motive over M_g given by the n-fold fibered power of the universal curve. We prove that there exists a direct sum decomposition

$$h(C_g^n/M_g) \cong \bigoplus_i \mathbf{V}_{\langle\lambda_i\rangle} \otimes \mathbb{L}^{m_i}, \tag{1}$$

   where $\mathbb{L}$ denotes the Lefschetz motive, and in particular we get upon taking Chow groups

$$\mathrm{CH}^k(C_g^n) \cong \bigoplus_i \mathrm{CH}^{k-m_i}(M_g, \mathbf{V}_{\langle\lambda_i\rangle}). \tag{2}$$

3. We construct an algebra of correspondences defined by tautological classes, which acts on the motives $h(C_g^n/M_g)$, and hence on the Chow groups $\mathrm{CH}^k(C_g^n)$. Using this algebra we obtain a *canonical* choice of decomposition (1), and a method for computing the projection of any class in $\mathrm{CH}^k(C_g^n)$ into any particular summand on the right hand side of (2).
4. We define subgroups $R^k(M_g, \mathbf{V}_{\langle\lambda\rangle}) \subseteq \mathrm{CH}^k(M_g, \mathbf{V}_{\langle\lambda\rangle})$ with the property that for any decomposition of $h(C_g^n/M_g)$ as in Eq. (1), we have

$$R^k(C_g^n) \cong \bigoplus_i R^{k-m_i}(M_g, \mathbf{V}_{\langle\lambda_i\rangle}).$$

   We call the groups $R^k(M_g, \mathbf{V}_{\langle\lambda\rangle})$ the *tautological groups of M_g with twisted coefficients.* Informally, all information about the tautological rings $R^\bullet(C_g^n)$ is contained in the groups $R^k(M_g, \mathbf{V}_{\langle\lambda\rangle})$.
5. The motives $\mathbf{V}_{\langle\lambda\rangle}$ come with a duality pairing $\mathbf{V}_{\langle\lambda\rangle} \otimes \mathbf{V}_{\langle\lambda\rangle} \to \mathbb{L}^{|\lambda|}$, which is the motivic avatar of the fact that all representations of the symplectic group are self-dual. We

prove that the socle pairing

$$R^k(C_g^n) \otimes R^{g-2+n-k}(C_g^n) \to R^{g-2+n}(C_g^n) \cong \mathbf{Q} \tag{3}$$

is a direct sum of pairings of the form

$$R^k(M_g, \mathbf{V}_{\langle\lambda\rangle}) \otimes R^{g-2+|\lambda|-k}(M_g, \mathbf{V}_{\langle\lambda\rangle}) \to R^{g-2+|\lambda|}(M_g, \mathbb{L}^{|\lambda|}) = R^{g-2}(M_g) \cong \mathbf{Q} \tag{4}$$

for $|\lambda| \leq n$. In particular, $R^\bullet(C_g^n)$ is a Gorenstein algebra—that is, (3) is a perfect pairing for all k—if and only if (4) is a perfect pairing for all k and all $|\lambda| \leq n$.

6. If we fix a genus and consider the *direct sum* over all partitions,

$$\bigoplus_\lambda R^\bullet(M_g, \mathbf{V}_{\langle\lambda\rangle}) \otimes \sigma_{\lambda^T},$$

where σ_{λ^T} denotes the representation of the symmetric group corresponding to the partition conjugate to λ, we obtain the structure of a *twisted commutative algebra*. The $\lambda = 0$ component of this twisted commutative algebra is just the tautological ring of M_g. We prove using the FZ relations that this twisted commutative algebra is finitely generated with an explicit bound for the degrees of the generators, which for $\lambda = 0$ specializes to the theorem of Ionel-Morita [31, 50].

7. For $g = 2, 3, 4$ we completely determine the groups $R^k(M_g, \mathbf{V}_{\langle\lambda\rangle})$ for all k and λ. A key input is that the twisted commutative algebra described in point (vi) above will in these low genera have only 0, 2 and 3 generators, respectively, by our generalization of the theorem of Ionel-Morita. As a consequence we can compute the ranks $R^k(C_g^n)$ for all k and n in these genera, and how these tautological groups decompose into irreducible representations of $\mathfrak{S}_n$. It also follows that $R^\bullet(C_g^n)$ is always a Gorenstein algebra in these genera.

The algebra of projectors described in point (iii) above seems like a particularly powerful tool for "zooming in" on a specific part of the tautological ring. As explained in Section 10.4 we expect that the Faber conjecture fails when $g = 5$ and $n = 8$ for the motives $\mathbf{V}_{\langle 2,2,2,2\rangle}$ and $\mathbf{V}_{\langle 3,2,2,1\rangle}$. Using our algebra of projectors we can project specific tautological classes onto these summands, which gives explicit classes which pair to zero with everything in complementary degree but which are conjecturally nonzero.

It would be interesting to try to extend our results from M_g to the Deligne-Mumford compactification $\overline{M}_g$. On the level of cohomology this would correspond to studying the forgetful maps $f: \overline{M}_{g,n} \to \overline{M}_g$ rather than $C_g^n \to M_g$. Then f is no longer smooth, but still proper, so by the decomposition theorem [3] the complex $Rf_*\mathbf{Q}$ is a direct sum of perverse sheaves. In fact, each of these perverse sheaves will be the pushforward along some gluing map

$$\Big(\prod_{v \in \mathrm{Vert}(\Gamma)} \overline{M}_{g(v),n(v)}\Big)/\mathrm{Aut}(\Gamma) \to \overline{M}_{g,n}$$

of the intermediate extension of a product of local systems associated to representations of the smaller symplectic groups $\mathrm{Sp}(2g(v))$. This suggests that one should try to define a subspace of tautological classes inside the intersection cohomology groups $IH^\bullet(\overline{M}_g, \mathbb{V}_{\langle\lambda\rangle})$ for each genus g and dominant weight λ, and that these tautological groups should "govern" all of the tautological groups $R^\bullet(\overline{M}_{g,n})$ much in the same way as the tautological groups of $\mathbb{V}_\lambda$ on M_g govern all the tautological groups $R^\bullet(C_g^n)$. Similarly there should be tautological classes inside $IH^\bullet(M_g^{\mathrm{ct}}, \mathbb{V}_{\langle\lambda\rangle}) = H^\bullet(M_g^{\mathrm{ct}}, \mathbb{V}_{\langle\lambda\rangle})$ which govern all the tautological rings $R^\bullet(M_{g,n}^{\mathrm{ct}})$. The result [60, Theorem 3.4] can be seen as calculating the tautological subspace of $H^\bullet(M_2^{\mathrm{ct}}, \mathbb{V}_{\langle\lambda\rangle})$ for all λ. Moreover, it should be possible to carry out the suggestions in this paragraph also on the level of Chow groups, using the intersection chow motives of Corti-Hanamura [7].

1.1. How to read this paper

As mentioned already, this paper is written in the language of Chow motives. Readers who would prefer not to know what a motive is should still be able to follow the arguments by translating the arguments to cohomology using the following table:

Chow motive $h(X/S)$ of a family $f\colon X \to S$	Complex $Rf_*\mathbf{Q}$ in the derived category of S
Decomposition $h(X/S) \cong \bigoplus_i h^i(X/S)$	Decomposition $Rf_*\mathbf{Q} \cong \bigoplus_i R^i f_*\mathbf{Q}[-i]$
Chow group $\mathrm{CH}^k(S, h^i(X/S))$	Cohomology group $H^{2k-i}(S, R^i f_*\mathbf{Q})$
Lefschetz motive $\mathbb{L}$	Constant sheaf $\mathbf{Q}$, considered as a complex concentrated in degree 2
Chow motive $\mathbf{V}_{\langle\lambda\rangle}$ over M_g	Local system $\mathbb{V}_{\langle\lambda\rangle}$ on M_g, considered as a complex concentrated in degree $\lvert\lambda\rvert$

Thus the only real complication is the indexing: the kth Chow group of the motive $\mathbf{V}_{\langle\lambda\rangle} \otimes \mathbb{L}^i$ corresponds to the $(2k - \lvert\lambda\rvert - 2i)$th cohomology group of the local system $\mathbb{V}_{\langle\lambda\rangle}$.

Sections 2–4 of this paper explain necessary preliminary material from the representation theory of the symplectic group and about Chow motives. In particular, Section 4 explains a result of Ancona which is used to lift our methods from cohomology to Chow groups. It could be a good idea for the reader to start from Section 5 and refer back to the previous sections only as needed. Section 5 provides the theoretical backbone to the article, and Section 6 provides some simple (hopefully instructive) example calculations. In Sections 7–10 the theory is applied and our main results are proven. The concluding Sections 11 and 12 explain the relationship between what we do and previous work of Looijenga, Hain, Morita, and Kawazumi.

1.2. Conventions

Representations of groups will be considered as left representations unless specified otherwise. However, if V is a left representation, then we consider its dual V^* as a right representation. (Recall that the dual of a left module over a noncommutative ring is a right module, and vice versa.)

Chow groups are always taken with rational coefficients.

We occasionally consider cohomology groups as well as Chow groups. Although we write cohomology with rational coefficients throughout, it will be clear that all results could have been carried out equally well in the étale setting, with coefficients in $\mathbf{Q}_\ell$.

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2. Chow motives

The results of this paper will be formulated in the language of Chow motives. The first parts of this section briefly recall standard definitions for the reader's convenience and to fix conventions. For a more detailed and motivated introduction see e.g., [66].

2.1. The category of Chow motives

Let S be a smooth connected scheme or Deligne-Mumford stack over a field k that we assume algebraically closed for simplicity. Let X and Y be smooth proper schemes over S.[(1)] We define a graded vector space of *correspondences over* S as follows: if X is connected and $X \to S$ is of relative dimension d then $\mathsf{Corr}_S(X,Y) = \mathrm{CH}^{d+\bullet}(X \times_S Y)$; in the general case we define $\mathsf{Corr}_S(\coprod_\alpha X_\alpha, Y) = \prod_\alpha \mathsf{Corr}_S(X_\alpha, Y)$. We write

$$f\colon X \vdash Y$$

to denote that f is a correspondence from X to Y. The composition of $f\colon X \vdash Y$ and $g\colon Y \vdash Z$ is defined by

$$g \circ f = (p_{13})_*(p_{12}^*(f) \cdot p_{23}^*(g)),$$

(note the reversed ordering!) where p_{ij} denotes the projection from $X \times_S Y \times_S Z$ onto the ith and jth factor of the fibered product. One checks that composition of correspondences is associative and that the diagonal, considered as a correspondence $X \vdash X$, acts as the identity id_X, so that Corr_S is a category.

We say that a correspondence $p\colon X \vdash X$ of degree 0 is *idempotent* if $p \circ p = p$. We also say that p is a *projector*.

We define the category Mot_S of *Chow motives over* S. The objects of Mot_S are triples (X, p, n) where X is smooth and proper over S, $p\colon X \vdash X$ is a projector, and $n \in \mathbf{Z}$. Morphisms are defined by

$$\mathsf{Mot}_S((X,p,n),(Y,q,m)) = q \circ \mathsf{Corr}_S^{m-n}(X,Y) \circ p \subset \mathsf{Corr}_S(X,Y),$$

where $\mathsf{Corr}_S^r(X,Y)$ denotes the degree r part of $\mathsf{Corr}_S(X,Y)$, and $q \circ \mathsf{Corr}_S^{m-n}(X,Y) \circ p$ denotes the joint image of the projectors p and q acting on $\mathsf{Corr}_S(X,Y)$ on the right and on the left, respectively.

The *Lefschetz motive over* S is defined as $(S, \mathrm{id}, -1)$ and will be denoted by $\mathbb{L}_S$. If S is clear from context we will omit the subscript and write $\mathbb{L}$.

We define a tensor product on motives as follows. If $M = (X,p,n)$ and $N = (Y,q,m)$ then $M \otimes N = (X \times_S Y, p \times q, n+m)$. This makes Mot_S a symmetric monoidal category with monoidal unit $\mathbf{1} = (S, \mathrm{id}, 0)$. The category is in fact rigid symmetric monoidal, i.e., every object has a dual: if X is of pure dimension d over S, then the dual of $M = (X,p,n)$ is

(1) If S is a Deligne-Mumford stack we do not impose the condition that X and Y are schemes, only that the maps to S are representable in schemes.

$M^* = (X, p^t, d-n)$, where p^t denotes the transpose correspondence. The category also has direct sums. The sum $(X, p, n) \oplus (Y, q, m)$ is the easiest to define when $n = m$, in which case it is given by $(X \sqcup Y, p \oplus q, n)$.

Let $\mathcal{V}_S$ be the category of smooth proper schemes over S. There is a contravariant functor $\mathcal{V}_S \to \mathsf{Mot}_S$ which is given on objects by $X \mapsto (X, \mathrm{id}, 0)$ and which maps an S-morphism $f: Y \to X$ to the class of the transpose of its graph in $\mathrm{CH}^{\dim_S(X)}(X \times_S Y)$. For $X \to S$ smooth and proper we denote by $h(X/S)$ the corresponding Chow motive over S. Note that $h(X/S)^* \cong h(X/S) \otimes \mathbb{L}^{-\dim_S X}$ (Poincaré duality).

2.2. Chow groups and cohomology groups of a Chow motive

Let M be a Chow motive over S. We define its Chow groups by $\mathrm{CH}^k(S, M) = \mathsf{Mot}_S(\mathbb{L}_S^k, M)$. We can make this definition more explicit as follows. Note that for X a smooth proper scheme over S we have $\mathrm{CH}^\bullet(X) = \mathsf{Corr}_S(S, X)$. As such, the algebra $\mathsf{Corr}_S(X, X)$ acts on the Chow groups of X on the left. Let $M = (X, p, n)$ be a Chow motive over S. Then its Chow groups are given by

$$\mathrm{CH}^k(S, M) = p \circ \mathrm{CH}^{k+n}(X).$$

Remark 2.1. – It is also true that $\mathrm{CH}^\bullet(X) = \mathsf{Corr}_S(X, S)$ (up to a degree shift), so that $\mathsf{Corr}_S(X, X)$ acts on the Chow groups of X on the right. One could also define $\mathrm{CH}^k(S, M) = \mathrm{CH}^{k+n}(X) \circ p^t$, where p^t denotes the transpose correspondence of p.

Let us suppose that S is a complex algebraic variety. There is a *Betti realization* functor $\mathsf{real} : \mathsf{Mot}_S \to D^b(S)$ into the bounded derived category of sheaves of $\mathbf{Q}$-vector spaces on S. For $\lambda: X \to S$ a smooth proper scheme over S we have

$$\mathsf{real}\, h(X/S) = R\lambda_* \mathbf{Q}.$$

The algebra $\mathsf{Corr}_S(X, X)$ acts on the complex $R\lambda_*\mathbf{Q}$, and for $p: X \vdash X$ idempotent we define $\mathsf{real}\,(X, p, n) = \mathrm{Im}(p_*: R\lambda_*\mathbf{Q} \to R\lambda_*\mathbf{Q})[2n]$, where $[2n]$ denotes the suspension functor in $D^b(S)$. There is a *cycle class map*

$$\mathrm{CH}^k(S, M) \to \mathbb{H}^{2k}(S, \mathsf{real}\, M)$$

(where $\mathbb{H}$ denotes hypercohomology) which on motives of the form $h(X/S)$ agrees with the usual cycle class map:

$$\mathrm{CH}^k(X) = \mathrm{CH}^k(S, h(X/S)) \to \mathbb{H}^{2k}(S, \mathsf{real}\, M) = \mathbb{H}^{2k}(S, R\lambda_*\mathbf{Q}) = H^{2k}(X, \mathbf{Q}).$$

Over an arbitrary field there is an analogous realization functor from Mot_S to the derived category of étale $\mathbf{Q}_\ell$-sheaves on S.

2.3. Künneth decomposition, the summands h^0 and h^{2d}

Let $\lambda\colon X \to S$ be smooth, proper and purely of relative dimension d. By Deligne's theorem [8] there is an isomorphism in $D^b(S)$:

$$R\lambda_*\mathbf{Q} \cong \bigoplus_{i=0}^{2d} R^i\lambda_*\mathbf{Q}[-i].$$

In particular, this decomposition implies that the Leray spectral sequence for λ degenerates, and $H^k(X,\mathbf{Q}) \cong \bigoplus_{p+q=k} H^p(S, R^q\lambda_*\mathbf{Q})$. It is expected that this decomposition always lifts to the category of Chow motives. Thus there should be an isomorphism

$$h(X/S) \cong \bigoplus_{i=0}^{2d} h^i(X/S)$$

for which $\mathsf{real}\, h^i(X/S) \cong R^i\lambda_*\mathbf{Q}[-i]$. In particular one would have

$$\mathrm{CH}^k(X) \cong \bigoplus_{i=0}^{2d} \mathrm{CH}^k(S, h^i(X/S)).$$

The summands h^0 and h^{2d} can easily be constructed unconditionally. Let us suppose that X is connected, and let $\mathfrak{z} \in \mathrm{CH}^d(X)$ be a cycle of degree 1 on each fiber of $X \to S$, e.g., a section. One checks that the two correspondences $X \vdash X$ given by

$$\pi_0 = [\mathfrak{z} \times X] \qquad \text{and} \qquad \pi_{2d} = [X \times \mathfrak{z}]$$

are idempotent. If we define $h^0(X/S) = (X, \pi_0, 0)$ and $h^{>0}(X/S) = (X, \mathrm{id}_X - \pi_0, 0)$ then

$$h(X/S) \cong h^0(X/S) \oplus h^{>0}(X/S)$$

which on realizations gives the decomposition

$$R\lambda_*\mathbf{Q} \cong R^0\lambda_*\mathbf{Q} \oplus \tau_{\geq 1}R\lambda_*\mathbf{Q},$$

where τ denotes a truncation functor in the derived category $D^b(S)$. Similarly we get decompositions $h(X/S) \cong h^{<2d}(X/S) \oplus h^{2d}(X/S)$ with realization $\tau_{\leq 2d-1}R\lambda_*\mathbf{Q} \oplus R^{2d}\lambda_*\mathbf{Q}[-2d]$.

LEMMA 2.2. – *Let X and $\mathfrak{z}$ be as above. Then $h^0(X/S) \cong \mathbf{1}$ and $h^{2d}(X/S) \cong \mathbb{L}^d$.*

Proof. – We prove only the second isomorphism. By definition we have

$$\mathsf{Mot}_S(\mathbb{L}^d, h^{2d}(X/S)) = \pi_{2d} \circ \mathrm{CH}^d(X)$$
$$\mathsf{Mot}_S(h^{2d}(X/S), \mathbb{L}^d) = \mathrm{CH}^0(X) \circ \pi_{2d}.$$

It is clear that $\pi_{2d} \circ \mathfrak{z} = \mathfrak{z}$ and $1 \circ \pi_{2d} = \pi_0 \circ 1 = 1$ (cf. Remark 2.1). As such the cycle $\mathfrak{z}$ and the fundamental class 1 define morphisms $\mathbb{L}^d \to h^{2d}(X/S) \to \mathbb{L}^d$. Moreover, their composition in $\mathsf{Mot}_S(\mathbb{L}^d, \mathbb{L}^d) = \mathrm{CH}^0(S)$ is given by $\lambda_*(\mathfrak{z}) = 1$, the identity. Their composition in $\mathsf{Mot}_S(h^{2d}(X/S), h^{2d}(X/S)) = \pi_{2d} \circ \mathrm{CH}^d(X \times_S X) \circ \pi_{2d}$ is given by the correspondence π_{2d}, which is also the identity. □

One might want to define a motive $h^\star(X/S) = (X, \mathrm{id} - \pi_0 - \pi_{2d})$ to get a decomposition $h(X/S) \cong h^0(X/S) \oplus h^\star(X/S) \oplus h^{2d}(X/S)$, where the Betti realization of $h^\star(X/S)$ is the complex $\tau_{\geq 1}\tau_{\leq 2d-1} R\lambda_*\mathbf{Q}$. Unfortunately the correspondence $\mathrm{id} - \pi_0 - \pi_{2d}$ is not in general idempotent, since π_0 and π_{2d} are not in general orthogonal when the base scheme S is nontrivial; this will be clear from the proof of the following lemma. To make the projectors orthogonal one needs to slightly modify the cycle $\mathfrak{z}$.

LEMMA 2.3. – *Let $\lambda: X \to S$ be as above. Let $\mathfrak{z} \in \mathrm{CH}^d(X)$ be a cycle of degree 1 on each fiber over S. Define $\mathfrak{z}' = \mathfrak{z} - \frac{1}{2}\lambda^*\lambda_*(\mathfrak{z}^2)$. Then $\mathfrak{z}'$ also has degree 1 on each fiber, the projectors $\pi_0 = [\mathfrak{z}' \times X]$, $\pi_{2d} = [X \times \mathfrak{z}']$ are orthogonal, and there is a decomposition*

$$h(X/S) \cong h^0(X/S) \oplus h^\star(X/S) \oplus h^{2d}(X/S)$$

with $h^0(X/S) = (X, \pi_0, 0)$, $h^\star(X/S) = (X, \mathrm{id}_X - \pi_0 - \pi_{2d}, 0)$ and $h^{2d}(X/S) = (X, \pi_{2d}, 0)$.

Proof. – We check that π_0 and π_{2d} are orthogonal. We have

$$\pi_0 \circ \pi_{2d} = (p_{13})_*(p_{12}^*(\pi_{2d}) \cdot p_{23}^*(\pi_0)) = (p_{13})_*(p_1^*(\mathfrak{z}') \cdot p_3^*(\mathfrak{z}')) = 0$$

and

$$\pi_{2d} \circ \pi_0 = (p_{13})_*(p_{12}^*(\pi_0) \cdot p_{23}^*(\pi_{2d})) = (p_{13})_*(p_2^*(\mathfrak{z}')^2).$$

From the cartesian diagram

$$\begin{tikzcd} X \times_S X \times_S X \arrow[r, "p_{13}"] \arrow[d, "p_2"] & X \times_S X \arrow[d, "\lambda\times\lambda"] \\ X \arrow[r, "\lambda"] & S \end{tikzcd}$$

we get $(p_{13})_*(p_2^*(\mathfrak{z}')^2)) = (\lambda \times \lambda)^*(\lambda_*(\mathfrak{z}')^2)$. But now

$$\lambda_*(\mathfrak{z}')^2 = \lambda_*(\mathfrak{z}^2 - \mathfrak{z} \cdot \lambda^*\lambda_*\mathfrak{z}^2 + \frac{1}{4}(\lambda^*\lambda_*\mathfrak{z}^2)^2) = \lambda_*(\mathfrak{z}^2) - \lambda_*(\mathfrak{z}^2) + 0 = 0. \qquad \square$$

REMARK 2.4. – The decomposition $h(X/S) \cong h^0(X/S) \oplus h^\star(X/S) \oplus h^{2d}(X/S)$ is not unique: it depends very much on the choice of a cycle $\mathfrak{z}$. Nevertheless each of the summands on the right hand side is determined up to a canonical isomorphism, independently of $\mathfrak{z}$. Indeed after Lemma 2.2 we only need to verify this for h^1, and a small verification shows that the diagonal in $X \times_S X$ composed with the respective projectors gives the required isomorphism.

2.4. Künneth decomposition for abelian schemes

The decomposition of Lemma 2.3 provides a Künneth decomposition for families of curves (or surfaces with no odd cohomology). The other case we will use in this paper is the existence of a motivic Künneth decomposition for abelian schemes:

THEOREM 2.5 (Shermenev, Deninger-Murre, Künnemann). – *Let $A \to S$ be an abelian scheme of relative dimension g. There exists a Künneth decomposition*

$$h(A/S) \cong \bigoplus_{i=0}^{2g} h^i(A/S)$$

in which we have $h^i(A/S) \cong \mathrm{Sym}^i h^1(A/S)$ *for all* i, *and* $\mathrm{Sym}^i h^1(A/S) = 0$ *for* $i > 2g$.

Shermenev's proof of this fact starts by choosing a curve C such that its jacobian maps surjectively onto A. This makes the decomposition highly noncanonical, and limits the construction to the absolute case—there is no reason for a general abelian scheme to be a quotient of the jacobian of a family of curves. By contrast, the constructions of Deninger-Murre [10] and Künnemann [39] use Fourier theory and are canonical and functorial.

The Deninger-Murre decomposition can be described using *Manin's identity principle* (see e.g., [66, 2.3]), which says that although a Chow motive M over S is not determined by its Chow groups, it is determined by its Chow groups after any base change; more precisely, the functor that takes a smooth proper morphism $f: T \to S$ to the Chow groups of the relative Chow motive f^*M over T, determines M completely. For an abelian scheme $A \to S$ we have the morphism $[N]: A \to A$ of multiplication by $N > 1$, and the Chow groups of A (and any base change of A) can be decomposed canonically into eigenspaces for N, for all N. The summand $h^i(A/S)$ in the Deninger-Murre decomposition corresponds to the N^i-eigenspace of the Chow groups of A.

PROPOSITION 2.6. – *Let* $C \to S$ *be a family of smooth curves, and* $J \to S$ *the jacobian. Then there exists an isomorphism of Chow motives* $h^1(C/S) \cong h^1(J/S)$, *where* $h^1(C/S)$ *is the summand of* $h(C/S)$ *described in Lemma 2.3 and* $h^1(J/S)$ *is the summand of* $h(J/S)$ *provided by the decomposition of Deninger-Murre.*

Sketch of proof. – After replacing S with a finite étale Galois cover we may assume that $C \to S$ has a section. It is enough to prove the isomorphism under this assumption; since we work with $\mathbf{Q}$-coefficients we may then take Galois invariants to obtain the conclusion over our original base scheme. The section defines an Abel-Jacobi map $C \to J$ and puts us in the situation considered by Shermenev [67]. Shermenev's construction provides a motivic Künneth decomposition $h(J/S) = \bigoplus_i h^i(J/S)$ for which it is clear from construction that $h^1(C/S) = h^1(J/S)$. Unfortunately the resulting motivic decomposition of $h(J/S)$ is in general different from that of Deninger-Murre.

The claim is now that even though the two direct sum decompositions of the Chow groups of J are different, they give rise to the same descending filtration of the Chow groups, so that the two associated graded objects are isomorphic. Since S is arbitrary it will then hold also after base change to an arbitrary smooth proper $S' \to S$, and from Manin's identity principle it will then follow that the motives $h^i(J/S)$ obtained from Shermenev's decomposition are isomorphic to those of Deninger-Murre. The fact that the two descending filtrations of Chow groups coincide is part of a theorem of Moonen-Polishchuk [47, Theorem 4]. □

3. Preliminaries from representation theory

We define a *partition* to be a non-increasing sequence of natural numbers which eventually reaches zero: $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0 \geq 0 \geq \cdots)$. The *weight* of a partition is defined as $|\lambda| = \sum_i \lambda_i$. The *length* of a partition is defined as $\ell(\lambda) = \max\{i : \lambda_i \neq 0\}$. Partitions are often identified with *Young diagrams*. Our convention for Young diagrams is that the numbers λ_i are the lengths of the rows in the diagram. We denote the conjugate partition of λ, obtained by reflecting the Young diagram across the diagonal, by λ^T.

3.1. Schur-Weyl duality

Let V be a vector space over $\mathbf{Q}$. (Everything that follows is true more generally over any field of characteristic zero.) Partitions with $\ell(\lambda) \leq \dim(V)$ are in a natural bijection with irreducible finite dimensional representations of $\mathrm{GL}(V)$ via the theory of highest weight vectors. We write V_λ for the representation of $\mathrm{GL}(V)$ corresponding to λ. For example, if $\lambda = (n \geq 0 \geq 0 \geq \cdots)$, then $V_\lambda = \mathrm{Sym}^n(V)$ and $V_{\lambda^T} = \bigwedge^n V$. If $\ell(\lambda) > \dim(V)$ then we define V_λ to be zero. The representations of the symmetric group are also indexed by partitions: the partitions with $|\lambda| = n$ are in natural bijection with the representations of the symmetric group $\mathfrak{S}_n$, which we denote by σ_λ. If $\lambda = (n \geq 0 \geq 0 \geq \cdots)$ then σ_λ is the trivial representation and σ_{λ^T} is the sign representation.

The vector space $V^{\otimes n}$ carries commuting left and right actions by $\mathrm{GL}(V)$ and $\mathfrak{S}_n$, respectively. *Schur-Weyl duality* in its most basic form is an expression of how to decompose $V^{\otimes n}$ into irreducible representations under this action of $\mathrm{GL}(V) \times \mathfrak{S}_n$:

$$V^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda \otimes \sigma_\lambda^*.$$

Although $\sigma_\lambda \cong \sigma_\lambda^*$, we dualize to emphasize that we are considering a right action.

Schur-Weyl duality can be formulated more abstractly in terms of mutual centralizers. Namely, $V^{\otimes n}$ admits commuting actions of $\mathrm{GL}(V)$ and the group algebra $\mathbf{Q}[\mathfrak{S}_n]$, and Schur-Weyl duality is equivalent to the claim that the centralizer of $\mathrm{GL}(V)$ in $\mathrm{End}_{\mathbf{Q}}(V^{\otimes n})$ equals the image of $\mathbf{Q}[\mathfrak{S}_n]$ in $\mathrm{End}_{\mathbf{Q}}(V^{\otimes n})$, and vice versa. A useful consequence of this more abstract viewpoint is that it produces for all λ an explicit idempotent endomorphism of $V^{\otimes n}$ whose image is exactly the summand $V_\lambda \otimes \sigma_\lambda^*$. Namely, the group algebra $\mathbf{Q}[\mathfrak{S}_n]$ contains a family of orthogonal idempotents called *Young symmetrizers*. If c_λ denotes a Young symmetrizer corresponding to the partition λ, then the image of c_λ is the summand $V_\lambda \otimes \sigma_\lambda^*$ of $V^{\otimes n}$.

3.2. Symplectic groups and Weyl's construction

Suppose that the vector space V is equipped with a symplectic form. Then partitions of length $\ell(\lambda) \leq \frac{1}{2}\dim(V)$ are in a bijection with irreducible finite dimensional representations of $\mathrm{Sp}(V)$, and we write $V_{\langle\lambda\rangle}$ for the representation of $\mathrm{Sp}(V)$ corresponding to λ. Similarly we set $V_{\langle\lambda\rangle} = 0$ if $\ell(\lambda) > \frac{1}{2}\dim(V)$. The decomposition of $V^{\otimes n}$ into irreducible representations of $\mathrm{Sp}(V) \times \mathfrak{S}_n$ is more complicated than that for $\mathrm{GL}(V)$. The first nontrivial example is the case $n = 2$:

$$V^{\otimes 2} = V_{\langle 2\rangle} \otimes \sigma_2^* \oplus V_{\langle 1,1\rangle} \otimes \sigma_{1,1}^* \oplus V_{\langle 0\rangle} \otimes \sigma_{1,1}^*.$$

The first two terms are exactly what one expects from Schur-Weyl duality. The third term arises because $\bigwedge^2 V$ is not irreducible: it contains the trivial representation spanned by the class of the symplectic form as a subrepresentation.

The example $n = 2$ generalizes to larger values of n as follows. We define

$$V^{\langle n\rangle} \subset V^{\otimes n}$$

to be the subspace of *traceless tensors*, i.e., the intersection of the kernels of all $\binom{n}{2}$ maps

$$V^{\otimes n} \to V^{\otimes(n-2)}$$

given by contracting with the symplectic form. Alternatively, we can think of $V^{\langle n\rangle}$ as a quotient of $V^{\otimes n}$, where we divide by the images of all $\binom{n}{2}$ maps $V^{\otimes(n-2)} \to V^{\otimes n}$ given by inserting the class of the symplectic form. The subspace $V^{\langle n\rangle}$ is clearly $\mathrm{Sp}(V)$-invariant, and Weyl proved that there is an isomorphism

$$V^{\langle n\rangle} = \bigoplus_{|\lambda|=n} V_{\langle\lambda\rangle} \otimes \sigma_\lambda^*$$

for all n. Thus we know how to decompose the subspace $V^{\langle n\rangle}$ into irreducible representations of $\mathrm{Sp}(V) \times \mathfrak{S}_n$. Moreover, we may write $V^{\otimes n}$ as the direct sum of $V^{\langle n\rangle}$ and the image of all the maps $V^{\otimes(n-2)} \to V^{\otimes n}$; we may write $V^{\otimes(n-2)}$ as the direct sum of $V^{\langle n-2\rangle}$ and the image of all maps $V^{\otimes(n-4)} \to V^{\otimes(n-2)}$, etc. This leads inductively to a decomposition of $V^{\otimes n}$ into irreducible $\mathrm{Sp}(V) \times \mathfrak{S}_n$-representations. All $V_{\langle\lambda\rangle}$ with $|\lambda| \leq n$ and $|\lambda| \equiv n \pmod 2$ will occur in this decomposition. We refer to this as *Weyl's construction* of the irreducible representations of $\mathrm{Sp}(V)$.

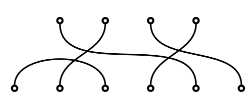
3.3. Brauer algebra

We now wish to give a version of Schur-Weyl duality for the symplectic group in terms of mutual centralizer algebras for the action of $\mathrm{Sp}(V)$ on $V^{\otimes n}$. The centralizer of $\mathrm{Sp}(V)$ acting on $V^{\otimes n}$ is larger than $\mathbf{Q}[\mathfrak{S}_n]$. It can be described as the algebra of endomorphisms of $V^{\otimes n}$ generated by $\mathbf{Q}[\mathfrak{S}_n]$ and the maps given by compositions

$$V^{\otimes n} \to V^{\otimes(n-2)} \to V^{\otimes n},$$

where the first map contracts two tensor factors using the symplectic form, and the second map inserts the form. Brauer [5] introduced a diagrammatic calculus which is useful for describing endomorphisms in this centralizer algebra. We give here a category-theoretic treatment of the Brauer algebra. Somewhat similar presentations can be found in [40, 65].

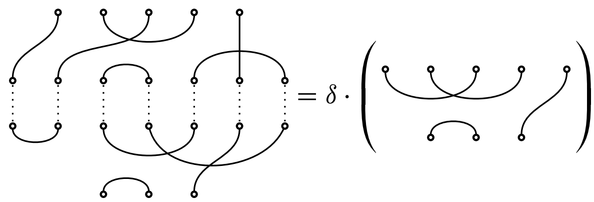
Let n and m be nonnegative integers. We define an (n,m)-*Brauer diagram* to be a diagram of two rows containing n and m dots, respectively, and $(n+m)/2$ strands connecting these dots pairwise. The set of (n,m)-Brauer diagrams is empty unless $n \equiv m \pmod 2$. Here is a (4,6)-Brauer diagram:



For any parameter $\delta \in \mathbf{Q}$, let $\mathfrak{Br}^{(\delta)}(n,m)$ be the $\mathbf{Q}$-vector space spanned by all (n,m)-Brauer diagrams. We define a composition map

$$\mathfrak{Br}^{(\delta)}(n,m) \otimes \mathfrak{Br}^{(\delta)}(m,k) \to \mathfrak{Br}^{(\delta)}(n,k)$$

which is defined on basis elements as follows: to compose an (n,m)-Brauer diagram and an (m,k)-Brauer diagram, connect the strands on the bottom of the first diagram with those on the top of the second diagram, erase any loops that are formed in the process, and multiply the result by δ to the power of the number of erased loops. The following example illustrates a composition $\mathfrak{Br}^{(\delta)}(5,7) \otimes \mathfrak{Br}^{(\delta)}(7,3) \to \mathfrak{Br}^{(\delta)}(5,3)$:



This composition defines in particular the structure of an associative algebra on $\mathfrak{Br}^{(\delta)}(n,n)$. This algebra is classically called the *Brauer algebra*.

DEFINITION 3.1. – Let V be an object of a symmetric monoidal category. A *dual* of V is an object V^* equipped with unit and counit maps $\mathbf{1} \to V \otimes V^*$ and $V^* \otimes V \to \mathbf{1}$ such that both the following compositions are identities:

$$V \cong \mathbf{1} \otimes V \to V \otimes V^* \otimes V \to V \otimes \mathbf{1} \cong V$$

and

$$V^* \cong V^* \otimes \mathbf{1} \to V^* \otimes V \otimes V^* \to \mathbf{1} \otimes V^* \cong V^*.$$

An object V is *self-dual* if it is equipped with a pair of maps $\mathbf{1} \to V \otimes V$ and $V \otimes V \to \mathbf{1}$ making V into its own dual. It is *symmetrically self-dual* if, in addition, the unit and counit are invariant under the flip map $V \otimes V \to V \otimes V$. If V is dualizable, then we define the *quantum dimension* of V to be the element of $\mathrm{End}(\mathbf{1})$ given by the composition

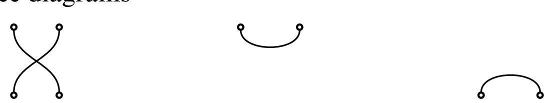
$$\mathbf{1} \to V \otimes V^* \cong V^* \otimes V \to \mathbf{1}.$$

The following proposition can be seen as part of the diagrammatic calculus of "string diagrams," describing morphisms in tensor categories (see e.g., [37, Chapter XIV]). In this calculus the precise form of the diagrams depend on the properties of the tensor category. For example, in a symmetric monoidal category strings are allowed to cross each other freely, but in a braided monoidal category the strings must be considered as braids. If we did not insist that V was *symmetrically* self-dual in the following proposition, we would need to equip the strands in the Brauer algebra with orientations or framings.

PROPOSITION 3.2. – *Let V be a symmetrically self-dual object of quantum dimension δ in a $\mathbf{Q}$-linear symmetric monoidal category $\mathcal{C}$. There is a natural map $\mathfrak{Br}^{(\delta)}(n,m) \to \mathrm{Hom}_{\mathcal{C}}(V^{\otimes n}, V^{\otimes m})$ which makes the following diagram commute:*

$$\begin{array}{ccc} \mathfrak{Br}^{(\delta)}(n,m) \otimes \mathfrak{Br}^{(\delta)}(m,k) & \longrightarrow & \mathfrak{Br}^{(\delta)}(n,k) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_{\mathcal{C}}(V^{\otimes n}, V^{\otimes m}) \otimes \mathrm{Hom}_{\mathcal{C}}(V^{\otimes m}, V^{\otimes k}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(V^{\otimes n}, V^{\otimes k}). \end{array}$$

The collection of maps $\mathfrak{Br}^{(\delta)}(n,m) \to \mathrm{Hom}_{\mathcal{C}}(V^{\otimes n}, V^{\otimes m})$ are completely determined by the images of the three diagrams



which generate the Brauer algebras in an appropriate sense; these diagrams are mapped to the flip map $V \otimes V \to V \otimes V$, the counit $V \otimes V \to \mathbf{1}$ and the unit $\mathbf{1} \to V \otimes V$, respectively.

REMARK 3.3. – The proposition can be formulated using the language of PROPs: the collection $\{\mathfrak{Br}^{(\delta)}(n,m)\}_{n,m\geq 0}$ is a PROP, and V is an algebra over this PROP in the category $\mathcal{C}$.

Let sV denote the symplectic vector space V considered as a $\mathbf{Z}/2$-graded vector space concentrated in *odd* degree. Let $\mathbf{1}$ be the monoidal unit in this category, i.e., the vector space $\mathbf{Q}$ placed in even degree. The symplectic form defines "contraction" and "insertion" maps

$$sV \otimes sV \to \mathbf{1} \qquad \text{and} \qquad \mathbf{1} \to sV \otimes sV.$$

Taking into account the Koszul sign rule for $\mathbf{Z}/2$-graded vector spaces, both these maps are now $\mathfrak{S}_2$-invariant; that is, by shifting V into odd degree, we have converted the symplectic form to a symmetric bilinear form. Equivalently, sV is symmetrically self-dual in the category of $\mathbf{Z}/2$-graded vector spaces. The quantum dimension of sV is $-2g$.

COROLLARY 3.4. – *Let V be a symplectic vector space of dimension $2g$. The map which sends an (n,m)-Brauer diagram to a morphism $(sV)^{\otimes n} \to (sV)^{\otimes m}$ makes the following diagram commute:*

$$\begin{tikzcd}
\mathfrak{Br}^{(-2g)}(n,m) \otimes \mathfrak{Br}^{(-2g)}(m,k) \arrow[r] \arrow[d] & \mathfrak{Br}^{(-2g)}(n,k) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Sp}(V)}((sV)^{\otimes n},(sV)^{\otimes m}) \otimes \mathrm{Hom}_{\mathrm{Sp}(V)}((sV)^{\otimes m},(sV)^{\otimes k}) \arrow[r] & \mathrm{Hom}_{\mathrm{Sp}(V)}((sV)^{\otimes n},(sV)^{\otimes k}),
\end{tikzcd}$$

and the vertical maps are surjective.

Proof. – After the previous proposition, we only need to explain surjectivity. Surjectivity is equivalent to the statement that the space of symplectic invariant tensors inside $(sV)^{\otimes 2n}$ is spanned by the classes obtained by inserting the symplectic form n times, which follows from Weyl's decomposition of $V^{\otimes 2n}$ into irreducible representations described in the previous section. Alternatively, surjectivity is part of the first fundamental theorem of invariant theory for the symplectic group. □

In particular, the Brauer algebra $\mathfrak{Br}^{(-2g)}(n,n)$ surjects onto the centralizer $\mathrm{End}_{\mathrm{Sp}(V)}((sV)^{\otimes n})$. Moreover, there is an isomorphism $\mathrm{End}_{\mathrm{Sp}(V)}((sV)^{\otimes n}) \cong \mathrm{End}_{\mathrm{Sp}(V)}(V^{\otimes n})$ given by desuspending and carefully inserting signs. This isomorphism is described explicitly by Hanlon and Wales [28, Theorem 2.10]. Let us explain why their result gives such an isomorphism. Hanlon and Wales define two versions of Brauer algebra, $\mathfrak{A}_f^{(x)}$ and $\mathfrak{B}_f^{(x)}$, for any natural number f and any parameter x in the ground field. These algebras act naturally on the fth tensor power of a vector space of dimension x equipped with a symmetric or antisymmetric bilinear form, respectively. They show by a direct calculation that there is an isomorphism $\mathfrak{A}_f^{(x)} \cong \mathfrak{B}_f^{(-x)}$ for all f, x: in our terms, this isomorphism arises from the fact that the functor $V \mapsto sV$ maps a vector space of dimension x to a space of dimension $-x$, and converts a symmetric bilinear form to an antisymmetric one and vice versa. Our algebra $\mathfrak{Br}^{(-2g)}(n,n)$ is identical with their $\mathfrak{A}_n^{(-2g)}$.

The Brauer algebra contains the group algebra $\mathbf{Q}[\mathfrak{S}_n]$—as it should, since the centralizer of $\mathrm{Sp}(V)$ acting on $V^{\otimes n}$ should contain the centralizer of $\mathrm{GL}(V)$—as the subalgebra consisting of (n,n)-Brauer diagrams in which all strands are vertical, i.e., go from the top row to the bottom row. The inclusion $\mathbf{Q}[\mathfrak{S}_n] \to \mathfrak{Br}^{(-2g)}(n,n)$ has a left inverse, given by mapping any diagram containing a horizontal strand to zero; one checks that the subspace spanned by all diagrams containing a horizontal strand is an ideal.

THEOREM 3.5 (Symplectic Schur-Weyl duality). – *Let V be a symplectic vector space of dimension $2g$.*

1. *The image of the Brauer algebra $\mathfrak{Br}^{(-2g)}(n,n)$ in $\mathrm{End}_{\mathbf{Q}}(V^{\otimes n})$ is the centralizer of $\mathrm{Sp}(V)$, and vice versa.*
2. *There is an isomorphism*
$$V^{\otimes n} \cong \bigoplus_{\substack{|\lambda|\leq n \\ |\lambda|\equiv 2 \pmod n}} V_{\langle\lambda\rangle} \otimes \beta^*_{\lambda,n},$$
where $\beta_{\lambda,n}$ denotes the simple module over the Brauer algebra $\mathfrak{Br}^{(-2g)}(n,n)$ corresponding to λ.
3. *For $|\lambda| = n$, the representation $\beta_{\lambda,n}$ coincides with the representation σ_{λ^T} of $\mathfrak{S}_n$, considered as a module over the Brauer algebra via the map $\mathfrak{Br}^{(-2g)}(n,n) \to \mathbf{Q}[\mathfrak{S}_n]$ which sends any diagram containing a horizontal strand to zero.*

Part (2) follows from (1), given a description of how the split semisimple algebra $\mathrm{End}_{\mathrm{Sp}(V)}(V^{\otimes n})$ decomposes into simple algebras [75, Corollary 3.5].

REMARK 3.6. – It may seem strange that the representation σ_{λ^T}, rather than σ_λ, appears in part (3) of Theorem 3.5. Indeed, we have seen from Weyl's construction that $V^{\otimes n}$, when decomposed into irreducible representations of $\mathrm{Sp}(V) \times \mathfrak{S}_n$, should contain the summands $V_{\langle\lambda\rangle} \otimes \sigma^*_\lambda$ for $|\lambda| = n$. But Theorem 3.5 says that $V^{\otimes n}$ contains the summands $V_{\langle\lambda\rangle} \otimes \beta^*_{\lambda,n}$, and that $\beta_{\lambda,n} \cong \sigma_{\lambda^T}$ when $|\lambda| = n$. So why isn't this a contradiction? The reason is that when $\mathfrak{S}_n$ acts on $V^{\otimes n}$ via the composition
$$\mathbf{Q}[\mathfrak{S}_n] \hookrightarrow \mathfrak{Br}^{(-2g)}(n,n) \to \mathrm{End}_{\mathbf{Q}}((sV)^{\otimes n}) \cong \mathrm{End}_{\mathbf{Q}}(V^{\otimes n}),$$
then this action is *not* equal to the standard action of $\mathfrak{S}_n$ on $V^{\otimes n}$ by permuting the factors; instead, one obtains the standard action twisted by the sign representation. So the isomorphism $\beta_{\lambda,n} \cong \sigma_{\lambda^T}$ does hold when $\beta_{\lambda,n}$ is considered as a $\mathbf{Q}[\mathfrak{S}_n]$-module by restriction of scalars, but this is not the same as the $\mathbf{Q}[\mathfrak{S}_n]$-module structure obtained by the natural action of $\mathfrak{S}_n$ on $V^{\otimes n}$.

The conventions are more natural when V is placed in odd degree: the composition $\mathbf{Q}[\mathfrak{S}_n] \to \mathfrak{Br}^{(-2g)}(n,n) \to \mathrm{End}_{\mathbf{Q}}((sV)^{\otimes n})$ does give the standard action of $\mathfrak{S}_n$ on $(sV)^{\otimes n}$, which now takes the Koszul sign rule into account. Thus $(sV)^{\otimes n}$ will contain $V_{\langle\lambda\rangle} \otimes \sigma^*_{\lambda^T}$ as a summand, placed in odd/even degree according to whether n is odd/even. We caution the reader that the calculations of this paper will require some care to be taken to tensor with the sign representation when appropriate, in particular when passing between Chow groups and cohomology groups.

REMARK 3.7. – If we want to decompose $V^{\otimes n}$ into irreducible representations of $\mathrm{Sp}(V) \times \mathfrak{S}_n$, then we may start from the usual Schur-Weyl duality (which gives a decomposition into irreducible representations of $\mathrm{GL}(V) \times \mathfrak{S}_n$), and apply a branching formula for $\mathrm{Sp}(V) \subset \mathrm{GL}(V)$. Equivalently we could start with the symplectic Schur-Weyl duality (which gives a decomposition into $\mathrm{Sp}(V) \times \mathfrak{Br}^{(-2g)}(n,n)$-representations) and try to determine how the modules $\beta_{\mu,n}$ over $\mathfrak{Br}^{(-2g)}(n,n)$ decompose into sums of Specht modules under restriction of scalars to $\mathbf{Q}[\mathfrak{S}_n]$: note that if

$$\mathrm{Res}^{\mathrm{GL}(V)}_{\mathrm{Sp}(V)} V_\lambda \cong \bigoplus_\mu a^\lambda_\mu \; V_{\langle\mu\rangle}$$

for some integers a^λ_μ, then $\beta_{\mu,n} \cong \bigoplus_{|\lambda|=n} a^\lambda_\mu \, \sigma_{\lambda^T}$ as $\mathbf{Q}[\mathfrak{S}_n]$-modules. Then the decomposition of $V^{\otimes n}$ reads

$$V^{\otimes n} \cong \bigoplus_{|\lambda|=n} \bigoplus_\mu a^\lambda_\mu \; V_{\langle\mu\rangle} \otimes \sigma^*_\lambda.$$

The discussion in the preceding paragraphs says that $a^\lambda_\mu \neq 0$ only for $|\mu| \equiv |\lambda| \pmod 2$ and $|\mu| < |\lambda|$, with the sole exception of $a^\lambda_\lambda = 1$. The problem of calculating the coefficients a^λ_μ was first solved by Littlewood [42] and Newell [53], and many subsequent authors have given methods for computing them.

3.4. Projectors

Given the above, it is natural to ask for an analogue of Young symmetrizers in the Brauer algebra. That is, one would like idempotents $\pi_{\lambda,n} \in \mathfrak{Br}^{(-2g)}(n,n)$ such that the image of $\pi_{\lambda,n}$ acting on $V^{\otimes n}$ is the irreducible summand $V_{\langle\lambda\rangle} \otimes \beta^*_{\lambda,n}$. The question of how to find such idempotents π_λ was posed already by Weyl. Nevertheless, no explicit construction was known until Nazarov [52] gave a simple formula describing $\pi_{\lambda,n}$ in the most interesting case $|\lambda| = n$. Although the statement of the result is elementary and involves only very classical representation theory, the proof proceeds through the theory of quantum groups.

The results of Section 10, and some of the examples in Section 6, rely on computer calculations which require us to have explicit formulas for the idempotents $\pi_{\lambda,n}$ for $|\lambda| = n$. However, the reader does not need to know the precise expression for $\pi_{\lambda,n}$ to follow the arguments, only that such a formula exists. Nevertheless we state Nazarov's theorem here for completeness. For a partition λ, we define the *row tableau* associated to λ to be the Young tableau given by filling in the numbers $1, 2, \ldots, n$ in the Ferrers diagram so that the first row gets the numbers $1, 2, \ldots, \lambda_1$, the second row gets the numbers $\lambda_1+1, \lambda_1+2, \ldots, \lambda_1+\lambda_2$, and so on. We define the *content* of a box in the ith row and jth column of the Ferrers diagram to be $j-i$. For $k \in \{1, \ldots, n\}$, we define the number $c_k(\lambda)$ to be the content of the box labeled "k" in the row tableau corresponding to λ.

For any $1 \leq i, j \leq n$, let B_{ij} be the element of $\mathfrak{Br}^{(-2g)}(n,n)$ corresponding to contracting and inserting the ith and jth tensor factors with the symplectic form. That is, it has $n-2$ vertical strands, and two horizontal ones: one connecting the ith and jth "inputs," and one connecting the ith and jth "outputs".

THEOREM 3.8 (Nazarov). – *For any partition λ of n, define*

$$\pi_{\lambda,n} = \prod_{k,l} \left(1 + \frac{B_{kl}}{2g+1+c_k(\lambda)+c_l(\lambda)}\right) \cdot c_{\lambda^T} \in \mathfrak{Br}^{(-2g)}(n,n).$$

Here the product ranges over all pairs $1 \le k < l \le n$ such that the boxes labeled k and l are in distinct rows of the row tableau associated to λ. Since the operators B_{kl} do not commute, this must be interpreted as an ordered *product: we order the terms in the product lexicographically by (k,l). Finally, c_{λ^T} is a Young symmetrizer. The image of $\pi_{\lambda,n}$ acting on $(sV)^{\otimes n}$ is the summand $V_{\langle\lambda\rangle} \otimes \sigma^*_{\lambda^T}$, which is placed in odd degree if n is odd and even degree if n is even.*

REMARK 3.9. – For our purposes it would be enough to have formulas for such idempotents in the smaller algebra $\mathrm{End}_{\mathrm{Sp}(V)}((sV)^{\otimes n})$, which a priori do not have to lift to an idempotent of $\mathfrak{Br}^{(-2g)}(n,n)$. This means that in principle we could have used results of Ram and Wenzl [64] instead of Nazarov's theorem.

4. A result of Ancona

4.1. Schur functors

Let $\mathcal{C}$ be a $\mathbf{Q}$-linear symmetric monoidal category, and let $M \in \mathrm{ob}\mathcal{C}$. Then $M^{\otimes n}$ has an action of the group algebra $\mathbf{Q}[\mathfrak{S}_n]$, in the sense that there is a homomorphism of $\mathbf{Q}$-algebras $\mathbf{Q}[\mathfrak{S}_n] \to \mathrm{Hom}_{\mathcal{C}}(M^{\otimes n}, M^{\otimes n})$.

Let us suppose moreover that $\mathcal{C}$ is pseudo-abelian, i.e., that every idempotent endomorphism in $\mathcal{C}$ has an image. For $|\lambda| = n$, let $\pi_\lambda \in \mathbf{Q}[\mathfrak{S}_n]$ be a Young symmetrizer corresponding to λ, and define $\mathsf{S}^\lambda(M)$ to be the image of the idempotent π_λ acting on $M^{\otimes n}$. We call S^λ the *Schur functor* corresponding to λ. Then there is a decomposition [9]

$$M^{\otimes n} = \bigoplus_{|\lambda|=n} \mathsf{S}^\lambda(M) \otimes \sigma^*_\lambda,$$

where σ_λ denotes the representation of the symmetric group corresponding to λ. When $\mathcal{C}$ is the category of finite dimensional $\mathbf{Q}$-vector spaces, this is the decomposition of $M^{\otimes n}$ given by Schur-Weyl duality, described in Section 3.

The key fact used in proving this result is that $\mathbf{Q}[\mathfrak{S}_n]$ is a semisimple algebra and in fact a product of matrix algebras over $\mathbf{Q}$: there is an isomorphism $\mathbf{Q}[\mathfrak{S}_n] \cong \prod_{|\lambda|=n} \mathrm{End}_{\mathbf{Q}}(\sigma_\lambda)$.

4.2. Brauer algebra action

We will need to generalize Proposition 3.2 to a weaker notion of symmetrically self-dual object. An object L of $\mathcal{C}$ is called *invertible* if the functor $-\otimes L$ is an equivalence of categories. If this is the case then L is dualizable, the quasi-inverse is given by tensoring with L^*, and the maps $\mathbf{1} \to L \otimes L^*$ and $L \otimes L^* \to \mathbf{1}$ are isomorphisms. We say that L is *even* or *odd* if $\mathfrak{S}_n$ acts on $\mathrm{Hom}_{\mathcal{C}}(L^{\otimes n}, L^{\otimes n}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$ by the trivial representation or the sign representation, respectively.

We say that $M \in \mathrm{ob}\mathcal{C}$ is *weakly self-dual* if there is an even invertible object $L \in \mathrm{ob}\mathcal{C}$, and unit and counit maps

$$L \to M \otimes M \qquad M \otimes M \to L$$

such that the compositions

$$M \otimes L \to M \otimes M \otimes M \to L \otimes M$$

and

$$L \otimes M \to M \otimes M \otimes M \to M \otimes L$$

both equal the swap map. This implies that $M^* \cong M \otimes L^*$. We call M *weakly symmetrically self-dual* if moreover the unit and counit maps are invariant under the swap map $M \otimes M \to M \otimes M$. We omit the proof of the following result, which generalizes Proposition 3.2 and is an exercise in the diagrammatic calculus for rigid symmetric monoidal categories.

PROPOSITION 4.1. – *Let M be a weakly symmetrically self-dual object of $\mathcal{C}$ of quantum dimension δ. There is an action of $\mathfrak{Br}^{(\delta)}(n,n)$ on $M^{\otimes n}$, under which the subalgebra $\mathbf{Q}[\mathfrak{S}_n]$ acts on $M^{\otimes n}$ in the usual way, and a Brauer diagram of the form*  *acts as the composition*

$$M \otimes M \to L \to M \otimes M.$$

A more general statement is that any element of $\mathfrak{Br}^{(\delta)}(m,n)$ gives a well defined morphism in $\mathrm{Hom}_{\mathcal{C}}(M^{\otimes m} \otimes L^{n-m}, M^{\otimes n}) = \mathrm{Hom}_{\mathcal{C}}(M^{\otimes m}, M^{\otimes n} \otimes L^{m-n})$, in a way compatible with composition.

REMARK 4.2. – Proposition 4.1 does not in general lead to a decomposition of $M^{\otimes n}$ into summands indexed by irreducible representations of the Brauer algebra, even over $\overline{\mathbf{Q}}$. The reason is that the algebra $\mathfrak{Br}^{(\delta)}(n,n)$ is not in general semisimple, which was the crucial property of $\mathbf{Q}[\mathfrak{S}_n]$ used for defining the decomposition of $M^{\otimes n}$ in terms of Schur functors. In the case of $\mathfrak{Br}^{(-2g)}(n,n)$, which acts naturally on $V^{\otimes n}$ for V a symplectic vector space of dimension $2g$, what *does* hold is that $\mathrm{End}_{\mathrm{Sp}(2g)}(V^{\otimes n})$, i.e., the image of $\mathfrak{Br}^{(-2g)}(n,n)$ in $\mathrm{End}_{\mathbf{Q}}(V^{\otimes n})$, is a product of matrix algebras over $\mathbf{Q}$.

4.3. Self-products of abelian schemes

Let $f\colon A \to S$ be an abelian scheme of relative dimension g, where we assume S smooth and connected. Let $\mathbb{V}$ be the local system $R^1 f_*\mathbf{Q}$ on S of rank $2g$. Then $\mathbb{V}$ is defined by a homomorphism $\pi_1(S, x_0) \to \mathrm{Sp}(2g, \mathbf{Q})$. The Brauer algebra $\mathfrak{Br}^{(-2g)}(n,n)$ acts on the n-fold tensor power of the defining representation V of $\mathrm{Sp}(2g)$, and hence also on $\mathbb{V}^{\otimes n}$. As explained in Theorem 3.5 the Brauer algebra action gives rise to a decomposition

$$V^{\otimes n} \cong \bigoplus_{\substack{|\lambda| \le n \\ |\lambda| \equiv n \pmod 2}} V_{\langle\lambda\rangle} \otimes \beta^*_{\lambda,n},$$

which then gives us also a decomposition of $\mathbb{V}^{\otimes n}$, i.e., $\mathbb{V}^{\otimes n} \cong \bigoplus_{\substack{|\lambda| \le n \\ |\lambda| \equiv n \pmod 2}} \mathbb{V}_{\langle\lambda\rangle} \otimes \beta^*_{\lambda,n}$.

The next result is a special case of the main theorem of [2]. See also [46].

THEOREM 4.3. – *The above decomposition of $\mathbb{V}^{\otimes n}$ lifts to the category of Chow motives over S:*

$$h^1(A/S)^{\otimes n} \cong \bigoplus_{\substack{|\lambda| \le n \\ |\lambda| \equiv n \pmod 2}} h^1(A/S)_{\langle\lambda\rangle} \otimes \mathbb{L}^{(n-|\lambda|)/2} \otimes \beta^*_{\lambda,n}.$$

We note that the action of $\mathfrak{Br}^{(-2g)}(n,n)$ lifts to an action on $h^1(A/S)^{\otimes n}$. This follows from Proposition 4.1 given that the cup product $h^1(A/S) \otimes h^1(A/S) \to \mathbb{L}$ makes $h^1(A/S)$ weakly symmetrically self-dual, and that $\dim h^1(A/S) = -2g$. To see that $\dim h^1(A/S) = -2g$, observe that the quantum dimension of an object is preserved by any strict symmetric monoidal functor. We apply this to $\omega\colon \mathsf{Mot}_S \to \mathsf{grVect}_{\mathbf{Q}}$ given by $\omega(M) = \bigoplus_i \mathcal{H}^i(\mathsf{real}\, M)_{x_0}$, where $x_0 \in S$ is an arbitrary point. Since $\mathsf{Mot}_S(\mathbf{1},\mathbf{1}) = \mathsf{grVect}_{\mathbf{Q}}(\mathbf{1},\mathbf{1}) = \mathbf{Q}$ we have $\dim h^1(A/S) = \dim \omega(h^1(A/S))$. But $\omega(h^1(A/S))$ is the $2g$-dimensional vector space $H^1(A_{x_0},\mathbf{Q})$ placed in degree 1, so its dimension in the sense of graded vector spaces is $-2g$.

As explained in Remark 4.2, the obstruction to obtaining a result like Theorem 4.3 in a general rigid symmetric monoidal category is that the algebra $\mathfrak{Br}^{(-2g)}(n,n)$ does not split as a product of matrix algebras; only its quotient $\mathrm{End}_{\mathrm{Sp}(2g)}(V^{\otimes n})$ does. The key point is then that the action of $\mathfrak{Br}^{(-2g)}(n,n)$ on $h^1(A/S)^{\otimes n}$ factors through $\mathrm{End}_{\mathrm{Sp}(2g)}(V^{\otimes n})$. In Ancona's paper [2] this is proven using O'Sullivan's results on symmetrically distinguished cycles on abelian varieties [55]. An alternative proof (cf. [1], [40, Theorem 4.8]) proceeds by using invariant theory to show that the kernel of the homomorphism of PROPs of Proposition 3.4 is the PROP-ideal generated by a single element of $\mathfrak{Br}^{(-2g)}(g+1,g+1)$ whose vanishing is equivalent to $\bigwedge^{2g+2}(V) = 0$. Then the fact that this single relation holds also on the level of Chow groups is equivalent to the fact that $h^1(A/S)$ is a finite dimensional motive in the sense of Kimura. Either way it is clear that the result is at present quite special to abelian varieties.

5. The Künneth decomposition of the tautological ring

Let $p\colon C_g \to M_g$ be the universal genus g curve, and C_g^n the n-fold fibered power of C_g over M_g. There are n natural line bundles L_i on C_g^n; the fiber of L_i over a moduli point is given by the cotangent space of the curve at the ith marking. We denote the first Chern class of L_i by ψ_i. Thus ψ_i is pulled back from C_g along the map $C_g^n \to C_g$ that forgets all markings except the ith.

We make the definition $\kappa_d = p_*\psi_1^{d+1} \in \mathrm{CH}^d(M_g)$. In particular, $\kappa_{-1} = 0$ and $\kappa_0 = (2g-2)$. We denote by the same symbol κ_d also the pullback of this class to C_g^n.

For any distinct elements $i, j \in \{1,\dots,n\}$ we denote by $\Delta_{ij} \in \mathrm{CH}^1(C_g^n)$ the class of the diagonal locus where the ith and jth marked points coincide with each other.

Definition 5.1. – The *tautological ring* $R^\bullet(C_g^n)$ is the subring of $\mathrm{CH}^\bullet(C_g^n)$ generated by all ψ-classes, κ-classes and diagonal classes. The *tautological cohomology ring* $RH^\bullet(C_g^n)$ is the image of the tautological ring inside $H^\bullet(C_g^n,\mathbf{Q})$ under the cycle class map. (The grading of $RH^\bullet(C_g^n)$ is twice that of $R^\bullet(C_g^n)$, so that $RH^k(C_g^n) \subset H^k(C_g^n,\mathbf{Q})$.)

The generators for the tautological rings satisfy the following relations for all i, j and k:

$$\begin{aligned} \Delta_{ij}\Delta_{ik} &= \Delta_{ij}\Delta_{jk}, \\ \Delta_{ij}\psi_i &= \Delta_{ij}\psi_j, \\ \Delta_{ij}^2 &= -\Delta_{ij}\psi_i. \end{aligned} \tag{5}$$

The first two are geometrically obvious, and the third one is a consequence of the excess intersection formula.

DEFINITION 5.2. – Let $S_n^\bullet$ be the commutative graded $\mathbf{Q}$-algebra generated by classes Δ_{ij} and ψ_i of degree 1 (where i and j range from 1 to n and are distinct) and κ_d of degree d for all $d \geq 1$, modulo the above three relations.

REMARK 5.3. – By a "commutative graded" algebra (as opposed to a "graded commutative" algebra) we mean an algebra in which $x \cdot y = y \cdot x$ for all x, y, regardless of their degree; we do not impose the Koszul sign rule $x \cdot y = (-1)^{|x|\cdot|y|} y \cdot x$.

For each $g \geq 2$, there is a natural surjection

$$S_n^\bullet \to R^\bullet(C_g^n),$$

and describing the tautological ring $R^\bullet(C_g^n)$ is equivalent to describing the kernel of this surjection. The algebra $S_n^\bullet$ plays the same role for the study of $R^\bullet(C_g^n)$ as the *strata algebra* does for the study of $R^\bullet(\overline{M}_{g,n})$, cf. e.g., [58, 0.3].

REMARK 5.4. – As mentioned in the introduction, tautological classes are usually considered on the Deligne-Mumford spaces. In that case the tautological rings $R^\bullet(\overline{M}_{g,n})$ can be defined, following Faber and Pandharipande [15], as the smallest collection of unital subrings of $\mathrm{CH}^\bullet(\overline{M}_{g,n})$ closed under pushforward along the gluing maps

$$\overline{M}_{g,n+2} \to \overline{M}_{g+1,n} \qquad \text{and} \qquad \overline{M}_{g,n+1} \times \overline{M}_{g',n'+1} \to \overline{M}_{g+g',n+n'}$$

and the forgetful maps

$$\overline{M}_{g,n+1} \to \overline{M}_{g,n}.$$

Although it is not imposed in the definition, it turns out that the tautological rings are also closed under pullback along the same maps. A similar characterization can be given of the tautological rings $R^\bullet(C_g^n)$. For any function $\phi: \{1,\ldots,n\} \to \{1,\ldots,m\}$ there is a map $C_g^m \to C_g^n$,

$$(C, x_1, x_2, \ldots, x_m) \mapsto (C, x_{\phi(1)}, \ldots, x_{\phi(n)}).$$

We call all maps of this form *tautological*. Then it is not hard to see that the system of tautological rings $R^\bullet(C_g^n)$ can be defined to be the smallest collection of unital subrings closed under pushforward along all tautological maps, and it turns out a posteriori that the tautological rings are also closed under pullback along the same maps.

5.1. The Künneth decomposition of the universal curve

As explained in Section 2.3, any choice of a cycle $\mathfrak{z} \in \mathrm{CH}^1(C_g)$ of degree 1 on each fiber of $p: C_g \to M_g$ gives rise to a decomposition of the relative Chow motive:

$$h(C_g/M_g) = h^0(C_g/M_g) \oplus h^1(C_g/M_g) \oplus h^2(C_g/M_g).$$

Since $\mathrm{CH}^1(C_g) = \mathbf{Q}\{\kappa_1, \psi_1\}$, and κ_1 vanishes on the fibers of p, the only possibilities we have are $\mathfrak{z} = \frac{1}{2g-2}\psi_1 + (\text{const.}) \cdot \kappa_1$. Regardless of the constant we get $\mathfrak{z}' = \mathfrak{z} - \frac{1}{2} p^* p_* \mathfrak{z}^2 =$

$\frac{1}{2g-2}\psi_1 - \frac{1}{2(2g-2)^2}\kappa_1$. Hence without making any choices we get projectors π_0, π_1 and π_2 acting on $h(C_g/M_g)$, defined by

$$\begin{aligned}\pi_0 &= \frac{1}{2g-2}\psi_1 - \frac{1}{2(2g-2)^2}\kappa_1,\\ \pi_1 &= \Delta_{12} - \frac{1}{2g-2}(\psi_1+\psi_2) + \frac{1}{(2g-2)^2}\kappa_1,\\ \pi_2 &= \frac{1}{2g-2}\psi_2 - \frac{1}{2(2g-2)^2}\kappa_1.\end{aligned}$$

We have isomorphisms $\mathbf{1} \cong h^0(C_g/M_g)$ and $\mathbb{L} \cong h^2(C_g/M_g)$, where $\mathbf{1}$ and $\mathbb{L}$ denote the unit object and the Lefschetz motive in the category of Chow motives over M_g. Thus the interesting motive is $h^1(C_g/M_g)$.

We may form the Chow groups of these relative motives: there is an isomorphism

$$\mathrm{CH}^k(C_g) = \mathrm{CH}^k(M_g, h(C_g/M_g)) = \bigoplus_{i=0}^{2} \mathrm{CH}^k(M_g, h^i(C_g/M_g)),$$

where

$$\begin{aligned}\mathrm{CH}^k(M_g, h^0(C_g/M_g)) &= \mathrm{Im}(\pi_0\colon \mathrm{CH}^k(C_g) \to \mathrm{CH}^k(C_g)) \cong \mathrm{CH}^k(M_g)\\ \mathrm{CH}^k(M_g, h^1(C_g/M_g)) &= \mathrm{Im}(\pi_1\colon \mathrm{CH}^k(C_g) \to \mathrm{CH}^k(C_g))\\ \mathrm{CH}^k(M_g, h^2(C_g/M_g)) &= \mathrm{Im}(\pi_2\colon \mathrm{CH}^k(C_g) \to \mathrm{CH}^k(C_g)) \cong \mathrm{CH}^{k-1}(M_g).\end{aligned}$$

The isomorphism $\mathrm{CH}^k(M_g, h^0(C_g/M_g)) \cong \mathrm{CH}^k(M_g)$ is induced by the pullback p^*, and the isomorphism $\mathrm{CH}^k(M_g, h^2(C_g/M_g)) \cong \mathrm{CH}^{k-1}(M_g)$ by the proper pushforward p_*. Informally, the Chow groups of $h^1(C_g/M_g)$ capture the parts of the Chow groups of C_g that do not come from the base M_g.

Now let us consider the n-fold fibered power $C_g^n \to M_g$. Then $h(C_g^n/M_g) = h(C_g/M_g)^{\otimes n}$, so our decomposition yields an equally canonical isomorphism

$$h(C_g^n/M_g) = \bigoplus_{i_1,\dots,i_n \in \{0,1,2\}} \bigotimes_{j=1}^{n} h^{i_j}(C_g/M_g).$$

We call this the *relative Künneth decomposition* of $h(C_g^n/M_g)$. By extension, we will also refer to $\mathrm{CH}^k(C_g^n/M_g) = \bigoplus_{i_1,\dots,i_n\in\{0,1,2\}} \mathrm{CH}^k(M_g, \bigotimes_{j=1}^n h^{i_j}(C_g/M_g))$ as the relative Künneth decomposition of the Chow groups of C_g^n.

For any $i_1,\dots,i_n \in \{0,1,2\}$ we get a projector $\pi_{i_1}\times\cdots\times\pi_{i_n}$ acting on $h(C_g^n/M_g)$ with image $\bigotimes_{j=1}^n h^{i_j}(C_g/M_g)$. In particular this projector acts by correspondences on $\mathrm{CH}^k(C_g^n)$ with image $\mathrm{CH}^k(M_g, \bigotimes_{j=1}^n h^{i_j}(C_g/M_g))$. We write $\pi_1^{\times n}$ for the projector $\pi_1\times\pi_1\times\cdots\times\pi_1$.

5.2. Tautological maps

Let us consider how the decomposition just defined behaves under the tautological maps between the moduli spaces C_g^n.

5.2.1. *Cross product*. – The isomorphism $h(C_g^n/M_g) \otimes h(C_g^m/M_g) \cong h(C_g^{n+m}/M_g)$ yields *cross product* maps

$$\mathrm{CH}^k(C_g^n) \otimes \mathrm{CH}^l(C_g^m) \to \mathrm{CH}^{k+l}(C_g^{n+m});$$

explicitly, $\alpha \times \beta = \mathrm{pr}_1^*(\alpha) \cdot \mathrm{pr}_2^*(\beta)$, where pr_1 and pr_2 denote projections onto the first n and last m factors, respectively.

Since the relative Künneth decomposition of C_g^{n+m} is the tensor product of the relative Künneth decompositions of C_g^n and C_g^m, it follows that the cross product maps are compatible with the projectors π_i in a strong sense: for $i_1, \dots, i_n \in \{0, 1, 2\}$ and $j_1, \dots, j_m \in \{0, 1, 2\}$ we have

$$(\pi_{i_1} \times \cdots \times \pi_{i_n}) \circ \alpha \times (\pi_{j_1} \times \cdots \times \pi_{j_m}) \circ \beta = (\pi_{i_1} \times \cdots \times \pi_{i_n} \times \pi_{j_1} \times \cdots \times \pi_{j_m}) \circ (\alpha \times \beta).$$

5.2.2. *Forgetful maps*. – Let $p\colon C_g^{n+m} \to C_g^n$ be the map that forgets the last m markings. Considering p as a correspondence gives maps of Chow motives

$$h(C_g^n/M_g) \to h(C_g^{n+m}/M_g) \qquad \text{and} \qquad h(C_g^{n+m}/M_g) \to h(C_g^n/M_g) \otimes \mathbb{L}^m,$$

which upon taking Chow groups gives the maps

$$p^*\colon \mathrm{CH}^k(C_g^n) \to \mathrm{CH}^k(C_g^{n+m}) \qquad \text{and} \qquad p_*\colon \mathrm{CH}^k(C_g^{n+m}) \to \mathrm{CH}^{k-m}(C_g^n).$$

Now the map $h(C_g^n/M_g) \to h(C_g^{n+m}/M_g)$ coincides with the composition $h(C_g^n/M_g) \cong h(C_g^n/M_g) \otimes h^0(C_g/M_g)^{\otimes m} \subset h(C_g^{n+m}/M_g)$, and $h(C_g^{n+m}/M_g) \to h(C_g^n/M_g) \otimes \mathbb{L}^m$ coincides with the composition $h(C_g^{n+m}/M_g) \twoheadrightarrow h(C_g^n/M_g) \otimes h^2(C_g/M_g)^{\otimes m} \cong h(C_g^n/M_g) \otimes \mathbb{L}^m$. It follows that the maps p^* and p_* are also compatible with the relative Künneth decomposition of Chow groups:

- The map p^* is given by mapping each summand $(\pi_{i_1} \times \cdots \times \pi_{i_n}) \circ \mathrm{CH}^k(C_g^n)$ isomorphically onto the summand $(\pi_{i_1} \times \cdots \times \pi_{i_n} \times \pi_0 \times \cdots \times \pi_0) \circ \mathrm{CH}^k(C_g^{n+m})$. This can also be seen from the fact that p^* is given by cross product with the class $1 \in \mathrm{CH}^0(C_g^m)$.
- The map p_* maps each summand $(\pi_{i_1} \times \cdots \times \pi_{i_n} \times \pi_2 \times \cdots \times \pi_2) \circ \mathrm{CH}^k(C_g^{n+m})$ isomorphically onto the summand $(\pi_{i_1} \times \cdots \times \pi_{i_n}) \circ \mathrm{CH}^{k-m}(C_g^n)$, and p_* vanishes on all summands not of this form.

5.2.3. *Diagonals*. – The diagonal $C_g \to C_g^2$, considered as a correspondence, defines a map of Chow motives $h(C_g/M_g) \otimes h(C_g/M_g) \to h(C_g/M_g)$. This is the cup product on the level of Chow motives. Forming the relative Künneth decomposition on both sides, we see that the cup product is the sum of maps $h^i(C_g/M_g) \otimes h^j(C_g/M_g) \to h^k(C_g/M_g)$.

We caution the reader that this is *not* in general a multiplicative decomposition, in the sense of [72]: that is, the maps $h^i(C_g/M_g) \otimes h^j(C_g/M_g) \to h^k(C_g/M_g)$ are *not* only nonzero for $i + j = k$. To see this, note that if $\delta \subset C_g^3$ denotes the small diagonal, considered as a correspondence $C_g^2 \vdash C_g$, then the decomposition is multiplicative if and only if

$$\pi_k \circ \delta \circ (\pi_i \times \pi_j) = 0$$

for $i + j \neq k$. Now we have

$$\begin{aligned}
\pi_k \circ \delta \circ (\pi_i \times \pi_j) &= (p_{126})_*(p_{13}^*(\pi_i) \cdot p_{24}^*(\pi_j) \cdot \Delta_{345} \cdot p_{56}^*(\pi_k)) \\
&= (p_{456})_*(\Delta_{123} \cdot p_{14}^*(\pi_i^t) \cdot p_{25}^*(\pi_j^t) \cdot p_{46}^*(\pi_k)) \\
&= (\pi_{2-i} \times \pi_{2-j} \times \pi_k) \circ \Delta_{123}.
\end{aligned}$$

Thus we get a more symmetric condition for the decomposition to be multiplicative: we must have $(\pi_a \times \pi_b \times \pi_c) \circ \Delta_{123} = 0$ for $a + b + c \neq 4$. In particular, e.g., the nonvanishing of the Gross-Schoen cycle (Example 6.4) implies that the decomposition is not multiplicative.

However, let us also remark that when $g = 2$, we do have that $(\pi_a \times \pi_b \times \pi_c) \circ \Delta_{123} = 0$ for $a+b+c \neq 4$, and the decomposition is multiplicative. More generally, it follows from the results of [71] that the decomposition is multiplicative over the moduli space of *hyperelliptic* curves of arbitrary genus.

In any case, failure of decomposition to be multiplicative implies that the cup product in the algebra $\mathrm{CH}^\bullet(C_g^n)$ will look somewhat strange with respect to the relative Künneth decomposition of the Chow groups of $\mathrm{CH}^\bullet(C_g^n)$. The situation is analogous to what happens in topology, when one has a multiplicative spectral sequence $E_r^{pq} \implies H^\bullet$, and the cup product on the E_∞ page of the spectral sequence is different from the cup product in the algebra $H^\bullet$. On the level of Betti realizations, this is more than an analogy. The cohomology groups $H^\bullet(C_g^n, \mathbf{Q})$ carry a *Leray filtration*, and the associated graded $\mathfrak{gr}_L H^\bullet(C_g^n, \mathbf{Q})$ is isomorphic to the E_∞ page of the Leray spectral sequence for $C_g^n \to M_g$. Our canonical decomposition of the Chow motive $h(C_g^n/M_g)$ gives, on the level of cohomology, an isomorphism of $\mathbf{Q}$-vector spaces $H^\bullet(C_g^n, \mathbf{Q}) \cong \mathfrak{gr}_L H^\bullet(C_g^n, \mathbf{Q})$. But this is not an isomorphism of algebras: the multiplication in $\mathfrak{gr}_L H^\bullet(C_g^n, \mathbf{Q})$ is defined by using only the maps $h^i(C_g/M_g) \otimes h^j(C_g/M_g) \to h^k(C_g/M_g)$ for $i + j = k$, and discarding all other parts of the cup product $h(C_g/M_g)^{\otimes 2} \to h(C_g/M_g)$.

5.3. Decomposition into representations of the symplectic group

Let $J_g \to M_g$ be the universal jacobian. By Proposition 2.6 there is an isomorphism of Chow motives over M_g:

$$h^1(C_g/M_g) \cong h^1(J_g/M_g).$$

It follows that Theorem 4.3 gives us a decomposition

$$h^1(C_g/M_g)^{\otimes n} \cong \bigoplus_{\substack{|\lambda| \leq n \\ |\lambda| \equiv n \pmod 2}} h^1(C_g/M_g)_{\langle\lambda\rangle} \otimes \mathbb{L}^{n-|\lambda|} \otimes \beta^*_{\lambda,n}.$$

We denote the motive $h^1(C_g/M_g)_{\langle\lambda\rangle}$ by $\mathbf{V}_{\langle\lambda\rangle}$. We often write $\mathbf{V}$ for the motive $\mathbf{V}_{\langle 1\rangle}$.

We also denote by $\mathbf{V}^{\langle n\rangle}$ the summand of $\mathbf{V}^{\otimes n}$ given by $\bigoplus_{|\lambda|=n} \mathbf{V}_{\langle\lambda\rangle} \otimes \sigma^*_{\lambda^T}$, and we refer to this as the *primitive part* of $\mathbf{V}^{\otimes n}$.

One can make the action of the Brauer algebra $\mathfrak{Br}^{(-2g)}(n,n)$ on $\mathbf{V}^{\otimes n}$ more explicit. Let B be an (n,n)-Brauer diagram. Label the nodes in the Brauer diagram along the top row as $1, \dots, n$ and along the bottom row as $n+1, \dots, 2n$. Write $(ij) \in B$ to denote that the ith and jth row are connected by a strand. Then

$$\prod_{(ij)\in B} p_{ij}^*(\pi_1) \in \mathrm{CH}^n(C_g^{2n})$$

is a well defined correspondence $C_g^n \vdash C_g^n$, where we consider π_1 as a cycle in $\mathrm{CH}^1(C_g^2)$ and p_{ij} denotes the projection onto the ith and jth factor. This correspondence gives a map $h(C_g^n/M_g) \to h(C_g^n/M_g)$ that preserves the summand $h^1(C_g/M_g)^{\otimes n}$, and we obtain a well defined action of $\mathfrak{Br}^{(-2g)}(n,n)$ on $h^1(C_g/M_g)^{\otimes n}$. This action agrees with the one defined in Proposition 4.1.

On the level of Chow groups, we can also describe the action of a Brauer diagram as follows. An $(n, n-2)$-Brauer diagram in which the ith and j dot on the top row are connected by a strand, and all others are vertical, gives rise to the following morphism of Chow groups:

$$\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes n}) \hookrightarrow \mathrm{CH}^\bullet(C_g^n) \xrightarrow{\Delta_{ij}^*} \mathrm{CH}^\bullet(C_g^{n-1}) \xrightarrow{p_*} \mathrm{CH}^{\bullet-1}(C_g^{n-2}),$$

where p denotes the projection that forgets the marked point corresponding to the diagonal Δ_{ij}. The image of the composition of these morphisms actually lands inside the summand $\mathrm{CH}^{\bullet-1}(M_g, \mathbf{V}^{\otimes n-2}) \subset \mathrm{CH}^{\bullet-1}(C_g^{n-2})$. Similarly, an $(n-2, n)$-Brauer diagram in which the ith and jth dots on the bottom row are connected by a strand gives rise to the following morphism of Chow groups:

$$\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes(n-2)}) \hookrightarrow \mathrm{CH}^\bullet(C_g^{n-2}) \xrightarrow{p^*} \mathrm{CH}^\bullet(C_g^{n-1}) \xrightarrow{(\Delta_{ij})_*} \mathrm{CH}^{\bullet+1}(C_g^n) \xrightarrow{\pi_1^{\times n}} \mathrm{CH}^{\bullet+1}(M_g, \mathbf{V}^{\otimes n}).$$

5.4. Decomposing the tautological ring

We have explained that $h(C_g^n/M_g)$ is a direct sum of terms of the form $h^0(C_g/M_g)^{\otimes n_0} \otimes h^1(C_g/M_g)^{\otimes n_1} \otimes h^2(C_g/M_g)^{\otimes n_2}$, with $n_0+n_1+n_2 = n$. Since we also have $h^0(C_g/M_g) \cong \mathbf{1}$ and $h^2(C_g/M_g) \cong \mathbb{L}$, and $h^1(C_g/M_g)^{\otimes n_1}$ is a direct sum of terms of the form $\mathbf{V}_{\langle\lambda\rangle} \otimes \mathbb{L}^{n_1-|\lambda|}$, we conclude that $h(C_g^n/M_g)$ is a direct sum of motives of the form $\mathbf{V}_{\langle\lambda\rangle}$ and their Tate twists.

THEOREM 5.5. – *Let n be arbitrary, and consider $C_g^n \to M_g$. Choose any decomposition*

$$h(C_g^n/M_g) \cong \bigoplus_i \mathbf{V}_{\langle\lambda_i\rangle} \otimes \mathbb{L}^{m_i}$$

in Mot_{M_g}. Then under the equality $\mathrm{CH}^k(C_g^n) \cong \mathrm{CH}^k(M_g, h(C_g^n/M_g))$ we have the following compatibility:

$$\begin{array}{ccc} \mathrm{CH}^k(C_g^n) & \cong & \bigoplus_i \mathrm{CH}^{k-m_i}(M_g, \mathbf{V}_{\langle\lambda_i\rangle}) \\ \cup\!\shortmid & & \cup\!\shortmid \\ R^k(C_g^n) & \cong & \bigoplus_i R^{k-m_i}(M_g, \mathbf{V}_{\langle\lambda_i\rangle}), \end{array}$$

where both horizontal arrows are induced by our choice of decomposition of $h(C_g^n/M_g)$.

Proof. – Consider a summand $\mathbf{V}_{\langle\lambda\rangle} \otimes \mathbb{L}^{m_1}$ of $h(C_g^{n_1}/M_g)$ and a summand $\mathbf{V}_{\langle\lambda\rangle} \otimes \mathbb{L}^{m_2}$ of $h(C_g^{n_2}/M_g)$. Then there exists a correspondence $\pi \in \mathrm{CH}(C_g^{n_1+n_2})$—not a correspondence of degree 0, in general—which maps the first summand isomorphically onto the second, considered as a correspondence $C_g^{n_1} \vdash C_g^{n_2}$. Moreover, π can be built out of the projectors onto the Künneth components of $h(C_g^n/M_g)$ and the correspondences given by Brauer diagrams. As such, π is actually a tautological class.

Now $\mathrm{CH}^\bullet(M_g, \mathbf{V}_{\langle\lambda\rangle} \otimes \mathbb{L}^{m_1})$ is a summand of $\mathrm{CH}^\bullet(C_g^{n_1})$ and as such there is a well defined subspace of tautological classes inside it, which we denote $R^\bullet(M_g, \mathbf{V}_{\langle\lambda\rangle} \otimes \mathbb{L}^{m_1})$. Similarly for the other summand. Now the correspondence π which gives the isomorphism between the two summands is a tautological class, and in particular it maps tautological classes to tautological classes and gives an isomorphism (not preserving the grading unless $m_1 = m_2$), $R^\bullet(M_g, \mathbf{V}_{\langle\lambda\rangle} \otimes \mathbb{L}^{m_1}) \cong R^\bullet(M_g, \mathbf{V}_{\langle\lambda\rangle} \otimes \mathbb{L}^{m_2})$.

Finally since the projectors onto each summand $\mathbf{V}_{\langle\lambda_i\rangle} \otimes \mathbb{L}^{m_i}$ of $h(C_g^n/M_g)$ are all given by tautological classes, the tautological ring $R^\bullet(C_g^n)$ is the direct sum of its projections onto each of the summands in the decomposition of $\mathrm{CH}^\bullet(C_g^n)$. □

5.5. Curves with rational tails

Let X be a smooth projective variety, and $X[n]$ the Fulton-MacPherson compactification of the configuration space of n distinct ordered points on X [17]. A result of Li [41] expresses the Chow motive $h(X[n])$ as a direct sum of Chow motives of the form $h(X)^{\otimes i} \otimes \mathbb{L}^j$ for $0 \leq i \leq n$; this can be seen by an inductive argument from the blow-up formula and the construction of $X[n]$ as an iterated blow-up of the cartesian product X^n.

The analogous statement remains true (with the same proof) for a family $X \to S$ of smooth projective varieties, and the relative Chow motive of the relative Fulton-MacPherson compactification. In particular we may consider the universal family $C_g \to M_g$ over the moduli space of curves. In this case the relative configuration space of n distinct ordered points is the space $M_{g,n}$, and the relative Fulton-MacPherson compactification of $M_{g,n}$ is the moduli space $M_{g,n}^{\mathrm{rt}}$ of curves with *rational tails*, which is an iterated blow-up of C_g^n. By "relative compactification" we mean that the map $M_{g,n}^{\mathrm{rt}} \to M_g$ is proper.

It follows from the above considerations that the results of this section remain valid when C_g^n is replaced with $M_{g,n}^{\mathrm{rt}}$. There will in particular exist a decomposition of Chow motives

$$h(M_{g,n}^{\mathrm{rt}}/M_g) \cong \bigoplus_i \mathbf{V}_{\langle\lambda_i\rangle} \otimes \mathbb{L}^{m_i},$$

and moreover, there is a canonical choice of such decomposition in which each correspondence projecting onto a summand is a tautological class (cf. [41, Theorem 3.2]). It follows in particular that

$$R^\bullet(M_{g,n}^{\mathrm{rt}}) \cong \bigoplus_i R^{\bullet - m_i}(M_g, \mathbf{V}_{\langle\lambda_i\rangle}).$$

The results of [41] give explicit formulas expressing the motive $h(M_{g,n}^{\mathrm{rt}}/M_g)$ in terms of motives $\mathbf{V}^{\otimes i} \otimes \mathbb{L}^j$ and hence in terms of Tate twists of motives $\mathbf{V}_{\langle\lambda\rangle}$. For an $\mathfrak{S}_n$-equivariant version (formulated in that paper only in terms of cohomology) see [18].

5.6. Tautological cohomology groups

For a partition λ we let $\mathbb{V}_{\langle\lambda\rangle}$ be the $\mathbf{Q}$-local system on M_g defined by the representation of $\mathrm{Sp}(2g)$ of highest weight λ. Then the Chow motive $\mathbf{V}_{\langle\lambda\rangle}$ has as its Betti realization $\mathbb{V}_{\langle\lambda\rangle}[-|\lambda|]$, i.e., the local system $\mathbb{V}_\lambda$, considered as a complex concentrated in cohomological degree $|\lambda|$, and there is a cycle class map

$$\mathrm{CH}^k(M_g, \mathbf{V}_{\langle\lambda\rangle}) \to H^{2k-|\lambda|}(M_g, \mathbb{V}_{\langle\lambda\rangle}).$$

We denote by $RH^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle})$ the image of $R^\bullet(M_g, \mathbf{V}_{\langle\lambda\rangle})$ under the cycle class map. For $\lambda = 0$ we get the usual tautological cohomology groups of M_g. A folklore conjecture says that any homological equivalence between tautological classes is a rational equivalence, which would imply that

$$R^k(M_g, \mathbf{V}_{\langle\lambda\rangle}) \to RH^{2k-|\lambda|}(M_g, \mathbb{V}_{\langle\lambda\rangle})$$

is always an isomorphism.

We caution the reader that since the Betti realization of $\mathbf{V}$ is not the local system $\mathbb{V}$, but $\mathbb{V}[-1]$, there are many opportunities to get confused about the Koszul sign rule when comparing results in cohomology and in Chow.

All the results of this section are valid mutatis mutandis also on the level of cohomology. One can either see this formally by applying the Betti realization functor or by repeating the proofs in the cohomological setting. For example, let us state the cohomological version of Theorem 5.5:

THEOREM 5.6. – *Let n be arbitrary, and consider $f: C_g^n \to M_g$. Choose a decomposition*

$$Rf_*\mathbf{Q} \cong \bigoplus_i \mathbb{V}_{\langle\lambda_i\rangle}[-m_i]$$

in $D_c^b(M_g)$. Under the equality $H^k(C_g^n, \mathbf{Q}) \cong \mathbb{H}^k(M_g, Rf_*\mathbf{Q})$ we have the following compatibility:

$$\begin{array}{ccc} H^k(C_g^n, \mathbf{Q}) & \cong & \bigoplus_i H^{k-m_i}(M_g, \mathbb{V}_{\langle\lambda_i\rangle}) \\ \cup\!\!| & & \cup\!\!| \\ RH^k(C_g^n) & \cong & \bigoplus_i RH^{k-m_i}(M_g, \mathbb{V}_{\langle\lambda_i\rangle}), \end{array}$$

where both horizontal arrows are induced by the choice of decomposition of $Rf_*\mathbf{Q}$.

6. Examples

6.1. Example: ψ_1^n

Let us consider the class $\psi_1^n \in \mathrm{CH}^n(C_g^1)$. Its image under the projectors π_0, π_1 and π_2 is given by

$$\pi_0 \circ \psi_1^n = (p_2)_*\Big(\frac{1}{2g-2}\psi_1^{n+1} - \frac{1}{2(2g-2)^2}\kappa_1\psi_1^n\Big) = \frac{1}{2g-2}\kappa_n - \frac{1}{2(2g-2)^2}\kappa_1\kappa_{n-1},$$

$$\begin{aligned}\pi_1 \circ \psi_1^n &= (p_2)_*\Big(\Delta_{12}\psi_1^n - \frac{1}{2g-2}\psi_1^n\psi_2 - \frac{1}{2g-2}\psi_1^{n+1} + \frac{1}{(2g-2)^2}\kappa_1\psi_1^n\Big) \\ &= \psi_1^n - \frac{1}{2g-2}\kappa_{n-1}\psi_1 - \frac{1}{2g-2}\kappa_n + \frac{1}{(2g-2)^2}\kappa_1\kappa_{n-1},\end{aligned}$$

$$\pi_2 \circ \psi_1^n = (p_2)_*\Big(\frac{1}{2g-2}\psi_1^n\psi_2 - \frac{1}{2(2g-2)^2}\kappa_1\psi_1^n\Big) = \frac{1}{2g-2}\kappa_{n-1}\psi_1 - \frac{1}{2(2g-2)^2}\kappa_1\kappa_{n-1}.$$

Here $p_2: C_g^2 \to C_g^1$ forgets the *first* marked point. These are thus the projections of the class $\psi_1^n \in \mathrm{CH}^n(C_g)$ into the three summands $\mathrm{CH}^n(M_g, h^0(C_g/M_g))$, $\mathrm{CH}^n(M_g, h^1(C_g/M_g))$ and $\mathrm{CH}^n(M_g, h^2(C_g/M_g))$, respectively. We can make some simple observations/sanity checks:

- The three classes sum to ψ_1^n.
- The class in $\mathrm{CH}^n(M_g, h^0(C_g/M_g))$ is pulled back from $\mathrm{CH}^n(M_g)$.
- When $n = 1$, the class in $\mathrm{CH}^1(M_g, h^2(C_g/M_g))$ restricts to a cycle of degree 1 in CH^1 of a fiber of $\lambda: C_g \to M_g$.
- The classes $\pi_0 \circ \psi_1^n$ and $\pi_1 \circ \psi_1^n$ push forward to zero under $\lambda: C_g \to M_g$. The class $\pi_2 \circ \psi_1^n$ has the same pushforward as ψ_1^n.

We note that $\pi_1 \circ \psi_1^n$ vanishes for $n = 0$ and $n = 1$, using that $\kappa_0 = 2g - 2$. This could also have been seen from the fact that

$$\begin{aligned}\mathrm{CH}^k(C_g) &\cong \mathrm{CH}^k(M_g, \mathbf{1}) \oplus \mathrm{CH}^k(M_g, \mathbf{V}_{\langle 1\rangle}) \oplus \mathrm{CH}^k(M_g, \mathbb{L})\\ &= \mathrm{CH}^k(M_g) \oplus \mathrm{CH}^k(M_g, \mathbf{V}) \oplus \mathrm{CH}^{k-1}(M_g)\end{aligned}$$

which (using Harer's calculation of the Picard groups of M_g and C_g [29]) implies that $\mathrm{CH}^0(M_g, \mathbf{V}) = \mathrm{CH}^1(M_g, \mathbf{V}) = 0$. The first of these classes which can be nontrivial is thus $\pi_1 \circ \psi_1^2 \in \mathrm{CH}^2(M_g, \mathbf{V})$. This class vanishes for $g \leq 4$, but is nonzero if $g \geq 5$.

To prove that the class is nonzero for $g \geq 5$ it is easier to work in cohomology. For large g, nontriviality is a consequence of Harer stability and the Mumford conjecture [45]: as $g \to \infty$, $H^\bullet(C_g^1)$ stabilizes to a polynomial ring in the κ-classes and ψ_1 [44, Proposition 2.1]. More precisely, $H^4(C_g^1)$ is in the stable range when $g \geq 7$, so there are no relations between the generators in this degree and $\pi_1 \circ \psi_1^2$ must be nonzero. Nontriviality for $g = 5, 6$ can be checked e.g., by multiplying the class with ψ_1^2 and pushing it down to M_g. One computes that

$$\lambda_*(\psi_1^2 \cdot (\pi_1 \circ \psi_1^2)) = \kappa_3 - \frac{2}{2g-2}\kappa_2\kappa_1 + \frac{1}{(2g-2)^2}\kappa_1^3.$$

This class can then be multiplied with κ_1^{g-5} to get a class in the socle of the tautological ring, which can be verified to be nonzero by integrating it against $\lambda_g\lambda_{g-1}$. (We discuss the $\lambda_g\lambda_{g-1}$-pairing more in Section 7.)

The vanishing for $g \leq 4$ can be proven by standard methods for computing tautological rings and a dimension count. Let us consider the case $g = 4$: in this case one only needs to know that $R^1(M_4) \cong R^2(M_4) \cong \mathbf{Q}$ and that $R^2(C_4^1) \cong \mathbf{Q}^2$. Now we have the relative Künneth decomposition

$$R^2(C_4) = R^2(M_4) \oplus R^2(M_4, \mathbf{V}) \oplus R^1(M_4),$$

where the first and last terms are one-dimensional since $R^1(M_4) \cong R^2(M_4) \cong \mathbf{Q}$; consequently, the middle term has to vanish since $R^2(C_4)$ is two-dimensional. But $\pi_1 \circ \psi_1^2$ is an element of $R^2(M_4, \mathbf{V})$, so it must vanish.

6.2. Example: the diagonal

Let us decompose the class $\Delta_{12} \in \mathrm{CH}^1(C_g^2)$ into summands. One finds exactly three nonzero terms:

$$\begin{aligned}(\pi_2 \times \pi_0) \circ \Delta_{12} &= \frac{1}{2g-2}\psi_1 - \frac{1}{2(2g-2)^2}\kappa_1 && \in \mathrm{CH}^1(M_g, \mathbb{L} \otimes \mathbf{1}),\\ (\pi_1 \times \pi_1) \circ \Delta_{12} &= \Delta_{12} - \frac{1}{2g-2}(\psi_1 + \psi_2) + \frac{1}{(2g-2)^2}\kappa_1 && \in \mathrm{CH}^1(M_g, \mathbf{V} \otimes \mathbf{V}),\\ (\pi_0 \times \pi_2) \circ \Delta_{12} &= \frac{1}{2g-2}\psi_2 - \frac{1}{2(2g-2)^2}\kappa_1 && \in \mathrm{CH}^1(M_g, \mathbf{1} \otimes \mathbb{L}),\end{aligned}$$

where $h^0(C_g/M_g) = \mathbf{1}$, $h^1(C_g/M_g) = \mathbf{V}$, $h^2(C_g/M_g) = \mathbb{L}$. Thus the terms in the decomposition of Δ_{12} are given exactly by the projectors π_i themselves, considered as classes on C_g^2.

Note that $\mathbf{V} \otimes \mathbf{V} \cong \mathbf{V}_{\langle 2\rangle} \oplus \mathbf{V}_{\langle 1,1\rangle} \oplus \mathbb{L}$. In fact we have $\mathrm{CH}^1(M_g, \mathbf{V}_{\langle 2\rangle}) = \mathrm{CH}^1(M_g, \mathbf{V}_{\langle 1,1\rangle}) = 0$, and the class π_1 is a generator for the summand $\mathrm{CH}^1(M_g, \mathbb{L}) \cong \mathrm{CH}^0(M_g) \cong \mathbf{Q}$.

The fact that $\mathrm{CH}^1(M_g, \mathbf{V}_{\langle 2\rangle}) = \mathrm{CH}^1(M_g, \mathbf{V}_{\langle 1,1\rangle}) = 0$ can be proven by a dimension count argument much like the one from the previous example, using $\mathrm{CH}^1(C_g^2) = \mathbf{Q}\{\kappa_1, \Delta_{12}, \psi_1, \psi_2\}$, which follows from Harer's calculation of the Picard group of $M_{g,n}$ [29]. Now in the relative Künneth decomposition of $\mathrm{CH}^1(C_g^2)$ we find the summands $\mathrm{CH}^1(M_g, \mathbf{1} \otimes \mathbf{1}) \cong \mathbf{Q}\{\kappa_1\}$, $\mathrm{CH}^1(M_g, \mathbf{1} \otimes \mathbb{L})$, $\mathrm{CH}^1(M_g, \mathbb{L} \otimes \mathbf{1})$ and $\mathrm{CH}^1(M_g, \mathbb{L}) \subset \mathrm{CH}^1(M_g, \mathbf{V}\otimes\mathbf{V})$. All these last three terms are nonzero since $\mathrm{CH}^1(M_g, \mathbb{L}) \cong \mathrm{CH}^0(M_g) \cong \mathbf{Q}$.

6.3. Example: the Faber-Pandharipande cycle

We consider the class $\Delta_{12}\psi_1 \in \mathrm{CH}^2(C_g^2)$. Applying the operator $\pi_1^{\times 2}$ gives the class

$$\begin{aligned}\pi_1^{\times 2} \circ \Delta_{12}\psi_1 = \Delta_{12}\psi_1 - \frac{1}{2g-2}\psi_1\psi_2 - \frac{1}{2g-2}(\psi_1^2 + \psi_2^2) + \frac{1}{(2g-2)^2}\kappa_1(\psi_1 + \psi_2) \\ + \frac{1}{(2g-2)^2}\kappa_2 - \frac{1}{(2g-2)^3}\kappa_1^2\end{aligned} \tag{6}$$

which now defines an element of $\mathrm{CH}^2(M_g, \mathbf{V}^{\otimes 2})$. Since the class (6) is $\mathfrak{S}_2$-invariant, it is in fact a class in $\mathrm{CH}^2(M_g, \mathrm{Sym}^2\mathbf{V})$. We call this class the *unrefined Faber-Pandharipande cycle*.

Now we have a decomposition $\mathrm{Sym}^2\mathbf{V} = \mathbf{V}_{\langle 1,1\rangle} \oplus \mathbb{L}$. So (6) can be written as the sum of a class in $\mathrm{CH}^2(M_g, \mathbf{V}_{\langle 1,1\rangle})$ and a class in $\mathrm{CH}^2(M_g, \mathbb{L}) \cong \mathrm{CH}^1(M_g)$. As explained in Subsection 3.3, Nazarov's theorem gives a general method to write down a projector acting on $\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes n})$ whose image is a particular summand $\mathrm{CH}^\bullet(M_g, \mathbf{V}_{\langle\lambda\rangle}) \otimes \sigma_{\lambda^T}^*$, where λ is a partition of n. Although Nazarov's theorem is overkill in this case, where one could quite easily figure out the right projector by hand, the result is that the correspondence

$$\pi = \frac{1}{2}(b_{13}b_{24} + b_{14}b_{23}) + \frac{1}{2g}b_{12}b_{34} \in \mathrm{CH}^2(C_g^4)$$

acts on $\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes 2})$ with image $\mathrm{CH}^\bullet(M_g, \mathbf{V}_{\langle 1,1\rangle}) \otimes \sigma_2^* \cong \mathrm{CH}^\bullet(M_g, \mathbf{V}_{\langle 1,1\rangle})$. Here b_{ij} denotes the class $\Delta_{ij} - \frac{1}{2g-2}(\psi_i + \psi_j) + \frac{1}{(2g-2)^2}\kappa_1$, i.e., the pullback of the correspondence defining the projector π_1.

Applying π gives the class

$$\begin{aligned}\Delta_{12}\psi_1 - \frac{1}{2g-2}\psi_1\psi_2 - \frac{1}{2g-2}(\psi_1^2 + \psi_2^2) + \frac{1}{(2g-2)^2}\kappa_1(\psi_1 + \psi_2) + \frac{1}{(2g-2)^2}\kappa_2 \\ - \frac{1}{(2g-2)^3}\kappa_1^2 - \frac{2g-1}{2g(2g-2)}\kappa_1\left(\Delta_{12} - \frac{1}{2g-2}(\psi_1 + \psi_2) + \frac{1}{(2g-2)^2}\kappa_1\right)\end{aligned} \tag{7}$$

which now defines an element of $\mathrm{CH}^2(M_g, \mathbf{V}_{\langle 1,1\rangle})$. We call (7) the *refined Faber-Pandharipande cycle*. This is the projection of $\Delta_{12}\psi_1$ onto the summand $\mathrm{CH}^2(M_g, \mathbf{V}_{\langle 1,1\rangle})$. Comparing the expressions for the refined and unrefined Faber-Pandharipande cycles, we see that we have subtracted the term

$$\frac{2g-1}{2g(2g-2)}\kappa_1 b_{12}.$$

This term is thus the projection of $\Delta_{12}\psi_1$ onto the summand $\mathrm{CH}^2(M_g, \mathbb{L}) \subset \mathrm{CH}^2(M_g, \mathrm{Sym}^2\mathbf{V})$. We remark that since we know from the previous example that b_{12} lies in the summand $\mathrm{CH}^1(M_g, \mathbb{L}) \subset \mathrm{CH}^1(M_g, \mathbf{V}^{\otimes 2})$, it follows that $\kappa_1 b_{12}$ indeed gives a class in $\mathrm{CH}^2(M_g, \mathbb{L}) \subset \mathrm{CH}^2(M_g, \mathbf{V}^{\otimes 2})$.

Let X be a curve of genus g. It is interesting to consider the image of the Faber-Pandharipande cycle in $\mathrm{CH}^2(X^2)$. In this case, removing terms which obviously vanish leaves

$$\Delta_{12}\psi_1 - \frac{1}{2g-2}\psi_1\psi_2 \in \mathrm{CH}^2(X^2).$$

This simplified expression is what is more commonly referred to as the Faber-Pandharipande cycle. Green and Griffiths [22] proved that the Faber-Pandharipande cycle is nonzero in $\mathrm{CH}^2(X^2)$ for a very general curve X of genus ≥ 4 over the complex numbers. The third author gave a different proof of this result [76] valid also in positive characteristic. Such a result is rather subtle since the Faber-Pandharipande cycle is not only homologically trivial but Abel-Jacobi trivial.

Reasoning as in Example 6.1, one can show that the refined Faber-Pandharipande cycle is zero for $g = 2, 3$ but that it is nonzero for all $g \geq 4$. Thus the refined Faber-Pandharipande cycle is nonzero precisely in those genera where it is nonzero in $\mathrm{CH}^2(X^2)$ for a generic curve X. By contrast the unrefined Faber-Pandharipande cycle is nonzero also for $g = 3$, even though $R^2(M_3, \mathbf{V}_{\langle 1,1\rangle}) = 0$. This illustrates the utility of Nazarov's refined projectors when trying to determine precisely which of these local systems have nonzero tautological groups.

6.4. Example: the Gross-Schoen cycle

Let us consider the class $\Delta_{123} \in \mathrm{CH}^2(C_g^3)$. We can apply $\pi_1^{\times 3}$ to this class to get

$$\begin{aligned}\pi_1^{\times 3} \circ \Delta_{123} = \Delta_{123} &- \frac{1}{2g-2}((\Delta_{12}+\Delta_{23})\psi_1 + (\Delta_{13}+\Delta_{23})\psi_2 + (\Delta_{12}+\Delta_{13})\psi_3) \\ &+ \frac{1}{(2g-2)^2}(\psi_1^2+\psi_2^2+\psi_3^2) + \frac{2}{(2g-2)^2}(\psi_1\psi_2+\psi_1\psi_3+\psi_2\psi_3) \\ &+ \frac{1}{(2g-2)^2}\kappa_1(\Delta_{12}+\Delta_{13}+\Delta_{23}) - \frac{3}{(2g-2)^3}\kappa_1(\psi_1+\psi_2+\psi_3) \\ &- \frac{1}{(2g-2)^3}\kappa_2 + \frac{3}{(2g-2)^4}\kappa_1^2.\end{aligned}$$

We call this the *Gross-Schoen cycle*. Being $\mathfrak{S}_3$-invariant, it defines a class in the summand $\mathrm{CH}^2(M_g, \mathrm{Sym}^3\mathbf{V}) \subset \mathrm{CH}^2(M_g, \mathbf{V}^{\otimes 3})$. There is now a decomposition

$$\mathrm{Sym}^3\mathbf{V} \cong \mathbf{V}_{\langle 1,1,1\rangle} \oplus \mathbf{V}_{\langle 1\rangle} \otimes \mathbb{L},$$

and one could try to define a "refined" Gross-Schoen cycle by projecting onto the summand $\mathrm{CH}^2(M_g, \mathbf{V}_{\langle 1,1,1\rangle})$, just as we did for the Faber-Pandharipande cycle in the previous example. However, the difference between the refined and unrefined Gross-Schoen cycles would then be an element of $\mathrm{CH}^2(M_g, \mathbf{V}_{\langle 1\rangle} \otimes \mathbb{L}) = \mathrm{CH}^1(M_g, \mathbf{V})$, which always vanishes. By using Nazarov's theorem one can construct a "refined projector" onto $\mathrm{CH}^\bullet(M_g, \mathbf{V}_{\langle 1,1,1\rangle})$; the image of Δ_{123} under this refined projector agrees with the image under the naive projector $\pi_1^{\times 3}$, which is a nontrivial consistency check.

The cycle originally considered by Gross and Schoen in [23] is related to ours as follows. Let X be a smooth curve and $\mathfrak{z}$ a degree 1 zero-cycle on X. Then they studied the cycle

$$\Delta_{123} - \mathfrak{z}_1\Delta_{23} - \mathfrak{z}_2\Delta_{13} - \mathfrak{z}_3\Delta_{12} + \mathfrak{z}_2\mathfrak{z}_3 + \mathfrak{z}_1\mathfrak{z}_3 + \mathfrak{z}_1\mathfrak{z}_2 \in \mathrm{CH}^2(X^3),$$

where $\mathfrak{z}_i$ denotes the pullback of $\mathfrak{z}$ from the projection map onto the ith factor. Considering our cycle in $\mathrm{CH}^2(X^3)$ and removing terms which obviously vanish gives

$$\begin{aligned}\Delta_{123} - \frac{1}{2g-2}((\Delta_{12}+\Delta_{23})\psi_1 + (\Delta_{13}+\Delta_{23})\psi_2 + (\Delta_{12}+\Delta_{13})\psi_3)\\ + \frac{2}{(2g-2)^2}(\psi_1\psi_2+\psi_1\psi_3+\psi_2\psi_3),\end{aligned}$$

which does *not* coincide with the usual Gross-Schoen cycle for $\mathfrak{z} = \frac{1}{2g-2}\psi$. However, the difference between the two cycles is given by a sum of Faber-Pandharipande cycles. In particular, our Gross-Schoen cycle and the usual one will have the same image under the Abel-Jacobi map, since the Faber-Pandharipande cycle is Abel-Jacobi trivial.

The Gross-Schoen cycle defines a nontrivial class in $\mathrm{CH}^2(M_g, \mathbf{V}_{\langle 1,1,1\rangle})$ for all $g \geq 3$. Nonvanishing of the Gross-Schoen cycle is equivalent to nonvanishing of the *Ceresa cycle* [6], which is known to be nonzero in $\mathrm{CH}^2(X^3)$ if X is a very general curve of genus $g \geq 3$. See [16] for this result in positive characteristic.

7. Consequences for Gorenstein conjectures

By a theorem of Looijenga [43], it is known that

$$R^{g-2+n}(C_g^n) \cong \mathbf{Q}$$

and that the tautological ring vanishes above this degree. More precisely, Looijenga proved the vanishing and that this group is at most 1-dimensional, and Faber [14] found an example of a nonzero tautological class in this degree. The top nonzero degree of the tautological ring is called the *socle*.

This isomorphism can be made explicit in the following way. We define a map

$$R^{g-2}(M_g) \to \mathbf{Q}$$

by

$$\alpha \mapsto \int_{\overline{M}_g} \overline{\alpha}\cdot\lambda_g\lambda_{g-1},$$

where $\overline{\alpha}$ denotes the closure of an algebraic cycle representing the class α, and λ_i denotes the ith Chern class of the Hodge bundle. We recall that the Hodge bundle is the locally free sheaf of rank g whose fiber at a moduli point $[C]$ is the space of holomorphic differentials on C. A priori the integral would seem to not be well defined, since it depends on the choice of an algebraic cycle representing α, but the integral is in fact well defined since multiplication by $\lambda_g\lambda_{g-1}$ kills everything supported on the Deligne-Mumford boundary. For $n > 0$, one has an isomorphism

$$R^{g-2+n}(C_g^n) \to R^{g-2}(M_g)$$

given by pushforward (with inverse given by pullback and multiplication by $\frac{1}{(2g-2)^n}\psi_1\psi_2\cdots\psi_n$).

All in all, this means that we have a pairing, which we denote by brackets:

$$\begin{aligned} R^k(C_g^n)\otimes R^{g-2+n-k}(C_g^n) &\longrightarrow \mathbf{Q}\\ \alpha\otimes\beta &\longmapsto \langle\alpha,\beta\rangle\end{aligned}$$

given by cup product, pushing down to M_g, and integrating against $\lambda_g\lambda_{g-1}$. Arbitrary integrals of tautological classes on $\overline{M}_{g,n}$ can be calculated algorithmically and efficiently [11, 35], in particular integrals over $\overline{M}_g$ of top degree classes in $R^{g-2}(M_g)$ paired with $\lambda_g\lambda_{g-1}$.

Lemma 7.1. – *Let $\alpha \in R^k(C_g^n)$ and $\beta \in R^{g-2+n-k}(C_g^n)$, and let $i_1,\dots,i_n \in \{0,1,2\}$. There is an equality*

$$\big\langle(\pi_{i_1}\times\cdots\times\pi_{i_n})\circ\alpha,\beta\big\rangle = \big\langle\alpha,(\pi_{2-i_1}\times\cdots\times\pi_{2-i_n})\circ\beta\big\rangle.$$

Proof. – Consider α as a morphism $\mathbf{1} \to h(C_g^n/M_g)\otimes\mathbb{L}^{-k}$, and β as a morphism $h(C_g^n/M_g))\otimes\mathbb{L}^{-k} \to \mathbb{L}^{2-g}$. The composition $\beta\circ\alpha \in \mathsf{Mot}_{M_g}(\mathbf{1},\mathbb{L}^{2-g}) = \mathrm{CH}^{g-2}(M_g)$ is the product $\alpha\cdot\beta$ pushed forward to M_g. Now if π is any correspondence $C_g^n \vdash C_g^n$ then

$$\beta\circ(\pi\circ\alpha) = (\beta\circ\pi)\circ\alpha.$$

But $\beta\circ\pi = \pi^t\circ\beta$ by Remark 2.1. Since $(\pi_{i_1}\times\cdots\times\pi_{i_n})^t = (\pi_{2-i_1}\times\cdots\times\pi_{2-i_n})$ we are done. □

As explained in Subsection 5.2.3, the cup product in $\mathrm{CH}^\bullet(C_g^n)$ (and then also in $R^\bullet(C_g^n)$) is in general not so easily described in terms of the relative Künneth decomposition of these algebras. The next proposition shows, however, that the *socle pairing* on $R^\bullet(C_g^n)$ takes a very simple form with respect to the Künneth decomposition.

Proposition 7.2. – *The Gram matrix describing the socle pairing in the algebra $R^\bullet(C_g^n)$ is block-diagonal with respect to the relative Künneth decomposition of $R^\bullet(C_g^n)$. More precisely, the summand*

$$R^k(M_g,h^{i_1}(C_g/M_g)\otimes\cdots\otimes h^{i_n}(C_g/M_g)) \subset R^k(C_g^n)$$

pairs to zero with all summands in complementary degree except for

$$R^{g-2+n-k}(M_g,h^{2-i_1}(C_g/M_g)\otimes\cdots\otimes h^{2-i_n}(C_g/M_g)) \subset R^{g-2+n-k}(C_g^n).$$

Proof. – Take $\alpha \in R^k(C_g^n)$. Suppose it lies in the summand

$$R^k(M_g,h^{i_1}(C_g/M_g)\otimes\cdots\otimes h^{i_n}(C_g/M_g));$$

equivalently, $(\pi_{i_1}\times\cdots\times\pi_{i_n})\circ\alpha = \alpha$. For β in complementary degree we have

$$\langle\alpha,\beta\rangle = \big\langle(\pi_{i_1}\times\cdots\times\pi_{i_n})\circ\alpha,\beta\big\rangle = \big\langle\alpha,(\pi_{2-i_1}\times\cdots\times\pi_{2-i_n})\circ\beta\big\rangle$$

by the previous lemma. But if β lies in any summand except for

$$R^{g-2+n-k}(M_g,h^{2-i_1}(C_g/M_g)\otimes\cdots\otimes h^{2-i_n}(C_g/M_g)),$$

then $(\pi_{2-i_1}\times\cdots\times\pi_{2-i_n})\circ\beta = 0$, hence $\langle\alpha,\beta\rangle = 0$, as claimed. □

Remark 7.3. – The previous proposition shows that the socle pairing in $R^\bullet(C_g^n)$ depends only on the cup product maps $h^i(C_g/M_g)\otimes h^{2-i}(C_g/M_g) \to h^2(C_g/M_g) \cong \mathbb{L}$. Since for $i = 0$ or $i = 2$ these maps are given by the canonical isomorphisms $\mathbf{1}\otimes\mathbb{L}\to\mathbb{L}$ and $\mathbb{L}\otimes\mathbf{1}\to\mathbb{L}$, the socle pairing in fact only depends nontrivially on the maps

$$R^k(M_g,h^1(C_g/M_g)^{\otimes m})\otimes R^{g-2+m-k}(M_g,h^1(C_g/M_g)^{\otimes m}) \to R^{g-2+m}(M_g,\mathbb{L}^{\otimes m}) \cong R^{g-2}(M_g).$$

REMARK 7.4. – On the level of Betti realizations, Proposition 7.2 says that even though the algebras $H^\bullet(C_g^n, \mathbf{Q})$ and $\mathfrak{gr}_L H^\bullet(C_g^n, \mathbf{Q})$ (and then also $RH^\bullet(C_g^n, \mathbf{Q})$ and $\mathfrak{gr}_L RH^\bullet(C_g^n, \mathbf{Q})$) have very different cup product, both algebras $RH^\bullet(C_g^n, \mathbf{Q})$ and $\mathfrak{gr}_L RH^\bullet(C_g^n, \mathbf{Q})$ will have identical socle pairings.

In particular, the algebra $RH^\bullet(C_g^n)$ is Gorenstein (i.e., satisfies Poincaré duality) if and only if the same holds for the algebra $\mathfrak{gr}_L RH^\bullet(C_g^n)$. That said, Proposition 7.2 is not actually needed to prove this last fact. Since the Leray filtration is compatible with cup product, we know a priori that the Gram matrix describing the socle pairing for the algebra $RH^\bullet(C_g^n)$ is block-triangular, and that the diagonal blocks coincide with the Gram matrix for the socle pairing in the algebra $\mathfrak{gr}_L RH^\bullet(C_g^n)$. In particular both matrices have the same rank. In these terms, Proposition 7.2 says that even though the intersection pairing for $RH^\bullet(C_g^n)$ is a priori only block-triangular, it turns out to actually be block-diagonal.

THEOREM 7.5. – *Fix a genus g. The following are equivalent:*

1. *All algebras $R^\bullet(C_g^n)$ for $n \geq 0$ are Gorenstein.*
2. *For each partition λ, the pairing*
$$R^k(M_g, \mathbf{V}_{\langle\lambda\rangle}) \otimes R^{g-2+|\lambda|-k}(M_g, \mathbf{V}_{\langle\lambda\rangle}) \to R^{g-2+|\lambda|}(M_g, \mathbb{L}^{|\lambda|}) \cong R^{g-2}(M_g) \cong \mathbf{Q}$$
is perfect.

The pairing in (2) comes from the map of motives $\mathbf{V}_{\langle\lambda\rangle} \otimes \mathbf{V}_{\langle\lambda\rangle} \to \mathbb{L}^{|\lambda|}$ given by the fact that $\mathbf{V}_{\langle\lambda\rangle}$ is self-dual.

Proof. – If we decompose $h(C_g^n/M_g)$ as a direct sum of motives $\mathbf{V}_{\langle\lambda\rangle} \otimes \mathbb{L}^m$, then the socle pairing in $R^\bullet(C_g^n)$ is the direct sum of the various pairings
$$R^k(M_g, \mathbf{V}_{\langle\lambda\rangle}) \otimes R^{g-2+|\lambda|-k}(M_g, \mathbf{V}_{\langle\lambda\rangle}) \to R^{g-2}(M_g) \cong \mathbf{Q}.$$
Thus the socle pairing in the tautological ring of C_g^n is perfect if and only if the same holds for the pairing for each of the motives $\mathbf{V}_{\langle\lambda\rangle}$. □

A variant of the preceding theorem, with the same proof, is as follows:

THEOREM 7.6. – *Fix a genus g. The following are equivalent:*

1. *All algebras $R^\bullet(C_g^n)$ for $0 \leq n \leq N$ are Gorenstein.*
2. *For each partition λ with $|\lambda| \leq N$, the pairing*
$$R^k(M_g, \mathbf{V}_{\langle\lambda\rangle}) \otimes R^{g-2+|\lambda|-k}(M_g, \mathbf{V}_{\langle\lambda\rangle}) \to R^{g-2}(M_g) \cong \mathbf{Q}$$
is perfect.

We also wish to mention the following result, which was proven by somewhat different arguments in [59, 70].

THEOREM 7.7. – *The following statements are equivalent:*

1. *All algebras $R^\bullet(C_g^n)$ for $0 \leq n \leq N$ are Gorenstein.*
2. *The algebra $R^\bullet(M_{g,N}^{\mathrm{rt}})$ is Gorenstein.*

This shows that the Faber conjecture for the spaces $M^{\mathrm{rt}}_{g,n}$ can also be equivalently reformulated in terms of the motives $\mathbf{V}_{\langle\lambda\rangle}$. We saw in Section 5.5 that the tautological groups of $M^{\mathrm{rt}}_{g,n}$ can be expressed in terms of the tautological groups of the motives $\mathbf{V}_{\langle\lambda\rangle}$; however, there does not appear to be an analogue of Theorem 7.2 for the spaces $M^{\mathrm{rt}}_{g,n}$: the socle pairing for $M^{\mathrm{rt}}_{g,n}$ is block upper triangular with respect to the natural decomposition, but not in general block diagonal.

7.1. Symmetric powers

We may also consider the *symmetric powers* of the universal curve.

DEFINITION 7.8. – We define $C_g^{(n)}$ to be the nth symmetric power of the universal curve over M_g; that is, $C_g^{(n)} = C_g^n/\mathfrak{S}_n$. We define its tautological ring by $R^\bullet(C_g^{(n)}) = R^\bullet(C_g^n)^{\mathfrak{S}_n}$.

LEMMA 7.9. – *Suppose that the Chow groups $\mathrm{CH}^\bullet(C_g^n)$ are decomposed as a direct sum of summands $\mathrm{CH}^\bullet(M_g, \mathbf{V}_{\langle\lambda\rangle})$. The only local systems occuring in the subspace $\mathrm{CH}^\bullet(C_g^{(n)}) \subseteq \mathrm{CH}^\bullet(C_g^n)$ are those of the form $\lambda = (1,1,1,\ldots)$, i.e., those that occur as summands of the symmetric powers of $\mathbf{V}$.*

Proof. – Consider first the summand $\mathrm{CH}^k(M_g, \mathbf{V}^{\otimes n})$. The $\mathfrak{S}_n$-invariants in this subspace are $\mathrm{CH}^k(M_g, \mathrm{Sym}^n\mathbf{V})$, which proves the lemma in this case. In general for $n = n_0 + n_1 + n_2$, the summand

$$\mathrm{CH}^k(M_g, h^0(C_g/M_g)^{\otimes n_0} \otimes h^1(C_g/M_g)^{\otimes n_1} \otimes h^2(C_g/M_g)^{\otimes n_2}),$$

together with its conjugates under the action of $\mathfrak{S}_n$, can be written as the induced representation

$$\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n_0}\times\mathfrak{S}_{n_1}\times\mathfrak{S}_{n_2}} \mathrm{CH}^k(M_g, \mathbf{V}^{\otimes n_1} \otimes \mathbb{L}^{n_2}).$$

In particular, the $\mathfrak{S}_n$-invariants in this induced representation are isomorphic to

$$\mathrm{CH}^{k-n_2}(M_g, \mathbf{V}^{\otimes n_1})^{\mathfrak{S}_{n_1}} = \mathrm{CH}^{k-n_2}(M_g, \mathrm{Sym}^{n_1}\mathbf{V})$$

by Frobenius reciprocity. □

THEOREM 7.10. – *Fix a genus g. The following are equivalent:*

1. *For all $n \geq 0$, $R^\bullet(C_g^{(n)})$ is a Gorenstein ring.*
2. *For some $n \geq g$, $R^\bullet(C_g^{(n)})$ is a Gorenstein ring.*

Proof. – The ring $R^\bullet(C_g^{(n)})$ is Gorenstein if and only if all motives $\mathbf{V}_{\langle\lambda\rangle}$ occuring in the decomposition of $h(C_g^{(n)}/M_g)$ have the property that the pairing

$$R^k(M_g, \mathbf{V}_{\langle\lambda\rangle}) \otimes R^{g-2+|\lambda|-k}(M_g, \mathbf{V}_{\langle\lambda\rangle}) \to \mathbf{Q}$$

is perfect. But the motives $\mathbf{V}_{\langle\lambda\rangle}$ occuring in the decomposition of the nth symmetric power are exactly those with $\lambda = (1,1,\ldots)$, where $|\lambda| \leq n$, by the previous lemma. The result follows from the fact that the motive $\mathbf{V}_{\langle 1,1,1,\ldots\rangle}$ is zero if the number of 1's is greater than g. □

More generally, we can consider the "partial symmetric powers," i.e., the tautological rings of $C_g^{n+k}/\mathfrak{S}_n$, the n-fold symmetric power of the universal curve over C_g^k. For $k = 1$ these rings were considered in [77], where they were proven to be intimately related to the tautological ring of the universal jacobian variety over $C_g = M_{g,1}$. The previous theorem admits a variant for the partial symmetric powers as well.

THEOREM 7.11. – *Fix a genus g. The following are equivalent:*

1. *For all $n \geq 0$, $R^\bullet(C_g^{n+k}/\mathfrak{S}_n)$ is a Gorenstein ring.*
2. *For some $n \geq g + k$, $R^\bullet(C_g^{n+k}/\mathfrak{S}_n)$ is a Gorenstein ring.*

Proof. – The relative Chow motive of $C_g^{n+k}/\mathfrak{S}_n$ over M_g is the tensor product $h(C_g^{(n)}/M_g) \otimes h(C_g^k/M_g)$. Since $h(C_g^k/M_g)$ only contains motives $\mathbf{V}_\lambda$ with $|\lambda| \leq k$, and $h(C_g^{(n)}/M_g)$ only contains motives $\mathbf{V}_\lambda$ with $|\lambda| \leq g$ (by the argument of the preceding proof), the result follows. ☐

REMARK 7.12. – In [77, Theorem 7.15] it is proven that if $R^\bullet(C_g^{n+1}/\mathfrak{S}_n)$ is Gorenstein for some $n \geq 2g-1$ then $R^\bullet(C_g^{n+1}/\mathfrak{S}_n)$ is Gorenstein for all $n \geq 0$. The proof uses the relationship with the tautological ring of the universal jacobian J_g over C_g, and that $C_g^{n+1}/\mathfrak{S}_n$ is a projective bundle over J_g for $n \geq 2g-1$. Thus the arguments here re-prove this result with a slightly better lower bound.

8. Twisted commutative algebras and tautological rings

In the next sections we will analyze the structure of the collection of tautological rings $R^\bullet(C_g^n)$ for fixed g but varying n. When we consider the direct sum $\bigoplus_n R^\bullet(C_g^n)$ we obtain the structure of a *twisted commutative algebra*.

DEFINITION 8.1. – A *twisted associative algebra* is an $\mathbf{N}$-graded unital associative algebra (say over $\mathbf{Q}$)

$$A = \bigoplus_{n\geq 0} A(n)$$

together with an action of the symmetric group $\mathfrak{S}_n$ on the summand $A(n)$, such that the multiplication

$$A(n) \otimes A(m) \to A(n+m)$$

is equivariant for the action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on both sides. We say that $A(n)$ is the *arity n component* of A.

DEFINITION 8.2. – Let $A = \bigoplus_{n\geq 0} A(n)$ be a twisted associative algebra. We say that A is a *twisted commutative algebra* if the diagram

$$\begin{array}{ccc} A(n) \otimes A(m) & \longrightarrow & A(n+m) \\ \downarrow & & \downarrow \\ A(m) \otimes A(n) & \longrightarrow & A(m+n) \end{array}$$

commutes for all $n, m \geq 0$, where the horizontal maps are given by multiplication, the left vertical map swaps the two factors, and the right map is given by acting via the "box permutation" swapping the first n and the last m elements.

REMARK 8.3. – Let us make three remarks concerning the definition.

1. In all our examples, we will have what should more properly be called a "twisted commutative graded algebra"—each summand $A(n)$ is itself $\mathbf{Z}$-graded, and the multiplication respects the $\mathbf{Z}$-grading.
2. The notion of a twisted commutative *graded* algebra is potentially ambiguous: in the commutativity condition, should the Koszul sign rule be applied to the map $A(n) \otimes A(m) \to A(m) \otimes A(n)$ that swaps the two factors? In fact we will require both possible conventions in this paper: when working with Chow groups we do not impose the Koszul sign rule, but when working with cohomology groups we do impose it. This is because the Chow ring $\mathrm{CH}^\bullet(X)$ of an algebraic variety X is commutative in the strict sense, whereas the cohomology ring $H^\bullet(X)$ is commutative in the graded sense. We will pass over this ambiguity in silence for the rest of the paper; this should not cause any confusion.
3. There are many equivalent ways to axiomatize the notion of a twisted commutative algebra. Here is an alternative one: let B be the symmetric monoidal category of finite sets and bijections, with monoidal structure given by disjoint union. A twisted commutative algebra is a lax symmetric monoidal functor $B \to \mathsf{Vect}_{\mathbf{Q}}$ (or to the category of graded $\mathbf{Q}$-vector spaces, with or without the Koszul sign rule).

For more on twisted commutative algebras see e.g., [20] or [36, Chapitre 4].

Our main example will be the following. Fix a genus $g \geq 2$. The direct sum $\bigoplus_{n\geq 0} \mathrm{CH}^\bullet(C_g^n)$ is an example of a twisted commutative algebra. The multiplication

$$\mathrm{CH}^k(C_g^n) \otimes \mathrm{CH}^l(C_g^m) \to \mathrm{CH}^{k+l}(C_g^{n+m})$$

is given by the cross product, as defined in 5.2.1. More generally, for any partition $\{1, \dots, n\} = T \sqcup T'$ we have maps $\mathrm{CH}^k(C_g^T) \otimes \mathrm{CH}^l(C_g^{T'}) \to \mathrm{CH}^{k+l}(C_g^n)$. We will refer to maps of this form, too, as cross product maps; this should not cause any confusion.

We now have the following proposition, which in a sense explains why we will find the notion of a twisted commutative algebra useful. We have defined maps $R^\bullet(C_g^n) \to R^\bullet(M_g, \mathbf{V}^{\otimes n}) \to R^\bullet(M_g, \mathbf{V}^{\langle n\rangle})$; recall that $\mathbf{V}^{\langle n\rangle}$ denotes the "primitive part" of $\mathbf{V}^{\otimes n}$ and was defined in Subsection 5.3. These maps are not in any sense ring homomorphisms (in fact there is no ring structure on the latter two spaces). Nevertheless these will define homomorphisms of *twisted* commutative algebras, when we consider all n simultaneously:

PROPOSITION 8.4. – *Fix $g \geq 2$, and consider the following commutative diagram:*

$$\begin{tikzcd} & \bigoplus_{n\geq 0} \mathrm{CH}^\bullet(C_g^n) \arrow[r, two heads] & \bigoplus_{n\geq 0} \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes n}) \arrow[r, two heads] & \bigoplus_{n\geq 0} \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\langle n\rangle}) \\ \bigoplus_{n\geq 0} S_n^\bullet \arrow[ur] \arrow[dr, two heads] & & & \\ & \bigoplus_{n\geq 0} R^\bullet(C_g^n) \arrow[r, two heads] \arrow[uu, hook] & \bigoplus_{n\geq 0} R^\bullet(M_g, \mathbf{V}^{\otimes n}) \arrow[r, two heads] \arrow[uu, hook] & \bigoplus_{n\geq 0} R^\bullet(M_g, \mathbf{V}^{\langle n\rangle}). \arrow[uu, hook] \end{tikzcd}$$

Each entry in this diagram is a twisted commutative algebra, and the arrows in this diagram are morphisms of twisted commutative algebras.

Let us remind the reader of the definitions needed to make sense of Proposition 8.4. $S_n^\bullet$ was defined in Definition 5.2; it is the graded polynomial algebra on classes ψ_i, Δ_{ij} and κ_i, modulo the geometrically obvious relations of Eq. (5).

The maps $\mathrm{CH}^\bullet(C_g^n) \to \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes n})$ are given by the projectors $\pi_1^{\times n}$. The same is true for the maps $R^\bullet(C_g^n) \to R^\bullet(M_g, \mathbf{V}^{\otimes n})$.

The maps $\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes n}) \to \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\langle n\rangle})$ and the twisted commutative algebra structure on $\bigoplus_{n\geq 0} \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\langle n\rangle})$ are both defined by the fact that $\mathbf{V}^{\langle n\rangle}$ is in a canonical way a direct summand of $\mathbf{V}^{\otimes n}$. As such, the natural projection $\mathbf{V}^{\otimes n} \to \mathbf{V}^{\langle n\rangle}$ defines the map $\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes n}) \to \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\langle n\rangle})$. The multiplication in the twisted commutative algebra $\bigoplus_{n\geq 0} \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\langle n\rangle})$ is defined by using the composition

$$\mathbf{V}^{\langle n\rangle} \otimes \mathbf{V}^{\langle m\rangle} \hookrightarrow \mathbf{V}^{\otimes n} \otimes \mathbf{V}^{\otimes m} = \mathbf{V}^{\otimes (n+m)} \twoheadrightarrow \mathbf{V}^{\langle n+m\rangle}$$

to define a product $\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\langle n\rangle}) \otimes \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\langle m\rangle}) \to \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\langle n+m\rangle})$. This is associative: the diagram

$$\begin{tikzcd} \mathbf{V}^{\langle n\rangle} \otimes \mathbf{V}^{\langle m\rangle} \otimes \mathbf{V}^{\langle k\rangle} \arrow[r] \arrow[d] & \mathbf{V}^{\langle n+m\rangle} \otimes \mathbf{V}^{\langle k\rangle} \arrow[d] \\ \mathbf{V}^{\langle n\rangle} \otimes \mathbf{V}^{\langle m+k\rangle} \arrow[r] & \mathbf{V}^{\langle n+m+k\rangle} \end{tikzcd}$$

commutes, since both compositions coincide with the map given by

$$\mathbf{V}^{\langle n\rangle} \otimes \mathbf{V}^{\langle m\rangle} \otimes \mathbf{V}^{\langle k\rangle} \hookrightarrow \mathbf{V}^{\otimes (n+m+k)} \twoheadrightarrow \mathbf{V}^{\langle n+m+k\rangle}.$$

Proof. – (of Proposition 8.4.) That the map $\bigoplus_{n\geq 0} S_n^\bullet \to \bigoplus_{n\geq 0} \mathrm{CH}^\bullet(C_g^n)$ is a homomorphism of twisted commutative algebras is obvious. That the maps $\pi_1^{\times n}\colon \mathrm{CH}^\bullet(C_g^n) \to \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes n})$ are homomorphisms with respect to the cross product is explained in 5.2.1. That the maps $\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes n}) \to \mathrm{CH}^\bullet(M_g, \mathbf{V}^{\langle n\rangle})$ are homomorphisms with respect to the cross product is also clear, since the multiplication in the twisted commutative algebra $\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\langle n\rangle})$ was defined by lifting elements to $\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes n})$, and using the multiplication in the twisted commutative algebra "upstairs" to multiply. □

DEFINITION 8.5. – Let $S \to R_g \to R'_g \to R''_g$ be the four twisted commutative algebras linked by the chain of surjections

$$\bigoplus_{n\geq 0} S_n^\bullet \to \bigoplus_{n\geq 0} R^\bullet(C_g^n) \to \bigoplus_{n\geq 0} R^\bullet(M_g, \mathbf{V}^{\otimes n}) \to \bigoplus_{n\geq 0} R^\bullet(M_g, \mathbf{V}^{\langle n\rangle}).$$

PROPOSITION 8.6. – *The twisted commutative algebra S is the free twisted commutative algebra generated by the elements $\kappa_d \in S_0^d$ for $d \geq 1$, $\psi_1^m \in S_1^m$ for $m \geq 0$, and $\Delta_{12\cdots n}\psi_1^m \in S_n^{n-1+m}$ for $m \geq 0$.*

Proof. – It is straightforward that every monomial in $S_n^\bullet$ can be uniquely reduced modulo the relations of Eq. (5) to a product

$$\prod_{i=1}^{m} \kappa_{d_i} \cdot \prod_{j=1}^{k} \Delta_{P_j} \psi_{P_j}^{e_j}$$

where $(d_1, \dots, d_m) \in \mathbf{Z}_{>0}^m$, $(e_1, \dots, e_k) \in \mathbf{Z}_{\geq 0}^k$, $P_1, \dots, P_k$ is some partition of the set $\{1, \dots, n\}$ into nonempty blocks, and ψ_{P_j} denotes ψ_a for any $a \in P_j$ (this is also observed in [34, Lemma 5]). For example, the monomial $\kappa_1^2 \Delta_{13}\Delta_{14}\psi_3^2 \in S_4^\bullet$ would correspond to $(d_1, d_2) = (1,1)$, $P_1 = \{2\}$, $e_1 = 0$, $P_2 = \{1,3,4\}$, $e_2 = 2$. But such a product is exactly the same as a cross product of the generators for the twisted commutative algebra S. □

DEFINITION 8.7. – For $n \geq 0$ and $r \geq n-1$ we put

$$D_{n,r} = \begin{cases} \kappa_r & n = 0 \\ \Delta_{12\cdots n}\psi_1^{1-n+r} & n \geq 1. \end{cases}$$

Note that $D_{1,r} = \psi_1^r$, $D_{0,-1} = \kappa_{-1} = 0$, and $D_{0,0} = \kappa_0 = 2g-2$. The previous proposition can be stated in a more compact form in terms of this notation: specifically, that the twisted commutative algebra S is freely generated by $\mathfrak{S}_n$-invariant classes $D_{n,r}$ placed in arity n and degree r, where $n = 0$ and $r \geq 1$ or $n \geq 1$ and $r \geq n-1$.

The fact that the classes $D_{n,r}$ generate S implies that their images generate the twisted commutative algebras R_g, R'_g and R''_g, since the map from S to these algebras is surjective.

PROPOSITION 8.8. – *Fix $g \geq 2$ and consider the surjections $R_g \to R'_g \to R''_g$. The kernels of these maps are ideals in the respective twisted commutative algebras.*

1. *The kernel of $R_g \to R'_g$ is the ideal generated by $1 \in \mathrm{CH}^0(C_g^1)$ and $\psi_1 \in \mathrm{CH}^1(C_g^1)$.*
2. *The kernel of $R_g \to R''_g$ is the ideal generated by $1 \in \mathrm{CH}^0(C_g^1)$, $\psi_1 \in \mathrm{CH}^1(C_g^1)$ and $\Delta_{12} \in \mathrm{CH}^1(C_g^2)$. Equivalently, the kernel of $R'_g \to R''_g$ is the ideal generated by $\pi_1^{\times 2}\Delta_{12} = \Delta_{12} - \frac{1}{2g-2}(\psi_1 + \psi_2) + \frac{1}{(2g-2)^2}\kappa_1$.*

Proof. – (1) The kernel of $\mathrm{CH}^k(C_g^n) \to \mathrm{CH}^k(M_g, \mathbf{V}^{\otimes n})$ equals the image of the projectors $\pi_{i_1} \times \cdots \times \pi_{i_n}$ where $(i_1, i_2, \dots, i_n) \neq (1, 1, \dots, 1)$; equivalently, the image of all projectors

$$\mathrm{id} \times \mathrm{id} \times \cdots \times \pi_i \times \cdots \times \mathrm{id}$$

(i.e., all factors except one are given by the identity correspondence, the diagonal), where $i = 0, 2$. By $\mathfrak{S}_n$-symmetry, let's assume that all factors except the first are given by the identity. Let $\alpha \in \mathrm{CH}^\bullet(C_g^n)$. One checks that

$$(\pi_0 \times \mathrm{id} \times \cdots \times \mathrm{id}) \circ \alpha = 1 \times \alpha' - \frac{1}{2(2g-2)^2}\kappa_1 \times 1 \times \alpha''$$

and

$$(\pi_2 \times \mathrm{id} \times \cdots \times \mathrm{id}) \circ \alpha = \psi_1 \times \alpha'' - \frac{1}{2(2g-2)^2}\kappa_1 \times 1 \times \alpha'',$$

where $\alpha' = (p_{23\cdots n})_*(\psi_1 \cdot \alpha)$ and $\alpha'' = (p_{23\cdots n})_*(\alpha)$. Hence both projectors map all cycles α into the ideal generated by $\psi_1 \in \mathrm{CH}^1(C_g^1)$ and $1 \in \mathrm{CH}^1(C_g^1)$. Conversely, one checks that π_1 annihilates both 1 and ψ_1.

(2) The map $\mathrm{CH}^k(M_g, \mathbf{V}^{\otimes n}) \to \mathrm{CH}^k(M_g, \mathbf{V}^{\langle n\rangle})$ is defined by the projection $\mathbf{V}^{\otimes n} \to \mathbf{V}^{\langle n\rangle}$, and the kernel of $\mathbf{V}^{\otimes n} \to \mathbf{V}^{\langle n\rangle}$ is spanned by the image of all $\binom{n}{2}$ maps $\mathbf{V}^{\otimes(n-2)} \otimes \mathbb{L} \to \mathbf{V}^{\otimes n}$ given by $(n-2,n)$-Brauer diagrams of the form considered in the second half of Section 5.3. But it is clear from the description in Section 5.3 that an element of $\mathrm{CH}^k(M_g, \mathbf{V}^{\otimes n})$ is in the image of one of the maps $\mathrm{CH}^{k-1}(M_g, \mathbf{V}^{\otimes(n-2)}) \to \mathrm{CH}^k(M_g, \mathbf{V}^{\otimes n})$ precisely if it can be written in a nontrivial way as a cross product with $\pi_1^{\times 2}\Delta_{12}$. □

REMARK 8.9. – Let us emphasize that the word "ideal" in the preceding proposition should be understood in the sense of twisted commutative algebras; that is, the smallest twisted commutative submodule containing the given elements. In particular, the ring structures of (say) the individual tautological rings $R^\bullet(C_g^n)$ are not what is important.

COROLLARY 8.10. – *The twisted commutative algebra R_g'' is generated by the images of the elements $D_{n,r}$ such that $n = 0$ and $r \geq 1$, or $n \geq 1$ and $r \geq \max(n-1, 2)$.*

Proof. – We have seen that S is generated by the classes $D_{n,r}$ for $n = 0$ and $r \geq 1$ or $n \geq 1$ and $r \geq n-1$. Since the generators $D_{1,0}$, $D_{1,1}$ and $D_{2,1}$ go to zero under $S \to R_g''$ by Proposition 8.8, we deduce that R_g'' is generated by the images of the remaining generators. □

COROLLARY 8.11. – *The arity n component $R_g''(n)$ vanishes in degrees below $\frac{2n}{3}$.*

Proof. – Every generator $D_{n,r}$ fulfills this bound, since $\max(n-1,2) \geq \frac{2n}{3}$ for all natural numbers n (with equality only for $n = 3$). Since the bound is linear, and degrees and arities are both additive under cross product, the result follows. □

REMARK 8.12. – The cohomology groups of the spaces C_g^n also form twisted commutative algebras, and so do the cohomology groups of the local systems $\mathbb{V}^{\langle n\rangle}$ on M_g. In particular we have a chain of surjections of twisted commutative algebras in graded vector spaces:

$$\bigoplus_{n\geq 0} H^\bullet(C_g^n, \mathbf{Q}) \to \bigoplus_{n\geq 0} H^{\bullet-n}(M_g, \mathbb{V}^{\otimes n}) \to \bigoplus_{\lambda} H^{\bullet-n}(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes \sigma_{\lambda^T}^*.$$

If we consider $\bigoplus_{n\geq 0}\mathrm{CH}^\bullet(C_g^n)$, $\bigoplus_{n\geq 0}\mathrm{CH}^\bullet(M_g, \mathbf{V}^{\otimes n})$ and $\bigoplus_\lambda \mathrm{CH}^\bullet(M_g, \mathbf{V}_{\langle\lambda\rangle}) \otimes \sigma_{\lambda^T}^*$ also as twisted commutative algebras in graded vector spaces, but with doubled degrees, then they map compatibly to the cohomological versions of these twisted commutative algebras under the cycle class map. We also get twisted commutative algebras of tautological classes

$$\bigoplus_{n\geq 0} RH^\bullet(C_g^n, \mathbf{Q}) \to \bigoplus_{n\geq 0} RH^{\bullet-n}(M_g, \mathbb{V}^{\otimes n}) \to \bigoplus_{\lambda} RH^{\bullet-n}(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes \sigma_{\lambda^T}^*.$$

There is a natural "suspension" operation on twisted graded commutative algebras [61, 4.1] which has the effect of shifting the grading on the arity n component by n and tensoring with the sign representation of $\mathfrak{S}_n$. In this way one can get rid of both the annoying degree shift which appears in cohomology and the conjugate of the partition λ: one finds that there is a natural structure of twisted commutative algebra on

$$\bigoplus_{\lambda} H^{\bullet}(M_g, \mathbb{V}_\lambda) \otimes \sigma_\lambda^*$$

with a subalgebra $\bigoplus_\lambda RH^{\bullet}(M_g, \mathbb{V}_\lambda) \otimes \sigma_\lambda^*$ of tautological classes.

9. A consequence of the FZ relations

In this section we will recall the *FZ relations* between tautological classes in $R^{\bullet}(C_g^n)$, and draw some simple consequences from them. In particular, we will prove an analogue of the following theorem, which was conjectured by Faber [12] and proved independently by Ionel [31] and Morita [50]. (Morita's proof was only valid in cohomology, but Ionel's proof worked in Chow, too.)

THEOREM 9.1 (Ionel, Morita). – *The tautological ring $R^{\bullet}(M_g)$ is generated by the classes κ_r for which $3r < g + 1$.*

This theorem is a direct consequence of the FZ relations. We will see that the FZ relations can be used to prove the following stronger result:

THEOREM 9.2. – *Fix a genus $g \geq 2$. The twisted commutative algebra $R_g = \bigoplus_{n \geq 0} R^{\bullet}(C_g^n)$ is generated by the classes $D_{n,r}$ for which $3r - n < g + 1$.*

This implies in particular the result of Ionel-Morita, since the arity 0 component of R_g is the tautological ring $R^{\bullet}(M_g)$, and $D_{0,r}$ is the kappa class κ_r.

9.1. The FZ relations

In the early 2000s, Faber and Zagier (in unpublished work) formulated a conjectural infinite family of relations in the tautological ring $R^{\bullet}(M_g)$. These relations were proven using the geometry of stable quotients by Pandharipande and Pixton [57]. Around the same time, Pixton found a generalization of this conjecture to incorporate also marked points and an extension of these relations to the Deligne-Mumford boundary. These extended FZ relations on $\overline{M}_{g,n}$ were subsequently proven in cohomology by Pandharipande-Pixton-Zvonkine [58] and on the level of Chow rings by Janda [34, 33].

The FZ relations on C_g^n take a simpler form than on $\overline{M}_{g,n}$. Let us state the result in this case, following [34, Section 4].

Let

$$A(z) = \sum_{i \geq 0} \frac{(6i)!}{(2i)!(3i)!} z^i \qquad B(z) = \sum_{i \geq 0} \frac{(6i)!}{(2i)!(3i)!} \frac{(6i+1)}{(6i-1)} z^i.$$

We introduce a sequence of further power series C_n by

$$C_1 = \frac{B}{A}; \qquad C_{n+1} = (12z^2 \frac{d}{dz} - 4nz) C_n. \tag{8}$$

We note that C_n is a multiple of z^{n-1}. We will also define

$$C_0 = \log(A),$$

which is a multiple of z^1. Then we have

$$C_1 = -1 + 144z + 2^5 3^3 z^2 \frac{d}{dz} C_0, \tag{9}$$

so that the coefficients C_1 (and hence also the higher C_n) are in fact recursively expressed in terms of those of C_0, except for low order terms.

For any power series $F(z) = \sum_{i \geq 0} a_i z^i$ in $\mathbf{Q}[[z]]$, we define bracket operators

$$\{F\}_\kappa = \sum_{i \geq 0} \kappa_i a_i z^i$$

and

$$\{F\}_{\Delta_S} = \sum_{i \geq 0} (-1)^{|S|-1} \Delta_S \psi_S^{i-|S|+1} a_i z^i$$

for any $S \subseteq \{1, \ldots, n\}$; here ψ_S denotes ψ_j for any $j \in S$.

We use $[F]_{z^r}$ to denote the coefficient of z^r in a power series.

Theorem 9.3 (Janda, Pixton-Pandharipande, Pixton-Pandharipande-Zvonkine). *For any r such that $3r - g - 1 - n$ is a nonnegative even integer, the expression*

$$\Big[\exp(-\{\log(A)\}_\kappa) \sum_{P \text{ partition of } n} \prod_{S \in P} \{C_{|S|}\}_{\Delta_S} \Big]_{z^r}$$

vanishes in $\mathrm{CH}^r(C_g^n)$.

Definition 9.4. – We denote the above expression $[\exp(-\{\log(A)\}_\kappa) \sum_P \prod_{S \in P} \{C_{|S|}\}_{\Delta_S}]_{z^r}$ by $\mathsf{FZ}_{g,n,r}$.

Lemma 9.5 (Ionel). – *All coefficients $[C_n]_{z^r}$, for $n = 0$ and $r \geq 1$ or $n \geq 1$ and $r \geq n - 1$, are strictly positive rational numbers, except the constant term of C_1 which is negative.*

Proof. – The case $n = 0$ is [31, Lemma 3.6], since the coefficients of C_0 are (up to rescaling) the numbers she denotes $c_{k,k}$. The case $n = 1$ follows from this by the differential Equation (9); in fact, it is even stated in Ionel's lemma, since the coefficients of C_1 are (up to rescaling) the numbers she calls $c_{k,k-1}$. The differential Equation (8) says that

$$[C_{n+1}]_{z^{r+1}} = (12r - 4n)[C_n]_{z^r}$$

for $n \geq 1$, and one checks that $12r - 4n$ is strictly positive in all cases of interest except $n = 1$, $r = 0$, where it is negative: consequently, all coefficients of C_n for $n \geq 2$, $r \geq n - 1$ are strictly positive, too. □

We may now prove Theorem 9.2.

Proof. – (of Theorem 9.2) We know that the twisted commutative algebra R_g is generated by the images of the classes $D_{n,r} \in S_g$. Consider some generator $D_{n,r}$ for which $3r - g - 1 - n \geq 0$. If $3r - g - 1 - n$ is even then one of the terms in the relation $\mathsf{FZ}_{g,n,r}$ equals

$$(-1)^{n-1} [C_n]_{z^r} \cdot D_{n,r},$$

and all other terms are products of generators with smaller r. By Lemma 9.5, $[C_n]_{z^r}$ is nonzero, and this relation can be used to express the class $D_{n,r}$ in terms of "simpler" generators.

If $3r-g-1-n$ is odd, we may instead consider the FZ relation $\psi_{n+1}\cdot \mathsf{FZ}_{g,n+1,r}$, and push forward along the map forgetting the last marked point to get a codimension r relation on C_g^n. When we push down a monomial in the kappa, diagonal and psi-classes from C_g^{n+1} to C_g^n, we get a multiple of $D_{n,r}$ exactly when we push down $D_{n+1,r+1}=\Delta_{12\cdots n+1}\psi_1^{r-n+1}$ (which pushes forward to $D_{n,r}$) and when we push down $D_{n,r}\times D_{1,1}=\Delta_{12\cdots n}\psi_1^{r-n+1}\psi_{n+1}$, which pushes forward to $(2g-2)D_{n,r}$. Thus the resulting relation on C_g^n will have as one of its terms

$$(-1)^n\big([C_{n+1}]_{z^r}-(2g-2)[C_n]_{z^r}[C_1]_{z^0}\big)\cdot D_{n,r},$$

and all other terms are products of generators with smaller values of r. By Lemma 9.5 the coefficients $[C_{n+1}]_{z^r}$ and $[C_n]_{z^r}$ are positive and the coefficient $[C_1]_{z^0}$ is negative, so the coefficient behind $D_{n,r}$ is nonzero and we may use this relation to eliminate the generator $D_{n,r}$. □

10. Low genus calculations

In this section of the paper we will completely calculate the groups $R^k(M_g,\mathbf{V}_{\langle\lambda\rangle})$ for all k and λ when $g\le 4$.

THEOREM 10.1. – *Recall the twisted commutative algebra $R''_g=\bigoplus_\lambda R^\bullet(M_g,\mathbf{V}_{\langle\lambda\rangle})\otimes\sigma^*_{\lambda^T}$, defined for any $g\ge 2$.*

1. *The twisted commutative algebra R''_2 is trivial. Equivalently, $R^k(M_2,\mathbf{V}_{\langle\lambda\rangle})=0$ unless $k=0$, $\lambda=0$, for which $R^0(M_2,\mathbf{V}_{\langle 0\rangle})=R^0(M_2)\cong\mathbf{Q}$.*
2. *The twisted commutative algebra R''_3 is generated by κ_1 and the Gross-Schoen cycle. We have*
$$R^0(M_3,\mathbf{V}_{\langle 0\rangle})\cong R^1(M_3,\mathbf{V}_{\langle 0\rangle})\cong R^2(M_3,\mathbf{V}_{\langle 111\rangle})\cong\mathbf{Q},$$
and all other tautological groups of all other motives $\mathbf{V}_{\langle\lambda\rangle}$ on M_3 vanish. The group $R^1(M_3,\mathbf{V}_{\langle 0\rangle})$ is spanned by κ_1 and the group $R^2(M_3,\mathbf{V}_{\langle 111\rangle})$ is spanned by the Gross-Schoen cycle.
3. *The twisted commutative algebra R''_4 is generated by κ_1, the Gross-Schoen cycle, and the Faber-Pandharipande cycle. The complete list of motives $\mathbf{V}_{\langle\lambda\rangle}$ on M_4 with nontrivial tautological groups are*
$$\begin{aligned}R^0(M_4,\mathbf{V}_{\langle 0\rangle})\cong R^1(M_4,\mathbf{V}_{\langle 0\rangle})\cong R^2(M_4,\mathbf{V}_{\langle 0\rangle})&\cong\mathbf{Q},\\ R^2(M_4,\mathbf{V}_{\langle 111\rangle})\cong R^3(M_4,\mathbf{V}_{\langle 111\rangle})&\cong\mathbf{Q},\\ R^2(M_4,\mathbf{V}_{\langle 11\rangle})&\cong\mathbf{Q},\\ R^4(M_4,\mathbf{V}_{\langle 2211\rangle})&\cong\mathbf{Q}.\end{aligned}$$
The group $R^k(M_4,\mathbf{V}_{\langle 0\rangle})$ is spanned by κ_1^k. The group $R^2(M_4,\mathbf{V}_{\langle 111\rangle})$ is spanned by the Gross-Schoen cycle, and $R^3(M_4,\mathbf{V}_{\langle 111\rangle})$ by the product of κ_1 and the Gross-Schoen cycle. The group $R^2(M_4,\mathbf{V}_{\langle 11\rangle})$ is spanned by the Faber-Pandharipande cycle. Finally, $R^4(M_4,\mathbf{V}_{\langle 2211\rangle})$ is spanned by the cross product of two Gross-Schoen cycles; that is, the projection of $\Delta_{123}\Delta_{456}$ into the summand $\mathrm{CH}^4(M_4,\mathbf{V}_{\langle 2,2,1,1\rangle})\otimes\sigma_{4,2}$ gives a generator.

In all cases, Poincaré duality holds, in the sense that

$$R^k(M_g, \mathbf{V}_{\langle\lambda\rangle}) \otimes R^{g-2+|\lambda|-k}(M_g, \mathbf{V}_{\langle\lambda\rangle}) \to R^{g-2}(M_g, \mathbf{V}_{\langle 0\rangle}) \cong \mathbf{Q}$$

is a perfect pairing.

The proof of this theorem occupies the rest of this section. In all genera, the strategy of the proof will be the same:

— Using Corollary 8.10 and Theorem 9.2 we get a finite list of generators for the twisted commutative algebra R''_g. Thus we have reduced the problem to finding the complete set of relations between these generators.

— We use the FZ relations to obtain relations between the generators. Since we are working in the twisted commutative algebra R''_g, it is enough to consider the FZ relations modulo the equivalence relation $\equiv$, which often dramatically simplifies the relations. In this way we find that all but a finite list of twisted tautological classes are zero, and we are done if we can prove nonvanishing of each of these.

— Using Nazarov's Theorem 3.8 and our Theorem 5.5, we can represent each of the remaining potentially nonzero twisted tautological classes by an explicit class in $R^\bullet(C_g^n)$, so that we reduce the problem to proving that a finite number of tautological classes (without twisted coefficients) in C_g^n are nonzero. This is now done by a standard argument: we multiply with some other class in complementary degree to land in the top degree part of the tautological ring, and then push down to get an element in the top degree of the tautological ring of M_g, whose structure we understand completely.

To formulate the calculations, it will be convenient to introduce the following notation: for $x, y \in \mathrm{CH}^k(C_g^n)$, we write $x \equiv y$ to denote that x and y have the same image in $\mathrm{CH}^k(M_g, \mathbf{V}^{\langle n\rangle})$. Equivalently by Theorem 8.8, $x \equiv y$ if x and y are equivalent modulo the twisted commutative algebra-ideal generated by $1 \in \mathrm{CH}^0(C_g^1)$, $\psi_1 \in \mathrm{CH}^1(C_g^1)$ and $\Delta_{12} \in \mathrm{CH}^1(C_g^2)$.

REMARK 10.2. – We caution the reader that the relation $\equiv$ does *not* respect the multiplication in the rings $R^\bullet(C_g^n)$, and is *not* preserved when pushing forward a relation along a diagonal inclusion. That is, if we are given an element $\mathsf{R} \in S_n^k$ such that $\mathsf{R} \equiv 0$ in $R^k(C_g^n)$, then it does not follow e.g., that $\psi_1 \cdot \mathsf{R} \equiv 0$ in $R^{k+1}(C_g^n)$, nor that the pushforward of R to $R^{k+1}(C_g^{n+1})$ along a diagonal inclusion vanishes modulo $\equiv$. One must therefore be careful to first multiply or push forward and only afterward reduce modulo $\equiv$.

10.1. Genus two

By Corollary 8.10 and Proposition 9.2, *all* of the generators $D_{n,r}$ go to zero in R''_2. So R''_2 is the free twisted commutative algebra on *no* generators, i.e., it contains only the unit element in arity 0. This proves the genus 2 case of Theorem 10.1.

This result was previously obtained (in a different form) in [69].

The analogous statement is also true in genus one: $R^k(M_{1,1}, \mathbf{V}_{\langle a\rangle}) \cong \mathbf{Q}$ for $k = a = 0$, and vanishes otherwise; this reformulates a result from [68]. This statement is not hard to prove in our framework, but we have chosen to simplify the exposition by only talking about tautological groups $R^\bullet(M_g, \mathbf{V}_{\langle\lambda\rangle})$ on moduli spaces of unpointed curves.

10.2. Genus three

By Corollary 8.10 and Proposition 9.2, the twisted commutative algebra R''_3 is generated by the images of $D_{0,1}$ and $D_{3,0}$, i.e., κ_1 and the Gross-Schoen cycle. Thus R''_3 is completely determined if we can find the complete set of relations between these generators. We claim that the product of any two generators vanishes. We have three products we need to check are zero:

1. The relation $\kappa_1^2 = 0$ in $R^2(M_3, \mathbf{V}_{\langle 0\rangle})$ is well known and is a very special case of Looijenga's theorem (see Section 7).
2. Modulo the equivalence relation $\equiv$, the relation $\mathsf{FZ}_{3,3,3}$ simplifies to
$$18432\Delta_{123}\psi_1 - 960\Delta_{123}\kappa_1 \equiv 0.$$
That is, this is the expression obtained from $\mathsf{FZ}_{3,3,3}$ by removing all terms which are divisible (in the twisted commutative algebra R_3) by $1 \in R^0(C_g^1)$, $\psi_1 \in R^1(C_g^1)$ and $\Delta_{12} \in R^1(C_g^2)$. Note that this is the relation we used to show that the generator $\Delta_{123}\psi_1$ can be expressed in terms of simpler generators in the proof of Theorem 9.2.

   Now consider instead the pushforward of $\mathsf{FZ}_{3,2,2}$ along a diagonal inclusion $C_g^2 \hookrightarrow C_g^3$. Modulo $\equiv$, that relation simplifies to
$$-1152\Delta_{123}\psi_1 + 240\Delta_{123}\kappa_1 \equiv 0.$$
It is now clear that we obtain $\Delta_{123}\kappa_1 \equiv 0$, so that the product of the Gross-Schoen cycle and κ_1 vanishes in R''_g.
3. Observe first of all that modulo the relation $\equiv$, the only nonzero monomials in S_6^4 (see Definition 5.2) are $\Delta_{123}\Delta_{456}$ and its $\mathfrak{S}_6$-conjugates.

   For distinct elements $i, j \in \{1, \dots, 6\}$, consider the pushforward of the relation $\mathsf{FZ}_{3,5,3}$ along the corresponding diagonal inclusion. The observation just made, and the fact that $\mathsf{FZ}_{3,5,3}$ is $\mathfrak{S}_5$-invariant, implies that the resulting relation takes the form
$$\sum_{\substack{S \sqcup T = \{1,\dots,6\} \\ |S|=|T|=3 \\ i,j\in S}} \Delta_S\Delta_T \equiv 0$$
(up to a nonzero constant), as all other terms in the pushforward of $\mathsf{FZ}_{3,5,3}$ vanish modulo $\equiv$. We think of these relations as $\binom{6}{2} = 15$ equations in $\frac{1}{2}\binom{6}{3} = 10$ unknowns $\Delta_S\Delta_T$. It is a simple matter of linear algebra to check that the matrix of equations has full rank, so that $\Delta_S\Delta_T \equiv 0$ for all S, T.

We should also verify that κ_1 and the Gross-Schoen cycle are both nonzero in genus three, and that Poincaré duality holds. Nonvanishing of κ_1 is well known. The square of the Gross-Schoen cycle, pushed down to M_3, equals $\frac{7}{4}\kappa_1$ (as one can verify on a computer). This proves both nonvanishing of the Gross-Schoen cycle and Poincaré duality, since the pairing $R^2(M_3, \mathbf{V}_{\langle 1,1,1\rangle}) \otimes R^2(M_3, \mathbf{V}_{\langle 1,1,1\rangle}) \to R^1(M_3, \mathbf{V}_{\langle 0\rangle}) \cong \mathbf{Q}$ is exactly given by multiplying the two cycles and pushing the result down to M_3. This proves the genus 3 case of Theorem 10.1.

10.3. Genus four

By Corollary 8.10 and Proposition 9.2 we now have three generators for the twisted commutative algebra R''_4: κ_1, the Gross-Schoen cycle, and the Faber-Pandharipande cycle.

We claim that κ_1^2 and the product of κ_1 and the Gross-Schoen cycle are both nonzero in R''_4. Moreover, let us consider the product of two Gross-Schoen cycles. If we consider the subspace of the twisted commutative algebra S spanned by all possible products of two classes $D_{3,0}$, then this decomposes as a representation of $\mathfrak{S}_6$ as $\sigma_{4,2} \oplus \sigma_6 = \mathrm{Ind}_{(\mathfrak{S}_3)^2 \rtimes \mathfrak{S}_2}^{\mathfrak{S}_6} \mathbf{1}$. We claim that the representation σ_6 goes to zero in R''_4, but that the representation $\sigma_{4,2}$ survives to $R''_4(6)$.

Finally, we also claim that all other products of generators for the twisted commutative algebra R''_4 vanish. More precisely, we need to check the following relations:

1. $\kappa_1^3 \equiv 0$,
2. $\kappa_1^2 \Delta_{123} \equiv 0$,
3. $\kappa_1 \Delta_{12} \psi_1 \equiv 0$,
4. $\sum_{\substack{S \sqcup T = \{1,\ldots,6\} \\ |S|=|T|=3}} \Delta_S \Delta_T \equiv 0$,
5. $\Delta_{123}\Delta_{456}\Delta_{789} \equiv 0$,
6. $\Delta_{123}\Delta_{45}\psi_4 \equiv 0$,
7. $\Delta_{123}\Delta_{456}\kappa_1 \equiv 0$,
8. $\Delta_{12}\psi_1\Delta_{34}\psi_3 \equiv 0$.

(1) It's well known that $\kappa_1^3 = 0$.

(2) Modulo $\equiv$ the only nonzero monomials in S_3^4 are $\Delta_{123}\psi_1^2$, $\Delta_{123}\psi_1\kappa_1$, $\Delta_{123}\kappa_1^2$, and the $\mathfrak{S}_3$-conjugates of $\psi_1^2\Delta_{23}\psi_2$. The relation $\mathsf{FZ}_{4,1,2}$ reduces to $\psi_1^2 \equiv 0$, which also implies $\psi_1^2\Delta_{23}\psi_2 \equiv 0$. This leaves us with three potentially nonzero monomials. But $\mathsf{FZ}_{4,3,4}$, the pushforward along a diagonal of $\mathsf{FZ}_{4,2,3}$, and the pushforward of $\mathsf{FZ}_{4,1,2}$, give us three linearly independent linear relations between these monomials modulo $\equiv$.

(3) The only nonzero monomials in S_2^3 modulo $\equiv$ are $\Delta_{12}\psi_1^2$ and $\Delta_{12}\kappa_1\psi_1$. The relations $\mathsf{FZ}_{4,2,3}$ and the pushforward of the relation $\mathsf{FZ}_{4,1,2}$ along the diagonal give two distinct linear relations between these monomials modulo $\equiv$, so both are zero modulo $\equiv$.

(4) The relation $\psi_7 \cdot \mathsf{FZ}_{4,7,4}$, pushed down along the forgetful map that forgets the 7th marked point, reduces to this expression modulo $\equiv$. Alternatively, since the expression is $\mathfrak{S}_6$-invariant, its image in R''_4 lands in the summand $R^\bullet(M_4, \mathbf{V}_{\langle 1,1,1,1,1,1\rangle}) \otimes \sigma_6^*$. But the motive $\mathbf{V}_{\langle 1,1,1,1,1,1\rangle}$ is zero.

(5) Modulo $\equiv$, the only nonzero monomials in S_9^6 are the $\mathfrak{S}_9$-conjugates of $\Delta_{123}\Delta_{456}\Delta_{789}$. For any four indices i, j, k, l we may consider the pushforward of $\mathsf{FZ}_{4,7,4}$ along the i,j-th and k,l-th diagonal, to get a relation which up to a scalar must equal

$$\sum_{\substack{S \sqcup T \sqcup U = \{1,\ldots,9\} \\ |S|=|T|=|U|=3 \\ i,j \in S \; k,l \in T}} \Delta_S \Delta_T \Delta_U \equiv 0.$$

This gives $\frac{1}{2}\binom{9}{2,2,5} = 378$ relations between $\frac{1}{3!}\binom{9}{3,3,3} = 280$ unknowns, and one can check using a computer that the resulting matrix has full rank; in particular, $\Delta_{123}\Delta_{456}\Delta_{789} \equiv 0$.

(6) Modulo $\equiv$, there are exactly 11 nonzero monomials in S_5^4: Δ_{12345}, and the $\mathfrak{S}_5$-conjugates of $\Delta_{123}\Delta_{45}\psi_4$. The relation $\mathsf{FZ}_{4,5,4}$, and the 10 different pushforwards of the relation $\mathsf{FZ}_{4,4,3}$ along a diagonal inclusion, give 11 linear relations between these monomials modulo $\equiv$. The resulting 11×11 matrix is invertible and we conclude that $\Delta_{123}\Delta_{45}\psi_4 \equiv 0$.

(7) The nonzero monomials in S_6^5 are Δ_{123456}, $\Delta_{123}\Delta_{456}\psi_1$, $\Delta_{123}\Delta_{456}\kappa_1$, $\Delta_{12}\Delta_{3456}\psi_1$, and their $\mathfrak{S}_6$-conjugates. This implies that the relation $\mathsf{FZ}_{4,5,4}$, pushed forward along the i,j-th diagonal and multiplied with κ_1, must be equal to

$$\sum_{\substack{S\sqcup T=\{1,\dots,6\}\\ |S|=|T|=3\\ i,j\in S}} \Delta_S\Delta_T\kappa_1 \equiv 0$$

up to a nonzero scalar, since all the monomials involved in this relation must involve κ_1 and have i,j in the same diagonal block. But this implies $\Delta_{123}\Delta_{456}\kappa_1 \equiv 0$ by the same calculation as for relation (3) in genus 3.

(8) The nonzero monomials in S_4^4 are $\Delta_{12}\psi_1\Delta_{34}\psi_3$, $\psi_1^2\Delta_{234}$, $\Delta_{1234}\psi_1$, $\Delta_{1234}\kappa_1$ and their $\mathfrak{S}_4$-conjugates. Since $\psi_1^2 \equiv 0$, as observed in (2) above, we will also have $\psi_1^2\Delta_{234} \equiv 0$. Moreover, the relation $\mathsf{FZ}_{4,4,3}$ simplifies to $\Delta_{1234} \equiv 0$, so that also $\Delta_{1234}\kappa_1 \equiv 0$. This leaves only four potentially nonzero monomials. The relations given by $\psi_1\cdot\mathsf{FZ}_{4,4,3}$, $\Delta_{12}\cdot\mathsf{FZ}_{4,4,3}$, $\Delta_{13}\cdot\mathsf{FZ}_{4,4,3}$ and $\Delta_{1,4}\cdot\mathsf{FZ}_{4,4,3}$ give four linearly independent relations between these monomials, and we conclude that they all vanish modulo $\equiv$.

We should also prove that all these cycles are nonzero and that Poincaré duality holds. We use that the relation $3\kappa_1^2 = -32\kappa_2$ holds in $R^2(M_4) = \mathbf{Q}\{\kappa_1^2\}$. The Gross-Schoen cycle squared, pushed down to M_4, equals $\frac{3}{2}\kappa_1$. This shows both that the Gross-Schoen cycle is nonzero and that its product with κ_1 is nonzero. The Faber-Pandharipande cycle squared, pushed down to M_4, equals $\frac{19}{96}\kappa_1^2 - \frac{4}{3}\kappa_2$, which is then also nonzero. The projection of $\Delta_{123}\Delta_{456}$ onto $R^4(M_4,\mathbf{V}_{\langle 2,2,1,1\rangle})\otimes\sigma_{4,2}$, squared and pushed down to M_4, equals $\frac{19877}{29160}\kappa_1^2 - \frac{25}{729}\kappa_2$, which is then also nonzero.

This settles the genus 4 case, and hence concludes the proof of Theorem 10.1.

10.4. Genus five

Let us briefly comment on the situation in genus 5. The generators for the twisted commutative algebra R_5'' are κ_1, ψ_1^2, $\Delta_{12}\psi_1$, Δ_{123}, and Δ_{1234}. For $n \le 7$ we find that the algebra $R^\bullet(C_5^n)$ is Gorenstein, and we can compute the groups $R^\bullet(M_5,\mathbf{V}_{\langle\lambda\rangle})$ for $|\lambda|\le 7$ by methods like those used in lower genera. One finds the following table of results, in which the classes in the right hand column project onto generators for the tautological groups listed in the left hand column.

$R^0(M_5)\cong R^1(M_5)\cong R^2(M_5)\cong R^3(M_5)\cong\mathbf{Q}$	
$R^2(M_5,\mathbf{V}_{\langle 1\rangle})\cong\mathbf{Q}$	ψ_1^2
$R^2(M_5,\mathbf{V}_{\langle 1,1\rangle})\cong R^3(M_5,\mathbf{V}_{\langle 1,1\rangle})\cong\mathbf{Q}$	$\Delta_{12}\psi_1, \Delta_{12}\psi_1\kappa_1$
$R^2(M_5,\mathbf{V}_{\langle 1,1,1\rangle})\cong R^3(M_5,\mathbf{V}_{\langle 1,1,1\rangle})\cong R^4(M_5,\mathbf{V}_{\langle 1,1,1\rangle})\cong\mathbf{Q}$	$\Delta_{123}, \Delta_{123}\kappa_1, \Delta_{123}\kappa_1^2$
$R^3(M_5,\mathbf{V}_{\langle 1,1,1,1\rangle})\cong R^4(M_5,\mathbf{V}_{\langle 1,1,1,1\rangle})\cong\mathbf{Q}$	$\Delta_{1234}, \Delta_{1234}\kappa_1$
$R^4(M_5,\mathbf{V}_{\langle 1,1,1,1,1\rangle})\cong\mathbf{Q}$	$\Delta_{123}\Delta_{45}\psi_4$
$R^4(M_5,\mathbf{V}_{\langle 2,1,1,1\rangle})\cong\mathbf{Q}$	$\Delta_{123}\Delta_{45}\psi_4$
$R^4(M_5,\mathbf{V}_{\langle 2,2,1\rangle})\cong\mathbf{Q}$	$\Delta_{123}\Delta_{45}\psi_4$
$R^4(M_5,\mathbf{V}_{\langle 2,2,1,1\rangle})\cong\mathbf{Q}$	$\Delta_{123}\Delta_{456}$

$R^5(M_5, \mathbf{V}_{\langle 2,2,1,1\rangle}) \cong \mathbf{Q}$ $\Delta_{123}\Delta_{456}\kappa_1$

$R^5(M_5, \mathbf{V}_{\langle 2,2,1,1,1\rangle}) \cong \mathbf{Q}$ $\Delta_{123}\Delta_{4567}$

$R^5(M_5, \mathbf{V}_{\langle 2,2,2,1\rangle}) \cong \mathbf{Q}$ $\Delta_{123}\Delta_{4567}$

For $n = 8$ the Faber conjecture predicts the vanishing results

$$\Delta_{1234}\Delta_{5678} \equiv \Delta_{123}\Delta_{456}\Delta_{78}\psi_7 \equiv 0.$$

Assuming that the FZ relations are all relations between tautological classes, one finds that $\Delta_{1234}\Delta_{5678}$ and $\Delta_{123}\Delta_{456}\Delta_{78}\psi_7$ should both have nonzero image in $R_5''(8)$, and we expect that

$$R^6(M_5, \mathbf{V}_{\langle 2,2,2,2\rangle}) \cong R^6(M_5, \mathbf{V}_{\langle 3,2,2,1\rangle}) \cong \mathbf{Q}.$$

Either of these nonvanishings would imply that $R^\bullet(C_5^8)$ is not Gorenstein. Proving them seems like a hard problem; nevertheless, we consider this to be progress in trying to find a counterexample to the Faber conjecture. Trying to prove that a specific cohomology group (or cohomology class) does not vanish is a far more appealing problem than, say, trying to prove that the rank of $R^6(C_5^8)$ is greater than 35166. Moreover, our approach relates the Faber conjecture to actively studied questions about *modified diagonals*, see e.g., [54, 73, 48].

11. Relation to work of Looijenga

11.1. The theorems of Harer and Madsen-Weiss

Let $M_{g,\vec{1}}$ denote the moduli space parametrizing smooth genus g curves equipped with a marked point and a nonzero tangent vector at the marking. The (analytifications of the) spaces M_g and $M_{g,\vec{1}}$ are both $K(\pi, 1)$ spaces in the orbifold sense, meaning in particular that their cohomology is given by the group cohomology of their fundamental groups. Whereas the (orbifold) fundamental group of M_g is the mapping class group of a closed genus g surface, the fundamental group of $M_{g,\vec{1}}$ is the mapping class group of a genus g surface with a parametrized boundary component. As such, there is a "stabilization" map

$$H_k(M_{g,\vec{1}}, \mathbf{Z}) \to H_k(M_{g+1,\vec{1}}, \mathbf{Z})$$

which on the level of fundamental groups is given by gluing a torus with two boundary components onto the boundary of the genus g surface. See also [26, Section 4] for how to define these stabilization maps algebro-geometrically.

The celebrated stability theorem of Harer [30] asserts that the stabilization map is an isomorphism for $g \gg k$.

THEOREM 11.1 (Harer). – *$H_k(M_{g,\vec{1}}, \mathbf{Z}) \to H_k(M_{g+1,\vec{1}}, \mathbf{Z})$ is an isomorphism for $k \leq \frac{2}{3}(g-1)$.*

If we are interested primarily in closed surfaces, we also have a stabilization result for the forgetful map $M_{g,\vec{1}} \to M_g$ that forgets the marking and the tangent vector:

THEOREM 11.2 (Harer). – *The map $H_k(M_{g,\vec{1}}, \mathbf{Z}) \to H_k(M_g, \mathbf{Z})$ is an isomorphism for $k \leq \frac{2}{3}g$.*

The original bounds for the stable range of Harer have successively been improved by multiple people to obtain these results, see [4, 74]. We note that to obtain stability with integer coefficients in Theorem 11.2 it is crucial that M_g is considered as a stack—if we work with its coarse moduli space, the result is only valid with $\mathbf{Q}$-coefficients.

It is a formidable problem to actually compute the stable homology of M_g. With $\mathbf{Q}$-coefficients, an answer was conjectured by Mumford and proven by Madsen-Weiss [45]:

THEOREM 11.3 (Madsen-Weiss). – *The map* $\mathbf{Q}[\kappa_1, \kappa_2, \kappa_3, \dots] \to H^\bullet(M_g, \mathbf{Q})$ *is an isomorphism in the stable range, i.e., in degrees up to* $\frac{2}{3}(g-1)$.

REMARK 11.4. – If we formally denote the value of the stable cohomology by $H^\bullet(M_\infty, \mathbf{Q})$, then the statement is that $H^\bullet(M_\infty, \mathbf{Q})$ is a polynomial algebra in the κ classes. Since the tautological ring of M_g is defined as the algebra generated by the κ classes, it therefore makes sense to say that *the tautological cohomology of* M_g *is the image of the stable cohomology in the unstable cohomology.*

11.2. Twisted coefficients

One can also ask whether homological stability holds with coefficients in a local system $\mathbb{V}_{\langle\lambda\rangle}$. In this case, stabilization should be interpreted as appending an integer $\lambda_{g+1} = 0$ to the weight vector $\lambda_1 \geq \cdots \geq \lambda_g \geq 0$. The analogue of Harer stability holds in this case, too, by a theorem of Ivanov [32]:

THEOREM 11.5 (Ivanov). – *The map* $H^k(M_g, \mathbb{V}_{\langle\lambda\rangle}) \to H^k(M_{g+1}, \mathbb{V}_{\langle\lambda\rangle})$ *is an isomorphism for* $g \gg k, |\lambda|$.

We should remark that Ivanov's statement was not specifically about the local systems $\mathbb{V}_{\langle\lambda\rangle}$; his theorem is valid for a more general notion of coefficient system of finite degree which makes sense over an arbitrary base ring, and the local systems $\mathbb{V}_{\langle\lambda\rangle}$ are an example of such.

Ivanov's theorem did not actually calculate the stable cohomology with twisted coefficients. Rationally, the stable cohomology with coefficients in $\mathbb{V}_{\langle\lambda\rangle}$ was calculated by Looijenga [44], in a paper that strongly influenced our way of thinking on these subjects.

The first step in Looijenga's calculation of $H^\bullet(M_\infty, \mathbb{V}_{\langle\lambda\rangle})$ is to compute the stable cohomology of the spaces C_g^n. His result can be reformulated in a rather appealing way in terms of twisted commutative algebras:

THEOREM 11.6 (Looijenga). – *The twisted commutative algebra* $\bigoplus_{n\geq 0} H^\bullet(C_\infty^n, \mathbf{Q})$ *is free on* $\mathfrak{S}_n$*-invariant generators* $D_{n,r}$ *in arity* n *and cohomological degree* $2r$ *for* $n = 0$, $r \geq 1$ *and* $n \geq 1, r \geq n-1$. *In other words, the map* $S_n^\bullet \to H^{2\bullet}(C_g^n)$ *is an isomorphism in the stable range.*

We note that this theorem contains in particular the Madsen-Weiss theorem, by restricting to the case $n = 0$ (in which case the generators $D_{0,r}$ are kappa classes), even though Looijenga's paper predates the Madsen-Weiss theorem. Thus Looijenga's theorem was rather that the stable cohomology of M_g with twisted coefficients is a free module over the stable cohomology with constant coefficients with explicitly given generators; plugging in the Madsen-Weiss theorem gives the above result.

To compute $H^\bullet(M_\infty, \mathbb{V}_{\langle\lambda\rangle})$ from Theorem 11.6, one notes that there is a surjection of twisted commutative algebras $\bigoplus_{n\geq 0} H^\bullet(C^n_\infty, \mathbf{Q}) \to \bigoplus_{n\geq 0}\bigoplus_{|\lambda|=n} H^{\bullet-n}(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes \sigma_{\lambda^T}$, whose kernel is the ideal generated by the classes $D_{1,0}$, $D_{1,1}$ and $D_{2,1}$. Thus one finds:

THEOREM 11.7 (Looijenga). – *The twisted comm. algebra $\bigoplus_\lambda H^{\bullet-|\lambda|}(M_\infty, \mathbb{V}_{\langle\lambda\rangle}) \otimes \sigma_{\lambda^T}$ is the free twisted commutative algebra on $\mathfrak{S}_n$-invariant generators $D_{n,r}$ in arity n and cohomological degree $2r$ for $n = 0$, $r \geq 1$ and $n \geq 1, r \geq \max(2, n-1)$.*

By decomposing this free twisted commutative algebra into irreducible representations of $\mathfrak{S}_n$, one finds a calculation of the stable cohomology $H^\bullet(M_\infty, \mathbb{V}_{\langle\lambda\rangle})$ for any λ. Looijenga does not state his result in these terms: he defines a certain algebra $B^\bullet_n$ which he decomposes into irreducible representations of $\mathfrak{S}_n$, and this algebra (tensored with the polynomial ring in the kappa classes) is the arity n component of the free twisted commutative algebra in the previous result.

The conclusion is in any case the following. For constant coefficents, the stable cohomology of M_g is a *free polynomial algebra* on the κ-classes. The image of the stable cohomology inside the unstable cohomology can be defined to be the *tautological cohomology of M_g*. If we consider instead the stable cohomology with all possible twisted coefficients, i.e., the direct sum $\bigoplus_\lambda H^{\bullet-|\lambda|}(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes \sigma_{\lambda^T}$, then this is a *free twisted commutative algebra*, and the image of the stable cohomology inside the unstable cohomology is now exactly what we defined to be the *tautological cohomology of M_g with twisted coefficients*.

12. The "primary approximation" to the cohomology of the moduli space

Prior to this paper, Hain [25] proposed a definition of tautological cohomology groups $RH^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle})$ of M_g with coefficients in a symplectic representation, which is a priori different from ours. In this section we will show that the two definitions coincide. In case $\lambda = 0$, this gives a new proof of a theorem of Kawazumi and Morita [38]. We note that Hain asked in loc. cit. whether his construction could be lifted to the level of Chow groups; our constructions provide such a lifting.

Let $\mathcal{O}(\mathrm{Sp}(2g))$ be the algebraic coordinate ring of the symplectic group over $\mathbf{Q}$. By the Peter-Weyl theorem, there is an isomorphism of $\mathrm{Sp}(2g) \times \mathrm{Sp}(2g)$-bimodules

$$\mathcal{O}(\mathrm{Sp}(2g)) \cong \bigoplus_\lambda V_{\langle\lambda\rangle} \otimes V^*_{\langle\lambda\rangle},$$

where the sum runs over all irreducible representations of the symplectic group. We consider $V_{\langle\lambda\rangle}$ to have a left action and $V^*_{\langle\lambda\rangle}$ with a right action. Using the left action of $\mathrm{Sp}(2g)$ on $\mathcal{O}(\mathrm{Sp}(2g))$, we may consider it as defining a local system of algebras $\mathbf{O}(\mathrm{Sp}(2g))$ on M_g. Taking its cohomology, we get that

$$\mathsf{T}_g \overset{\mathrm{def}}{=} H^\bullet(M_g, \mathbf{O}(\mathrm{Sp}(2g))) = \bigoplus_\lambda H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes V^*_{\langle\lambda\rangle}$$

is in a natural way a commutative $\mathbf{Q}$-algebra. See [26, Section 9.5] or [25].

REMARK 12.1. – A perhaps more down to earth way to understand this multiplication is as follows. Suppose that we have $V_{\langle\lambda\rangle} \otimes V_{\langle\mu\rangle} \supset V_{\langle\nu\rangle}$. Then we get a multiplication map

$$H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes H^\bullet(M_g, \mathbb{V}_{\langle\mu\rangle}) \to H^\bullet(M_g, \mathbb{V}_{\langle\nu\rangle})$$

which however depends nontrivially on the choice of an intertwiner $V_{\langle\lambda\rangle} \otimes V_{\langle\mu\rangle} \to V_{\langle\nu\rangle}$. What is instead completely well defined is the map

$$\mathrm{Hom}_{\mathrm{Sp}(2g)}(V_{\langle\lambda\rangle} \otimes V_{\langle\mu\rangle}, V_{\langle\nu\rangle}) \otimes H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes H^\bullet(M_g, \mathbb{V}_{\langle\mu\rangle}) \to H^\bullet(M_g, \mathbb{V}_{\langle\nu\rangle}),$$

which (using the canonical identification $\mathrm{Hom}(M,N) = N \otimes M^*$) can be thought of equivalently as an $\mathrm{Sp}(2g)$-equivariant map

$$H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes V^*_{\langle\lambda\rangle} \otimes H^\bullet(M_g, \mathbb{V}_{\langle\mu\rangle}) \otimes V^*_{\langle\mu\rangle} \to H^\bullet(M_g, \mathbb{V}_{\langle\nu\rangle}) \otimes V^*_{\langle\nu\rangle}.$$

Let us now consider the Gross-Schoen cycle as a class $\alpha \in H^1(M_g, \mathbb{V}_{\langle 1,1,1\rangle})$. We have a vector subspace $\alpha \otimes V^*_{\langle 1,1,1\rangle} \subset \mathsf{T}_g$, and therefore by the universal property of a polynomial algebra a morphism of graded commutative rings

$$\bigwedge^\bullet V^*_{\langle 1,1,1\rangle} \to \mathsf{T}_g.$$

There is an inclusion $V^*_{\langle 2,2\rangle} \subset \bigwedge^2 V^*_{\langle 1,1,1\rangle}$. Since α is tautological, every class in the image of this homomorphism is tautological; it follows from this that the summand $V^*_{\langle 2,2\rangle} \subset \bigwedge^2 V^*_{\langle 1,1,1\rangle}$ lies in the kernel, since one can compute from our results (or rather the work of Looijenga) that $RH^2(M_g, \mathbb{V}_{\langle 2,2\rangle}) = 0$. We denote the algebra $\bigwedge^\bullet V^*_{\langle 1,1,1\rangle}/(V^*_{\langle 2,2\rangle})$ by A_g. It follows that there exists an $\mathrm{Sp}(2g)$-equivariant ring homomorphism

$$\varphi \colon \mathsf{A}_g \to \mathsf{T}_g.$$

DEFINITION 12.2. – The *H-tautological ring* is the subring $\mathsf{R}_g \subset \mathsf{T}_g$ given as the image of φ. By decomposing the H-tautological ring into irreducible summands for its natural action of $\mathrm{Sp}(2g)$ we get a subspace of H-tautological classes inside $H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle})$ for any partition λ.

REMARK 12.3. – When $g = 2$ the local system $\mathbb{V}_{\langle 1,1,1\rangle}$ vanishes (and so does the Gross-Schoen cycle), and the H-tautological ring consists only of the unit element in $H^0(M_2, \mathbb{V}_{\langle 0\rangle})$.

REMARK 12.4. – The definition above may seem very ad hoc—why should the Gross-Schoen cycle play a more distinguished role than any other tautological class? A more “invariant” definition is that the H-tautological ring is the subring of $H^\bullet(M_g, \mathcal{O}(\mathrm{Sp}(2g)))$ generated by all normal functions over M_g [27].

REMARK 12.5. – It is a striking fact that unlike the usual tautological ring of M_g or C_g^n, the H-tautological ring is generated by a single algebraic cycle class.

Restricting to symplectic invariants, we get a map

$$\varphi^{\mathrm{Sp}(2g)} \colon \mathsf{A}_g^{\mathrm{Sp}(2g)} \to \mathsf{T}_g^{\mathrm{Sp}(2g)} = H^\bullet(M_g, \mathbf{Q}).$$

This morphism is exactly what Morita calls the *primary approximation* to the cohomology ring of M_g. Morita originally described it in rather different terms [49]; this re-interpretation is due to Hain. A theorem of Kawazumi and Morita [38] asserts that the image of $\varphi^{\mathrm{Sp}(2g)}$ is the tautological cohomology ring of M_g. We will prove a more general result below.

REMARK 12.6. – The above map can also be understood in terms of relative Malcev completion [24]. Hain constructs a Lie algebra $\mathfrak{u}_g$ of mixed Hodge structures with $\mathrm{Sp}(2g)$-action (the Lie algebra of the pro-unipotent radical of the relative completion of the mapping class group) and an $\mathrm{Sp}(2g)$-equivariant map $H^\bullet(\mathfrak{u}_g) \to H^\bullet(M_g, \mathcal{O}(\mathrm{Sp}(2g)))$. The results of [24] (and subsequent improvements) show that the weights of $\mathfrak{u}_g$ are negative and that $\mathrm{Gr}^W_{-1}\mathfrak{u}_g \cong V^*_{\langle 1,1,1\rangle}$, $\mathrm{Gr}^W_{-2}\mathfrak{u}_g \cong V^*_{\langle 2,2\rangle}$. It follows that the algebra A_g is the pure part $\bigoplus_k W_k H^k(\mathfrak{u}_g)$ of the Chevalley-Eilenberg cohomology of $\mathfrak{u}_g$, so the H-tautological ring can also be defined as the image of the lowest weight part of the cohomology of $\mathfrak{u}_g$.

LEMMA 12.7. – *Let n, m be integers with $n \geq 0$, $m \geq 0$ and $n + 2m - 2 > 0$. Construct a $(3 \cdot (n + 2m - 2), n)$-Brauer diagram as follows: for $i = 1, \dots, n + 2m - 3$, draw a horizontal strand connecting the $(3i)$th node on the top row to the $(3i + 1)$st. Of the remaining $n + 2m$ nodes on the top row, pick n of them arbitrarily and connect them to the nodes along the bottom row, and connect the remaining $2m$ nodes arbitrarily to each other by m horizontal strands. Consider the resulting map*

$$H^{n+2m-2}(M_g, \mathbb{V}^{\otimes 3(n+2m-2)}) \to H^{n+2m-2}(M_g, \mathbb{V}^{\otimes n}).$$

The image of $\pi_1^{\times 3}(\Delta_{123})^{\times(n+2m-2)}$ under this map is the class $\pi_1^{\times n}(\Delta_{12\cdots n}\psi_1^m)$ if $n > 0$, and κ_{m-1} if $n = 0$.

Proof. – This is an easy consequence of the discussion in Section 5.3. Namely, to compute the image of $\pi_1^{\times 3}(\Delta_{123})^{\times(n-2+2m)}$ we start with the cycle $\Delta_{123}\Delta_{456}\Delta_{789}\cdots$, restrict to a suitable diagonal locus—specifically, the diagonals are specified by the horizontal strands in the Brauer diagram—and then project away from the markings corresponding to these diagonals. This gives $\Delta_{12\cdots n}\psi_1^m$ if $n > 0$, and κ_{m-1} if $n = 0$, after which we should apply $\pi_1^{\times n}$, which gives the result. $\square$

THEOREM 12.8. – *The space of H-tautological classes inside $H^\bullet(M_g, \mathcal{O}(\mathrm{Sp}(2g)))$ coincides with the space $RH^\bullet(M_g, \mathcal{O}(\mathrm{Sp}(2g))) = \bigoplus_\lambda RH^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes V^*_\lambda$ of tautological classes in our sense.*

Proof. – We note first that $RH^\bullet(M_g, \mathcal{O}(\mathrm{Sp}(2g)))$ is a subalgebra of $H^\bullet(M_g, \mathcal{O}(\mathrm{Sp}(2g)))$. Indeed, consider the multiplication map

$$\mathrm{Hom}_{\mathrm{Sp}(2g)}(V_{\langle\lambda\rangle} \otimes V_{\langle\mu\rangle}, V_{\langle\nu\rangle}) \otimes H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes H^\bullet(M_g, \mathbb{V}_{\langle\mu\rangle}) \to H^\bullet(M_g, \mathbb{V}_{\langle\nu\rangle}).$$

Every element of $\mathrm{Hom}_{\mathrm{Sp}(2g)}(V_{\langle\lambda\rangle} \otimes V_{\langle\mu\rangle}, V_{\langle\nu\rangle})$ is given by Brauer diagrams. It follows that if we realize the cohomologies of the different local systems as summands of the cohomologies of fibered powers C_g^n, then the multiplication $H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes H^\bullet(M_g, \mathbb{V}_{\langle\mu\rangle}) \to H^\bullet(M_g, \mathbb{V}_{\langle\nu\rangle})$ is induced by an algebraic correspondence given by tautological cycles, for any choice of intertwiner $V_{\langle\lambda\rangle} \otimes V_{\langle\mu\rangle} \to V_{\langle\nu\rangle}$. This means in particular that the cup product maps tautological classes to tautological classes.

In particular, this means that every H-tautological class is a tautological class in our sense: since the H-tautological ring is generated by the Gross-Schoen cycle, it must be contained in the subalgebra of all tautological classes.

Conversely we need to prove that every tautological class in our sense can be written as a product of Gross-Schoen cycles. It is enough to prove this for the generators of the twisted commutative algebra R''_g, i.e., the images of the classes $\Delta_{12\cdots n}\psi_1^m$ and the κ-classes. If we consider the Brauer diagram of Lemma 12.7 as an element of $\mathrm{Hom}_{\mathrm{Sp}(2g)}((\mathbb{V}_{\langle 1,1,1\rangle})^{\otimes(n-2+2m)}, \mathbb{V}^{\langle n\rangle})$, then the image of this Brauer diagram and $n-2+2m$ copies of the Gross-Schoen cycle under the cup product map

$$\mathrm{Hom}_{\mathrm{Sp}(2g)}((\mathbb{V}_{\langle 1,1,1\rangle})^{\otimes(n-2+2m)}, \mathbb{V}^{\langle n\rangle}) \otimes H^1(M_g, (\mathbb{V}_{\langle 1,1,1\rangle})^{\otimes(n-2+2m)} \to H^{n-2+2m}(M_g, \mathbb{V}^{\langle n\rangle})$$

equal the image of $\Delta_{12\cdots n}\psi_1^m$ under the projection $H^{2(n-1+m)}(C_g^n, \mathbf{Q}) \to H^{n-2+2m}(M_g, \mathbb{V}^{\langle n\rangle})$, as Lemma 12.7 shows. The result follows. □

COROLLARY 12.9 (Kawazumi-Morita). – *The image of* $(\bigwedge^\bullet V_{\langle 1,1,1\rangle}/V_{\langle 2,2\rangle})^{\mathrm{Sp}(2g)} \to H^\bullet(M_g, \mathbf{Q})$ *is the tautological cohomology ring of* M_g.

Proof. – This is the case $\lambda = 0$ of the preceding theorem. □

Morita [51] has conjectured that the map $\varphi^{\mathrm{Sp}(2g)}$ is injective: that is, it defines an isomorphism between $\mathsf{A}_g^{\mathrm{Sp}(2g)}$ and the tautological ring $R^\bullet(M_g)$. Compared to other conjectural descriptions of the tautological ring, e.g., the conjecture that all relations are consequences of the FZ relations, this gives a remarkably quick and direct description of the tautological ring (even though it is not so easy to determine the structure of the algebra $\mathsf{A}_g^{\mathrm{Sp}(2g)}$). A natural generalization of Morita's conjecture is to ask whether $\varphi\colon \mathsf{A}_g \to \mathsf{R}_g$ is an isomorphism. A consequence of our results is that this is not the case, however:

PROPOSITION 12.10. – *The map φ is not injective when $g = 4$.*

Proof. – Using a computer algebra system, one can verify that the third exterior power $\bigwedge^3 V^*_{\langle 1,1,1\rangle}$ contains the irreducible representation $V^*_{\langle 3,2,2,2\rangle}$ as a summand. On the other hand the degree 3 part of the ideal $(V^*_{\langle 2,2\rangle})$ is a quotient of $V^*_{\langle 2,2\rangle} \otimes V^*_{\langle 1,1,1\rangle}$, which contains only representations of weight at most 7. It follows that A_4 has a nontrivial summand $V^*_{\langle 3,2,2,2\rangle}$ in degree 3. But our calculations of the tautological groups with twisted coefficients in genus four show that $RH^3(M_4, \mathbb{V}_{\langle 3,2,2,2\rangle}) = 0$, so this summand must be in the kernel of φ. □

As pointed out in the introduction of this section, Hain asked whether the construction of a tautological algebra inside $H^\bullet(M_g, \mathcal{O}(\mathrm{Sp}(2g)))$ could be lifted to the level of Chow groups, and our construction in this paper gives an affirmative answer to this question. However, there does not seem to be any sensible way to get the grading on the level of Chow groups, at least not without introducing fractional Tate twists. The source of the problem is that an intertwiner in $\mathrm{Hom}_{\mathrm{Sp}(2g)}(V_{\langle\lambda\rangle} \otimes V_{\langle\mu\rangle}, V_{\langle\nu\rangle})$ does not give rise to a morphism of Chow motives $\mathbf{V}_{\langle\lambda\rangle} \otimes \mathbf{V}_{\langle\mu\rangle} \to \mathbf{V}_{\langle\nu\rangle}$ unless $|\lambda| + |\mu| = |\nu|$; in general one only gets a morphism to a Tate twist of $\mathbf{V}_{\langle\nu\rangle}$. One option is to work instead with Chow motives with respect to ungraded correspondences—one can make sense of $\mathcal{O}(\mathrm{Sp}(2g))$ as a relative Chow motive over M_g with respect to ungraded correspondences—but the Chow groups of a motive with respect to ungraded correspondences only form a vector space, not a graded vector space, and so the grading needs to be put in "by hand". Alternatively, if we allow half-integer Tate twists, then we can replace $\mathbf{V}_{\langle\lambda\rangle}$ with $\mathbf{V}_{\langle\lambda\rangle} \otimes \mathbb{L}^{-|\lambda|/2}$ throughout, which will allow us to recover the cohomological grading (halved).

12.1. The twisted commutative algebra and the Peter-Weyl theorem

We have now seen two a priori completely different ways to build an algebra by considering all local systems $\mathbb{V}_{\langle\lambda\rangle}$ simultaneously: the commutative ring $\mathsf{T}_g = \bigoplus_\lambda H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes V^*_{\langle\lambda\rangle}$ and the twisted commutative algebra $\bigoplus_\lambda H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes \sigma^*_\lambda$ (see Remark 8.12). We now want to explain a connection between the two constructions.

Suppose that A is a ring in the category of graded $\mathrm{Sp}(2g)$-representations. We associate to A two twisted commutative algebras L_A and L°_A given by

$$n \mapsto L_A(n) = (A \otimes V^{\otimes n})^{\mathrm{Sp}(2g)}$$

and

$$n \mapsto L^\circ_A(n) = (A \otimes V^{\langle n\rangle})^{\mathrm{Sp}(2g)}.$$

(Recall that $V^{\langle n\rangle} = \bigoplus_{\lambda\vdash n} V_{\langle\lambda\rangle} \otimes \sigma^*_\lambda$.) The multiplication in L_A is given by

$$\begin{aligned} L_A(n) \otimes L_A(m) &= (A \otimes V^{\otimes n})^{\mathrm{Sp}(2g)} \otimes (A \otimes V^{\otimes m})^{\mathrm{Sp}(2g)} \subseteq (A \otimes V^{\otimes n} \otimes A \otimes V^{\otimes m})^{\mathrm{Sp}(2g)} \\ &\xrightarrow{\text{mult.}} (A \otimes V^{\otimes n} \otimes V^{\otimes m})^{\mathrm{Sp}(2g)} = L_A(n+m), \end{aligned}$$

and similarly for L°_A. We obtain two functors from the category of rings with action of $\mathrm{Sp}(2g)$ to the category of twisted commutative algebras.

We will apply our functors to the rings A_g, R_g and T_g defined above. (2) We find for example that

$$\begin{aligned} L^\circ_{\mathsf{T}_g}(n) &= \mathrm{Hom}_{\mathrm{Sp}(2g)}(H^\bullet(M_g, \mathcal{O}(\mathrm{Sp}(2g)), V^{\langle n\rangle}) \\ &\cong \bigoplus_\lambda \bigoplus_{\mu\vdash n} H^\bullet(M_g, \mathbb{V}_\lambda) \otimes \mathrm{Hom}_{\mathrm{Sp}(2g)}(V_\lambda, V_\mu) \otimes \sigma^*_\mu \\ &\cong \bigoplus_{\lambda\vdash n} H^\bullet(M_g, \mathbb{V}_{\langle\lambda\rangle}) \otimes \sigma^*_\lambda = H^\bullet(M_g, \mathbb{V}^{\langle n\rangle}). \end{aligned}$$

In particular, L_{R_g} and $L^\circ_{\mathsf{R}_g}$ are the cohomological analogues of the twisted commutative algebras denoted R'_g and R''_g in Section 8. So our twisted commutative algebras can be recovered functorially from Hain's tautological ring R_g, which explains how the two constructions are related.

A slightly more refined version of the above construction is possible. Recall that a twisted commutative algebra can be defined as a lax symmetric monoidal functor from $\coprod_{n\geq 0} \mathfrak{S}_n$ to graded vector spaces. We define instead a *twisted Brauer algebra* as a lax symmetric monoidal functor from $\mathfrak{Br}^{(-2g)}$ to graded vector spaces. Recall that the category $\mathfrak{Br}^{(-2g)}$ was defined in Section 3.3; it is the category whose objects are the natural numbers and morphisms $n \to m$ are (n,m)-Brauer diagrams, with symmetric monoidal structure given on objects by addition and on morphisms by disjoint union. There is an evident inclusion $\coprod_{n\geq 0} \mathfrak{S}_n \to \mathfrak{Br}^{(-2g)}$, by interpreting a bijection $[n] \to [n]$ as an (n,n)-Brauer diagram with only vertical strands, which defines a forgetful functor from twisted Brauer algebras to twisted commutative algebras.

(2) Strictly speaking these are rings in the category of representations of $\mathrm{Sp}(2g)^{\mathrm{op}}$, rather than $\mathrm{Sp}(2g)$: if we want to work with usual representations we should have $\bigwedge^\bullet V_{\langle 1,1,1\rangle}/(V_{\langle 2,2\rangle})$ rather than $\bigwedge^\bullet V^*_{\langle 1,1,1\rangle}/(V^*_{\langle 2,2\rangle})$. We will ignore this detail.

It is not hard to see that if A is a ring with $\mathrm{Sp}(2g)$-action, then L_A may in fact be considered as a twisted Brauer algebra. (3) This has some advantages. For example, we noted that the ring R_g is generated by a single algebraic cycle class (more precisely, by a single copy of the representation $V^*_{\langle 1,1,1\rangle}$), whereas the twisted commutative tautological algebras R_g, R'_g and R''_g had large numbers of generators. If we consider L_{R_g} as a twisted *Brauer* algebra rather than a twisted commutative algebra, it is in fact generated by a single element in arity 3 corresponding to the Gross-Schoen cycle. This shows that by considering twisted Brauer algebras one retains slightly more information.

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(3) One may also consider L°_A as a twisted Brauer algebra, but one does not gain anything in doing so: all maps $L^\circ_A(n) \to L^\circ_A(m)$ for $n \neq m$ given by (n,m)-Brauer diagrams are zero.

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EXISTENCE OF EXPANDERS OF THE HARMONIC MAP FLOW

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ABSTRACT. – We investigate the existence of weak expanding solutions of the harmonic map flow for maps with values into a smooth closed Riemannian manifold. We prove the existence of such solutions in the case the target manifold is isometrically embedded as a hypersurface of some Euclidean space and the initial condition is a Lipschitz map that is homotopic to a constant. Regularity is proved outside a compact set.

RÉSUMÉ. – Nous nous intéressons à l'existence de solutions faibles au flot d'applications harmoniques pour des applications à valeurs dans une variété riemannienne compacte lisse et sans bord. Nous montrons l'existence de telles solutions dans le cas où la variété cible peut-être plongée isométriquement comme une hypersurface de l'espace euclidien et si la condition initiale est une application lipschitzienne homotope à une constante. La question de la régularité est également traitée.

1. Introduction

In this paper we consider the Cauchy problem for the heat flow of harmonic maps $(u(t))_{t\geq 0}$ from $\mathbb{R}^n$, $n \geq 3$ to a closed smooth Riemannian manifold (N^{m-1}, g) isometrically embedded as a hypersurface in some Euclidean space $\mathbb{R}^m$, $m \geq 2$. More precisely, we study the parabolic system

$$\begin{cases} \partial_t u = \Delta u + A(u)(\nabla u, \nabla u), & \text{on } \mathbb{R}^n \times \mathbb{R}_+, \\ u|_{t=0} = u_0, \end{cases} \tag{1}$$

for a given map $u_0 : \mathbb{R}^n \to N$, where $A(u)(\cdot,\cdot) : T_uN \times T_uN \to (T_uN)^{\perp}$ denotes the second fundamental form of the embedding $N^{m-1} \hookrightarrow \mathbb{R}^m$ evaluated at u. Note that the equation (1) is equivalent to $\partial_t u - \Delta u \perp T_uN$ for a family of maps $(u(t))_{t\geq 0}$ which map into N. Recall that this evolution equation is invariant under the scaling

$$(u_0)_\lambda(x) := u_0(\lambda x), \quad x \in \mathbb{R}^n, \tag{2}$$

$$u_\lambda(x,t) := u(\lambda x, \lambda^2 t), \quad \lambda > 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{3}$$

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If u_0 is invariant under the above scaling, i.e., if u_0 is 0-homogeneous, solutions of the harmonic map flow which are invariant under scaling are potentially well-suited for smoothing out u_0 instantaneously. Such solutions are called expanding solutions or expanders. In this setting, it turns out that (1) is equivalent to a static equation, i.e., that it does not depend on time anymore. Indeed, if u is an expanding solution in the previous sense then the map $U(x) := u(x,1)$ for $x \in \mathbb{R}^n$, satisfies the elliptic system

$$\begin{cases} \Delta_f U + A(U)(\nabla U, \nabla U) = 0, & \text{on } \mathbb{R}^n, \\ \lim_{r\to+\infty} U(r,\omega) = u_0(\omega), & (r,\omega) \in \mathbb{R}_+ \times \mathbb{S}^{n-1}, \end{cases} \tag{4}$$

where f and Δ_f are defined by

$$f(x) := \frac{|x|^2}{4} + \frac{n}{2}, \quad x \in \mathbb{R}^n,$$
$$\Delta_f U := \Delta U + \nabla f \cdot \nabla U = \Delta U + \frac{r}{2}\partial_r U.$$

The function f is called the potential function and it is defined up to an additive constant. The choice of this constant is dictated by the requirement

$$\Delta_f f = f.$$

The operator Δ_f is called a weighted laplacian and it is unitarily conjugate to a harmonic oscillator $\Delta - |x|^2/16$.

Conversely, if U is a solution to (4) then the map $u(x,t) := U(x/\sqrt{t})$, for $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, is a solution to (1). Because of this equivalence, u_0 can be interpreted either as an initial condition or as a boundary data at infinity.

The interest in expanding solutions is basically due to two reasons. On the one hand, these scale invariant solutions are important with respect to the continuation of a weak harmonic map flow between two closed Riemannian manifolds. Indeed, by the work of Chen and Struwe [3], there always exists a weak solution of the harmonic map flow starting from a smooth map between two closed Riemannian manifolds. It turns out that such a flow is not always smooth and the appearance of singularities is caused by either non-constant 0-homogeneous harmonic maps defined from $\mathbb{R}^n$ to N, the so called tangent maps, or shrinking solutions (also called quasi-harmonic spheres) that are ancient solutions invariant under scaling. Expanding solutions can create an ambiguity in the continuation of the flow after it reaches a singularity by a gluing process. On the other hand, one might be interested in using the smoothing effect of the harmonic map flow. More precisely, it is tempting to attach a canonical map to any map between (stratified) manifolds with prescribed singularities. It turns out that 0-homogeneous maps are the building blocks of such singularities and expanding solutions are likely to be the best candidates to do this job.

In this paper, we investigate the question of existence of expanding solutions coming out of u_0 in the case where there is no topological obstruction, i.e., under the assumption that u_0 is homotopic to a constant when restricted to $\mathbb{S}^{n-1}$. Our main result is the following

THEOREM 1.1. – *Let $n \geq 3$ and $m \geq 2$ be two integers and let $u_0 : \mathbb{R}^n \to (N^{m-1}, g) \subset \mathbb{R}^m$ be a Lipschitz 0-homogeneous map such that its restriction to $\mathbb{S}^{n-1}$ is homotopic to a constant.*

Then there exists a weak expanding solution $u(\cdot,1)=:U(\cdot)$ *of the harmonic map flow coming out of* u_0 *weakly which is regular off a closed singular set with at most finite* $(n-2)$*-dimensional Hausdorff measure. Moreover, there exist a radius* $R=R(\|\nabla u_0\|_{L^2_{\mathrm{loc}}},n,m)>0$ *and a constant* $C=C(\|\nabla u_0\|_{L^2_{\mathrm{loc}}},n,m)>0$ *such that* U *is smooth outside* $B(0,R)$ *and,*

$$|\nabla U|(x)\leq \frac{C}{|x|},\quad |x|\geq R,$$
$$\|\nabla u(t)\|_{L^2(B(x_0,1))}\leq C\left(n,m,\|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)},t\right)\|\nabla u_0\|_{L^2(B(x_0,1))},\quad \forall x_0\in\mathbb{R}^n,$$
$$\|\partial_t u\|_{L^2((0,t),L^2_{\mathrm{loc}}(\mathbb{R}^n))}\leq C(n,m,t)\|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)},$$

where $\lim_{t\to 0} C\left(n,m,\|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)},t\right)=\lim_{t\to 0}C(n,m,t)=1.$

In particular, $u(\cdot,t)$ *tends to* u_0 *as* t *goes to* 0 *in the* $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ *sense and if* u_0 *is not harmonic then* $u(\cdot,t)$ *is not constant in time. Finally, one has the following convergence rate:*

$$|U(x)-u_0(x/|x|)|\leq C|x|^{-1},\quad |x|\geq R. \tag{5}$$

We remark that the regularity result of the theorem is reminiscent of and based on the fundamental work of Chen and Struwe [3] and Cheng [4]. We localize their approach to ensure the smoothness of the solution outside a closed ball since the local energy is decaying to 0 at infinity. This lets us establish a sharp convergence rate for Lipschitz maps. Theorem 1.1 and its proof provide the existence of a non constant in time (or equivalently non radial) expanding solution in case the initial map is not harmonic. Since the initial condition u_0 is allowed to have large local-in-space energy, it is likely that uniqueness will fail. In particular, the authors do not know if the solution produced by Theorem 1.1 coming out of a 0-homogeneous harmonic map will stay harmonic.

Let us make some comments about the proof of Theorem 1.1 before we describe its main steps. A direct perturbative approach is well-suited in case the target Riemannian manifold (N,g) has non-positive sectional curvature as shown by the second author [7] or if the $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ energy of u_0 is assumed to be arbitrarily small: see Section 2 for a proof. One more instance where such a direct approach works well is by imposing further symmetry on the initial condition u_0 and the target manifold N as initiated by Germain and Rupflin [9]. To conclude, the nonlinearity of the target manifold and the potential formation of finite time singularities are the two main obstacles to a direct perturbative approach in general.

To circumvent this issue, we follow Chen-Struwe's penalisation procedure [3] and we construct our expanding solution to the harmonic map flow as a limit of expanding solutions starting from the same initial condition u_0 of a so called homogeneous Chen-Struwe flow with parameter K, see Section 3.3 for more definitions. Before stating the main result about this flow, we recall some definitions.

Let $n\geq 3$ and let $u_0:\mathbb{R}^n\to(N,g)\subset\mathbb{R}^m$ be a 0-homogeneous map. Let us notice that (N,g) is not assumed to be a hypersurface of $\mathbb{R}^m$ at this stage.

A map $u : \mathbb{R}^n \times (0,T) \to \mathbb{R}^m$ is a weak solution of the Homogeneous Chen-Struwe flow with initial condition u_0 if it satisfies:

$$\begin{cases} \partial_t u = \Delta u - \dfrac{K}{t}\chi'\left(d_N^2(u)\right)\nabla\left(\dfrac{d_N^2}{2}\right)(u), \quad \text{on } \mathbb{R}^n \times \mathbb{R}_+, \\ u|_{t=0} = u_0, \quad \text{in the weak sense,} \end{cases} \tag{6}$$

where χ is a smooth, non-decreasing function such that $\chi(s) = s$ for $0 \leq s \leq \delta_N^2$ and $\chi(s) = 2\cdot\delta_N^2$ if $s \geq (2\cdot\delta_N)^2$ for some positive constant δ_N depending on the geometry of the embedding of N in $\mathbb{R}^m$. As in [3], such a function χ is designed to ignore the set of points of $\mathbb{R}^m \setminus N$ where the distance function d_N is not smooth. We notice that (6) differs from the flow introduced by Chen and Struwe by a factor of time t^{-1} in front of the parameter K: the main reason is to keep (6) invariant under parabolic rescalings as defined in (3).

Moreover, the pointwise energy associated to a solution $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ of the Homogeneous Chen-Struwe flow (6) is defined by

$$e_K(u)(x,t) := \frac{1}{2}\left(|\nabla u|^2(x,t) + \frac{K}{t}\chi\left(d_N^2(u(x,t))\right)\right), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{7}$$

For any expanding solution u_K of the Homogeneous Chen-Struwe flow with parameter K defined on $\mathbb{R}^n \times \mathbb{R}_+$, we denote its evaluation at time $t = 1$ by $U_K(x) := u_K(x,1)$, for $x \in \mathbb{R}^n$.

Formally speaking, the Homogeneous Chen-Struwe flow with parameter K is the gradient flow of the (total) energy of a solution u given by the integral of the pointwise energy e_K introduced in (7). As K goes to $+\infty$, the penalty term

$$K\int_{B(x_0,1)} \chi\left(d_N^2(u(x,t))\right)dx, \quad (x_0,t) \in \mathbb{R}^n \in \mathbb{R}_+$$

forces the unconstrained solutions u_K to take their values in N, at least heuristically.

The following theorem sums up the main properties of the expanding solutions to this Homogeneous Chen-Struwe flow we are able to construct and make precise these heuristics:

Theorem 1.2. – *Let $u_0 : \mathbb{R}^n \to (N^{m-1}, g) \subset \mathbb{R}^m$ be a 0-homogeneous map in $C^3_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ for $n \geq 3$ such that its restriction to $\mathbb{S}^{n-1}$ is homotopically trivial. Then for any $K > 0$, there exists a regular Chen-Struwe expanding solution u_K coming out of u_0*

$$\begin{cases} -\Delta_f U_K + K\chi'\left(d_N^2(U_K)\right)\nabla\left(\dfrac{d_N^2}{2}\right)(U_K) = 0, \\ \lim\limits_{t\to 0^+} u_K(\cdot,t) = u_0 \quad \textit{in the weak sense.} \end{cases} \tag{8}$$

Moreover, there exist a radius $R = R(\|\nabla u_0\|_{L^2_{\mathrm{loc}}}, n, m) > 0$ and a positive constant $C = C(\|\nabla u_0\|_{L^2_{\mathrm{loc}}}, n, m)$ such that,

$$\begin{aligned} &e_K(u_K)(x,1) := \frac{|\nabla U_K(x)|^2}{2} + \frac{K}{2}\chi\left(d_N^2(U_K(x))\right) \leq \frac{C}{|x|^2}, \quad |x| \geq R, \\ &\|e_K(u_K)(t)\|_{L^1(B(x_0,1))} \leq C\left(n,m,\|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)},t\right)\|\nabla u_0\|^2_{L^2(B(x_0,1))}, \quad \forall x_0 \in \mathbb{R}^n, \\ &\|\partial_t u_K\|_{L^2((0,t),L^2_{\mathrm{loc}}(\mathbb{R}^n))} \leq C(n,m,t)\|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}, \end{aligned} \tag{9}$$

where $\lim_{t\to 0} C\left(n,m,\|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)},t\right) = \lim_{t\to 0} C(n,m,t) = 1$.

Finally, $u_K(t)$ converges strongly to u_0 as t goes to 0 in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$.

The proof of Theorem 1.2 is divided into four steps explained just before Section 3.4. Each step is proved in Sections 3.4, 3.5, 4 and 5.

In order to prove Theorem 1.2, we reformulate (8) as a fixed point problem in the perspective of applying the Leray-Schauder Fixed Point theorem as has been done in [10] in the context of the Navier-Stokes equation: the main reason why we assume the target Riemannian manifold (N^{m-1}, g) to be isometrically embedded as a hypersurface of $\mathbb{R}^m$ is to make sense of this fixed point formulation.

In order to guess the qualitative properties of the space of perturbations, we investigate the properties of the first approximations in Section 3.2.

In the case of the target manifold being a round sphere another approximation of the harmonic map flow was introduced by Chen [2]. In our setting, it is given by the homogeneous Ginzburg-Landau flow with parameter $K > 0$

$$\begin{cases} \partial_t u = \Delta u + \dfrac{K}{t}(1-|u|^2)u, & \text{on } \mathbb{R}^n \times \mathbb{R}_+, \\ u|_{t=0} = u_0. \end{cases} \tag{10}$$

The reason why we introduce the factor t^{-1} in front of the term $K(1-|u|^2)u$ is to make the Ginzburg-Landau flow invariant under the same scaling (2) and (3) as the harmonic map flow. Despite its physical relevance, the homogeneous Ginzburg-Landau flow does not seem to give a precise estimate on the singular set of the limiting harmonic map as was noticed by Chen and Struwe. This is essentially due to the lack of a good Bochner formula which in turn is caused by the difficulty of controlling the vanishing set of an expanding solution a priori. Still, with the same methods one gets the following result:

THEOREM 1.3. – *Let $u_0 : \mathbb{R}^n \to \mathbb{S}^{m-1} \subset \mathbb{R}^m$ be a 0-homogeneous map in $C^3_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, $n \geq 3$, such that its restriction to the sphere $\mathbb{S}^{n-1}$ is homotopically trivial. Then for any $K > 0$, there exists a smooth Homogeneous Ginzburg-Landau expanding solution u_K coming out of u_0:*

$$\begin{cases} \Delta_f U_K + K(1-|U_K|^2)U_K = 0, \\ \lim_{t\to 0^+} u_K(\cdot, t) = u_0, & \textit{in the weak sense.} \end{cases} \tag{11}$$

Moreover, if

$$e_K(u_K)(x,t) := \frac{1}{2}\left(|\nabla u_K|(x,t)^2 + \frac{K}{2t}(1-|u_K(x,t)|^2)^2\right), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+,$$

denotes the pointwise energy associated to the solution u_K then:

$$\|e_K(u_K)(t)\|_{L^1(B(x_0,1))} \leq C\left(n, m, \|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}, t\right) \|\nabla u_0\|^2_{L^2(B(x_0,1))}, \quad \forall x_0 \in \mathbb{R}^n,$$
$$\|\partial_t u_K\|_{L^2((0,t), L^2_{\mathrm{loc}}(\mathbb{R}^n))} \leq C(n,m,t)\|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)},$$

where $\lim_{t\to 0} C\left(n, m, \|\nabla u_0\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}, t\right) = \lim_{t\to 0} C(n,m,t) = 1.$

In particular, $u_K(t)$ converges strongly to u_0 as t goes to 0 in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$.

We would like to relate our work to previous articles on this subject. To our knowledge, most of the literature concerns maps from $\mathbb{R}^n$ to an equator of a rotationally symmetric target manifold such as the works of Germain and Rupflin [9], Biernat and Bizon [1] and the more recent work due to Germain, Ghoul and Miura [8]. In particular, our setting includes theirs in the case the target is a sphere since a map from $\mathbb{S}^{n-1}$ with values in an equator is homotopic to a constant. Of course, since (1) reduces to an ODE in such a corotational setting, the above mentioned works obtain more quantitative results even if the question of regularity is not really addressed.

There are at least two other partial differential equations that motivate this work. Jia and Šverák [10] proved the existence of smooth expanding solutions of the Navier-Stokes equation. In this case, the homogeneity is of degree -1. To prove Theorem 1.2, we proceed similarly to their work by using the Leray-Schauder degree theory. For this, one needs a path of initial conditions $(u_0^\sigma)_{0\leq\sigma\leq 1)} : \mathbb{S}^{n-1} \to N$ connecting the restriction u_0^0 of u_0 to $\mathbb{S}^{n-1}$ to a suitable map u_0^1. A suitable map u_0^1 means here that there is an obvious solution coming out of u_0^1 for which there is a uniqueness result for small solutions lying in a suitable function space. In the case of the Navier-Stokes equation, the path is given for free since it suffices to contract the initial vector field to zero. In our case, the path is given by assumption. There is also a deep analogy with the Ricci flow that exhibits the same scale invariance. In the setting of the Ricci flow, u_0 is replaced by a metric cone $C(M)$ over a closed Riemannian manifold (M, g) endowed with its Euclidean cone metric $dr^2 + r^2g$ and the topological assumption on M similar to the triviality of the homotopy class of u_0 is that it is null cobordant. See [6] and [5] in the case (M, g) is a Riemmanian manifold with curvature operator larger than 1.

Finally, let us describe the content of each section.

In Section 2 we show a perturbation result for expanding solutions with small energy.

Section 3 defines the notion of homogeneous Chen-Struwe flow and investigates the existence of smooth expanding solutions to this flow as stated in Theorem 1.2. Section 3.2 analyzes carefully the properties of the first approximation and Section 3.3 reduces the analysis to a fixed point problem. Sections 3.4 and 3.5 prove this fixed point problem is well-posed in the context of Leray-Schauder degree theory. Before proceeding to a priori bounds on expanding solutions to the Homogeneous Chen-Struwe flow, Sections 4.1 and 4.2 establish a Bochner formula and a local entropy monotonicity formula that are crucial to prove an ε-regularity theorem in this setting: see Section 4.3. Then Section 5.1 starts proving a priori C^0 bounds of such expanding solutions. Sections 5.2 and 5.3 take care of bounding such expanding solutions at infinity a priori. Section 6 ends the proof of Theorem 1.2.

Finally, Section 7 investigates Taylor expansions at infinity of any expanding solutions of the harmonic map flow in terms of the jets of the initial condition at a formal level. This section is inspired by a similar expansion for asymptotically conical expanding solutions of the Ricci flow done by Lott and Wilson [14].

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2. Deformation theory of expanders (with small energy)

From now on, let $u : \mathbb{R}^n \to N$ be an expanding solution of the harmonic map flow, fixed once and for all. We consider the linearisation of equation (1) around u. It takes the following time-dependent form: if $u+h$ is an expanding solution then,

$$\partial_t(u+h) = \Delta(u+h) + A(u+h)(\nabla(u+h),\nabla(u+h))|,$$

is equivalent to

$$\begin{aligned}\partial_t h &= \Delta h + A(u+h)(\nabla(u+h),\nabla(u+h)) - A(u)(\nabla u,\nabla u)\\ &=: \Delta h + R(u,\nabla u,h,\nabla h),\end{aligned}$$

where

$$R(u,\nabla u,h,\nabla h) := A(u+h)(\nabla(u+h),\nabla(u+h)) - A(u)(\nabla u,\nabla u). \tag{12}$$

Our main result in this section is a uniqueness result for expanding solutions close to a constant map.

First, let us introduce the main function space for the initial conditions:

$$X_0 := \{u_0 : \mathbb{R}^n \to N \subset \mathbb{R}^m \quad | \quad \|u_0\|_{\mathrm{BMO}} < +\infty\},$$

where

$$\|u_0\|_{\mathrm{BMO}} := \sup_{x\in\mathbb{R}^n, r>0} ⨍_{B(x,r)} |u_0 - ⨍_{B(x,r)} u_0|(y)dy.$$

Secondly, the relevant function space X of solutions is defined by:

$$\begin{aligned}X := \; &\{u : \mathbb{R}^n\times\mathbb{R}_+ \to \mathbb{R}^m \quad | \quad \|u\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}_+)} + \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}_+} \sqrt{t}|\nabla u|(x,t)\\ &+ \sup_{(x,s)\in\mathbb{R}^n\times\mathbb{R}_+} ⫲\nabla u\|_{L^2(P(x,\sqrt{s}))} < +\infty\},\end{aligned}$$

where $P(x,r) := B(x,r)\times(0,r^2)$ and where

$$⫲h\|^p_{L^p(P(x,r))} := ⨍_{P(x,r)} r^2|h|^p(y,s)dyds, \quad x\in\mathbb{R}^n, \quad r>0, \quad p\geq 1,$$

where h is a tensor defined on $\mathbb{R}^n\times\mathbb{R}_+$. The number $⫲h\|^p_{L^p(P(x,r))}$ is the normalized p-norm of h on the parabolic neighborhood $P(x,r)$.

Finally, as in [11], we introduce a somewhat intermediate function space Y:

$$Y := \left\{R : \mathbb{R}^n \to \mathbb{R}^m \quad | \quad \sup_{t\in\mathbb{R}_+} t\|R(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{s\in\mathbb{R}_+} ⫲R\|_{L^1(P(x,\sqrt{s}))} < +\infty\right\}.$$

We are now in a position to state the main result of this section:

THEOREM 2.1. – *There exists $\varepsilon>0$ such that if $u : \mathbb{R}^n \to N$ is an expanding solution of the harmonic map flow coming out of $u_0 \in X_0$ with $\||\nabla u|^2\|_Y < \varepsilon$, then there is a neighborhood $\mathcal{U}_0$ of u_0 in the BMO-topology such that for any 0-homogeneous map $v_0 \in \mathcal{U}_0$, there exists an expanding solution of the harmonic map flow $v : \mathbb{R}^n \to N$ coming out of v_0. Besides, uniqueness holds in a small ball with center u with respect to the topology of X.*

Proof. – The proof follows closely the one in the paper [16] which in turn is motivated by the paper [11].

First of all, let us fix some map $v_0 \in \mathrm{BMO}(\mathbb{R}^n, N)$ and define the map $T : X \to X$ as follows:

$$T(h) := k_t * (v_0 - u_0) + \int_0^t k_{t-s} * R(u, \nabla u, h, \nabla h)(s) ds,$$

where $R(u, \nabla u, h, \nabla h)$ is defined by (12) and where k_t denotes the Euclidean heat kernel

$$k_t(x) := (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad t > 0, \quad x \in \mathbb{R}^n.$$

Claim 1. – *T is well-defined and the following estimates holds:*

$$\|T(h)\|_X \leq c \left(\|v_0 - u_0\|_{\mathrm{BMO}} + \left[\||\nabla u|^2\|_Y + \|u\|_X \|h\|_X + \|h\|_X^2\right] \|h\|_X\right),$$

for some uniform positive constant $c > 0$.

Indeed, the estimate

$$\|k_t * (v_0 - u_0)\|_X \leq C \|v_0 - u_0\|_{\mathrm{BMO}}$$

can be found at the beginning of Section 2 in [16] and it follows from Lemma 3.1 in [16] that

$$\| \int_0^t k_{t-s} * R(u, \nabla u, h, \nabla h)(s) ds\|_X \leq C \|R(u, \nabla u, h, \nabla h)\|_Y.$$

Now a standard computation shows that

$$\|R(u, \nabla u, h, \nabla h)\|_Y \leq C \|u\|_X \|h\|_X^2 + C \|h\|_X^3 + C \||\nabla u|^2\|_Y \|h\|_X.$$

Similarly one observes that

Claim 2. – *T is a contraction on a sufficiently small ball around u in X, i.e., there is some $q \in (0, 1)$ such that*

$$\|T(h_1) - T(h_2)\|_X \leq q \|h_1 - h_2\|_X,$$

for all h_1, h_2 in $B_X(u, \delta)$ if δ is small enough.

Hence we obtain the desired solution $v = u + h$ from the Banach fixed point theorem. It was also shown in [16], Proof of Theorem 1.3, that v indeed maps into N. □

3. Preliminaries

3.1. Geometry of hypersurfaces of Euclidean space

Let (N^{m-1}, g) be a closed Riemannian manifold which is isometrically embedded in the Euclidean space $\mathbb{R}^m$. Let us denote the distance function to N by d_N. Then there exists a tubular neighborhood $T_{2\delta_N}(N) := \{y \in \mathbb{R}^m \mid d_N(y) < 2\delta_N\}$ of N such that the projection map $\Pi_N : T_{2\delta_N}(N) \to N$ is well-defined and smooth. As in [3], let χ be a smooth, non-decreasing function such that $\chi(s) = s$ for $0 \leq s \leq \delta_N^2$ and $\chi(s) = 2 \cdot \delta_N^2$ if $s \geq (2 \cdot \delta_N)^2$.

We also recall the definition of the signed distance to N. Since (N^{m-1}, g) is closed and isometrically embedded in the Euclidean space $\mathbb{R}^m$, there is a smooth function $\rho : \mathbb{R}^m \to \mathbb{R}$ such that

$$\{x \in \mathbb{R}^m \mid \rho(x) < 0\} =: \Omega \quad \text{is an open bounded domain with smooth boundary } N,$$
$$\{x \in \mathbb{R}^m \mid \rho(x) = 0\} = N, \quad \text{and } \nabla\rho(x) \neq 0, \quad x \in \partial\Omega = N.$$

Such a function ρ is called a defining function for the domain Ω which is the "inside" of N, the set $\{x \in \mathbb{R}^m \mid \rho(x) > 0\}$ being the "outside" of N: see [Chapter 1, Section 1.2, [12]] for a systematic study. The signed distance of N denoted by $\bar{d}_N$ is then defined by:

$$\bar{d}_N(x) := \operatorname{sgn}(\rho(x)) d_N(x), \quad x \in \mathbb{R}^m. \tag{13}$$

[Theorem 1.2.6, [12]] ensures that $\bar{d}_N$ is a smooth function on a tubular neighborhood $T_{2\delta_N}$ of N. As a final remark, we notice that $\bar{d}_N^2 = d_N^2$.

Let us treat the case of a Euclidean sphere $N = \mathbb{S}^{m-1} \subset \mathbb{R}^m$. A defining function for the unit m-Euclidean ball $\mathbb{B}^m(0,1)$ centered at $0 \in \mathbb{R}^m$ is $\rho(x) = |x|^2 - 1$ for $x \in \mathbb{R}^m$. The signed distance of $\mathbb{S}^{m-1}$ is then $\bar{d}_{\mathbb{S}^{m-1}}(x) = |x| - 1$ and it is smooth on $\mathbb{R}^m \setminus \{0\}$.

3.2. Properties of the first approximation

Denote by $U_0(t)$ the caloric extension of the map u_0, i.e.,

$$U_0(x,t) := (k_t * u_0)(x), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+,$$

and denote by U_0 the map $U_0(\cdot, 1)$. By construction, U_0 is 0-homogeneous and hence

$$U_0(x,t) = U_0\left(\frac{x}{\sqrt{t}}, 1\right) = U_0\left(\frac{x}{\sqrt{t}}\right), \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_+.$$

LEMMA 3.1. – *Let $u_0 : \mathbb{R}^n \to (N, g) \subset \mathbb{R}^m$ be a Lipschitz 0-homogenous map. Then the caloric extension U_0 of u_0 satisfies*

$$\|U_0\|_{L^\infty} \leq \|u_0\|_{L^\infty}, \tag{14}$$

$$\sup_{x \in \mathbb{R}^n} (1 + |x|)|\nabla^k U_0|(x) \leq C(k, u_0), \quad \forall k \geq 1, \tag{15}$$

$$(1 + |x|) d_N(U_0(x)) \leq (1 + |x|)|U_0(x) - u_0(x/|x|)| \leq C(u_0). \tag{16}$$

Moreover, if u_0 is in $C^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, one has the improved decay

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{\frac{\min\{k,2\}}{2}} |\nabla^k U_0|(x) \leq C(u_0), \quad k \geq 1, \tag{17}$$

$$(1 + |x|^2) d_N(U_0(x)) \leq (1 + |x|^2)|U_0(x) - u_0(x/|x|)| \leq C(u_0). \tag{18}$$

If u_0 is in $C^3_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, with $n \geq 4$, one has

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^3)|\nabla^3 U_0|(x) + \sup_{|x| \geq R(u_0)} (1 + |x|^3)|\nabla(\bar{d}_N(U_0))|(x) \leq C(u_0), \tag{19}$$

for some positive radius $R(u_0)$.

Moreover, if $(u_0^\sigma)_{\sigma \in [0,1]}$ is a path of C^3 maps from $\mathbb{S}^{n-1}$ into N such that $u_0^0 = u_0$ and $u_0^1 \equiv P \in N$ then the constants $(C(u_0^\sigma))_{\sigma \in [0,1]}$, $(R(u_0^\sigma))_{\sigma \in [0,1]}$ in (16)*,* (17)*,* (18) *and* (19) *satisfy*

$$\lim_{\sigma \to 1} C(u_0^\sigma) = 0, \quad \sup_{\sigma \in [0,1]} R(u_0^\sigma) < +\infty.$$

Proof. – The first bound on the C^0 norm of U_0 follows easily from the maximum principle applied to the heat equation or by using the explicit formula in terms of the Euclidean heat kernel. The decay (15) on the first derivatives of U_0 is proved by using the corresponding decay of the derivatives of the initial condition u_0. The bound (16) uses the heat equation in its static form. Since $U_0(t)$ is 0-homogeneous as a time dependent function then the map $U_0(\cdot, 1) = U_0$ satisfies $\Delta_f U_0 = 0$, i.e.,

$$\frac{r}{2}\partial_r U_0 = -\Delta U_0 = O((1+r)^{-1}),$$

thanks to the previous estimate (15) on the second derivatives of U_0. In particular, this implies that

$$U_0(x) = u_0(x/|x|) + O((1+|x|)^{-1}),$$

as x tends to $+\infty$.

Therefore, $d_N(U_0(x)) \leq |U_0(x) - u_0(x/|x|)| = O((1+|x|)^{-1})$.

Now, if u_0 is in $C^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, the decays (17) on the first and the second derivatives of U_0 are proved by using the corresponding decay of the derivatives of the initial data u_0. Notice that if $k = 0, 1, 2$ then $\nabla^k u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if $n \geq 3$. Let us prove (17) for the first derivative $\nabla^3 U_0$ for instance. Since u_0 is not assumed to be in $C^3_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, one differentiates twice the map u_0 and once the heat kernel k_1 at time $t = 1$ in the integral representation formula of the caloric extension U_0 to get:

$$\begin{aligned}
|\nabla^3 U_0|(x) &\leq C_1 \int_{\mathbb{R}^n} |x-y| e^{-\frac{|x-y|^2}{4}} \frac{dy}{|y|^2} \leq C_2 \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{8}} \frac{dy}{|y|^2} = C_2 \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{8}} \frac{dy}{|x-y|^2} \\
&\leq C_2 \left(\int_{|y| \leq 2^{-1}|x|} e^{-\frac{|y|^2}{8}} \frac{dy}{|x-y|^2} + \int_{2^{-1}|x| \leq |y| \leq 2|x|} e^{-\frac{|y|^2}{8}} \frac{dy}{|x-y|^2} \right) \\
&\quad + C_2 \int_{|y| \geq 2|x|} e^{-\frac{|y|^2}{8}} \frac{dy}{|x-y|^2}.
\end{aligned}$$

The first integral on the righthand side of the previous inequality can be estimated as follows for some $x \in \mathbb{R}^n \setminus \{0\}$:

$$\int_{|y| \leq 2^{-1}|x|} e^{-\frac{|y|^2}{8}} \frac{dy}{|x-y|^2} \leq \frac{4}{|x|^2} \int_{|y| \leq 2^{-1}|x|} e^{-\frac{|y|^2}{8}} dy \leq \frac{C}{|x|^2},$$

by the triangular inequality $|x-y| \geq |x| - |y| \geq 2^{-1}|x|$ if $y \in B(0, 2^{-1}|x|)$.

The second one is handled as follows:

$$\begin{aligned}
\int_{2^{-1}|x| \leq |y| \leq 2|x|} e^{-\frac{|y|^2}{8}} \frac{dy}{|x-y|^2} &\leq e^{-\frac{|x|^2}{32}} \int_{2^{-1}|x| \leq |y| \leq 2|x|} \frac{dy}{|x-y|^2} \leq e^{-\frac{|x|^2}{32}} \int_{|x-y| \leq 3|x|} \frac{dy}{|x-y|^2} \\
&\leq C|x|^{n-2} e^{-\frac{|x|^2}{32}},
\end{aligned}$$

where C is a positive constant uniform in $x \in \mathbb{R}^n$ since $|x|^{-2} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Finally, the third integral can be estimated similarly.

We proceed similarly to (16) to prove (18), since $U_0(t)$ is 0-homogeneous as a time dependent function then the map U_0 satisfies $\Delta_f U_0 = 0$, i.e.,

$$\frac{r}{2}\partial_r U_0 = -\Delta U_0 = O((1+r^2)^{-1}), \tag{20}$$

thanks to the previous estimate on the second derivatives of U_0. In particular, this implies that

$$U_0(x) = u_0(x/|x|) + O((1+|x|^2)^{-1}),$$

as x tends to $+\infty$ and therefore $\bar{d}_N(U_0(x)) = O((1+|x|^2)^{-1})$ as x tends to $+\infty$.

It remains to prove (19). The first estimate on the third derivatives can be proved as we proceed for (17). Let us handle the second term on the lefthand side of (19).

First of all, since the function $\bar{d}_N$ is differentiable on the set $T_{2\delta_N}(N)$, estimate (18) ensures that $d_N(U_0) < 2\delta_N$ as soon as $|x| \geq R(u_0)$. Now observe that:

$$\left|\nabla\bar{d}_N(U_0(x)) - \nabla\bar{d}_N(u_0(x/|x|))\right| \leq C|U_0(x) - u_0(x/|x|)| \leq \frac{C(u_0)}{1+|x|^2},$$

by (18). Therefore we get the following intermediate estimate for $x \in \mathbb{R}^n \setminus \{0\}$:

$$\begin{aligned}
|\nabla(\bar{d}_N(U_0))|(x) &= |\nabla\bar{d}_N(U_0)(\nabla U_0)|(x)\\
&\leq |\nabla\bar{d}_N(u_0)(\nabla U_0)|(x) + \frac{C(u_0)}{1+|x|^2}|\nabla U_0|\\
&\leq |\nabla\bar{d}_N(u_0)(\nabla U_0)|(x) + \frac{C(u_0)}{1+|x|^3}\\
&\leq |\nabla\bar{d}_N(u_0)(\nabla u_0)|(x) + C(u_0)|\nabla(U_0 - u_0)|(x) + \frac{C(u_0)}{1+|x|^3}\\
&= C(u_0)|\nabla(U_0 - u_0)|(x) + \frac{C(u_0)}{1+|x|^3},
\end{aligned}$$

where we use (17) with $k = 1$ in the third line and $\bar{d}_N(u_0) = 0$ in the last line. To conclude, it suffices to show that $\nabla(U_0 - u_0)(x) = O((1+|x|^3)^{-1})$.

Let us remark that the radial derivative of U_0 hence the radial derivative of $U_0 - u_0$ is decaying as expected by (20). Now differentiate (20) once and observe that the hessian of $|x|^2$ is 2δ where δ denotes Euclidean metric. One gets:

$$\begin{aligned}
-\Delta\nabla U_0 &= \frac{r}{2}\partial_r\nabla U_0 + \frac{1}{2}\nabla U_0\\
&= \frac{1}{2}\partial_r\left(r\nabla U_0\right).
\end{aligned}$$

Estimate (19) on the third derivatives then implies that

$$\partial_r\left(r\nabla U_0\right) = O((1+|x|^3)^{-1}).$$

Integrating the previous differential equality along a ray gives:

$$r\nabla U_0(r,\omega) - \nabla^{\mathrm{sph}}u_0(\omega) = O((1+r)^{-2}), \quad (r,\omega) \in \mathbb{R}_+ \times \mathbb{S}^{n-1},$$

where $\nabla^{\mathrm{sph}}u_0$ denotes the spherical derivatives of u_0. This fact leads to the expected estimate. □

The previous Lemma 3.1 and the analysis to follow in the next sections show that if $n \geq 4$, the choice of the caloric extension of u_0 as a first approximation suffices. If $n = 3$, we consider a barycentric approximation of u_0 as follows: if $\eta : \mathbb{R}^n \to [0,1]$ denotes any smooth function such that $\eta \equiv 1$ if $|x| \geq 2$ and $\eta \equiv 0$ if $|x| \leq 1$, we define

$$U_0^b := (1-\eta)P + \eta u_0, \tag{21}$$

where $P \in N$ is fixed.

The properties of U_0^b can be summarized as follows:

Lemma 3.2. – *With the previous notations, assume that* u_0 *is in* $C^3_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$*, then the barycentric approximation satisfies*

$$\|U_0^b\|_{L^\infty} \leq |P| + \|u_0\|_{L^\infty}, \quad \forall x \in \mathbb{R}^n, \quad U_0^b(x) = u_0(x), \quad |x| \geq 2, \tag{22}$$

$$\sup_{x\in\mathbb{R}^n} (1+|x|^k)|\nabla^k U_0^b|(x) \leq C(k,u_0) < +\infty, \quad 0 \leq k \leq 3, \tag{23}$$

$$\sup_{x\in\mathbb{R}^n} (1+|x|^2)|\Delta_f U_0^b| \leq C(u_0) < +\infty, \tag{24}$$

$$d_N(U_0^b(x)) = 0, \quad \textit{if } |x| \leq 1 \textit{ or if } |x| \geq 2, \tag{25}$$

$$\sup_{x\in\mathbb{R}^n} (1+|x|^2) d_N(U_0^b(x)) \leq C(u_0), \tag{26}$$

where $|P|$ *denotes the euclidean norm of* $P \in \mathbb{R}^m$.

Moreover, if $(u_0^\sigma)_{\sigma\in[0,1]}$ *is a path of* C^3 *maps from* $\mathbb{S}^{n-1}$ *into* N *such that* $u_0^0 = u_0$ *and* $u_0^1 \equiv P \in N$ *then the constants* $(C(u_0^\sigma))_{\sigma\in[0,1]}$ *in* (23), (24) *and* (26) *satisfy* $\lim_{\sigma\to 1} C(u_0^\sigma) = 0$.

Proof. – By definition of the map U_0^b, $U_0^b \equiv u_0$ outside $B(0,2) \subset \mathbb{R}^n$. In particular, this implies (25). Moreover, by the triangular inequality:

$$|U_0^b|(x) \leq (1-\eta(x))|P| + \eta(x)\|u_0\|_{L^\infty}.$$

Since η takes its values into $[0,1]$, (22) follows. The estimate (23) comes from the fact that $\nabla\eta$ is compactly supported in $B(0,2)$ and the decay of the derivatives of u_0 outside the origin $0 \in \mathbb{R}^n$.

Now since u_0 is 0-homogeneous, $\Delta_f U_0^b = \Delta_f u_0 = \Delta u_0$ on $\mathbb{R}^n \setminus B(0,3)$ which decays quadratically. Therefore, (24) follows immediately.

In order to prove (26), let us remark that $(1+|x|^2)d_N(U_0^b(x)) = 0$ if $|x| \geq 2$ by (25). Similarly to the proof of (18), one observes that if $|x| \leq 2$,

$$\begin{aligned}(1+|x|^2)d_N(U_0^b(x)) &\leq (1+|x|^2)|U_0^b(x) - P| \\ &\leq (1+|x|^2)\eta(x)|P - u_0(x/|x|)|,\end{aligned}$$

which vanishes identically if $|x| \leq 1$. Therefore,

$$\sup_{x\in\mathbb{R}^n}(1+|x|^2)d_N(U_0^b(x)) \leq 5 \sup_{1\leq|x|\leq 2} |P - u_0(x/|x|)|,$$

which implies (26). □

Because of the similarities shared by Lemma 3.1 with Lemma 3.2, we will only consider the ansatz with the caloric extension in the next sections.

3.3. Fixed point formulation

Motivated by Lemmata 3.1 and 3.2, we introduce the Banach space

$$X := \left\{ V \in C^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m) \quad | \quad \sup_{x\in\mathbb{R}^n} (1+|x|)^2 |V(x)| + (1+|x|)^3 |\nabla V(x)| < +\infty \right\}.$$

In order to produce a solution to (8), we look for a solution u_K to the homogeneous Chen-Struwe equation of the form

$$\partial_t u_K = \Delta u_K - \frac{K}{t}\chi'\left(d_N^2(u_K)\right)\nabla\left(\frac{d_N^2}{2}\right)(u_K), \quad t > 0, \tag{27}$$

$$u_K(x,t) := U_0\left(\frac{x}{\sqrt{t}}\right) + V_K\left(\frac{x}{\sqrt{t}}\right), \tag{28}$$

where $U_0 : \mathbb{R}^n \to \mathbb{R}^m$ denotes the caloric extension of u_0 as defined in Section 3.2 and where $V_K \in X$. Notice that the time-dependent map $V_K(\cdot,t) := V_K(\cdot/\sqrt{t})$ will satisfy the bounds

$$(\sqrt{t}+|x|)^2|V_K(x,t)| + (\sqrt{t}+|x|)^3|\nabla V_K(x,t)| \leq Ct, \quad \forall x \in \mathbb{R}^n, \quad \forall t > 0. \tag{29}$$

As we want to use the Leray-Schauder Fixed Point Theorem in order to show Theorem 1.2, we need to reformulate this problem as follows. Let $\sigma \in [0,1]$ be a parameter and denote by $(u_0^\sigma)_{\sigma\in[0,1]}$ a path of C^3 maps such that $u_0^0 = u_0$ and $u_0^1 \equiv P \in N$. Note that this path is chosen inside the homotopy class of $[u_0] \in \pi_{n-1}(N)$.

Thus, solving (8) amounts to solving the static Chen-Struwe equation

$$\Delta_f V_K - K\chi'\left(d_N^2(U_0^\sigma + V_K)\right)\nabla\left(\frac{d_N^2}{2}\right)(U_0^\sigma + V_K) = 0, \quad V_K \in X. \tag{30}$$

If $V \in X$, $K > 0$ and $\sigma \in [0,1]$, we denote formally by $F_K^\sigma(V) \in X$ the solution to the problem

$$\begin{aligned}\Delta_f F_K^\sigma(V) - K\chi'(d_N^2(U_0^\sigma))d_{U_0^\sigma}\bar{d}_N(F_K^\sigma(V))\nabla\bar{d}_N(U_0^\sigma) \\ = -K\chi'(d_N^2(U_0^\sigma))d_{U_0^\sigma}\bar{d}_N(V)\nabla\bar{d}_N(U_0^\sigma) \\ + K\chi'\left(d_N^2(U_0^\sigma+V)\right)\nabla\left(\frac{d_N^2}{2}\right)(U_0^\sigma+V),\end{aligned}$$

where $d_p\bar{d}_N(v) = <\nabla\bar{d}_N(p), v>$ denotes the differential (when it makes sense) of the signed distance function $\bar{d}_N$ at the point $p \in \mathbb{R}^m$ evaluated at the vector $v \in T_p\mathbb{R}^m$.

REMARK 3.3. – The reason why we isolate the term $d_{U_0^\sigma}\bar{d}_N(F_K^\sigma(V))\nabla\bar{d}_N(U_0^\sigma)$ from the right hand side is that the map

$$V \in X \to d_{U_0^\sigma}\bar{d}_N(F_K^\sigma(V))\nabla\bar{d}_N(U_0^\sigma) \in X$$

is not a compact operator.

We proceed in three steps:

1. The map $F_K : X \times [0,1] \to X$ is a well-defined compact continuous map: this is the content of Section 3.4.
2. The Leray-Schauder degree of $I - F_K^\sigma : B_X(0,\varepsilon) \to B_X(0,\varepsilon)$ is 1 when σ is close to 1, for some positive ε: this is proved in Section 3.5.

3. (A priori estimates) There is a positive constant M (uniform in $\sigma \in [0,1]$) such that if $V \in X$ is such that $F_K^\sigma(V) = V$ then $\|V\|_X \leq M$: these are the contents of Sections 4 and 5.

3.4. F_K is a well-defined compact and continuous map

In this section, we prove that the map $F_K^\sigma : X \to X$ is compact and continuous.

Note that the map F_K^σ can also be formally interpreted as

$$
\begin{aligned}
F_K^\sigma(V)(x,t) = &-K\int_0^t\int_{\mathbb{R}^n} \mathscr{K}_{t-s}^\sigma(x,y)\frac{1}{s}\chi'(d_N^2(U_0^\sigma))d_{U_0^\sigma}\bar{d}_N(V)\nabla\bar{d}_N(U_0^\sigma)dyds\\
&+K\int_0^t\int_{\mathbb{R}^n} \mathscr{K}_{t-s}^\sigma(x,y)\frac{1}{s}\chi'\left(d_N^2(U_0^\sigma+V)\right)\nabla\left(\frac{d_N^2}{2}\right)(U_0^\sigma+V))(y,s)dyds,
\end{aligned}
$$

where $V(x,t) := V(x/\sqrt{t})$ with $V \in X$ and where $\mathscr{K}_t^\sigma \in \mathscr{L}(X,X)$ denotes the solution of

$$
\begin{aligned}
\partial_t\mathscr{K}_t^\sigma &= \Delta\mathscr{K}_t^\sigma - \frac{K}{t}\chi'(d_N^2(U_0^\sigma))d_{U_0^\sigma}\bar{d}_N(\mathscr{K}_t^\sigma)\nabla\bar{d}_N(U_0^\sigma),\\
\lim_{t\to 0^+}\mathscr{K}_t^\sigma &= \delta_0.
\end{aligned}
$$

The issue here is to make sense of this solution and therefore, we prefer to work with the static equation only. To do so, given $V \in X$, we first solve the following Dirichlet problem

$$
\begin{aligned}
\Delta_f W_R - K\chi'(d_N^2(U_0^\sigma))d_{U_0^\sigma}\bar{d}_N(W_R)\nabla\bar{d}_N(U_0^\sigma) &= Q(U_0^\sigma,V), \quad \text{on } B(0,R)\subset\mathbb{R}^n,\\
W_R &= 0, \quad \text{on } \partial B(0,R),
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
Q(U_0^\sigma,V) := &-K\chi'(d_N^2(U_0^\sigma))d_{U_0^\sigma}\bar{d}_N(V)\nabla\bar{d}_N(U_0^\sigma)\\
&+K\chi'\left(d_N^2(U_0^\sigma+V)\right)\nabla\left(\frac{d_N^2}{2}\right)(U_0^\sigma+V).
\end{aligned}
\tag{32}
$$

One can prove that such a solution W_R exists and is unique by the maximum principle.

We start with a lemma analyzing the behavior of the righthand side $Q(U_0^\sigma,V)$ for $V\in X$.

LEMMA 3.4. – *Let* $\sigma\in[0,1]$. *Then*

$$
\begin{aligned}
\|Q(U_0^\sigma,V)\|_X &\leq C(u_0^\sigma,K)(1+\|V\|_X)+C(K)\|V\|_X^2, \quad V\in B_X(0,1),\\
\|Q(U_0^\sigma,V_2)-Q(U_0^\sigma,V_1)\|_X &\leq C(u_0^\sigma,K)\|V_2-V_1\|_X + C(K)\|V_1+V_2\|_X\|V_2-V_1\|_X,
\end{aligned}
\tag{33}
$$

where $V_1,V_2\in B_X(0,1)$ *and where the constant* $C(u_0^\sigma,K)$ *satisfies* $\lim_{\sigma\to1}C(u_0^\sigma,K)=0$ *when* K *is fixed.*

Proof. – Since K is fixed and the expected estimates are depending on K a priori, we can assume $K=1$ to lighten the notations.

A Taylor expansion of degree 2 around U_0^σ of the righthand side of (32) gives:

$$\begin{aligned}\chi'\left(d_N^2(U_0^\sigma+V)\right)\nabla\left(\frac{d_N^2}{2}\right)(U_0^\sigma+V) &= \chi'(d_N^2(U_0^\sigma))\nabla\left(\frac{d_N^2}{2}\right)(U_0^\sigma)\\ &\quad+\chi''(d_N^2(U_0^\sigma))d_{U_0^\sigma}d_N^2(V)\nabla\left(\frac{d_N^2}{2}\right)(U_0^\sigma)\\ &\quad+\chi'(d_N^2(U_0^\sigma))\nabla^2_{U_0^\sigma}\left(\frac{d_N^2}{2}\right)(V)+V*V.\end{aligned}\tag{34}$$

where, if A and B are two tensors, $A*B$ denotes any contraction of linear combinations of the tensor product $A\otimes B$. Therefore, by using that

$$\chi'(d_N^2(U_0^\sigma))\nabla^2\left(\frac{d_N^2}{2}\right)(V)=\chi'(d_N^2(U_0^\sigma))\left(d_{U_0^\sigma}\bar{d}_N(V)\nabla\bar{d}_N(U_0^\sigma)+\bar{d}_N(U_0^\sigma)\nabla^2_{U_0^\sigma}\bar{d}_N(V)\right),$$

one gets,

$$Q(U_0^\sigma,V)=\chi'(d_N^2(U_0^\sigma))\bar{d}_N(U_0^\sigma)\nabla\bar{d}_N(U_0^\sigma)+\bar{d}_N(U_0^\sigma)O(V)+V*V.\tag{35}$$

By using Lemma 3.1 together with (35), one gets:

$$\begin{aligned}&\sup_{x\in\mathbb{R}^n}(1+|x|^2)|Q(U_0^\sigma,V)|(x)\\ &\qquad\leq C(u_0^\sigma)\left(1+\sup_{x\in\mathbb{R}^n}(1+|x|^2)|V|(x)\right)+C\left(\sup_{x\in\mathbb{R}^n}(1+|x|^2)|V|(x)\right)^2,\end{aligned}$$

if $V\in B_X(0,1)$. By differentiating (35) and by using Lemma 3.1 once more, one gets the first line of (33). The estimate on the second line of (33) can be proved similarly. □

In order to obtain a solution defined on all of $\mathbb{R}^n$, we first establish an a priori C^0 bound.

Proposition 3.5. – *The solution W_R defined above satisfies the following a priori weighted C^0 bound:*

$$\max_{B(0,R)} f|W_R|\leq C(n,m,K)\|Q(U_0^\sigma,V)\|_X.\tag{36}$$

In particular,

$$\|W_R\|_{C^{2,\beta}(B(0,R_0))}\leq C(\beta,n,m,R_0,K)\|Q(U_0^\sigma,V)\|_X,\quad\forall R_0<R,\quad\beta\in(0,1).$$

Proof. – Since $Q:=Q(U_0^\sigma,V)$ is C^1, the last assertion on the derivatives of W_R follows immediately by interior elliptic Schauder estimates in case the a priori C^0 bound holds. Therefore, it suffices to establish (36). For this we note that

$$\begin{aligned}\Delta_f|W_R|^2&\geq 2|\nabla W_R|^2+2K\chi'(d_N^2(U_0^\sigma))\left(d_{U_0^\sigma}\bar{d}_N(W_R)\right)^2-2|Q||W_R|\\ &\geq 2|\nabla W_R|^2-2|Q||W_R|.\end{aligned}$$

In particular, for $\varepsilon>0$ we consider the function $W_R^\varepsilon:=\sqrt{|W_R|^2+\varepsilon^2}$ and we get that W_R^ε satisfies

$$\Delta_f|W_R^\varepsilon|\geq -|Q|\geq -\|Q\|_X f^{-1},$$

since $Q \in X$. Now observe that the function f^{-1} is a good barrier function since

$$\begin{aligned}\Delta_f f^{-1} &= -f^{-2}\Delta_f f + 2|\nabla f|^2 f^{-3}\\ &= -f^{-1}\left(1 - 2f^{-2}\frac{|x|^2}{4}\right)\\ &\leq -Cf^{-1},\end{aligned}$$

for some positive constant C. Indeed, one can check that for $n \geq 2$ we have

$$\inf_{x\in\mathbb{R}^n}\left(1 - 2\frac{|x|^2}{4f^2(x)}\right) > 0.$$

Hence, for some sufficiently large constant A depending linearly on $\|Q\|_X$ and which is independent of ε,

$$\Delta_f\left(|W_R^\varepsilon| - Af^{-1}\right) > 0, \quad \text{on} \quad B(0,R).$$

The maximum principle forces the function $|W_R^\varepsilon| - Af^{-1}$ to attain its maximum at the boundary $\partial B(0,R)$, hence

$$\begin{aligned}\max_{B(0,R)} |W_R^\varepsilon| - Af^{-1} &= \max_{\partial B(0,R)} |W_R^\varepsilon| - Af^{-1}\\ &\leq \varepsilon.\end{aligned}$$

The result follows by letting ε go to 0. $\square$

By Proposition 3.5, one can extract a subsequence $(W_{R_k})_{k\geq 0}$ for a sequence of radii $(R_k)_{k\geq 0}$ going to $+\infty$, that converges in the $C^{2,\beta}_{\mathrm{loc}}$ topology, for any $\beta \in (0,1)$, to a vector field $W : \mathbb{R}^n \to \mathbb{R}^m$ satisfying:

$$\Delta_f W - K\chi'(d_N^2(U_0^\sigma))d_{U_0^\sigma}\bar{d}_N(W)\nabla\bar{d}_N(U_0^\sigma) = Q(U_0^\sigma, V), \quad \text{on } \mathbb{R}^n, \tag{37}$$

$$\sup_{\mathbb{R}^n} f|W| \leq C(n,m,K)\|Q(U_0^\sigma, V)\|_X. \tag{38}$$

This solution W is unique among regular solutions that converge to 0 at infinity by the maximum principle. Therefore, the map F_K^σ makes sense provided the gradient of $F_K^\sigma(V)$ decays like $f^{-3/2}$, i.e., $F_K^\sigma(V) \in X$. This is the content of the next proposition.

Proposition 3.6. – *The solution $F_K^\sigma(V)$ satisfies the weighted a priori C^1 bound*

$$\sup_{\mathbb{R}^n} f^{3/2}|\nabla F_K^\sigma(V)| \leq C(n,m,K)\|Q(U_0^\sigma, V)\|_X. \tag{39}$$

Proof. – Strictly speaking, the solution $F_K^\sigma(V) =: F(V)$ is in $C^{2,\beta}_{\mathrm{loc}}$ only. Since the bound we want to establish only involves the first derivatives of $F(V)$ and of $Q(U_0^\sigma, V)$, we can assume V and hence $F(V)$ and $Q(U_0^\sigma, V)$ are smooth. Moreover, by interior parabolic Schauder estimates applied to the corresponding time-dependent solution $F(V)(x,t) := F(V)(x/\sqrt{t})$ defined on $\mathbb{R}^n \times \mathbb{R}_+$, one gets the bounds

$$\sup_{x\in\mathbb{R}^n} f(x)\|F(V)\|_{C^{2,\beta}(B(x,1))} \leq C(n,m,\beta,K)\|Q(U_0^\sigma, V)\|_X. \tag{40}$$

We compute the evolution equation (in disguise) of the gradient $\nabla F(V)$:

$$\begin{aligned}\Delta_f \nabla F(V) &= \nabla(\Delta_f F(V)) - \frac{1}{2}\nabla F(V)\\ &= -\frac{1}{2}\nabla F(V) + K\nabla\left(\chi'(d_N^2(U_0^\sigma))d_{U_0^\sigma}\bar{d}_N(F(V))\nabla\bar{d}_N(U_0^\sigma)\right) + \nabla\left(Q(U_0^\sigma, V)\right),\end{aligned}$$

which implies, by (38) and the fact that $Q(U_0^\sigma, V) \in X$,

$$\begin{aligned}\Delta_f|\nabla F(V)|^2 &\geq 2|\nabla^2 F(V)|^2 - |\nabla F(V)|^2 + 2K\chi'(d_N^2(U_0^\sigma))\left[d_{U_0^\sigma}\bar{d}_N(\nabla F(V))\right]^2\\ &\quad - C\|Q(U_0^\sigma, V)\|_X f^{-3/2}|\nabla F(V)|\\ &\geq 2|\nabla^2 F(V)|^2 - |\nabla F(V)|^2 - C\|Q(U_0^\sigma, V)\|_X f^{-3/2}|\nabla F(V)|.\end{aligned} \tag{41}$$

Indeed,

$$\begin{aligned}&\left\langle\nabla\left(\chi'(d_N^2(U_0^\sigma))d_{U_0^\sigma}\bar{d}_N(F(V))\nabla\bar{d}_N(U_0^\sigma)\right), \nabla F(V)\right\rangle\\ &\qquad = \chi''(d_N^2(U_0^\sigma))\langle\nabla\bar{d}_N(U_0^\sigma), F(V)\rangle\cdot\langle\nabla\bar{d}_N(U_0^\sigma), \nabla_{\nabla(d_N^2(U_0^\sigma))}F(V)\rangle\\ &\qquad\quad + \chi'(d_N^2(U_0^\sigma))\left(\nabla U_0^\sigma * F(V) + \left[d_{U_0^\sigma}\bar{d}_N(F(V))\right]^2\right).\end{aligned}$$

Therefore, (38) and [(19), Lemma 3.1] implies the estimate (41).

Now, recall that $\Delta_f f^2 = 2f^2 + 2|\nabla f|^2 \geq 2f^2$ and multiply the differential inequality (41) by f^2 to absorb the term $-|\nabla F(V)|^2$ as follows

$$\begin{aligned}\Delta_f\left(f^2|\nabla F(V)|^2\right) &\geq 2f^2|\nabla^2 F(V)|^2 - 8f^{3/2}|\nabla^2 F(V)||\nabla F(V)|\\ &\quad + f^2|\nabla F(V)|^2 - C(K, \|Q(U_0^\sigma, V)\|_X)f^{-1/2}(f|\nabla F(V)|)\\ &\geq \left(1 - Cf^{-1}\right)f^2|\nabla F(V)|^2 - C\|Q(U_0^\sigma, V)\|_X f^{-1}\\ &\geq f^2|\nabla F(V)|^2 - C\|Q(U_0^\sigma, V)\|_X f^{-1},\end{aligned}$$

where we used Young inequality together with (40) to get the last inequality and C denotes a positive constant independent of V and $F(V)$ that may vary from line to line.

Observe that $\Delta_f \ln f = 1 - |\nabla \ln f|^2 \leq 1$ on $\mathbb{R}^n$.

By considering a function of the form $k^{-1}\ln f$ where k is a positive integer, one gets:

$$\begin{aligned}\Delta_f\left(f^2|\nabla F(V)|^2 - k^{-1}\ln f - Af^{-1}\right) &\geq f^2|\nabla F(V)|^2 - C\|Q(U_0^\sigma, V)\|_X f^{-1} - k^{-1} + ACf^{-1}\\ &\geq f^2|\nabla F(V)|^2 - k^{-1}\ln f - Af^{-1} - k^{-1}\end{aligned}$$

for any positive constant A larger than $C\|Q(U_0^\sigma, V)\|_X$, for some positive constant C uniform in $\sigma \in [0,1]$. Now, as we know that the function $f^2|\nabla F(V)|^2$ is bounded, the function $f^2|\nabla F(V)|^2 - k^{-1}\ln f - Af^{-1}$ goes to $-\infty$ as x goes to $+\infty$. Therefore, it attains its maximum. The maximum principle applied to the previous differential inequality implies that

$$f^2|\nabla F(V)|^2 - k^{-1}\ln f - Af^{-1} \leq k^{-1}, \quad \text{on } \mathbb{R}^n.$$

As A is independent of k, we can let k go to $+\infty$ and get the expected result. □

We now claim that the map F_K is a compact operator, the proof of its continuity being analogous:

Proposition 3.7. – *The map $F_K : X \times [0,1] \to X$ is a continuous and compact map.*

Proof. – We only prove the compactness of the map F_K^σ where $\sigma \in [0,1]$. Let $(V_i)_{i\geq 0}$ be a bounded sequence in X. According to (40), the sequence $(F_K^\sigma(V_i))_i$ subconverges in the $C^{2,\beta}_{\mathrm{loc}}$ topology to a map that belongs to X. In order to prove that the convergence holds in X, it is sufficient to prove that for any $V \in X$ and $i = 0, 1$ the following estimates hold:

$$\sup_{\mathbb{R}^n} f^{2+i/2}|\nabla^i(F_K^\sigma(V) - F_K^\sigma(0))| \leq C(K,n,m,\|V\|_X).$$

In the proof of these estimates, for the sake of clarity, we omit the dependence of F_K^σ and U_0^σ on K and σ.

Recall that $F_K^\sigma(0) =: F(0)$ is the unique solution in X of:

$$\begin{aligned}\Delta_f F(0) - K\chi'(d_N^2(U_0))d_{U_0}\bar{d}_N(F(0))\nabla\bar{d}_N(U_0) &= Q(U_0,0)\\ &= K\chi'(d_N^2(U_0))\nabla\left(\frac{d_N^2}{2}\right)(U_0).\end{aligned}$$

Therefore, $G(V) := F(V) - F(0) \in X$ is a solution of

$$\Delta_f G(V) - K\chi'(d_N^2(U_0^\sigma))d_{U_0^\sigma}\bar{d}_N(G(V))\nabla\bar{d}_N(U_0^\sigma) = Q(U_0,V) - Q(U_0,0).$$

Now, a Taylor expansion of degree 2 as in (34), using the very definition (32) of $Q(U_0,V)$, shows that the differential of Q with respect to the variable V satisfies:

$$\begin{aligned}D_2Q(U_0,0)(V) &= K\chi'(d_N^2(U_0^\sigma))\bar{d}_N(U_0)\nabla^2\bar{d}_N(U_0)(V)\\ &\quad + K\chi''(d_N^2(U_0^\sigma))d_{U_0^\sigma}d_N^2(V)\nabla\left(\frac{d_N^2}{2}\right)(U_0^\sigma).\end{aligned} \tag{42}$$

Notice that the second term on the righthand side of (42) is compactly supported in a ball of $\mathbb{R}^n$ whose radius is independent of $\sigma \in [0,1]$ by the definition of the function χ and Lemma 3.1.

In particular, by Taylor's Theorem together with the estimates of Lemma 3.1,

$$\sup_{\mathbb{R}^n} f^{2+i/2}|\nabla^i(Q(U_0,V) - Q(U_0,0))| \leq C(K,n,m,\|V\|_X), \quad i = 0,1.$$

As in the proof of Proposition 3.5, one can use a barrier function of the form f^{-2} in order to prove that $G(V)$ decays like f^{-2} uniformly with respect to σ and V in a fixed ball in X. Similarly to the proof of Proposition 3.6, one proves the expected decay on the gradient of $G(V)$ at infinity. $\square$

3.5. Well-posedness of the homogeneous Chen-Struwe equation for small initial data

In this subsection, we assume that σ is close to 1 and hence U_0^σ is close to a constant map in the sense that

$$\|U_0^\sigma - P\|_{L^\infty} + \sup_{x\in\mathbb{R}^n}(1+|x|)|\nabla U_0^\sigma(x)| \leq C(u_0^\sigma),$$

where $P \in \mathbb{S}^{m-1}$ and where $\lim_{\sigma\to 1} C(u_0^\sigma) = 0$. We show that for every ε small enough the map $I - F_K^\sigma : B_X(0,\varepsilon) \to B_X(0,\varepsilon)$ has Leray-Schauder degree one at the origin. In other words, we show that F_K^σ has a unique fixed point in $B_X(0,\varepsilon)$ if ε and $1-\sigma$ are sufficiently small.

In order to see this, we note that Lemma 3.4 together with Propositions 3.5 and 3.6 shows that for all $V \in B_X(0,\varepsilon)$, we have

$$\|F_K^{\sigma}(V)\|_X \leq C(u_0^{\sigma}) + C\varepsilon^2$$

and therefore we have indeed that F_K^{σ} maps $B_X(0,\varepsilon)$ into itself, provided that $1-\sigma$ and ε are chosen small enough.

In order to show that F_K^{σ} is also a contraction on $B_X(0,\varepsilon)$, we invoke Lemma 3.4 together with Propositions 3.5 and 3.6 again to prove that

$$\|F_K^{\sigma}(V_1) - F_K^{\sigma}(V_2)\|_X \leq C\varepsilon\|V_1 - V_2\|_X$$

and hence this shows the contraction property if we choose ε small enough. Altogether, this implies the desired result about the Leray-Schauder degree.

4. An ε-regularity theorem for Chen-Struwe expanding solutions

We emphasize on the fact that this section does not require the target manifold (N, g) to be embedded as a hypersurface of Euclidean space.

4.1. A Bochner formula

It is a straightforward adaptation from [3] to get the following crucial Bochner formula:

PROPOSITION 4.1 (Bochner formula). – *Let $u : \mathbb{R}^n \times (0,T) \to \mathbb{R}^m$ be a smooth solution to the Homogeneous Chen-Struwe flow:*

$$\partial_t u - \Delta u + \frac{K}{t}\chi'\left(d_N^2(u)\right)\nabla\left(\frac{d_N^2}{2}\right)(u) = 0. \tag{43}$$

Then, the pointwise energy $e_K(u) := \frac{1}{2}\left(|\nabla u|^2 + \frac{K}{t}\chi\left(d_N^2(u)\right)\right)$ satisfies the inequality

$$(\partial_t - \Delta)e_K(u) \leq Ce_K(u)^2,$$

on $\mathbb{R}^n \times (0,T)$, for some positive constant C independent of K.

Proof. – We first remark that if $d_N(u) \leq 2\cdot\delta_N$ then,

$$\left|\nabla\left(\frac{d_N^2}{2}\right)(u)\right|^2 = d_N^2(u).$$

Now,

$$\frac{1}{2}(\partial_t - \Delta)\chi\left(d_N^2(u)\right) = -\frac{K}{t}\chi'^2 d_N^2(u) - \nabla\left(\chi'\nabla\left(\frac{d_N^2}{2}\right)(u)\right)\cdot\nabla u,$$
$$\frac{1}{2}(\partial_t - \Delta)|\nabla u|^2 = -\frac{1}{t}\nabla\left(K\chi'\nabla\left(\frac{d_N^2}{2}\right)(u)\right)\cdot\nabla u - |\nabla^2 u|^2.$$

These computations lead to the estimate

$$\begin{aligned}(\partial_t-\Delta)e_K(u)+|\nabla^2u|^2+\frac{K^2}{t^2}\chi'^2d_N^2(u)&=-\frac{2}{t}\nabla\left(K\chi'\nabla\left(\frac{d_N^2}{2}\right)(u)\right)\cdot\nabla u\\&\quad-\frac{K}{t^2}\chi\left(d_N^2(u)\right)\\&\le-\frac{2}{t}\nabla\left(K\chi'\nabla\left(\frac{d_N^2}{2}\right)(u)\right)\cdot\nabla u,\end{aligned}$$

which implies the expected result if $d_N(u)>2\cdot\delta_N$. If $d_N(u)\le 2\cdot\delta_N$, by using the fact that χ' is nonnegative we obtain

$$(\partial_t-\Delta)e_K(u)+|\nabla^2u|^2+\frac{K^2}{t^2}\chi'^2d_N^2(u)\le\frac{1}{2t^2}K^2d_N^2(u)+c|\nabla u|^4$$

for some uniform positive constant c independent of $K>0$.

Therefore, in all cases, this gives the expected estimate. □

4.2. An energy inequality and a local entropy monotonicity formula

We define the L^2_{loc} norm at scale $R>0$ of a map $u:\mathbb{R}^n\to\mathbb{R}^m$ in $H^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^m)$ as follows:

$$\|\nabla u\|^2_{L^2_{\mathrm{loc},R}}:=\sup_{x_0\in\mathbb{R}^n}\fint_{B(x_0,R)}|\nabla u|^2(y)dy.$$

It follows easily that

$$c_n^{-1}\left(\frac{R_1}{R_2}\right)^n\|\nabla u\|^2_{L^2_{\mathrm{loc},R_1}}\le\|\nabla u\|^2_{L^2_{\mathrm{loc},R_2}}\le c_n\left(\frac{R_2}{R_1}\right)^n\|\nabla u\|^2_{L^2_{\mathrm{loc},R_1}},\quad 0<R_1\le R_2. \tag{44}$$

Moreover, we use the shorthand notation $\|\nabla u\|^2_{L^2_{\mathrm{loc}}}$ for $\|\nabla u\|^2_{L^2_{\mathrm{loc},1}}$.

Finally, we define the rescaled energy with parameter $K>0$ for a solution u to the Homogeneous Chen-Struwe flow with parameter $K>0$ by

$$\begin{aligned}E_{K,x_0}(u(t))&:=\fint_{B(x_0,1)}\left(\frac{|\nabla u|^2}{2}+\frac{K}{2t}\chi\left(d_N^2(u)\right)\right)dx,\quad t>0,\\E_{K,\mathrm{loc}}(u(t))&:=\sup_{x_0\in\mathbb{R}^n}E_{K,x_0}(u(t)),\quad t>0.\end{aligned}$$

THEOREM 4.2. – *Let $u_0:\mathbb{R}^n\to(N,g)\subset\mathbb{R}^m$ be in $H^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^m)$. Let $(u(t))_{t>0}$ be a smooth solution to the Homogeneous Chen-Struwe flow with parameter $K>0$ coming out of u_0 such that $(E_{K,x_0}(u(t)))_{t>0}$ is continuous at $t=0$ for every $x_0\in\mathbb{R}^n$. Then,*

$$E_{K,x_0}(u(t))\le\left(1+C\left(n,m,\|\nabla u_0\|_{L^2_{\mathrm{loc}}},t\right)\right)\|\nabla u_0\|^2_{L^2(B(x_0,1))},\quad\forall x_0\in\mathbb{R}^n, \tag{45}$$

$$E_{K,\mathrm{loc}}(u(t))\le\left(1+c_n\left(e^{c_nt}-1\right)\right)\|\nabla u_0\|^2_{L^2_{\mathrm{loc}}},\quad t>0, \tag{46}$$

where $\lim_{t\to0}C\left(n,m,\|\nabla u_0\|_{L^2_{\mathrm{loc}}},t\right)=0$.

Moreover, the following estimate holds (it is uniform in K)

$$\sup_{x_0\in\mathbb{R}^n}\int_{B(x_0,1)\times(0,t)}\left(|\partial_su|^2+\frac{K}{2s^2}\chi\left(d_N^2(u)\right)\right)dxds\le\left(1+c_n\left(e^{c_nt}-1\right)\right)\|\nabla u_0\|^2_{L^2_{\mathrm{loc}}},\quad t>0.$$

In particular, if u_0 is 0-homogeneous and if u is an expanding solution coming out of u_0 smoothly, then:

$$E_{K,\mathrm{loc}}(u) \leq (1 + c_n\,(e^{c_n} - 1))\,\|\nabla u_0\|^2_{L^2_{\mathrm{loc}}}. \tag{47}$$

Proof. – We proceed analogously to what is done to establish an energy estimate in this setting. We multiply the Homogeneous Chen-Struwe flow equation by $\phi^2_{x_0}\partial_t u$ where $\phi_{x_0} : \mathbb{R}^n \to \mathbb{R}_+$ is a smooth function with compact support in $B(x_0, 2)$ which equals 1 on $B(x_0, 1)$ and whose gradient is less than c, and then we integrate by parts to get

$$\begin{aligned}
\int_{\mathbb{R}^n} |\partial_t u|^2 \phi^2_{x_0} dx &= \int_{\mathbb{R}^n} <\Delta u, \partial_t u> \phi^2_{x_0} dx - \frac{K}{2t}\int_{\mathbb{R}^n} \partial_t\left(\chi\left(d^2_N(u)\right)\right)\phi^2_{x_0} dx \\
&= -\partial_t \int_{\mathbb{R}^n} e_K(u)\phi^2_{x_0} dx + 2<\nabla_{\nabla\phi_{x_0}} u, \phi_{x_0}\partial_t u>_{L^2} \\
&\quad -\frac{K}{2t^2}\int_{\mathbb{R}^n} \chi\left(d^2_N(u)\right)\phi^2_{x_0} dx \\
&\leq -\partial_t \int_{\mathbb{R}^n} e_K(u)\phi^2_{x_0} dx + \frac{1}{2}\|\phi_{x_0}\partial_t u\|^2_{L^2} \\
&\quad +2\|\nabla_{\nabla\phi_{x_0}} u\|^2_{L^2} - \frac{K}{2t^2}\int_{\mathbb{R}^n} \chi\left(d^2_N(u)\right)\phi^2_{x_0} dx.
\end{aligned}$$

By integrating with respect to time we then obtain

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^n\times(0,t)} &\left(|\partial_s u|^2 + \frac{K}{s^2}\chi\left(d^2_N(u)\right)\right)\phi^2_{x_0} dx ds + E_{K,x_0}(u(t)) \\
&\leq \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_0|^2\phi^2_{x_0} dx + 2c^2\int_0^t \|\nabla u(s)\|^2_{L^2_{\mathrm{loc},2}} ds,
\end{aligned}$$

where we used Young's inequality together with the L^2_{loc} continuity of $(u(t))_{t>0}$ at $t = 0$. Therefore, by remark (44), one gets in particular:

$$⨍_{B(x_0,1)} |\nabla u(t)|^2 dx \leq \|\nabla u_0\|^2_{L^2_{\mathrm{loc},1}} + c_n\int_0^t \|\nabla u(s)\|^2_{L^2_{\mathrm{loc},1}} dt,$$

which implies:

$$\|\nabla u(t)\|^2_{L^2_{\mathrm{loc}}} \leq \|\nabla u_0\|^2_{L^2_{\mathrm{loc}}} + c_n\int_0^t \|\nabla u(s)\|^2_{L^2_{\mathrm{loc}}} ds,$$

where c_n is a positive constant that can vary from line to line depending on the dimension n only.

The result now follows from Gronwall's inequality. □

In the following we let $\chi_R : \mathbb{R}^n \to [0, 1]$ be a cut-off function such that $\chi_R \equiv 1$ outside $B(0, R)$, $\chi_R \equiv 0$ in $B(0, R/2)$ and whose gradient satisfies $|\nabla\chi_R| = O(R^{-1})$. Then one has the following localized version of Theorem 4.2 at infinity.

Proposition 4.3. – *Let $u_0 : \mathbb{R}^n \to N$ be in $H^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$. Let $(u(t))_{t>0}$ be a smooth solution to the Homogeneous Chen-Struwe flow coming out of u_0 such that $(E_{K,x_0}(u(t))_{t>0}$ is continuous at $t = 0$ for every $x_0 \in \mathbb{R}^n$. Then, if $R > 0$,*

$$\|(\nabla u(t))\chi_R\|_{L^2_{\mathrm{loc}}} \leq C(n,T)\left(\|(\nabla u_0)\chi_R\|_{L^2_{\mathrm{loc}}} + \frac{\|\nabla u_0\|_{L^2_{\mathrm{loc}}}}{R}\right), \quad 0 < t \leq T.$$

In particular, if u_0 is 0-homogeneous and if u is an expanding solution coming out of u_0 smoothly, then we have

$$\|(\nabla u)\chi_R\|_{L^2_{\mathrm{loc}}} \leq C(n)\left(\|(\nabla u_0)\chi_R\|_{L^2_{\mathrm{loc}}} + \frac{\|\nabla u_0\|_{L^2_{\mathrm{loc}}}}{R}\right). \tag{48}$$

Proof. – The proof follows along the lines of the proof of Theorem 4.2. Let ϕ_{x_0} be the cut-off function defined as previously and let us multiply the Homogeneous Chen-Struwe flow equation by $\phi^2_{x_0}\chi^2_R\partial_t u$. Then we integrate by parts in space and we integrate with respect to time. We get

$$\begin{aligned}\|(\nabla u(t))\chi_R\|^2_{L^2_{\mathrm{loc}}} &\leq \|(\nabla u_0)\chi_R\|^2_{L^2_{\mathrm{loc}}} + c_n\int_0^t \|(\nabla u(s))\chi_R\|^2_{L^2_{\mathrm{loc}}}\,ds + \frac{c_n}{R^2}\int_0^t \|\nabla u(s)\|^2_{L^2_{\mathrm{loc}}}\,ds\\ &\leq \|(\nabla u_0)\chi_R\|^2_{L^2_{\mathrm{loc}}} + c_n\int_0^t \|(\nabla u(s))\chi_R\|^2_{L^2_{\mathrm{loc}}}\,ds + \frac{c_n}{R^2}\left(e^{c_n t} - 1\right)\|\nabla u_0\|^2_{L^2_{\mathrm{loc}}},\end{aligned}$$

where we used the estimate (46) in the last line. A straightforward application of the Gronwall inequality leads to the expected result. □

In order to get a local entropy monotonicity formula, we need to localize the arguments in [3] since in our case the energy is infinite. For this purpose, let $z_0 := (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$ and let $\phi_{x_0} : \mathbb{R}^n \to \mathbb{R}_+$ be a smooth function with compact support in $B(x_0, 2)$ which equals 1 on $B(x_0, 1)$ and whose gradient is less than c. For $R \in (0, 2^{-1}\cdot\sqrt{t_0})$, define as in [Chap. 7, [13]]:

$$\begin{aligned}\Phi(u, z_0, R) &:= R^2\int_{\mathbb{R}^n} e_K(u)G_{z_0}\phi^2_{x_0}dx\ |_{t_0-R^2},\\ \Psi(u, z_0, R) &:= \int_{t_0-4R^2}^{t_0-R^2} e_K(u)G_{z_0}\phi^2_{x_0}dxdt,\end{aligned}$$

where

$$G_{z_0}(x,t) := \frac{1}{(4\pi|t-t_0|)^{\frac{n}{2}}}\exp\left(-\frac{|x-x_0|^2}{4|t-t_0|}\right), \quad x \in \mathbb{R}^n, \quad t < t_0$$

denotes the backward heat kernel on $\mathbb{R}^n$. With the notations from the previous sections: $G_{z_0}(x,t) = k_{t_0-t}(x, x_0)$. We start with a Pohozaev identity like:

Proposition 4.4 (Pohozaev identity). – *Let $u : \mathbb{R}^n \times (0,T) \to \mathbb{R}^m$ be a smooth solution to the Homogeneous Chen-Struwe flow (with parameter $K > 0$). Then, for any C^1 vector field $\zeta : \mathbb{R}^n \times (0,T) \to \mathbb{R}^m$ compactly supported in space,*

$$\begin{aligned}<\partial_t u, \nabla_\zeta u>_{L^2(\mathbb{R}^n\times[t_1,t_2])} &= <e_K(u), \operatorname{div}\zeta>_{L^2(\mathbb{R}^n\times[t_1,t_2])}\\ &\quad - \frac{1}{2}<\mathcal{L}_\zeta(\mathrm{eucl}), \nabla u\otimes\nabla u>_{L^2(\mathbb{R}^n\times[t_1,t_2])},\end{aligned}$$

where $\mathcal{L}_\zeta(\mathrm{eucl})$ denotes the Lie derivative of the Euclidean metric along the vector field ζ:

$$\frac{1}{2}<\mathcal{L}_\zeta(\mathrm{eucl}), \nabla u \otimes \nabla u> := \nabla_i \zeta_j \nabla_i u_k \nabla_j u_k.$$

And, for any C^1 function $\theta : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ compactly supported in space, and $0 < t_1 < t_2 < T$,

$$\int_{L^2(\mathbb{R}^n\times[t_1,t_2])} \left(|\partial_t u|^2 + \frac{K}{2t^2}\chi\left(d_N^2(u)\right)\right)\theta dxdt + \left[\int_{\mathbb{R}^n} e_K(u)\theta dx\right]_{t_1}^{t_2}$$
$$= \int_{\mathbb{R}^n\times[t_1,t_2]} e_K(u)\partial_t\theta - <\nabla_{\nabla\theta}u, \partial_t u>dxdt.$$

Proof. – Define $u_\tau(x,t) := u(x + \tau\zeta(x,t), t + \tau\theta(x,t))$ where $\zeta(\cdot,t)$ is a smooth vector field compactly supported in space for τ small and $\theta(\cdot,t)$ is a smooth function compactly supported in space. Then, $\partial_\tau u_\tau|_{\tau=0} = \nabla_\zeta u + \theta\partial_t u$. Then, on one hand:

$$\int_{\mathbb{R}^n\times[t_1,t_2]} <\partial_t u, \nabla_\zeta u>dxdt = \int_{\mathbb{R}^n\times[t_1,t_2]} <\Delta u, \nabla_\zeta u>dxdt$$
$$- \int_{\mathbb{R}^n\times[t_1,t_2]} <\frac{K}{t}\chi'\left(d_N^2(u)\right)\nabla\left(\frac{d_N^2}{2}\right)(u), \nabla_\zeta u>dxdt.$$

Now, by integrating by parts:

$$\int_{\mathbb{R}^n\times[t_1,t_2]} <\Delta u, \nabla_\zeta u>dxdt = \frac{1}{2}\int_{\mathbb{R}^n\times[t_1,t_2]} |\nabla u|^2 \operatorname{div}\zeta - \mathcal{L}_\zeta(\nabla u, \nabla u)dxdt$$

and, similarly,

$$-\int_{\mathbb{R}^n\times[t_1,t_2]} <\frac{K}{t}\chi'\left(d_N^2(u)\right)\nabla\left(\frac{d_N^2}{2}\right)(u), \nabla_\zeta u>dxdt = \frac{1}{2}\int_{\mathbb{R}^n\times[t_1,t_2]} \frac{K}{t}\chi\left(d_N^2(u)\right)\operatorname{div}\zeta dxdt.$$

Therefore,

$$<\partial_t u, \nabla_\zeta u>_{L^2(\mathbb{R}^n\times[t_1,t_2])} = <e_K(u), \operatorname{div}\zeta>_{L^2(\mathbb{R}^n\times[t_1,t_2])} - \frac{1}{2}<\mathcal{L}_\zeta(\mathrm{eucl}), \nabla u\otimes\nabla u>_{L^2(\mathbb{R}^n\times[t_1,t_2])}.$$

On the other hand:

$$\int_{\mathbb{R}^n\times[t_1,t_2]} <\partial_t u, \theta\partial_t u>dxdt = \int_{\mathbb{R}^n\times[t_1,t_2]} <\Delta u, \theta\partial_t u>dxdt$$
$$- \int_{\mathbb{R}^n\times[t_1,t_2]} <\frac{K}{t}\chi'\left(d_N^2(u)\right)\nabla\left(\frac{d_N^2}{2}\right)(u), \theta\partial_t u>dxdt$$
$$= -\int_{\mathbb{R}^n\times[t_1,t_2]} \partial_t e_K(u)\theta dxdt - \int_{\mathbb{R}^n\times[t_1,t_2]} <\nabla_{\nabla\theta}u, \partial_t u>dxdt$$
$$- \int_{\mathbb{R}^n\times[t_1,t_2]} \frac{K}{2t^2}\chi\left(d_N^2(u)\right)\theta dxdt$$
$$= -\left[\int_{\mathbb{R}^n} e_K(u)\theta dx\right]_{t_1}^{t_2} - \int_{\mathbb{R}^n\times[t_1,t_2]} <\nabla_{\nabla\theta}u, \partial_t u>dxdt$$
$$+ \int_{\mathbb{R}^n\times[t_1,t_2]} e_K(u)\partial_t\theta dxdt - \int_{\mathbb{R}^n\times[t_1,t_2]} \frac{K}{2t^2}\chi\left(d_N^2(u)\right)\theta dxdt.$$

□

We are now in a position to prove a local entropy monotonicity formula.

PROPOSITION 4.5 (A local entropy monotonicity formula). – *Let $u_0 : \mathbb{R}^n \to (N, g) \subset \mathbb{R}^m$ be in $H^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$. Let $(u(t))_{t>0}$ be a smooth solution to the Homogeneous Chen-Struwe flow coming out of u_0 such that $(E_{K,x_0}(u(t)))_{t>0}$ is continuous at $t = 0$ for every $x_0 \in \mathbb{R}^n$. Then, for any $z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$ and $0 < R \le R_0 < \sqrt{t_0} \le 1$,*

$$\Phi(u, z_0, R) \le \Phi(u, z_0, R_0) + c(R_0 - R)\|\nabla u_0\|^2_{L^2(B(x_0,2))},$$
$$\Psi(u, z_0, R) \le \Psi(u, z_0, R_0) + c(R_0 - R)\|\nabla u_0\|^2_{L^2(B(x_0,2))}.$$

Proof. – Choose $\zeta(x,t) := G_{z_0}(x,t)\phi^2_{x_0}(x) \cdot (x - x_0)$ for $0 < t < t_0$ and $x \in \mathbb{R}^n$. Then,

$$\begin{aligned}
\operatorname{div} \zeta &= \phi^2_{x_0} \operatorname{div}(G_{z_0} \cdot (x - x_0)) + <\nabla \phi^2_{x_0}, G_{z_0} \cdot (x - x_0)> \\
&= \phi^2_{x_0} \left(-\frac{|x - x_0|^2}{2|t - t_0|} + n \right) G_{z_0} + <\nabla \phi^2_{x_0}, G_{z_0} \cdot (x - x_0)>, \\
\nabla_i (G_{z_0}(x - x_0))_j &= \left(-\frac{(x - x_0)_i (x - x_0)_j}{2|t - t_0|} + \delta_{ij} \right) G_{z_0}, \\
\frac{1}{2}<\mathcal{L}_\zeta(\mathrm{eucl}), \nabla u \otimes \nabla u> &:= \left(-\frac{|\nabla_{x-x_0} u|^2}{2|t - t_0|} + |\nabla u|^2 \right) G_{z_0} \phi^2_{x_0} + <\nabla_{\nabla \phi^2_{x_0}} u, \nabla_{x-x_0} u> G_{z_0}.
\end{aligned}$$

Therefore, by applying Proposition 4.4, one gets

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^n \times [t_1,t_2]} \left[<\partial_t u, \nabla_{x-x_0} u> - \left(-\frac{|x - x_0|^2}{2|t - t_0|} + n \right) e_K(u) - \frac{|\nabla_{x-x_0} u|^2}{2|t - t_0|} + |\nabla u|^2 \right] G_{z_0} \phi^2_{x_0} dx dt \\
&= \frac{1}{2} \int_{\mathbb{R}^n \times [t_1,t_2]} \left(<\nabla \phi^2_{x_0}, x - x_0> e_K(u) - <\nabla_{\nabla \phi^2_{x_0}} u, \nabla_{x-x_0} u> \right) G_{z_0} dx dt.
\end{aligned}$$

Now, by using Proposition 4.4 with $\theta(x,t) := (t_0 - t) G_{z_0} \phi^2_{x_0}$ for $0 < t < t_0$ and $x \in \mathbb{R}^n$, one obtains

$$\begin{aligned}
\partial_t \theta &= \left(\frac{n-2}{2} - \frac{|x - x_0|^2}{4|t - t_0|} \right) G_{z_0} \phi^2_{x_0}, \\
\nabla \theta &= -\frac{x - x_0}{2} G_{z_0} \phi^2_{x_0} + (t_0 - t) G_{z_0} \nabla \phi^2_{x_0},
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}^n \times [t_1,t_2]} |\partial_t u|^2 (t_0 - t) G_{z_0} \phi^2_{x_0} dx dt + \left[(t_0 - t) \int_{\mathbb{R}^n} e_K(u) G_{z_0} \phi^2_{x_0} dx \right]_{t_1}^{t_2} \\
&\quad \le \int_{\mathbb{R}^n \times [t_1,t_2]} \left\{ e_K(u) \left(\frac{n-2}{2} - \frac{|x - x_0|^2}{4|t - t_0|} \right) + \frac{1}{2} <\partial_t u, \nabla_{x-x_0} u> \right\} G_{z_0} \phi^2_{x_0} dx dt \\
&\qquad - \int_{\mathbb{R}^n \times [t_1,t_2]} (t_0 - t) <\partial_t u, \nabla_{\nabla \phi^2_{x_0}} u> G_{z_0} dx dt.
\end{aligned}$$

Subtracting the two previous identities yields

$$\begin{aligned}
&\int_{\mathbb{R}^n\times[t_1,t_2]}(t_0-t)\left|\partial_t u-\nabla_{\frac{x-x_0}{2(t_0-t)}}u\right|^2 G_{z_0}\phi_{x_0}^2\,dx\,dt+\left[(t_0-t)\int_{\mathbb{R}^n}e_K(u)G_{z_0}\phi_{x_0}^2\,dx\right]_{t_1}^{t_2}\\
&\leq-\int_{\mathbb{R}^n\times[t_1,t_2]}\frac{K}{2t}\chi\left(d_N^2(u)\right)G_{z_0}\phi_{x_0}^2\,dx\,dt-\int_{\mathbb{R}^n\times[t_1,t_2]}(t_0-t)<\partial_t u,\nabla_{\nabla\phi_{x_0}^2}u>G_{z_0}\,dx\,dt\\
&\quad-\frac{1}{2}\int_{\mathbb{R}^n\times[t_1,t_2]}\left(<\nabla\phi_{x_0}^2,x-x_0>e_K(u)-<\nabla_{\nabla\phi_{x_0}^2}u,\nabla_{x-x_0}u>\right)G_{z_0}\,dx\,dt\\
&\leq-\frac{1}{2}\int_{\mathbb{R}^n\times[t_1,t_2]}\left((t_0-t)<\nabla_{\nabla\phi_{x_0}^2}u,\partial_t u-\nabla_{\frac{x-x_0}{2(t_0-t)}}u>+<\nabla\phi_{x_0}^2,x-x_0>e_K(u)\right)G_{z_0}\,dx\,dt\\
&\leq\frac{1}{2}\int_{\mathbb{R}^n\times[t_1,t_2]}(t_0-t)\left|\partial_t u-\nabla_{\frac{x-x_0}{2(t_0-t)}}u\right|^2 G_{z_0}\phi_{x_0}^2\,dx\,dt+c\int_{\mathrm{supp}(\nabla\phi_{x_0})\times[t_1,t_2]}e_K(u)\,dx\,dt.
\end{aligned}$$

On $\mathrm{supp}(\nabla\phi_{x_0})$ we have $G_{z_0}(x,t)\leq C$ for all $t\in[0,t_0)$ and therefore the last term can be bounded with the help of Theorem 4.2 as follows

$$\int_{\mathrm{supp}(\nabla\phi_{x_0})\times[t_1,t_2]}e_K(u)\,dx\,dt\leq c\|\nabla u_0\|^2_{L^2(B(x_0,2))}.$$

This in turn implies the expected monotonicity result for Φ. The monotonicity result for Ψ follows from the one for Φ. □

4.3. An ε-regularity theorem

THEOREM 4.6. – *Let $u_0:\mathbb{R}^n\to(N,g)\subset\mathbb{R}^m$ be a 0-homogeneous Lipschitz map. Then there exist a radius $R=R(\|\nabla u_0\|_{L^2_{\mathrm{loc}}},n,m)>0$ and a constant $C=C(\|\nabla u_0\|_{L^2_{\mathrm{loc}}},n,m)>0$ such that if u is a smooth solution of the Homogeneous Chen-Struwe flow with parameter $K>0$ coming out of u_0 and such that $(E_{K,x_0}(u(t)))_{t>0}$ is continuous at $t=0$ for any $x_0\in\mathbb{R}^n$, u satisfies*

$$e_K(u)(x,1)\leq\frac{C}{|x|^2},\quad |x|\geq R.$$

Moreover, there exists a constant $\varepsilon_0>0$ depending on n and N only such that if for some $R\in(0,\min\{\sqrt{t_0},1\})$, $z_0=(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}_+$, u satisfies

$$\Psi(u,z_0,R)<\varepsilon_0,$$

then

$$\sup_{P_{\delta R}(z_0)}e_K(u)\leq C(\delta R)^{-2},$$

for some universal positive constant C and some positive constant δ depending on n, m, $\|\nabla u_0\|_{L^2_{\mathrm{loc}}}$ and $\min\{R,1\}$.

Proof. – The proof is a straightforward adaptation of the corresponding proof in the case of the Chen-Struwe flow. We mention the main steps: see [Lemma 2.4, [3]] for more details.

First of all, let $(x_1,t_1)=:z_1\in\mathbb{R}^n\times\mathbb{R}_+$, $0<R<2^{-1}\sqrt{t_1}$ and let $r_1:=2\delta R$ with $\delta\in(0,1/4)$ where δ will be defined later. Let $r,\sigma\in[0,r_1)$ such that $r+\sigma<r_1$ and

let $z_0 := (x_0, t_0) \in P_r(z_1)$. Thanks to the monotonicity formula from Proposition 4.5, then one shows that for a given positive ε, there exists a positive $\delta(\varepsilon)$ such that

$$\sigma^{-n} \int_{P_\sigma(z_0)} e_K(u) dx dt \leq c\Psi(u, z_1, R) + c((R-\sigma)+\varepsilon)\|\nabla u_0\|^2_{L^2(B(x_1,2))}. \tag{49}$$

Now, by smoothness of the solution, there exist $\sigma_0 \in [0, r_1)$ and $(x_0, t_0) \in \overline{P_{\sigma_0}(z_1)}$ such that

$$(r_1 - \sigma_0)^2 e_K(u)(x_0, t_0) = \max_{0 \leq \sigma \leq r_1} (r_1 - \sigma)^2 \sup_{P_\sigma(z_1)} e_K(u).$$

If one defines $\rho_0 := \frac{1}{2}(r_1 - \sigma_0)$, $r_0 := \sqrt{e_0}\rho_0$, and

$$v(x,t) := u\left(\frac{x}{\sqrt{e_0}} + x_0, \frac{t}{e_0} + t_0\right), \quad (x,t) \in P_{r_0}(0,0),$$

then v satisfies

$$\begin{aligned} (\partial_t - \Delta)v &= -\frac{K}{e_0 t_0 + t}\chi'\left(d_N^2(v)\right)\nabla\left(\frac{d_N^2}{2}\right)(v) = 0,\\ e_K(v)(0,0) &= 1, \qquad \sup_{P_{r_0}(0,0)} e_K(v) \leq 4. \end{aligned}$$

By Proposition 4.1 we obtain

$$(\partial_t - \Delta)e_K(v) \leq 4Ce_K(v), \quad \text{on } P_{r_0}(0,0)$$

and Moser's Harnack inequality together with (49) shows that $r_0 \leq 1$ if $\Psi(u, z_1, R)$ is small enough (independently of K).

The final step consists in applying Moser's Harnack inequality again to v in order to get

$$\begin{aligned} \max_{0\leq\sigma\leq r_1}(r_1-\sigma)^2 \sup_{P_\sigma} e_K(u) &\leq 4\rho_0^2 e_0 = 4r_0^2 \\ &\leq c\rho_0^{-n}\int_{P_{\rho_0}(x_0,t_0)} e_K(u) dx dt \\ &\leq c\Psi(u, z_1, R) + c(R+\varepsilon)\|\nabla u_0\|^2_{L^2(B(x_1,2))}. \end{aligned}$$

The result follows by noticing that if u_0 is Lipschitz then by the chain rule

$$\|\nabla u_0\|_{L^2(B(x_1,2))} \leq \frac{C}{1+|x_1|}.$$

In particular, thanks to Proposition 4.3, if x_1 is sufficiently far from the origin, by choosing $t_1 := 1$, $R := 1/4$, both $\Psi(u, z_1, 1/4)$ and $\|\nabla u_0\|_{L^2(B(x_1,2))}$ can be made arbitrarily small, independently of K.

The second statement can be proved similarly. □

5. A priori estimates for Chen-Struwe expanding solutions

5.1. C^0 a priori estimates

We start by establishing an a priori C^0 bound for Chen-Struwe expanding solutions U_K^σ uniform in $K > 0$ and $\sigma \in [0, 1]$.

PROPOSITION 5.1. – *There is a positive constant M uniform in $\sigma \in [0, 1]$ and $K > 0$ such that if $V \in X$ satisfies $F_K^\sigma(V) = V$, then $\|V\|_{C^0} \leq M$.*

Proof. – As $F_K^\sigma(V) = V$ we know that $U^\sigma := U_0^\sigma + V$ solves the static Homogeneous Chen-Struwe flow

$$\Delta_f U^\sigma = K\chi'\left(d_N^2(U^\sigma)\right)\nabla\left(\frac{d_N^2}{2}\right)(U^\sigma).$$

In particular,

$$\Delta_f |U^\sigma|^2 \geq 2|\nabla U^\sigma|^2 + 2<K\chi'\left(d_N^2(U^\sigma)\right)\nabla\left(\frac{d_N^2}{2}\right)(U^\sigma), U^\sigma>.$$

Next we fix a radius $R > 0$ and consider $\max_{B(0,R)} |U^\sigma|$. If this maximum is attained at an interior point x_R of $B(0, R)$, then we consider two cases.

Either $d_N(U^\sigma(x_R)) \leq 2 \cdot \delta_N$ which implies that $\max_{B(0,R)} |U^\sigma|$ is uniformly bounded by the triangular inequality.

Or $d_N(U^\sigma(x_R)) > 2 \cdot \delta_N$ and this implies by the strong maximum principle applied to the previous differential inequality that $|U^\sigma|$ is constant and that $\nabla U^\sigma = 0$ on a neighborhood of x_R. Therefore, U^σ is constant on a neighborhood of x_R. By connectedness, U^σ is constant on $B(0, R)$. As U^σ converges to u_0^σ at infinity, the term $\sup_{\partial B(0,R)} \bar{d}_N(U^\sigma)$ goes to 0 as R goes to $+\infty$. Consequently, this case is impossible if R is large enough.

This discussion ends the proof of the C^0 estimate. Moreover, this last fact also yields the desired estimate if the maximum is attained on the boundary. □

By interior parabolic Schauder estimates, one has the following corollary.

COROLLARY 5.2. – *For any $k \geq 0$, there is a positive constant $M(K, k)$ uniform in $\sigma \in [0, 1]$ and $K > 0$ such that if $V \in X$ is a fixed point of the map F_K^σ then $\|V\|_{C^k} \leq M(K, k)$.*

REMARK 5.3. – The constants $M(K, k)$ in Corollary 5.2 may depend on K.

5.2. A priori C^0 estimate at infinity

The purpose of this section is to establish a priori C^0 weighted estimates for Chen-Struwe expanding solutions that are uniform in $\sigma \in [0, 1]$. The bounds might depend on the parameter K.

PROPOSITION 5.4. – *There is a positive constant M uniform in $\sigma \in [0, 1]$ such that if $V \in X$ is a fixed point of F_K^σ then $\|fV\|_{C^0} \leq M$.*

REMARK 5.5. – Because of Proposition 5.1, it suffices to show this a priori bound outside a ball of radius independent of $\sigma \in [0, 1]$.

Proof. – Since V is fixed point of the map F_K^σ it follows that

$$\Delta_f V = K\chi'(d_N^2(U_0^\sigma + V))\nabla\left(\frac{d_N^2}{2}\right)(U_0^\sigma + V) \tag{50}$$

$$= K\bar{d}_N(U_0^\sigma + V)\chi'(d_N^2(U_0^\sigma + V))\nabla\bar{d}_N(U_0^\sigma + V) \tag{51}$$

$$= O(f^{-1/2}), \tag{52}$$

where $O(\cdot)$ is uniform in $\sigma \in [0,1]$ and where we used Theorem 4.6 in order to ensure that $\bar{d}_N(U_0^\sigma + V) = O(f^{-1/2})$. Therefore, by using $f^{-1/2}$ as a barrier, one gets a first a priori bound on the decay of V, namely there exists a positive constant M independent of $\sigma \in [0,1]$ such that

$$\|f^{1/2}V\|_{C^0} \leq M. \tag{53}$$

Now, we use the special structure of the nonlinearities of equation (50) together with the previous a priori estimate (53) and Lemma 3.1. Outside a ball of radius sufficiently large (but independent of V and $\sigma \in [0,1]$), one has

$$\Delta_f |V|^2 = 2|\nabla V|^2 + 2K\bar{d}_N(U_0^\sigma + V)<\nabla\bar{d}_N(U_0^\sigma + V), V>,$$

$$|d_{U_0^\sigma+V}\bar{d}_N(V) - d_{U_0^\sigma}\bar{d}_N(V)| \leq C(N, \|U_0^\sigma\|_{L^\infty})|V|^2,$$

$$|\bar{d}_N(U_0^\sigma + V) - \bar{d}_N(U_0^\sigma) - d_{U_0^\sigma}\bar{d}_N(V)| \leq C(N, \|U_0^\sigma\|_{L^\infty})|V|^2,$$

which implies:

$$\begin{aligned}
\Delta_f |V|^2 &\geq 2|\nabla V|^2 - C(N, \|U_0^\sigma\|_{L^\infty})K\bar{d}_N(U_0^\sigma)|V| + 2K\left(d_{U_0^\sigma}\bar{d}_N(V)\right)^2 - C(N, \|U_0^\sigma\|_{L^\infty})K|V|^3 \\
&\geq 2|\nabla V|^2 - C(N, \|U_0^\sigma\|_{L^\infty})K\bar{d}_N(U_0^\sigma)|V| - C(N, \|U_0^\sigma\|_{L^\infty})K|V|^3 \\
&\geq 2|\nabla V|^2 - O(f^{-1})|V|.
\end{aligned}$$

In particular, by the Kato inequality,

$$\Delta_f |V| \geq -O(f^{-1}),$$

when $|V|$ does not vanish.

In general, one can use the regularization of $|V|$ of the form $V_\varepsilon := \sqrt{|V|^2 + \varepsilon^2}$ where ε is positive which satisfies the same differential inequality

$$\Delta_f V_\varepsilon \geq -O(f^{-1}).$$

Now, as $\Delta_f f^{-1} = -(1 + o(1))f^{-1}$, one can use f^{-1} as a barrier function as follows

$$\Delta_f(V_\varepsilon - Af^{-1}) > 0,$$

outside a sufficiently large ball $B(0,R)$ independent of $\sigma \in [0,1]$ and for any sufficiently large constant A. In particular, as V_ε is bounded independently of $\varepsilon \in (0,1]$ and of $\sigma \in [0,1]$ (and of K), one can choose a constant A sufficiently large such that on the boundary $\partial B(0,R)$, $\sup_{\partial B(0,R)} V_\varepsilon - Af^{-1} < 0$. By applying the maximum principle to the function $V_\varepsilon - Af^{-1}$, one gets

$$\sup_{\mathbb{R}^n \setminus B(0,R)} \left\{V_\varepsilon - Af^{-1}\right\} \leq \limsup_{+\infty} \left\{V_\varepsilon - Af^{-1}\right\} = \varepsilon.$$

Since A and R can be chosen independently of $\varepsilon \in (0,1]$, one can pass to the limit in the previous inequality as ε goes to 0 to get the expected result. □

5.3. Weighted C^1 estimate

In this section, we prove a priori C^1 weighted estimates for Chen-Struwe expanding solutions that are uniform in the parameter $\sigma \in [0,1]$. As in Section 5.2, the bounds might depend on the parameter K.

PROPOSITION 5.6. – *There is a positive constant M uniform in $\sigma \in [0,1]$ such that if $V \in X$ is a fixed point of F_K^σ then $\|f^{3/2}\nabla V\|_{C^0} \leq M$.*

REMARK 5.7. – Because of Proposition 5.1, it suffices to show this a priori bound outside a ball of radius independent of $\sigma \in [0,1]$.

Proof. – We first establish the evolution equation satisfied by the gradient ∇V. Since $U := U_0^\sigma + V$ is an expanding solution of the homogeneous Chen-Struwe flow and because of the previous remark, the gradient ∇V satisfies the following equation outside a sufficiently large ball independent of σ

$$\Delta_f \nabla V = -\frac{\nabla V}{2} + K\nabla\left(\nabla\left(\frac{d_N^2}{2}\right)(U)\right).$$

More precisely, in coordinates, this gives:

$$\begin{aligned}\Delta_f \nabla_i V_j &= -\frac{\nabla_i V_j}{2} + K\nabla_i\left(\nabla_j\left(\frac{d_N^2}{2}\right)(U)\right)\\ &= -\frac{\nabla_i V_j}{2} + K\nabla_i\left(\bar{d}_N(U)\nabla_j \bar{d}_N(U)\right)\\ &= -\frac{\nabla_i V_j}{2} + K\left(\nabla_i(\bar{d}_N(U))\cdot(\nabla_j\bar{d}_N)(U) + \bar{d}_N(U)\cdot\nabla_i(\nabla_j\bar{d}_N(U))\right).\end{aligned}$$

By Taylor expansion of order 2 together with Proposition 5.4, Theorem 4.6 and Lemma 3.1 we have

$$\begin{aligned}|\nabla(\bar{d}_N(U) - \bar{d}_N(U_0^\sigma) - <\nabla\bar{d}_N(U_0^\sigma), V>)| &\leq C(N)(|\nabla V||V| + |\nabla U_0^\sigma||V|^2),\\ &\leq O(f^{-1})|\nabla V| + O(f^{-3/2}),\end{aligned}$$

and,

$$\begin{aligned}(\nabla(\nabla\bar{d}_N(U_0^\sigma)))(V) &= O(f^{-3/2}),\\ \bar{d}_N(U)\cdot\nabla_i(\nabla_j\bar{d}_N(U)) &= O(f^{-3/2}),\\ \nabla\bar{d}_N(U) &= \nabla\bar{d}_N(U_0^\sigma) + O(V) = \nabla\bar{d}_N(U_0^\sigma) + O(f^{-1}).\end{aligned}$$

Now, we can end the argument by discarding the nonnegative terms that are quadratic in the gradient ∇V as follows

$$\begin{aligned}\Delta_f|\nabla V|^2 \geq\ & 2|\nabla^2 V|^2 - (1 + O(f^{-1}))|\nabla V|^2 - O(f^{-3/2})|\nabla V|\\ &+2K\nabla(\bar{d}_N(U_0^\sigma))<\nabla\bar{d}_N(U), \nabla V> + 2K<\nabla\bar{d}_N(U_0^\sigma)(\nabla V), \nabla\bar{d}_N(U)(\nabla V)>\\ \geq\ & 2|\nabla^2 V|^2 - (1 + O(f^{-1}))|\nabla V|^2 - O(f^{-3/2})|\nabla V|\\ &+2K<\nabla\bar{d}_N(U_0^\sigma)(\nabla V), \nabla\bar{d}_N(U_0^\sigma)(\nabla V)>\\ \geq\ & 2|\nabla^2 V|^2 - (1 + O(f^{-1}))|\nabla V|^2 - O(f^{-3/2})|\nabla V|,\\ \geq\ & 2|\nabla^2 V|^2 - |\nabla V|^2 - O(f^{-3/2})|\nabla V|,\end{aligned}$$

where we used in the last line the fact that $\nabla V = O(f^{-1/2})$ a priori, thanks to Theorem 4.6.

By considering the function $f^{1/2}|\nabla V|_{\varepsilon} - Af^{-1}$ where $|\nabla V|_{\varepsilon}$ denotes a regularization of the norm of the gradient $|\nabla V|$ and where A is a positive constant large enough depending eventually on K but independent of $\sigma \in [0,1]$ and $\varepsilon \in (0,1]$, one can prove the expected a priori estimate on ∇V with the help of the maximum principle. □

5.4. L^2_{loc} convergence at $t = 0$

In this section, we investigate the L^2_{loc} continuity at $t = 0$ of the expanding solution of the homogeneous Chen-Struwe equation we produced in the previous sections. Since $U(t) = k_t * u_0 + V(t)$ with $\nabla^i V(x,1) = O((1+|x|)^{-2-i})$ with $i = 0,1$, i.e., $V \in X$. It suffices to prove the L^2 convergence on a ball $B(0,R)$ centered at the origin with radius R. We claim that:

$$\lim_{t\to 0} \|\nabla V\|_{L^2(B(0,R))}(t) = 0.$$

Indeed, since $V \in X$, ∇V decays at least quadratically, i.e.,

$$\nabla V(x,t) = O\left(\frac{1}{\sqrt{t}}\frac{1}{\left(\frac{|x|}{\sqrt{t}}+1\right)^2}\right).$$

Therefore,

$$\|\nabla V\|^2_{L^2(B(0,R))}(t) \le C(n,u_0)\int_{B(0,R)} \frac{t}{(|x|+\sqrt{t})^4}dx = C(n,u_0)\int_0^R \frac{t}{(r+\sqrt{t})^4} r^{n-1}dr,$$

for some positive constant $C(n,u_0)$.

Now, if $n \ge 5$,

$$\|\nabla V\|^2_{L^2(B(0,R))}(t) \le C(n,u_0,R)t \to 0, \quad \text{as } t \to 0^+.$$

If $n = 4$,

$$\|\nabla V\|^2_{L^2(B(0,R))}(t) \le C(n,u_0)\ln\left(\frac{R}{\sqrt{t}}+1\right)t \to 0, \quad \text{as } t \to 0^+.$$

If $n = 3$,

$$\|\nabla V\|^2_{L^2(B(0,R))}(t) \le C(n,u_0)\sqrt{t} \to 0, \quad \text{as } t \to 0^+.$$

5.5. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. – Let $K > 0$ and let $u_0 : \mathbb{R}^n \to N^{m-1} \subset \mathbb{R}^m$ be a 0-homogeneous map u_0 as in the statement of Theorem 1.2.

Thanks to Propositions 3.5 and 3.6, the map $F_K : X\times[0,1] \to X$ is a well-defined compact continuous map and Proposition 3.7 ensures that F_K is a compact and continuous map.

Moreover, the Leray-Schauder degree of $I - F_K^{\sigma} : B_X(0,\varepsilon) \to B_X(0,\varepsilon)$ is 1 when σ is close to 1, for some positive ε by Lemma 3.4 combined with Section 3.5.

Finally, there is a positive constant M (uniform in $\sigma \in [0,1]$) such that if $V \in X$ is such that $F_K^{\sigma}(V) = V$ then $\|V\|_X \le M$ by the combination of Propositions 5.1, 5.4 and 5.6 proved in Section 5 with the help of Section 4.

As a consequence of the Leray-Schauder fixed point theorem for each positive K, the map $F_K^0 : X \to X$ has a fixed point $V_K \in X$, i.e., the map $U_K := U_0 + V_K$ is a smooth solution to (8) by Section 3.3.

Finally, Section 5.4 ensures that the time-dependent expanding solution $u_K(t)$ converges strongly to u_0 as t goes to 0 in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$. □

6. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1:

Proof of Theorem 1.1. – Let $(u_0^\varepsilon)_{\varepsilon\in(0,1)}$ be a sequence of 0-homogeneous maps $u_0^\varepsilon : \mathbb{R}^n \to N \subset \mathbb{R}^m$ in $C^3_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ converging to u_0 in the C^0 topology as ε goes to 0, such that

$$\limsup_{\varepsilon\to 0} \mathrm{Lip}(u_0^\varepsilon) \leq \mathrm{Lip}(u_0). \tag{54}$$

Let $K > 0$ and let $(U_K^\varepsilon)_{\varepsilon\in(0,1)}$ be a sequence of smooth Chen-Struwe expanding solutions with fixed parameter K coming out of u_0^ε given by Theorem 1.2. By Theorem 1.2, we know there exist a radius $R = R(\|\nabla u_0^\varepsilon\|_{L^2_{\mathrm{loc}}}, n, m) > 0$ and a constant $C = C(\|\nabla u_0^\varepsilon\|_{L^2_{\mathrm{loc}}}, n, m) > 0$ such that,

$$|e_K(U_K^\varepsilon)|(x) \leq \frac{C}{|x|^2}, \quad |x| \geq R, \tag{55}$$

$$\|e_K(u_K^\varepsilon)(t)\|_{L^1(B(x_0,1))} \leq C\left(n, m, \|\nabla u_0^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}, t\right) \|\nabla u_0^\varepsilon\|^2_{L^2(B(x_0,1))}, \quad \forall x_0 \in \mathbb{R}^n, \tag{56}$$

$$\|\partial_t u_K^\varepsilon\|_{L^2((0,t),L^2_{\mathrm{loc}}(\mathbb{R}^n))} \leq C(n,m,t)\|\nabla u_0^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}, \tag{57}$$

where $\lim_{t\to 0} C\left(n, m, \|\nabla u_0^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}, t\right) = \lim_{t\to} C(n,m,t) = 1$. According to (54), there exist constants C and R uniform in ε such that the previous inequalities hold.

Fix $\varepsilon \in (0,1)$. As in [3], there exists a subsequence (still denoted by $(u_K^\varepsilon)_{K>0}$) converging weakly to a map $u^\varepsilon : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ as K tends to $+\infty$ such that

$$\begin{aligned}
u^\varepsilon(\lambda x, \lambda^2 t) &= u^\varepsilon(x,t), \quad \forall \lambda > 0, \quad \text{a.e. } (x,t) \in \mathbb{R}^n \times \mathbb{R}_+,\\
\nabla u_K^\varepsilon &\rightharpoonup \nabla u^\varepsilon, \quad \text{weakly* in } L^\infty(\mathbb{R}_+, L^2_{\mathrm{loc}}(\mathbb{R}^n)),\\
\partial_t u_K^\varepsilon &\rightharpoonup \partial_t u^\varepsilon, \quad \text{weakly in } L^2_{\mathrm{loc}}(\mathbb{R}_+, L^2_{\mathrm{loc}}(\mathbb{R}^n)),\\
u_K^\varepsilon &\to u^\varepsilon, \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^n),\\
u_K^\varepsilon &\to u^\varepsilon, \quad \text{in } C^{0,\beta}_{\mathrm{loc}}(\mathbb{R}^n \setminus B(0,R)), \text{ for all } \beta \in (0,1).
\end{aligned}$$

The last point is due to Arzela-Ascoli theorem together with the estimate (55). Moreover, thanks to (56), $u^\varepsilon \in N$ a.e..

This implies in particular that U^ε is a Lipschitz function outside $B(0,R)$. To sum it up, we have obtained that

$$|\operatorname{Lip}(U^\varepsilon)|(x) \le \frac{C}{|x|}, \quad |x| \ge R, \tag{58}$$

$$\|\nabla u^\varepsilon(t)\|_{L^2(B(x_0,1))} \le C\left(n, m, \|\nabla u_0^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}, t\right)\|\nabla u_0^\varepsilon\|_{L^2(B(x_0,1))}, \quad \forall x_0 \in \mathbb{R}^n, \tag{59}$$

$$\|\partial_t u^\varepsilon\|_{L^2((0,t),L^2_{\mathrm{loc}}(\mathbb{R}^n))} \le C(n,m,t)\|\nabla u_0^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}. \tag{60}$$

In particular, if one shows that $u^\varepsilon(\cdot,t) := U^\varepsilon(\cdot/\sqrt{t})$ converges weakly to u_0^ε then,

$$\liminf_{t\to 0}\|\nabla u^\varepsilon(t)\|_{L^2(B(x_0,1))} \ge \|\nabla u_0^\varepsilon\|_{L^2(B(x_0,1))}.$$

By combining this fact with (57), one ends up by proving that $u^\varepsilon(t)$ converges to u_0^ε in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ (in the strong sense).

Claim 3. – *$u^\varepsilon(t) \rightharpoonup u_0$ as $t \to 0$.*

To prove this statement, let $\psi_{x_0} : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map with compact support in $B(x_0,1)$ and let $0<s<t$. Then, for $K>0$:

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} <u_K^\varepsilon(t) - u_K^\varepsilon(s), \psi_{x_0}> dx\right| &= \left|\int_s^t \int_{\mathbb{R}^n} <\partial_\tau u_K^\varepsilon, \psi_{x_0}> dx\, d\tau\right| \\
&\le \int_s^t \int_{B(x_0,1)} |\partial_\tau u_K^\varepsilon||\psi_{x_0}|\, dx\, d\tau \\
&\le \left(\int_s^t \int_{B(x_0,1)} |\partial_\tau u_K^\varepsilon|^2 dx\right)^{1/2} \left(\int_s^t \int_{B(x_0,1)} \psi_{x_0}^2\, dx\, d\tau\right)^{1/2} \\
&\le C(n,m,t)\|\nabla u_0^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}\sqrt{t-s}\,\|\psi_{x_0}\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

where we used the a priori uniform bound (60) in the last line. Now, by letting s go to 0, one gets:

$$\left|\int_{\mathbb{R}^n} <u_K^\varepsilon(t) - u_0^\varepsilon, \psi_{x_0}> dx\right| \le C(n,m,t)\|\nabla u_0^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}\sqrt{t}\,\|\psi_{x_0}\|_{L^2(\mathbb{R}^n)},$$

as $u_K^\varepsilon(t)$ converges to u_0 weakly as t goes to 0. By letting K go to $+\infty$, one has:

$$\left|\int_{\mathbb{R}^n} <u^\varepsilon(t) - u_0^\varepsilon, \psi_{x_0}> dx\right| \le C(n,m,t)\|\nabla u_0^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}^n)}\sqrt{t}\,\|\psi_{x_0}\|_{L^2(\mathbb{R}^n)},$$

which proves the expected convergence as t goes to 0. This ends the proof of the claim.

The fact that U^ε is regular off a singular closed (hence compact by (58)) set of finite $(n-2)$ Hausdorff dimensional measure follows from [Sect. III, [3]] and [4]. Finally, the fact that u^ε solves the harmonic map flow follows from [Sect. III, [3]] as well.

The same strategy can be applied now to the sequence of expanding solutions $(u^\varepsilon)_{\varepsilon\in(0,1)}$ of the Harmonic map flow by using (58), (59), (60) together with (54).

The remaining statement to prove concerns the convergence rate at infinity (5). By using the evolution equation (4), it is sufficient to prove the following claim:

CLAIM 4. – *Let U be a solution of* (4). *Assume U is smooth on $B(x_0, 2r)$ for some positive radius r. Then,*

$$\sup_{B(x_0,r/2)} |\nabla^2 U|^2 \leq C\left(1 + \frac{1}{r^2} + \frac{|x_0|}{r} + \sup_{B(x_0,r)} |\nabla U| + \sup_{B(x_0,r)} |\nabla U|^2\right) \sup_{B(x_0,r)} |\nabla U|^2,$$

where C is a positive constant independent of U, r and x_0.

Proof of Claim 4. – We proceed in the spirit of Shi's estimates for the Ricci flow [15]. We compute the evolution equation of the first two derivatives of U

$$\begin{aligned}\Delta_f \nabla U &= -\frac{\nabla U}{2} + A(U) * \nabla^2 U * \nabla U + D_U A * \nabla U^{*3},\\ \Delta_f \nabla^2 U &= -\nabla^2 U + \nabla^3 U * \nabla U + \nabla^2 U^{*2} + \nabla^2 U * \nabla U^{*2} + \nabla U^{*4}.\end{aligned}$$

In particular, by using Young's inequality

$$\begin{aligned}\Delta_f |\nabla U|^2 &\geq 2|\nabla^2 U|^2 - |\nabla U|^2 - |\nabla^2 U||\nabla U|^2 - c|\nabla U|^4\\ &\geq |\nabla^2 U|^2 - c(1 + |\nabla U|^2)|\nabla U|^2,\end{aligned}$$

where c denotes a positive constant depending on the geometry of N only that can vary from line to line. Similarly, one has:

$$\Delta_f |\nabla^2 U|^2 \geq |\nabla^3 U|^2 - c(1 + |\nabla U|^2)|\nabla^2 U|^2 - c|\nabla^2 U|^3 - c(1 + |\nabla U|^2)|\nabla U|^4.$$

Now, let a be a positive constant to be defined later and consider the auxiliary function $F := (|\nabla U|^2 + a^2)|\nabla^2 U|^2$. The function F satifies the following differential inequality

$$\begin{aligned}\Delta_f F &\geq |\nabla^2 U|^4 - c(1 + |\nabla U|^2)F - 8|\nabla^2 U|^2|\nabla U||\nabla^3 U|\\ &\quad + |\nabla^3 U|^2(|\nabla U|^2 + a^2) - c|\nabla^2 U|^3(|\nabla U|^2 + a^2)\\ &\quad - c|\nabla U|^4(|\nabla U|^2 + a^2)(1 + |\nabla U|^2)\\ &\geq \frac{1}{2}|\nabla^2 U|^4 - c(1 + |\nabla U|^2)F + |\nabla^3 U|^2(a^2 - c|\nabla U|^2)\\ &\quad - c|\nabla^2 U|^3(|\nabla U|^2 + a^2) - c|\nabla U|^4(|\nabla U|^2 + a^2)(1 + |\nabla U|^2).\end{aligned}$$

If $a^2 := c \sup_{B(x_0,1)} |\nabla U|^2$ where c is a positive constant sufficiently large, then

$$\begin{aligned}\Delta_f F &\geq \frac{F^2}{2a^4} - c(1 + a^2)F - ca^{-1}F^{3/2} - c(1 + a^2)a^6\\ &\geq \frac{F^2}{4a^4} - c(1 + a^2)F - c(1 + a^2)a^6,\end{aligned}$$

by Young's inequality.

Let $\phi : \mathbb{R}^n \to [0, 1]$ be a smooth positive function with compact support defined by $\phi(x) := \psi(r_{x_0}/r)$ where $r_{x_0}(x) := |x - x_0|$ and where $r > 0$ and $\psi : [0, +\infty[\to [0, 1]$ is a smooth positive function satisfying

$$\psi|_{[0,1/2]} \equiv 1, \quad \psi|_{[1,+\infty[} \equiv 0, \quad \psi' \leq 0, \quad \frac{\psi'^2}{\psi} \leq c, \quad \psi'' \geq -c.$$

Define the (last) auxiliary function $G := \phi F$ and consider a point $x_1 \in B(x_0, 1)$ such that $G(x_1) = \max_{B(x_0,1)} G$. By the previous differential inequality satisfied by F evaluated at x_1 together with the maximum principle

$$0 = \nabla G = F\nabla\phi + \phi\nabla F, \tag{61}$$

$$0 \geq \phi\Delta_f G \geq \frac{G^2}{4a^4} - ca^2 G - c(1+a^2)a^6 - 2G\frac{|\nabla\phi|^2}{\phi} + G(\Delta\phi + \langle\nabla f, \nabla\phi\rangle). \tag{62}$$

Now,

$$\nabla\phi = \frac{\psi'}{r}\nabla r_{x_0}, \quad \Delta\phi = \frac{\psi''}{r^2} + \frac{\psi'}{r}\Delta r_{x_0}.$$

Hence,

$$2\frac{|\nabla\phi|^2}{\phi} - \Delta_f\phi = \frac{1}{r^2}\left[\frac{2\psi'^2}{\psi} - \psi''\right] - \frac{\psi'}{r}(\langle\nabla f, \nabla r_{x_0}\rangle + \Delta r_{x_0}).$$

Also,

$$\Delta r_{x_0} \leq \frac{n-1}{r}, \quad \text{on } B(x_0, r)\backslash B(x_0, r/2).$$

Coming back to inequality (62), one gets

$$\begin{aligned} 0 &\geq \frac{G^2}{4a^4} - c\left(a^2 + \frac{1}{r^2} + \frac{\sup_{B(x_0,r)}|\nabla f|}{r}\right)G - a^6 \\ &\geq \frac{G^2}{4a^4} - c\left(1 + a^2 + \frac{1}{r^2} + \frac{|x_0|}{r}\right)G - c(1+a^2)a^6, \end{aligned}$$

by the very definition of f together with the triangular inequality. The expected estimate on the second derivatives of U follows immediately. □

We are now in a position to prove the convergence rate as stated in (5). Indeed, by Theorem 4.6 together with Claim 4 applied to a point $x_0 \in \mathbb{R}^n$ sufficiently far from the origin $0 \in \mathbb{R}^n$ and to a radius $r := |x_0|/2$, one gets in particular that the Laplacian of U decays at least linearly at infinity. Consequently, by Equation (4), the radial derivative of U decays at least quadratically. By integrating along radial lines, U approaches u_0 at a linear rate:

$$|U(x) - u_0(x/|x|)| = O(|x|^{-1}),$$

for every x far from the origin. □

REMARK 6.1. – It does not seem straightforward to improve the convergence rate in case u_0 is at least $C^3_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. The main reason is the lack of an ε-regularity theorem that detects the smoothness of the map u_0 at infinity.

7. Taylor expansions at infinity for expanders of the harmonic map flow

We gather necessary conditions at infinity on an expanding solution of the Homogeneous Ginzburg-Landau flow (10) or the harmonic map flow smoothly coming out of a 0-homogeneous map $u_0 : \mathbb{S}^{n-1} \to \mathbb{S}^{m-1} \subset \mathbb{R}^m$ in $C^\infty(\mathbb{R}^n \setminus \{0\})$. A similar treatment could be done for the Homogeneous Chen-Struwe flow for a general target closed manifold (N, g) isometrically embedded in some Euclidean space $\mathbb{R}^m$.

Let U be an expanding solution to the Homogeneous Ginzburg-Landau flow with parameter $K > 0$ coming out of the map u_0.

Let us assume that there are smooth maps $u_i : \mathbb{S}^{n-1} \to \mathbb{R}^m$, $i = 1, ..., k$, such that

$$U(x) = \sum_{i=0}^{k} \frac{u_i(x/|x|)}{|x|^{2i}} + O(|x|^{-2k-2}),$$

as x goes to $+\infty$ for every nonnegative integer k. This expansion is assumed to hold in the smooth sense. Then, on one hand,

$$\begin{aligned}
\Delta_f U &= \sum_{i=0}^{k} \left(|x|^{-2i-2}\Delta_{\mathbb{S}^{n-1}} u_i + \Delta_f(|x|^{-2i})u_i\right) + O(|x|^{-2k-2})\\
&= \sum_{i=0}^{k} \left(|x|^{-2i-2}\Delta_{\mathbb{S}^{n-1}} u_i + \left(2i(2(i+1)-n)|x|^{-2} - i\right)|x|^{-2i}u_i\right) + O(|x|^{-2k-2})\\
&= \sum_{i=1}^{k} |x|^{-2i}\left(\Delta_{\mathbb{S}^{n-1}} u_{i-1} - iu_i + 2(i-1)(2i-n)u_{i-1}\right) + O(|x|^{-2k-2}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(1-|U|^2)U &= \left(1 - \left|\sum_{i=0}^{k}|x|^{-2i}u_i + O(|x|^{-2k-2})\right|^2\right)\left(\sum_{i=0}^{k}|x|^{-2i}u_i + O(|x|^{-2k-2})\right)\\
&= -\left(\sum_{i=1}^{k}|x|^{-2i}\left(\sum_{j=0}^{i}<u_j, u_{i-j}>\right)\right)\left(\sum_{i=0}^{k}|x|^{-2i}u_i\right) + O(|x|^{-2k-2})\\
&= -\sum_{i=0}^{k}|x|^{-2i}\left(\sum_{j=0}^{i}a_j u_{i-j}\right) + O(|x|^{-2k-2}),
\end{aligned}$$

where

$$a_0 := 0, \quad a_i := \sum_{j=0}^{i}<u_j, u_{i-j}>, \quad i \geq 1.$$

Therefore, by identifying terms by terms, for $i \geq 1$,

$$\Delta_{\mathbb{S}^{n-1}} u_{i-1} - iu_i + 2(i-1)(2i-n)u_{i-1} - K\sum_{j=1}^{i} a_j u_{i-j} = 0,$$

which gives for $i = 1$:

$$\Delta_{\mathbb{S}^{n-1}} u_0 - u_1 - 2K<u_1, u_0>u_0 = 0,$$

that is:

$$u_1 = \Delta_{\mathbb{S}^{n-1}} u_0 - \frac{2K}{2K+1}<\Delta_{\mathbb{S}^{n-1}} u_0, u_0>u_0.$$

In general, if $i \geq 2$, one has:

$$\begin{aligned} i u_i + 2K<u_i, u_0>u_0 &= \Delta_{\mathbb{S}^{n-1}} u_0 + 2(i-1)(2i-n)u_{i-1} \\ &\quad - K\sum_{j=1}^{i-1} a_j u_{i-j} - K\sum_{j=1}^{i-1}<u_j, u_{i-j}>u_0, \end{aligned}$$

which determines u_i.

Let U be an expanding solution to the harmonic map flow $K > 0$ coming out of the map u_0. Then, since the maps $(u_i)_{i\geq 0}$ are spherical,

$$\begin{aligned} |\nabla U|^2 &= \left|\nabla\left(\sum_{i=0}^{k}|x|^{-2i}u_i\right)\right|^2 + O(|x|^{-2k-2}) \\ &= \left|\sum_{i=0}^{k}(-2i)|x|^{-2i-1}u_i\right|^2 + \left|\sum_{i=0}^{k}|x|^{-2i-1}(\nabla^{\mathbb{S}^{n-1}}u_i)\right|^2 + O(|x|^{-2k-2}) \\ &= \sum_{i=0}^{k-1}|x|^{-2i-2}\left(\sum_{j=0}^{i}4j(i-j)<u_j, u_{i-j}> + <\nabla^{\mathbb{S}^{n-1}}u_j, \nabla^{\mathbb{S}^{n-1}}u_{i-j}>\right) + O(|x|^{-2k-2}), \end{aligned}$$

which implies:

$$|\nabla U|^2 U = \sum_{i=0}^{k}|x|^{-2i}\left(\sum_{l=0}^{i}b_l u_{i-l}\right) + O(|x|^{-2k-2}),$$

where,

$$b_0 := 0, \quad b_{i+1} := \sum_{j=0}^{i}4j(i-j)<u_j, u_{i-j}> + <\nabla^{\mathbb{S}^{n-1}}u_j, \nabla^{\mathbb{S}^{n-1}}u_{i-j}>, \quad i \geq 0.$$

By identification,

$$\Delta_{\mathbb{S}^{n-1}}u_{i-1} - iu_i + 2(i-1)(2i-n)u_{i-1} = -\sum_{l=0}^{i}b_l u_{i-l}, \quad i \geq 1.$$

For instance, if $i = 1$,

$$u_1 = \Delta_{\mathbb{S}^{n-1}}u_0 + |\nabla^{\mathbb{S}^{n-1}}u_0|^2 u_0, \tag{63}$$

which can be understood as the formal limit as K goes to $+\infty$ of the sequence of the corresponding coefficients $(u_1^K)_{K>0}$ of the Taylor expansion derived previously for the Ginzburg-Landau Equation with parameter K. Indeed, as K goes to $+\infty$, one gets by (7) that:

$$\begin{aligned} \lim_{K\to+\infty} u_1^K &= \Delta_{\mathbb{S}^{n-1}}u_0 - <\Delta_{\mathbb{S}^{n-1}}u_0, u_0>u_0 \\ &= \Delta_{\mathbb{S}^{n-1}}u_0 + |\nabla^{\mathbb{S}^{n-1}}u_0|^2 u_0, \end{aligned}$$

which is exactly formula (63) since $0 = \Delta_{\mathbb{S}^{n-1}}|u_0|^2 = 2<\Delta_{\mathbb{S}^{n-1}}u_0, u_0> + 2|\nabla^{\mathbb{S}^{n-1}}u_0|^2$.

The same holds for the other coefficients $(u_k)_{k\geq 1}$.

Moreover, one can check that if u_0 is harmonic, i.e., if $\Delta_{\mathbb{S}^{n-1}}u_0 + |\nabla^{\mathbb{S}^{n-1}}u_0|^2 u_0 = 0$ then all the other terms vanish: $u_k = 0$, for all $k \geq 1$.

In particular, we get the following corollary:

Corollary 7.1. – *Let $u_0 : \mathbb{R}^n \to \mathbb{S}^{m-1}$ be a* 0*-homogeneous map of the harmonic map flow in $C^\infty(\mathbb{R}^n \setminus \{0\})$ and let $U : \mathbb{R}^n \to \mathbb{S}^{m-1}$ be an expanding solution of the harmonic map flow smoothly coming out of u_0. Then the convergence rate of U at infinity is faster than any polynomial rate.*

Remark 7.2. – As in [6], it can be shown that the convergence rate of a smooth expanding solution u to its initial condition u_0 is exactly $O\left(r^{-n}e^{-r^2/4}\right)$ if u_0 is harmonic, hence much faster than what Corollary 7.1 predicts. This decay reveals the role of the Hermite functions at infinity: they are intimately connected to the weighted Laplacian Δ_f.

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ASYMPTOTICS OF QUANTUM REPRESENTATIONS OF SURFACE GROUPS

BY Julien MARCHÉ AND Ramanujan SANTHAROUBANE

ABSTRACT. – For a banded link L in a surface times a circle, the Witten-Reshetikhin-Turaev invariants are topological invariants depending on a sequence of complex $2p$-th roots of unity $(A_p)_{p\in 2\mathbb{N}}$. We show that there exists a polynomial P_L such that these normalized invariants converge to $P_L(u)$ when A_p converges to u, for all but a finite number of u's in S^1. This is related to the AMU conjecture which predicts that non-simple curves have infinite order under quantum representations (for big enough levels). Estimating the degree of P_L, we exhibit particular types of curves which satisfy this conjecture. Along the way we prove the Witten asymptotic conjecture for links in a surface times a circle.

RÉSUMÉ. – Pour un entrelacs en bande L dans le produit d'une surface par un cercle, les invariants de Witten-Reshetikhin-Turaev sont des invariants topologiques dépendant d'une suite de racines $2p$-ièmes de l'unité $(A_p)_{p\in 2\mathbb{N}}$. Nous montrons qu'il existe un polynôme P_L tel que ces invariants normalisés convergent vers $P_L(u)$ quand A_p tend vers u, sauf pour un nombre fini de u dans S^1. Ceci est relié à la conjecture AMU qui prédit que les courbes non simples sont d'ordre infini dans la représentation quantique (en niveau assez grand). En estimant le degré de P_L, on exhibe certaines courbes qui satisfont la conjecture. En chemin, nous prouvons la conjecture asymptotique de Witten pour les entrelacs dans le produit d'une surface par un cercle.

1. Statement of the results

1.1. Motivation and Main result

This paper is concerned with invariants arising from Witten-Reshetikhin-Turaev SU(2) topological quantum field theories (TQFT) following the skein theoretical approach of [3]. Such a TQFT defines for M an oriented compact 3-manifold without boundary and $L \subset M$ a banded link, a sequence of invariants $Z_p(M,L)$ indexed by even integers $p = 2r \geq 6$. For a given p, the invariant $Z_p(M,L)$ belongs to the cyclotomic field [(1)] $K_p = \mathbb{Q}[A]/(\phi_{2p}(A))$,where ϕ_{2p} denotes the cyclotomic polynomial, that is the (monic)

(1) Indeed in a finite extension of it, but we will not need it here.

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minimal polynomial over $\mathbb{Q}$ of $e^{i\pi/p}$. To have a numerical invariant, one needs to specify an embedding of K_p into $\mathbb{C}$ or equivalently a $2p$-th primitive root of unity $A_p \in \mathbb{C}$. We will denote by $\mathrm{ev}_{A_p} Z_p(M,L) \in \mathbb{C}$ the associated numerical evaluation. An interesting question is to understand the asymptotic of the quantity $\mathrm{ev}_{A_p} Z_p(M,L)$ as $p \to \infty$ and as A_p converges to a given number on the unit circle. When $A_p = -e^{i\pi/p}$, this problem is called the Witten asymptotic expansion conjecture. Other limits have not been studied yet with the exception of some Seifert spaces studied by Lawrence and Zagier, see [12].

In this paper, we focus on the case $M = \Sigma \times S^1$ where Σ is a compact connected oriented closed surface. We look for a formula for the quantum invariant

$$\mathrm{tr}_p(L) = \frac{Z_p(\Sigma \times S^1, L)}{Z_p(\Sigma \times S^1, \emptyset)} \in K_p,$$

where $L \subset \Sigma \times S^1$ is a given banded link. Notice that the quantity $Z_p(\Sigma \times S^1, \emptyset)$ is the dimension of $V_p(\Sigma)$: the K_p-vector space associated to Σ by the Witten-Reshetikhin-Turaev TQFT. Moreover, $\dim V_p(\Sigma)$ is computed by the Verlinde formula and is polynomial in p with degree $0, 1, 3g-3$ if the genus of Σ is $g = 0, 1$ or $g \geq 2$ respectively. Hence the asymptotics of the quantity $\mathrm{ev}_{A_p} Z_p(\Sigma \times S^1, L)$ is determined by the asymptotics of $\mathrm{ev}_{A_p} \mathrm{tr}_p(L)$. The main result of this paper is that the asymptotics of $\mathrm{ev}_{A_p} \mathrm{tr}_p(L)$ is almost determined by the evaluation of a Laurent polynomial with integral coefficients depending only on L.

Theorem 1.1. – *Let L be a link in $\Sigma \times S^1$. There exist a Laurent polynomial $P_L \in \mathbb{Z}[A^{\pm 1}]$ and a finite set $\Omega_L \subset S^1$ such that for any sequence $\{A_p\}_{p \in 2\mathbb{N}}$ such that $A_p \underset{p\to\infty}{\longrightarrow} u \notin \Omega_L$, one has*

$$\mathrm{ev}_{A_p} \mathrm{tr}_p(L) = P_L(u) + O\Big(\frac{1}{p}\Big).$$

In particular the polynomial $P_L \in \mathbb{Z}[A^{\pm 1}]$ is well-defined and is a topological invariant of L.

The polynomial P_L can be viewed as a generalization of the Kauffman bracket of a link in S^3. Indeed if $L \subset B^3$ is a link inside a 3-ball embedded in $\Sigma \times S^1$, the polynomial P_L is nothing but the usual Kauffman bracket of L. For a more complicated link inside $\Sigma \times S^1$, this polynomial can be computed algorithmically, see Section 2. The existence of such a polynomial is not clear a priori, we note that Gilmer already built one in the case of a connected sum of $S^2 \times S^1$'s using other methods (see [7]). A similar comment can be made on Costantino's work (see [5]).

1.2. Cyclic expansions

In order to prove Theorem 1.1, we introduce the key notion of *cyclic expansion*. We will denote by $\mathcal{C}$ the vector space of maps $f : \mathbb{R} \to \mathbb{R}$ which are piecewise polynomial with compact support.

Definition 1.2. – We will say that the sequence $\mathrm{tr}_p(L)$ has a cyclic expansion if there exist $P_L \in \mathbb{Z}[A^{\pm 1}]$, an integer $\beta \geq 0$ and a family $f_0, \dots, f_{2\beta-1} \in \mathcal{C}$ such that

$$\mathrm{tr}_p(L) = P_L(A) + \frac{1}{p} \sum_{\alpha=0}^{2\beta-1} \sum_{n \in \mathbb{Z}} A^{2\beta n + \alpha} f_\alpha\left(\frac{n}{p}\right) + O\left(\frac{1}{p}\right). \tag{1}$$

We need to explain the meaning of $O(\frac{1}{p})$ in Equation (1). Indeed, we have to interpret both sides as elements of $K_p \otimes \mathbb{R}$ endowed with the norm $||x||_p = \inf\{\max_n |c_n|, x = \sum_{n=0}^{p-1} c_n A^n\}$.

The main technical result of this article is the following theorem.

THEOREM 1.3. – *For any banded link $L \subset \Sigma \times S^1$, the sequence $\mathrm{tr}_p(L)$ admits a cyclic expansion.*

The proof of this theorem consists in a careful counting of integral points in various polytopes related to TQFT. It is postponed to Section 3. The interest of having such a cyclic expansion is that the study of asymptotics is reduced to the following question.

Let f be in $\mathcal{C}$, β be a non zero integer and A_p be a convergent sequence of $2p$-th primitive roots of unity. What is the asymptotics of $\frac{1}{p}\sum_{n\in\mathbb{Z}} A_p^{2\beta n} f(\frac{n}{p})$ as p tends to infinity? This problem can be solved with elementary analytic tools as follows.

PROPOSITION 1.4. – *Let $f \in \mathcal{C}$ and β be a positive integer. Let A_k be a sequence of $2p_k$-th primitive roots of unity with p_k a strictly increasing sequence of even integers.*

1. *If $\lim_{k\to\infty} A_k = u$ with $u^{2\beta} \neq 1$ one has*
$$\frac{1}{p_k}\sum_{n\in\mathbb{Z}} A_k^{2\beta n} f\Big(\frac{n}{p_k}\Big) = O\Big(\frac{1}{p_k}\Big).$$
2. *If $\lim_{k\to\infty} A_k = u$ with $u^{2\beta} = 1$ we write $A_k = ue^{i\pi\theta_k}$ so that $\lim_{k\to\infty}\theta_k = 0$. If $p_k\theta_k$ diverges when k goes to infinity then*
$$\frac{1}{p_k}\sum_{n\in\mathbb{Z}} A_k^{2\beta n} f\Big(\frac{n}{p_k}\Big) = O\Big(\frac{1}{p_k\theta_k}\Big).$$
3. *In the same setting as (2) suppose that $p_k\theta_k$ does not diverge. As $A_k^{2p_k} = 1$, the sequence $\theta_k p_k$ takes discrete values and up to extracting a subsequence, one can suppose that it is constant equal to $\frac{\sigma}{\beta}$, i. e. $A_k = ue^{\frac{i\pi\sigma}{\beta p_k}}$ for some odd integer σ. Then, we have*
$$\frac{1}{p_k}\sum_{n\in\mathbb{Z}} A_k^{2\beta n} f\Big(\frac{n}{p_k}\Big) = \int_{\mathbb{R}} e^{2i\pi x\sigma} f(x)dx + O\Big(\frac{1}{p_k}\Big).$$

Proof. – We set $H_k = \frac{1}{p_k}\sum_{n\in\mathbb{Z}} A_k^{2\beta n} f(\frac{n}{p_k})$. We compute
$$(1-A_k^{2\beta})H_k = \frac{1}{p_k}\sum_{n\in\mathbb{Z}} A_k^{2\beta n}\left(f\Big(\frac{n}{p_k}\Big) - f\Big(\frac{n-1}{p_k}\Big)\right) = O\Big(\frac{1}{p_k}\Big).$$
The last equality is obtained by applying a Taylor expansion of f away from a finite number of values: this is possible since f is piecewise polynomial.

In the first case, since $u^{2\beta} \neq 1$, we deduce that $H_k = O(\frac{1}{p_k})$.

In the second case, we simply observe that $1 - A_k^{2\beta} \sim -2i\pi\beta\theta_k$ hence $H_k = O\big(\frac{1}{p_k\theta_k}\big)$ and we can conclude.

In the last case, we write $H_k = \frac{1}{p_k}\sum_{n\in\mathbb{Z}} e^{2i\pi\sigma\frac{n}{p_k}} f(\frac{n}{p_k})$ and recognize a Riemann sum. This gives $H_k = \int_{\mathbb{R}} e^{2i\pi\sigma x} f(x)dx + O\big(\frac{1}{p_k}\big)$. □

We observe that Proposition 1.4 and Theorem 1.3 imply directly Theorem 1.1.

1.3. Applications for the AMU conjecture for surface groups

The polynomial P_L associated to the cyclic expansion can be used for proving the AMU conjecture. Let Σ be a closed surface of genus at least 2, $p = 2r$ be an even integer and

$$\rho_p : \pi_1(\Sigma) \longrightarrow \prod_{n=1}^{r-1} \mathrm{PAut}(V_p(\Sigma, n)) \tag{2}$$

be the quantum representation (2) considered by Koberda and the second author in [9]. Here $V_p(\Sigma, n)$ denotes the vector space associated by the SU(2) Witten-Reshetikhin-Turaev TQFT to the surface Σ equipped with a banded point colored by n. It corresponds to the SU(2)-TQFT at level $k = r - 2$ in the geometric setting. Here, we use the notation of [3] with a shift of 1 concerning the color n. A consequence of the AMU conjecture stated by Andersen, Masbaum and Ueno in [2] is the following (see [10, Section 7] for more detail on this implication).

Conjecture (AMU conjecture for surface groups). – *If $\gamma \in \pi_1(\Sigma) \setminus \{1\}$ is not a power of a simple element then $\rho_p(\gamma)$ has infinite order for all p big enough.*

Here a non-trivial element of $\pi_1(\Sigma)$ is said simple if it is freely homotopic to a simple closed curve. We denote by $\left(\mathbb{Z}[A^{\pm 1}]\right)^{\times} = \{\pm A^m\}_{m\in\mathbb{Z}}$ the group of units in $\mathbb{Z}[A^{\pm 1}]$. Using cyclic expansions we define the following:

Definition 1.5. – Let $\gamma \in \pi_1(\Sigma)$ be represented by a loop $\gamma : S^1 \to \Sigma$. We define $\hat{\gamma} \subset \Sigma \times S^1$ as the knot $t \in S^1 \mapsto (\gamma(t), t) \in \Sigma \times S^1$ with arbitrary banded structure. For $n \in \mathbb{N} \setminus \{0\}$, we denote by $(\hat{\gamma}, n)$ the banded link $\hat{\gamma}$ colored by n and by $\mathrm{tr}_p(\hat{\gamma}, n)$ its normalized trace (observe that $n = 1$ is the trivial color and $n = 2$ the usual one). Because of the indeterminacy of the banded structure, the polynomial $P_{\gamma,n}$ is well-defined up to the multiplication by a unit.

Theorem 1.6. – *Let $\gamma \in \pi_1(\Sigma)$. If for some n, $P_{\gamma,n}$ is neither zero nor a unit, then the AMU conjecture for surface groups holds for γ.*

Proof. – We have by definition $\mathrm{tr}_p(\hat{\gamma}, n) = \frac{\mathrm{tr}\, \rho_{p,n}(\gamma)}{\dim V_p(\Sigma)}$ where $\rho_{p,n}$ is the n-th factor of the representation ρ_p defined in Equation 2. If $\rho_{p,n}(\gamma)$ has finite order, then its eigenvalues are roots of unity and we have for any primitive root of unity of order $2p$ the inequality

$$|\operatorname{ev}_{A_p} \mathrm{tr}_p(\hat{\gamma}, n)| \leq 1.$$

By Proposition 1.4, for all but a finite number of $u \in S^1$, one has

$$\lim_{A_p \to u} \operatorname{ev}_{A_p} \mathrm{tr}_p(\hat{\gamma}, n) = P_{\gamma,n}(u).$$

Hence, if $\rho_{p_k,n}$ has finite order for a sequence p_k going to infinity, then $|P_{\gamma,n}(u)| \leq 1$ for all $u \in S^1$. The theorem is then a direct consequence of Lemma 1.7. □

Lemma 1.7. – *Let $P \in \mathbb{Z}[A^{\pm 1}]$ such that* $\sup_{z\in S^1} |P(z)| \leq 1$. *Then one has $P = 0$ or $P \in \left(\mathbb{Z}[A^{\pm 1}]\right)^{\times}$.*

(2) Strictly speaking, the definition given in [9] was in SO(3)-TQFT but the exact same can be done in the SU(2) setting.

Proof. – We can write $P = \sum_l a_l A^l$ with the a_j's in $\mathbb{Z}$. Let us define the continuous function $f(t) = P(e^{2i\pi t})$. One has

$$\sum_l |a_l|^2 = \int_0^1 |f(t)|^2 \mathrm{dt} \leq 1.$$

Since the a_j's are in $\mathbb{Z}$ this implies that $P = 0$ or $P = \pm A^m$ for some $m \in \mathbb{Z}$. □

Notice that determining if the polynomials $P_{\gamma,n}$ belong to $\left(\mathbb{Z}[A^{\pm 1}]\right)^\times \cup \{0\}$ is similar to the problem of determining if a non-trivial knot in S^3 has non-trivial colored Jones polynomials, which is believed to be true. The following Theorem 1.10 will provide examples where the degree of the polynomial $P_{\gamma,3}$ can be controlled. This is reminiscent of the estimation of the colored Jones polynomials of alternating knots.

1.3.1. *A formula for P_L.* – We give here a formula for P_L in the case where L is a banded link in $\Sigma \times S^1$ whose projection on the first factor is a multicurve.

Proposition 1.8. – *Let $p : \Sigma \times S^1 \to \Sigma$ be the first projection map and consider a collection of disjoint and non-parallel annuli $T_1, \ldots, T_n \subset \Sigma$. Let $L \subset \Sigma \times S^1$ be a banded link projecting to $\bigcup_{j=1}^n T_j$ and set $L_j = L \cap p^{-1}(T_j)$. We have $P_L = \prod_{j=1}^n P_{L_j}$ so that one can reduce to the case where L is inside an annulus times a circle. We put $P_L = 1$ if L is the empty link.*

Let L be a banded link in $U = S^1 \times S^1 \times [0,1]$ which is a union of banded knots $\gamma_i \subset S^1 \times S^1 \times \{t_i\}$ for $0 < t_0 < \cdots < t_k < 1$. Let $x_0, \ldots, x_k \in H_1(U,\mathbb{Z})$ be the corresponding homology classes. We have

$$P_L = 2 \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_k = \pm 1 \text{ s.t.} \\ x_0 + \varepsilon_1 x_1 + \cdots \varepsilon_k x_k = 0}} A^{2\,\mathrm{Area}(x_0, \varepsilon_1 x_1, \ldots, \varepsilon_k x_k)},$$

where $\mathrm{Area}(y_0, \ldots, y_k) = 0$ *is the area of the polygon in $H_1(U,\mathbb{R})$ whose sides are the vectors $y_0, \ldots, y_k$.*

This formula will be useful for proving that $P_{\gamma,3}$ is non-trivial for some curves $\gamma \in \pi_1(\Sigma)$. In order to describe them, we introduce the notion of Euler incompressibility.

Definition 1.9. – A cycle in a graph G is called *Eulerian* if it visits every edge at most once. A graph G embedded in Σ is said *Euler-incompressible* when no Eulerian cycle of G bounds a disk in Σ.

Theorem 1.10. – *Let Σ be a closed surface of genus $g \geq 2$ and $\gamma : S^1 \to \Sigma$ be an embedding with N transverse double points whose image $\Gamma = \gamma(S^1)$ is Euler-incompressible. Then, up to multiplication by a unit, we have*

$$P_{\gamma,3} = \sum_{i=-4N}^{4N} c_i A^i,$$

where c_{-4N} and c_{4N} are not zero. In particular, it is not trivial if $N > 0$ and Theorem 1.6 applies.

Figure 1 shows an example of curve which fulfills the assumptions of Theorem 1.10. Note that this loop is filling the surface in the sense that the complement of its image is a disjoint union of topological disks. According to Kra's Theorem [11, Theorem 1.1], this loop is sent to a pseudo-Anosov element in the Birman Exact Sequence. To the authors' knowledge, this is the first example of a pseudo-Anosov element in the mapping class group of a genus $g \geq 2$ surface satisfying the AMU conjecture. Moreover we note that any loop whose image's complementary is a single disk fulfills the assumptions of Theorem 1.10.

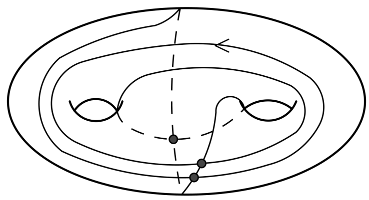


FIGURE 1. An Euler incompressible filling curve

1.4. Geometric interpretation of the cyclic expansion

The standard root of unity $A_p = -e^{i\pi/p}$ is generally used in TQFT as the vector space $V_p(\Sigma)$ is Hermitian in that case. Most of the results or conjectures about the asymptotics of TQFT concern this setting. The Witten conjecture is the most well-known and our work is related to the case when the underlying manifold is $\Sigma \times S^1$. Consider for more generality an odd integer σ and replace A_p with A_p^σ. Observe that for these roots to be primitive of order $2p$ we need that $2p$ is coprime to σ which we assume from now on.

Suppose that $L \subset \Sigma \times S^1$ is a banded link having a cyclic expansion as in Equation (1). Then this expansion governs the asymptotics in the sense that we have

$$\lim_{p\to\infty} \mathrm{ev}_{A_p^\sigma} \operatorname{tr}_p(L) = P_L(-1) + \int_{\mathbb{R}} e^{2i\pi x\sigma} \left(\sum_{\alpha=0}^{2\beta-1} (-1)^\alpha f_\alpha(x) \right) dx.$$

Let $X(\Sigma \times S^1)$ be the space of conjugacy classes of irreducible representations $\rho : \pi_1(\Sigma \times S^1) \to \mathrm{SU}_2$. Denoting by t the generator of $\pi_1(S^1)$, such a representation has to satisfy $\rho(t) = \pm 1$ as t is central. Hence $X(\Sigma \times S^1)$ is a union of two copies of $X(\Sigma)$ (the space of conjugacy classes of irreducible representations $\rho : \pi_1(\Sigma) \to \mathrm{SU}_2$), defined in the same way. This manifold has dimension $6g - 6$ and is endowed with a natural symplectic form ω and volume form $\nu = \frac{\omega^{3g-3}}{(3g-3)!}$. We set $\nu_g = \int_{X(\Sigma)} \nu$.

When $\sigma = 1$, the Witten conjecture predicts the following asymptotics where $L = L_1 \cup \cdots \cup L_k$.

$$\lim_{p\to\infty} \mathrm{ev}_{A_p} \operatorname{tr}_p(L) = \frac{1}{2\nu_g} \int_{X(\Sigma\times S^1)} \prod_{i=1}^{k} (-\operatorname{tr}\rho(L_i)) d\nu(\rho).$$

This formula was proved in [13] in the case where L lies inside $\Sigma \times [0,1] \subset \Sigma \times S^1$. This formula has also an intersection with [1], where special links in finite order mapping tori were studied. Theorem 1.11 will cover the general case where L and σ are arbitrary. Unfortunately, the geometric meaning of the formula is less clear when $\sigma > 1$.

THEOREM 1.11. – *Let σ be an odd integer and set $A_p = -e^{i\pi/p}$. If $L \subset \Sigma \times S^1$ projects to Σ without crossings, we have the following formula.*

$$\lim_{p\to\infty} \mathrm{ev}_{A_p^\sigma} \operatorname{tr}_p(L) = \frac{1}{2\nu_g} \int_{X(\Sigma\times S^1)} \prod_{i=1}^{k} (-\operatorname{tr}\rho(L_i)^\sigma) d\nu(\rho).$$

This formula extends to all links using the Kauffman relation, however we do not know a direct expression of $\lim\limits_{p\to\infty} \mathrm{ev}_{A_p^\sigma} \operatorname{tr}_p(L)$ for general banded links. The fact the σ exponent moved from A_p to $\rho(L_i)$ is a striking phenomenon which deserves further study. The theorem will be a rather direct consequence of Theorem 3.6 proved in Section 3.4.

2. Skein computations

2.1. Computing the polynomial P_L

The computation of the polynomial P_L associated to a banded link $L \subset \Sigma \times S^1$ is better understood in terms of skein modules. For any compact oriented manifold M (maybe with boundary), we denote by $\mathscr{K}(M)$ the Kauffman bracket skein module with coefficients in $\mathbb{Z}[A^{\pm 1}]$. We recall that it is the free $\mathbb{Z}[A^{\pm 1}]$-module generated by the set of isotopy classes of banded links in the interior of M quotiented by the following relations. First

$$L_\times = AL_\infty + A^{-1}L_0,$$

where $L_\times, L_0, L_\infty$ are any three banded links in M which are the same outside a small 3-ball but differ inside as in Figure 2. In this case, the triple $L_\times, L_0, L_\infty$ is called a Kauffman triple. The second relation satisfied in $\mathscr{K}(M)$ is $L \cup D = -(A^2 + A^{-2})\,L$ where L is any link in M and D is a trivial banded knot.

$L_\times$ L_0 L_∞

FIGURE 2. Kauffman triple

PROPOSITION 2.1. – *Let $L_\times$, L_0, L_∞ be a Kauffman triple in $\Sigma \times S^1$. If the sequences $\operatorname{tr}_p(L_0)$ and $\operatorname{tr}_p(L_\infty)$ have a cyclic expansion then the sequence $\operatorname{tr}_p(L_\infty)$ has a cyclic expansion. Moreover,*

$$P_{L_\times} = AP_{L_\infty} + A^{-1}P_{L_0}.$$

Proof. – Remark that the Witten-Reshetikhin-Turaev invariants satisfy the skein relation: $\operatorname{tr}_p(L_\times) = A\operatorname{tr}_p(L_\infty) + A^{-1}\operatorname{tr}_p(L_0)$. Therefore, it is enough to prove that cyclic expansions behave well under multiplication by $A^{\pm 1}$ and finite sum. For $\beta \geq 0$ an integer, $f = (f_0, \dots, f_{2\beta-1}) \in \mathcal{C}^{2\beta}$ we define

$$H_p(\beta, f) = \frac{1}{p} \sum_{\alpha=0}^{2\beta-1} \sum_{n \in \mathbb{Z}} A^{\alpha+2n\beta} f_\alpha\Big(\frac{n}{p}\Big).$$

Let $\beta > 0$ be an integer and let $f = (f_0, \dots, f_{2\beta-1}) \in \mathcal{C}^{2\beta}$. Applying a Taylor expansion to the functions $f_0, \dots, f_{2\beta-1}$ we have the following $(1 - A^{2\beta})H_p(\beta, f) = O(\frac{1}{p})$. Hence the sequence $A^{\pm 1} H_p(\beta, f)$ admits a cyclic expansion. This says that if a sequence admits a cyclic expansion then the same sequence multiplied by $A^{\pm 1}$ also admits a cyclic expansion.

Let $\beta' > 0$ be an integer divisible by β and set $\delta = \beta'/\beta$. One can make the Euclidean division $n = \delta m + \alpha'$ and write

$$H_p(\beta, f) = \frac{1}{p} \sum_{\alpha=0}^{2\beta-1} \sum_{\alpha'=0}^{\delta-1} \sum_{m \in \mathbb{Z}} A^{\alpha+2\beta\alpha'+2m\beta'} f_\alpha\Big(\frac{\delta m + \alpha'}{p}\Big).$$

By the previous argument and the estimation $f_\alpha(\frac{\delta m+\alpha'}{p}) = f_\alpha(\frac{\delta m}{p}) + O(\frac{1}{p})$, we find that there exists $g \in \mathcal{C}^{2\beta'}$ such that $H_p(f, \beta) = H_p(g, \beta') + O(\frac{1}{p})$. From this we can deduce that the sum of two sequences admitting cyclic expansions also admits a cyclic expansion. $\square$

This proposition means that the polynomial P_L extends to a $\mathbb{Z}[A^{\pm 1}]$-linear map $\eta : \mathcal{K}(\Sigma \times S^1) \to \mathbb{Z}[A^{\pm 1}]$ defined by $\eta(L) = P_L$. If L sits inside a ball $B \subset \Sigma \times S^1$, the polynomial P_L is just the Kauffman bracket of L. Hence the map η defines a section of the natural inclusion map $\mathcal{K}(B) \to \mathcal{K}(\Sigma \times S^1)$. We will construct η by giving its value on a $\mathbb{Z}[A^{\pm 1}]$-span of $\mathcal{K}(\Sigma \times S^1)$. That this map is well-defined is a non-trivial consequence of the existence of TQFT invariants and properties of the cyclic expansions.

Before that, we need to recall the results concerning the multiplicative structure of the skein module of the torus times an interval.

2.1.1. *Review of the skein module of the torus times an interval.* – We denote by T the torus $S^1 \times S^1$. It was proven by Frohman-Gelca and Sallenave (see [6, 14]) that the skein module of the torus T is isomorphic to the symmetric part of the quantum torus. Formally, we define the quantum torus as the non-commutative $\mathbb{Z}[A^{\pm 1}]$-algebra $\mathcal{T} = \mathbb{Z}[A^{\pm 1}]\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM - A^2 ML)$. Let σ be the involution of $\mathcal{T}$ defined by $\sigma(M^m L^l) = M^{-m} L^{-l}$.

PROPOSITION 2.2. – *For any $(m, l) \in \mathbb{Z}^2$ we set*

$$(m, l)_T = (-1)^{l+m} A^{ml} (M^m L^l + M^{-m} L^{-l}) \in \mathcal{T}^\sigma.$$

There is an isomorphism of algebras $\Upsilon : \mathcal{K}(T \times [0,1]) \xrightarrow{\sim} \mathcal{T}^\sigma$ which maps the standard banded curve in $T \times [0,1]$ with slope (m, l) to the element $(m, l)_T$ when $\gcd(m, l) = 1$.

We have for any $a, b, c, d \in \mathbb{Z}$ the following product-to-sum formula:

$$(a, b)_T (c, d)_T = A^{ad-bc}(a+c, b+d)_T + A^{-ad+bd}(a-c, b-d)_T.$$

For technical reasons, we will also need the following normalization:

$$\langle l, m\rangle_T = M^m L^l + M^{-m} L^{-l}.$$

REMARK 2.3. – This proposition implies that $\mathscr{K}(T \times [0,1])$ is generated as a $\mathbb{Z}[A^{\pm 1}]$-module by $\{\langle m,l\rangle_T \mid (m,l) \in \mathbb{Z}^2\} \cup \{\emptyset\}$.

2.1.2. *Weighted multicurves.* – We introduce weighted multicurves as a set of generators of $\mathscr{K}(\Sigma \times S^1)$ over $\mathbb{Z}[A^{\pm 1}]$.

DEFINITION 2.4. – Let $k \geq 0$, a k-multicurve is a collection of k disjoint essential simple oriented and non pairwise parallel closed curves.

DEFINITION 2.5. – Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$ be a k-multicurve. A weight on γ is an element $w = (a_1, b_1, \ldots, a_k, b_k) \in \mathbb{Z}^{2k}$ thought as an assignment of a pair (a_i, b_i) for each connected component γ_i of γ. A pair (γ, w) will be called a weighted multicurve.

Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$ be a k-multicurve on Σ with weight w. For $1 \leq j \leq k$, we choose a diffeomorphism between S^1 and γ_j respecting the orientation. Denote by T the torus $S^1 \times S^1$: we can embed T in $\Sigma \times S^1$ by sending the first factor to γ_j. This embedding extends to an embedding $\Phi_j : T \times [0,1] \to \Sigma \times S^1$ respecting the orientation of $\Sigma \times S^1$.

DEFINITION 2.6. – The skein associated to the weighted multicurve (γ, w) is the element $[\gamma, w] = \bigcup_{j=1}^{k} \Phi_j(\langle a_j, b_j\rangle_T) \in \mathscr{K}(\Sigma \times S^1)$ where $w = (a_1, b_1, \ldots, a_k, b_k)$ and $\langle a, b\rangle_T$ is the skein element defined in Subsection 2.1.1.

PROPOSITION 2.7. – *The set of weighted multicurves* $[\gamma, w]$ *spans the* $\mathbb{Z}[A^{\pm 1}]$*-module* $\mathscr{K}(\Sigma \times S^1)$.

Proof. – Let $L \subset \Sigma \times S^1$ be a banded link. Set $J = \{e^{i\pi t} \mid t \in [0,1]\}$ and $J^* = S^1 \setminus J$. There is a finite set $\{p_1, \ldots, p_n\}$ of banded points in Σ (perhaps empty) so that one has up to isotopy $L \cap (\Sigma \times J^*) = \{p_1, \ldots, p_n\} \times J^*$. Let L' be the intersection of L with $\Sigma \times J$. We have the following picture:

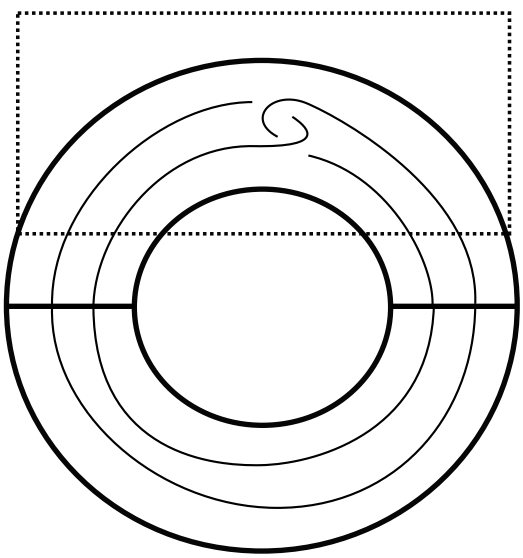


FIGURE 3. A banded link in $\Sigma \times S^1$

Applying the skein relations we can reduce to the case where the projection of L' on Σ is a finite disjoint union of banded simple closed curves and banded points. Hence we can find an integer $k \geq 0$ and a k-multicurve $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$ in Σ such that, up to isotopy, $L \subset \tilde{\gamma} \times S^1$ where $\tilde{\gamma} \subset \Sigma$ is a tubular neighborhood of γ. This says that L is in the image of the canonical map $\mathcal{K}(T \times [0,1])^{\otimes k} \to \mathcal{K}(\Sigma \times S^1)$ induced by the inclusion $\tilde{\gamma} \times S^1 \hookrightarrow \Sigma \times S^1$. Finally from Remark 2.3 we conclude that L is a $\mathbb{Z}[A^{\pm 1}]$-linear combination of weighted multicurves where the underlying multicurve is γ. ☐

Theorem 1.1 will follow from the following one which will be proved in Section 3.

THEOREM 2.8. – *For any weighted curve* $[\gamma, w]$ *the sequence* $\mathrm{tr}_p([\gamma, w])$ *has a cyclic expansion. Let k be the number of connected components of* γ. *One has* $\eta([\gamma, w]) = 2^k$ *if* $w = 0$ *and* 0 *otherwise.*

This theorem conjugated with the following lemma proves Proposition 1.8.

LEMMA 2.9. – *Let* T *be the standard torus and* α *be the 1-form on* $H_1(T, \mathbb{R})$ *given by* $\alpha_x(y) = \det(x, y)$ *where* det *stands for the intersection product.*

Let $x_0, \ldots, x_k$ *be vectors in* $H_1(T, \mathbb{Z})$. *We denote by* $P(x_0, \ldots, x_k)$ *the concatenation of the segments generated by the vectors* $x_0, \ldots, x_k$. *For* $x \in H_1(T, \mathbb{Z})$, *we denote by* $(x)_T$ *the corresponding vector in* $\mathcal{K}(T \times [0,1])$. *We have then the following formula:*

$$(x_0)_T \cdots (x_k)_T = \sum_{\varepsilon_1, \ldots, \varepsilon_k = \pm 1} A^{\int_{P(x_0, \varepsilon_1 x_1, \ldots, \varepsilon_k x_k)} \alpha} (x_0 + \varepsilon_1 x_1 + \cdots \varepsilon_k x_k)_T.$$

Proof. – This is a generalization of the product-to-sum formula which can be proved by an immediate induction, observing that

$$\int_{P(x_0, \ldots, x_{k+1})} \alpha = \int_{P(x_0, \ldots, x_k)} \alpha + \det(x_0 + \cdots + x_k, x_{k+1}). \qquad ☐$$

The main application of this lemma will concern the case when $x_0 + \varepsilon_1 x_1 + \cdots \varepsilon_k x_k = 0$ in which case Stokes formula and the equality $d\alpha = 2 \det$ imply

$$\int_{P(x_0, \varepsilon_1 x_1, \ldots, \varepsilon_k x_k)} \alpha = 2 \operatorname{Area}(x_0, \varepsilon_1 x_1, \ldots, \varepsilon_k x_k).$$

2.2. Application to the AMU conjecture

In this section, we prove Theorem 1.10.

We define the degree of a non-zero Laurent polynomial by the following formula:

$$\deg P = \inf\{n \in \mathbb{N}, P(A) = \sum_{i=-n}^{n} c_i A^i\}.$$

We fix a surface Σ of genus g and start the proof with two technical lemmas.

Definition 2.10. – Let G be a graph embedded in Σ with one oriented edge. Consider a tubular neighborhood V of G and a map $f : V \to S^1$ which is constant equal to 1 out of the labeled edge and makes one positive turn along the oriented edge. We define the following banded graph:

$$\hat{G} = \{(x, f(x)), x \in V\} \subset \Sigma \times S^1.$$

Moreover we denote by $\partial\hat{G}$ the banded link in $\Sigma \times S^1$ defined by the boundary of $\hat{G}$.

Given a banded graph $G \subset \Sigma \times S^1$. We say that a component C of G bounds a disk if there is an embedded disk $D \subset \Sigma \times S^1$ such that $D \cap G = C$.

Lemma 2.11. – *Let G be a graph embedded in Σ with one oriented edge and suppose that the component of G containing the arrow is not the boundary of a disk. Then the degree of $\eta(\partial\hat{G})$ is bounded by twice the number of components of $\partial\hat{G}$ bounding a disk in $\Sigma \times S^1$.*

Proof. – Each trivial component produces a factor $-[2]$ by the Kauffman relations: this shows that the bound is optimal. Remove all these components from $\partial\hat{G}$. We are reduced to prove that the polynomial $\eta(\partial\hat{G})$ has degree zero. Represent the directed edge by 2 arrows on the multicurve $\partial G \subset \Sigma$. The lemma follows from a case by case study of the possible configurations of arrows.

Case 1. – The two arrows belong to the same component and cancel. This component is non-trivial as we removed them before starting. We conclude with the following observation: the polynomial associated to a multicurve in Σ is a constant. Indeed, the multicurve is a union of parallel copies of curves of type $(1,0)_T$. Lemma 2.9 gives $\eta(1,0)_T^n = \binom{n}{n/2}$ which is an integer.

Case 2. – The two arrows belong to the same component γ and add. This is indeed impossible by considering γ as a boundary curve of the component of the banded graph containing the arrows. By construction, the two arrows are in opposite directions relatively to this orientation and hence cannot add.

Case 3. – The two arrows belong to two non parallel components. Hence, one of them is non-trivial and its neighborhood has the form $x = (1,0)_T^n(1,1)_T(1,0)_T^m$. Invoking Lemma 2.9, there are no closed paths in the expansion of x, hence $\eta(x) = 0$.

Case 4. – The two arrows belong to two parallel and trivial components. Then these curves bound a trivial circle containing the arrow which is forbidden by assumption.

Case 5. – The two arrows belong to parallel and consecutive non-trivial components. Then this corresponds to $x = (0,1)_T^n(1,1)_T^2(0,1)_T^m$. Again by Lemma 2.9, any closed path in the expansion of x has vanishing area, hence $\eta(x)$ is an integer.

Case 6. – The two arrows belong to parallel and non-consecutive non-trivial components. Then, noticing that there exists an embedded arc which joins them, we are in the configuration $(1,1)_T(1,0)^n_T(1,1)_T$. We conclude as in the Case 5. $\square$

LEMMA 2.12. – *Let G be an embedded graph with one oriented edge. Let n and e be respectively the number of vertices and edges of G. Let*

- *u be the number of components of $\partial\hat{G}$ bounding simply connected components of $\hat{G}$*
- *v be the number of components of $\partial\hat{G}$ bounding a disk and not counted in u.*

Suppose that G is Euler-incompressible and at most quadrivalent. Then we have the following inequalities

$$e + u \leq 2n \quad \textit{and} \quad v \leq n.$$

Moreover $v - n = u + e - 2n = 0$ implies that G is a disjoint union of circles (none of them bound a disk in Σ since G is Euler-incompressible).

Proof. – Let $(G_i)_{i\in I}$ be the connected components of G with negative Euler characteristic.

Let n_i (resp. e_i) be the number of vertices (resp. edges) of G_i. As G_i is not a circle, one has $n_i > 0$. Since G_i is at most quadrivalent, we have $e_i \leq 2n_i$ and hence $-\chi(G_i) \leq n_i$. We compute

$$-\chi(G) = e - n = -u + \sum_{i\in I} -\chi(G_i) \leq -u + \sum_{i\in I} n_i \leq -u + n,$$

from which we conclude $e + u \leq 2n$.

Now let w_i be the number of boundary components of G_i and v_i be the number of boundary components of G_i bounding a disk in Σ. Since G_i is Euler-incompressible, we have $\sum_{i\in I} v_i = v$.

Consider the closed surface S_i obtained by gluing disks to the boundary components of G_i. If S_i is a sphere then any boundary component of G_i is an Eulerian cycle which implies $v_i = 0$ by Euler-incompressibility. The equation $v_i \leq n_i$ follows in that case. If S_i is not a sphere, we have $\chi(S_i) \leq 0$. From $e_i \leq 2n_i$ and $v_i \leq w_i$ we get

$$-n_i + v_i \leq n_i - e_i + v_i \leq \chi(S_i) = n_i - e_i + w_i \leq 0. \tag{3}$$

In any cases, $v_i \leq n_i$, and summing over $i \in I$ we get $v \leq n$.

It remains to prove the last part of the lemma. Suppose that $v - n = u + e - 2n = 0$ and fix $i \in I$. We have $v_i = n_i$. If S_i is a sphere we have $v_i = n_i = 0$: this is not possible since a graph should have at least one vertex. Therefore (3) implies $\chi(S_i) = 0$ and $w_i = v_i$. Hence G_i is embedded in a torus S_i in such a way that any of its boundary components bounds a disk in Σ. This allows to define an embedding $S_i \to \Sigma$: since the genus of Σ is at least two, this is not possible. We conclude that I is empty and G is a disjoint union of circles. $\square$

Proof of Theorem 1.10. – We denote by $[\hat{\gamma}, 3] \in \mathscr{K}(\Sigma \times S^1)$ the skein element obtained by coloring $\hat{\gamma}$ (see Definition 1.5) by the color 3 (2 in the [3] setting). In the setting of [3], this corresponds to the insertion of the idempotent f_2 in $\partial\hat{\gamma}$ as shown in Figure 4.

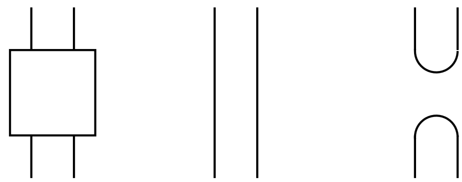
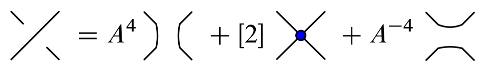


FIGURE 4. The idempotent f_2

For any banded graph $\Gamma \subset \Sigma \times S^1$, we will denote by $[\Gamma, 3]$ the element of $\mathscr{K}(\Sigma \times S^1) \otimes \mathbb{Q}(A)$ obtained by inserting the idempotent f_2 in $\partial\Gamma$ at all edges of Γ. Setting $[2] = A^2 + A^{-2}$, these skein elements satisfy the following skein relation:

(4) 

We extend the map $\eta : \mathscr{K}(\Sigma \times S^1) \to \mathbb{Z}[A^{\pm 1}]$ by tensoring with $\mathbb{Q}(A)$ so that we can evaluate banded colored graphs. The proof will consist in investigating all terms in the state sum suggested by Equation (4).

Set $\Gamma = \gamma(S^1) \subset \Sigma$ and orient the edge going through the base point. Let V be the set of vertices of Γ. Given $S : V \to \{-1, 0, 1\}$, we define Γ_S to be the graph obtained by transforming all vertices of Γ as follows: if a vertex evaluates to 1 under S it is replaced by a positive smoothing, if a vertex evaluates to -1 under S it is replaced by a negative smoothing and if a vertex evaluates to 0 under S it is not changed. Γ_S can be viewed as a quadrivalent graph in Σ with one directed edge. Consider the banded graph $\hat{\Gamma}_S$ in $\Sigma \times S^1$ defined by Definition 2.10.

For $S : V \to \{-1, 0, 1\}$, we denote by a_S, b_S, n_S respectively the number of preimages of $1, -1, 0$. Applying the rule (4) to all the crossings of γ gives

$$\eta([\hat{\gamma}, 3]) = \sum_{S:V \to \{-1,0,1\}} A^{4(a_S - b_S)} [2]^{n_S} \eta[\hat{\Gamma}_S, 3].$$

Consider one term of the sum associated to a map S. To avoid heavy notations, we remove the subscript S to a, b and n. We have the decomposition $N = n + a + b$ and n is the number of vertices of Γ_S. The original graph Γ was a quadrivalent graph with N vertices hence satisfying $\chi(\Gamma) = -N$. The graph Γ_S being obtained by smoothing $a + b$ vertices satisfies $\chi(\Gamma_S) = -N + a + b = -n$.

Computing $\eta[\hat{\Gamma}_S, 3]$ involves inserting idempotents in all edges of Γ_S. Let E_S be the set of edges of Γ_S. For $\xi : E_S \to \{0, 1\}$, we define $\Gamma_{S,\xi}$ to be the graph obtained by deleting all edges e such that $\xi(e) = 1$. We define $s_\xi = \mathrm{card}(\xi^{-1}(\{1\}))$. According to the rule of Figure 4, we have

$$\eta[\hat{\Gamma}_S, 3] = \sum_{\xi : E_S \to \{0,1\}} [2]^{-s_\xi} \eta(\partial\hat{\Gamma}_{S,\xi}).$$

Now fixing $\xi : E_S \to \{0, 1\}$ we want to bound $\deg A^{4a-4b}[2]^{n-s}\eta(\partial\hat{\Gamma}_{S,\xi})$ (here s stands for s_ξ). Note that $\chi(\Gamma_{S,\xi}) = -n + s$ and recall that as Γ is Euler-incompressible, this property still holds for $\Gamma_{S,\xi}$. Hence Lemma 2.11 applies for $\Gamma_{S,\xi}$ and we have

$$\deg A^{4a-4b}[2]^{n-s}\eta(\partial\hat{\Gamma}_{S,\xi}) \le 4|a - b| + 2(n - s + c), \tag{5}$$

where c is the number of components of $\partial\hat{\Gamma}_{S,\xi}$ which bound a disk in Σ. Let us write $c = u + v$, where u is the number of simply connected components of $\Gamma_{S,\xi}$.

We denote by T the left-hand side of Equation (5). Supposing that $a \geq b$, we can apply Lemma 2.12 to $\Gamma_{S,\xi}$. Hence (5) gives $T \leq 4N - 8b + 2(v-n) + 2(u-s) \leq 4N$ (we recall that $a+b+n = N$ and $\chi(\Gamma_{S,\xi}) = -n+s = n-e$). This is strictly less that $4N$ unless $b = 0$ and $n = 0$ (by Lemma 2.12), that is unless we smoothed all the vertices of Γ in the positive way. Doing so, Γ_S is a collection of non-trivial curves (since Γ is Euler-incompressible) colored by 3, one of them being oriented.

The computation can be done locally in an annulus times an interval. Expanding the idempotent f_2, an unoriented curve colored by 3 is $[(1,0),3] = (1,0)_T^2 - [\emptyset] = (2,0)_T + [\emptyset]$ and similarly, an unoriented curve colored by 3 is equal to $[(1,1),3] = (2,2)_T + [\emptyset]$. Hence expanding $\eta([(1,0),3]^n[(1,1),3][(1,0),3]^m)$ with the help of Lemma 2.9 gives a positive result. Repeating the argument for negative smoothing, we finally proved Theorem 1.10. $\square$

3. Proof of the cyclic expansion

3.1. Curve operators acting on the torus

Let $(e_l)_{l=1,\dots,r-1}$ be the basis of $V_p(T)$ obtained by filling T with $D^2\times S^1$ and coloring the core curve by l. Notice that we use the convention of [4], that is e_l corresponds to $(-1)^{l-1}u_{l-1}$ where u_l is the basis used in [3]. The algebra $\mathcal{K}(T\times[0,1])$ acts naturally on $V_p(T)$ by the axioms of a TQFT. Let us cover this action by an action of the quantum torus.

Define $E_p = \bigoplus_{l\in\mathbb{Z}/p\mathbb{Z}} K_p(\sqrt{2})\theta_l$ as a p-dimensional Hermitian vector space formally generated by θ_l where $\langle\theta_l,\theta_m\rangle = \delta^p_{l-m}$. By convention, we set $\delta^p_i = 1$ if $p|i$ and 0 otherwise. The quantum torus $\mathcal{T}$ acts on E_p by the formulas $M\theta_l = A^{2l}\theta_l$ and $L\theta_l = \theta_{l+1}$.

LEMMA 3.1. – *Let σ be the involution of E_p defined by $\sigma(\theta_l) = \theta_{-l}$. Then the map $V_p(T) \to E_p$ defined by $e_l \mapsto \frac{1}{\sqrt{2}}(\theta_l - \theta_{-l})$ is an isometry onto the σ-antisymmetric part which commutes with the action of $\mathcal{T}^\sigma$.*

Proof. – This is well-known, see for instance [4, Section 2]. Indeed, we get the expected formulas

$$\begin{aligned}(1,0)_T e_l &= (-M - M^{-1})e_l = (-A^{2l} - A^{-2l})e_l\\ (0,1)_T e_l &= (-L - L^{-1})e_l = -e_{l+1} - e_{l-1}.\end{aligned}\quad\square$$

LEMMA 3.2. – *We have the formula*

$$\langle (a,b)_T e_l, e_m\rangle = A^{2a(l+b)}(\delta^p_{l+b-m} - \delta^p_{l+b+m}) + A^{2a(-l+b)}(\delta^p_{l-b-m} - \delta^p_{l-b+m}).$$

Proof. – This is a direct computation, replacing e_l with $\frac{1}{\sqrt{2}}(\theta_l - \theta_{-l})$. $\square$

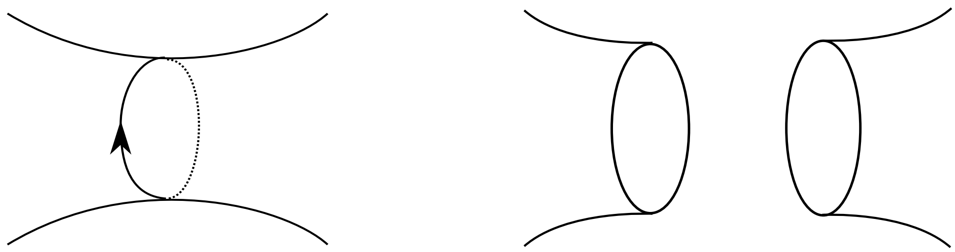


FIGURE 5. Surgery on Σ

3.2. Reduction to weighted multicurves

We remark that because of Proposition 2.7 and Proposition 2.1, it is enough to prove that any weighted multicurve has a cyclic expansion.

Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$ be a k-multicurve. We denote by Σ_γ the surface obtained by surgery on every component of γ. For any i, we add two marked points p_i, q_i from each side of the handle used in the surgery at γ_i as in Figure 5.

Finally, given a vector $x = (n_1, m_1, \dots, n_k, m_k) \in \{1, \dots, r-1\}^{2k}$ thought as a coloring of the points p_i, q_i, we set

$$C_{\gamma,p}(x) = \dim V_p(\Sigma_\gamma, n_1, m_1, \dots, n_k, m_k).$$

Proof of Theorem 2.8. – Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$ be a multicurve with weight $w = (a_1, b_1, \dots, a_k, b_k)$. Let γ_j' and γ_j'' be two parallel copies of γ_j such that $\Phi_j(T \times \{0,1\}) = \gamma_j' \times S^1 \cup \gamma_j'' \times S^1$. Consider the basis $(e_i)_{i=1,\dots,r-1}$ of $V_p(\gamma_j' \times S^1)$ obtained by filling $\gamma_j' \times S^1$ with $D^2 \times S^1$ (the same with γ_j'').

The TQFT axioms imply the following equation where n and m are elements of $\{1, \dots, r-1\}^k$.

$$\operatorname{tr}_p[\gamma, w] = \sum_{n,m} \left(\prod_{j=0}^{k} \langle \langle a_j, b_j \rangle_T \, e_{n_j}, e_{m_j} \rangle \right) \frac{C_{\gamma,p}(n,m)}{Z_p(\Sigma \times S^1)}.$$

Using Lemma 3.2, we have

$$\operatorname{tr}_p[\gamma, w] = \sum_{n,m} \prod_{j=1}^{k} \Big(A^{2n_j a_j} (\delta^p_{n_j+b_j-m_j} - \delta^p_{n_j+b_j+m_j}) + A^{-2n_j a_j} (\delta^p_{n_j-b_j-m_j} - \delta^p_{n_j-b_j+m_j}) \Big) \frac{C_{\gamma,p}(n,m)}{Z_p(\Sigma \times S^1)}. \tag{6}$$

Introducing signs ε and ζ in $\{\pm 1\}^k$, we can rewrite the sum in the following way

$$\operatorname{tr}_p[\gamma, w] = \sum_{n,m,\varepsilon,\zeta} \prod_j A^{2\varepsilon_j n_j a_j} \zeta_j \delta^p_{n_j+\varepsilon_j b_j - \zeta_j m_j} \frac{C_{\gamma,p}(n,m)}{Z_p(\Sigma \times S^1)}.$$

We conclude using Proposition 3.4 where x stands for the $2k$-tuple $(n_1, m_1, \dots, n_k, m_k)$, $l_j(x) = n_j - \zeta_j m_j$, $\alpha_j = -\varepsilon_j b_j$ for $j = 1, \dots, k$, $D = \gcd(a_1, \dots, a_k)$ and $l_0(x) = \frac{1}{D} \sum_j n_j a_j$. $\square$

3.3. Counting points in polytopes

DEFINITION 3.3. – Let N be a positive integer and Λ be an affine lattice of $\mathbb{R}^N$. We denote by $\mathrm{covol}(\Lambda)$ the volume for the Lebesgue measure of $\mathbb{R}^E/\vec{\Lambda}$ where $\vec{\Lambda}$ is the vectorial part of the affine lattice Λ.

PROPOSITION 3.4. – *Let D be an integer, $L = (l_0, \ldots, l_k)$ be a surjective linear map $L : \mathbb{Z}^{2k} \to \mathbb{Z}^{k+1}$, and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k$ some integers. Then setting*

$$F_p(l,\alpha) = \sum_{x \in \{1,\cdots,r-1\}^{2k}} A^{2Dl_0(x)} \prod_{j=1}^{k} \delta^p_{l_j(x)-\alpha_j} C_{\gamma,p}(x), \tag{7}$$

the quantity $\frac{F_p(l,\alpha)}{Z_p(\Sigma\times S^1)}$ has a cyclic expansion.

Proof. – Let H be a disjoint union of handlebodies bounding Σ_γ and Γ be a unitrivalent banded graph embedded in H such that $\Gamma \cap \Sigma = \partial\Gamma = \{p_1, q_1, \ldots, p_k, q_k\}$ and such that H retracts on Γ. We will denote by E the set of edges of Γ and identify an element of $\partial\Gamma$ with the edge incident to it.

Then it follows from [3, Theorem 4.11] that $C_{\gamma,p}(x)$ is the cardinality of the set

$$(\mathrm{int}\, rP) \cap \Lambda \cap \{c_i = x_i, i \in \partial\Gamma\} \subset \mathbb{R}^E,$$

where we have set

$$P = \{\tau : E \to [0,1], \tau_i \leq \tau_j + \tau_k, \tau_i + \tau_j + \tau_k \leq 2\}$$
$$\Lambda = \{c : E \to \mathbb{Z}, c_i + c_j + c_k \text{ odd}\}.$$

In these two expressions (i,j,k) runs over all triples of edges in E incident to a same trivalent vertex of Γ.

This comes from the fact that $V_p(\Sigma_\gamma, x)$ has a basis obtained by coloring the edges of Γ with c for $c \in r\,\mathrm{int}\,P \cap \Lambda$ satisfying $c_i = x_i$ for all $i \in \partial\Gamma$, see Theorem 4.11 in [3].

Observe that for all $i \in \mathbb{Z}$, $\delta^p_i = \sum_{s\in\mathbb{Z}} \delta_{i+ps}$ where $\delta_i = 1$ if $i = 0$ and 0 otherwise. Plugging this into Formula (7) gives $F_p(l,\alpha) = \sum_{s\in\mathbb{Z}^k} G_p(l, \alpha + ps)$ where G_p has the same definition as F_p with δ^p replaced with δ. We will see that this sum is actually finite so that we are reduced to study the cyclic expansion of G_p.

As an example of what will follow, taking $\gamma = \emptyset$, we have the equality $Z_p(\Sigma \times S^1) = \dim V_p(\Sigma) = \mathrm{card}\, P \cap \frac{1}{r}\Lambda$. Comparing the number of integral points with the volume gives the following estimate, proved in detail in Lemma 3.5:

$$Z_p(\Sigma \times S^1) = r^{\dim P} \frac{\mathrm{vol}\, P}{\mathrm{covol}\, \Lambda} + O(r^{\dim P - 1}).$$

Here $\dim P = 3g-3$ if $g > 1$ and 1 if $g = 1$, $\mathrm{vol}(P)$ is computed from the Lebesgue measure on $\mathbb{R}^E$ and $\mathrm{covol}(\Lambda)$ is defined in Definition 3.3. One can compute $\mathrm{covol}(\Lambda) = 2^{2g-3}$ if $g > 1$ and 1 if $g = 1$. Notice that up to a normalization factor, the leading order in r is the volume of the moduli space $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}_2)/\mathrm{SU}_2$ with the volume form associated to the Atiyah-Bott-Goldman symplectic structure (see [8] and also [13]).

The proof of the general case will rely on the same kind of estimations. More precisely, let V_n be the affine subspace of $\mathbb{R}^E$ given by the equations $l_j(x) = \alpha_j$ for $j = 1, \dots, k$ and $l_0(x) = n$. Then we have $G_p(l, \alpha) = \sum_{n \in \mathbb{Z}} A^{2nD} g_r(n)$ where

$$g_r(n) = \operatorname{card}(\operatorname{int} rP) \cap V_n \cap \Lambda.$$

We would like to apply Lemma 3.5 to this situation in order to estimate $g_r(n)$. The lemma applies to the polytope $rP \cap V_n$ included in the euclidean space V_n provided that $rP \cap V_n$ has non-empty interior and $V_n \cap \Lambda$ is an affine lattice in V_n. If the first assumption does not hold, it implies that $V_n \cap \operatorname{int}(rP) = \emptyset$ and hence $g_r(n) = 0$. Next, the linear part $\vec{V}$ of V_n is by hypothesis the kernel of the linear forms $l_0, \dots, l_k$ which are integral and linearly independent. It follows that $\vec{V} \cap \vec{\Lambda}$ is a free abelian group of rank $N - k - 1$ where we have set $N = \operatorname{card} E = \dim P$. Hence, $V_n \cap \Lambda$ is a lattice in V_n if and only if it is non-empty.

We observe that this last condition only depends on n modulo 2. Indeed, if $c : E \to \mathbb{Z}$ satisfies the equation defining V_n and Λ modulo 2, we can add to c a function $d : \partial\Gamma \to \mathbb{Z}$ which satisfies $2l_0(d) = n - l_0(c)$ and $2l_j(d) = \alpha_j - l_j(c)$. Such a function exists by the assumption that L is surjective. It follows that we need to specify the parity of n. In the sequel, we will denote by $\nu \in \mathbb{Z}/2\mathbb{Z}$ a solution of the preceding system and restrict to those n's which are congruent to ν.

For $n = \nu \bmod 2$ we have

$$g_r(n) = r^{N-k-1} \frac{\operatorname{vol} P \cap \frac{1}{r} V_n}{\operatorname{covol} V_n \cap \Lambda} + O(r^{N-k-2}).$$

We have $\operatorname{covol} V_n \cap \Lambda = \operatorname{covol} \vec{V} \cap \vec{\Lambda}$, showing that this quantity does depend on n.

On the other hand, the function $\mathcal{V}(\alpha_0, \dots, \alpha_k) = \operatorname{vol}\{\tau \in P, l_j(\tau) = \alpha_j, j = 0, \dots, k\}$ is a piecewise polynomial function with compact support . It follows that

$$g_r(n) = \frac{r^{N-1-k}}{\operatorname{covol} \vec{V} \cap \vec{\Lambda}} \mathcal{V}\Big(\frac{n}{r}, \frac{\alpha_1}{r} + 2s_1, \dots, \frac{\alpha_k}{r} + 2s_k\Big) + O(r^{N-k-2}),$$

which can be written $g_r(n) = r^{N-1-k} f(\frac{n}{r}) + O(r^{N-k-2})$ for a piecewise polynomial function f with compact support. This finally proves the proposition. $\square$

Lemma 3.5. – *Let P be a polytope with non-empty interior in a Euclidean space E of dimension N and Λ be an affine lattice in E. We define the radius of Λ as the constant (independent of λ) $\rho(\Lambda) = \sup\{d(x, \lambda)$, where $x \in E$ satisfies $d(x, \lambda) \le d(x, \mu)\ \forall \mu \in \Lambda\}$.*

$$\left| \operatorname{card}(\operatorname{int} P \cap \Lambda) - \frac{\operatorname{vol} P}{\operatorname{covol} \Lambda} \right| \le \sum_F \frac{b_{c_F} \operatorname{vol} F\, \rho(\lambda)^{c_F}}{\operatorname{covol}(\Lambda)}.$$

In this formula F runs over the faces of P of positive codimension c_F and b_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. – For any $\lambda \in \Lambda$ define its Voronoi cell

$$V(\lambda) = \{x \in \mathbb{R}^N \text{ s.t. } d(x, \lambda) \le d(x, \mu)\ \forall \mu \in \Lambda\}.$$

Then we have $\bigcup_{\lambda \in \Lambda} V(\lambda) = E$, $\operatorname{vol} V(\lambda) = \operatorname{covol}(\Lambda)$ and $\operatorname{int} V(\lambda) \cap \operatorname{int} V(\mu) = \emptyset$ for $\lambda \ne \mu$. Consider $Q = \bigcup_{\lambda \in \operatorname{int} P \cap \Lambda} V(\lambda)$ such that $\operatorname{vol} Q = \operatorname{card}(\operatorname{int} P \cap \Lambda) \operatorname{vol} V(\lambda)$. Any point x in the symmetric difference $P\Delta Q$ is at distance at most $\rho = \rho(\Lambda)$ from the boundary of ∂P as we have $\rho = \sup\{d(x,\lambda), x \in V(\lambda)\}$.

Hence, $|\operatorname{vol} P - \operatorname{vol} Q| \leq \operatorname{vol} P\Delta Q \leq \operatorname{vol} N_\rho(\partial P)$, where we have set

$$N_\rho(\partial P) = \{x \in E, d(x, \partial P) \leq \rho\}.$$

Any point in $N_\rho(\partial P)$ has its closest point in ∂P belonging to some face F of P of positive codimension. This gives the following estimation, proving the lemma.

$$\operatorname{vol} N_\rho(\partial P) \leq \sum_F \operatorname{vol} F\, b_{c_F} \rho^{c_F}. \qquad \square$$

3.4. A geometric interpretation of the trace

Consider a surface Σ and a weighted multicurve (γ, w). The aim of this subsection is to provide a geometric interpretation for the evaluations $\operatorname{ev}_{A_p^\sigma} \operatorname{tr}_p[\gamma, w]$.

Recall that we have set $X(\Sigma)$ to be the space of conjugacy classes of irreducible representations $\rho : \pi_1(\Sigma) \to \mathrm{SU}_2$, ν the volume form and $\nu_g = \int_{X(\Sigma)} \nu$. Every component γ_i of γ defines a map $\theta_i : X(\Sigma) \to [0,1]$ by the formula

$$\operatorname{tr} \rho(\gamma_i) = 2\cos \pi\theta_i([\rho]). \tag{8}$$

Suppose that γ is a pants decomposition of Σ and denote by Γ the corresponding graph. Then, the functions are well-known to be action variables on $X(\Sigma)$, see [8, 13]. Precisely, these maps Poisson commute and the joint map $\Theta = (\theta_1, \dots, \theta_k) : X(\Sigma) \to \mathbb{R}^k$ has image the polytope P defined in the proof of Proposition 3.4. The Hamiltonian flows of the maps θ_i induce an action of $\mathbb{R}^E$ which acts transitively on the fibers of Θ. Moreover, the kernel of this action is precisely $\frac{1}{2}\vec{\Lambda}$.

We prove here the following theorem which implies Theorem 1.11 and extends the main result of [13].

THEOREM 3.6. – *Let σ be an odd integer and set $A_p = -e^{\frac{i\pi}{p}}$. Suppose that p goes to infinity such that σ and $2p$ are coprime. Then*

$$\lim_{p\to\infty} \operatorname{ev}_{A_p^\sigma} \operatorname{tr}_p[\gamma, w] = \frac{1}{\nu_g} \int_{X(\Sigma)} \prod_j \operatorname{tr} \rho(\gamma_j)^{a_j \sigma} d\nu(\rho)$$

if $\sum_j b_j$ is even and 0 *otherwise.*

Before entering into the proof, let us see how this theorem implies Theorem 1.11. For any banded link $L \subset \Sigma \times S^1$, we set $\Lambda_\sigma(L) = \lim_{p\to\infty} \operatorname{ev}_{A_p^\sigma} \operatorname{tr}_p(L)$. The compatibility with Kauffman relations implies that Λ_σ is a linear form on $\mathscr{K}(\Sigma\times S^1, -1) = \mathscr{K}(\Sigma \times S^1) \underset{\mathbb{Z}[A^{\pm 1}]}{\otimes} \mathbb{C}$. Here $\mathbb{Z}[A^{\pm 1}]$ acts on $\mathbb{C}$ by the formula $P(A).z = P(-1)z$.

On the other hand, let $\gamma = \gamma_1 \cup \dots \cup \gamma_k$ be a pants decomposition of Σ and $\mathscr{K}^\gamma(\Sigma \times S^1)$ be the sub-module generated by banded links living in N, the product of a neighborhood of γ with the circle. We denote by $\tau : X(\Sigma \times S^1) \to \{\pm 1\}$ the function satisfying $\rho(t) = \tau(\rho)\mathrm{Id}$. Any element of $\mathscr{K}^\gamma(\Sigma \times S^1)$ may be viewed as a function on the character variety $X(\Sigma \times S^1)$

which depends only on $\theta_1,\dots,\theta_k$ and τ. That is, we can associate to any banded link $L\subset N$ a function $F_L(\theta_1,\dots,\theta_k,\tau)$ satisfying

$$\prod_i -\operatorname{tr}\rho(L_i) = F_L(\theta_1(\rho),\dots,\theta_k(\rho),\tau(\rho)).$$

In this setting, we define a linear form $\Lambda'_\sigma : \mathscr{K}^\gamma(\Sigma\times S^1)\to\mathbb{C}$ by the formula $\Lambda'_\sigma(L) = \int F_L(\sigma\theta_1,\dots,\sigma\theta_k,\tau)d\mu$ where μ is the image of ν/ν_g by $(\theta_1,\dots,\theta_k,\tau)$. Hence, by construction, we have $\Lambda'_1(f) = \frac{1}{\nu_g}\int_{X(\Sigma\times S^1)} f d\nu$. Theorem 1.11 consists in proving the formula $\Lambda_\sigma(L) = \Lambda'_\sigma(L)$ for a banded link L projecting without crossing. On the other hand, weighted multicurves $[\gamma,w]$ with fixed γ are generators for $\mathscr{K}^\gamma(\Sigma\times S^1)$, hence it is sufficient to prove the equality $\Lambda_\sigma[\gamma,w] = \Lambda'_\sigma[\gamma,w]$ for all weights w.

Fix a weight $w = (a_1,b_1,\dots,a_k,b_k)$. In the torus $\gamma_j\times[0,1]$, the element $\langle a_j,b_j\rangle_T$ corresponds to the function mapping the representation $[\rho]$ to $\operatorname{tr}\rho({\gamma_j}^{a_j}t^{b_j})$. Hence, $F=\prod_j 2\cos(\pi a_j\theta_j)\tau^{b_j}$ and $\Lambda'_\sigma[\gamma,w] = \frac{1}{2\nu_g}\int_{X(\Sigma\times S^1)}\prod_j \operatorname{tr}\rho({\gamma_j}^{\sigma a_j})\tau^{\sigma b_j}\,d\nu(\rho)$.

As $X(\Sigma\times S^1)$ is a disjoint union of two copies of $X(\Sigma)$ defined by the equations $t=1$ and $t=-1$, the integral vanishes if $\sum_j b_j$ is odd, and otherwise reduces to the integral in the right hand side of the equation in Theorem 3.6. The conclusion follows.

Proof of Theorem 3.6. – With the notation of the proof of Proposition 2.8, we have the formula

$$\operatorname{ev}_{A_p^\sigma}\operatorname{tr}_p[\gamma,w] = \sum_{m,n,\varepsilon,\zeta} e^{\frac{2i\pi\sigma}{p}\sum_j\varepsilon_j a_j n_j}\prod_j \delta^p_{n_j+\varepsilon_j b_j-\zeta_j m_j}\frac{C_{\gamma,p}(m,n)}{Z_p(\Sigma\times S^1)}.$$

For $C_{\gamma,p}(m,n)$ to be non-zero we need to have $0<m_j,n_j<r$. At the same time there exists $k_j\in\mathbb{Z}$ such that $n_j-\zeta_j m_j+\varepsilon_j b_j = 2rk_j$. Hence if $\zeta_j=1$, we have necessarily $k_j=0$. If $\zeta_j=-1$, we have either $k_j=0$ and $n_j+m_j=-\varepsilon_j b_j$ or $k_j=1$ and $n_j+m_j=2r-\varepsilon_j b_j$. The number of solutions is bounded with respect to r, and such terms can be neglected in the sum. Hence, we can suppose that $\zeta_j=1$ and replace δ^p with δ. This gives

$$\operatorname{ev}_{A_p^\sigma}\operatorname{tr}_p[\gamma,w] = \sum_{n,\varepsilon} e^{\frac{2i\pi\sigma}{p}\sum_j\varepsilon_j a_j n_j}\prod_j\frac{C_{\gamma,p}(n+\varepsilon b,n)}{Z_p(\Sigma\times S^1)} + O\Big(\frac{1}{p}\Big),$$

where $n+\varepsilon b$ is a shorthand for the tuple $(n_j+\varepsilon_j b_j)$.

Let Γ be the graph adapted to γ introduced in the proof of Proposition 3.4. We also introduce the polytope P and the lattice Λ.

Applying Lemma 3.5, we get that $C_{\gamma,p}(m,n)$ is either zero or equivalent to $\frac{\operatorname{vol}(rP\cap V_{m,n})}{\operatorname{covol}(\Lambda\cap V_{m,n})}$, where $V_{m,n}$ is the affine space given by the equations $c_{p_i}=m_i$ and $c_{q_j}=n_j$ for $j=1,\dots,k$. Supposing non-triviality, we get further

$$C_{\gamma,p}(m,n) = r^{\operatorname{card}E-2k}\frac{\operatorname{vol}(P\cap V_{\frac{m}{r},\frac{n}{r}})}{\operatorname{covol}(\Lambda\cap V_{m,n})} + O(r^{\operatorname{card}E-2k-1}).$$

Denote by $\hat\Gamma$ the graph obtained from Γ by gluing the vertices p_j and q_j for $j=1,\dots,k$. The graph $\hat\Gamma$ has an associated polytope and affine lattice that we respectively denote by $\hat P$ and $\hat\Lambda$. Then, the same reasoning gives $Z_p(\Sigma\times S^1) = r^{\operatorname{card}E-k}\frac{\operatorname{vol}\hat P}{\operatorname{covol}\hat\Lambda} + O(r^{\operatorname{card}E-k-1})$.

As $\frac{m}{r}$ and $\frac{n}{r}$ are equal up to $O(\frac{1}{r})$, the volume $\operatorname{vol}(P\cap V_{\frac{m}{r},\frac{n}{r}})$ is equal to $\operatorname{vol}\hat P\cap\hat V_{\frac{n}{r}}$ up to $O(\frac{1}{r})$ where $\hat V_{\frac{n}{r}}$ is the subspace of $\mathbb{R}^{\hat E}$ defined by the equation $c_j=\frac{n_j}{r}$.

Let us deal with the term covol$(\Lambda \cap V_{m,n})$ by considering first the case of $\hat{\Gamma}$. Let $C_*(\hat{\Gamma}, \mathbb{Z}/2\mathbb{Z})$ be the cellular complex associated to $\hat{\Gamma}$. A coloring $c : \hat{E} \to \mathbb{Z}$ reduces modulo 2 to an element of $C_1(\hat{\Gamma}, \mathbb{Z}/2\mathbb{Z})$. In this setting, the condition defining the vectorial part of $\hat{\Lambda}$ is simply to be a cycle. It follows that the index of $\hat{\Lambda}$ in $\mathbb{Z}^{\hat{E}}$ is the cardinality of $C_1(\hat{\Gamma}, \mathbb{Z}/2\mathbb{Z})/\ker \partial$ which is $2^{\operatorname{card} \hat{E} - \dim H_1(\hat{\Gamma}, \mathbb{Z}/2\mathbb{Z})}$.

The same reasoning applies to Γ, supposing that $\Lambda \cap V_{m,n}$ is non-empty. The vectorial part of $\Lambda \cap V_{m,n}$ is a map $c : E \to \mathbb{Z}$ which vanishes at boundary points and such that $c(e_i) + c(e_j) + c(e_k)$ is even for all triples (i, j, k) incident to a same vertex. Its reduction modulo 2 is a cycle in $C_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$, hence, the index of $\Lambda \cap V_{m,n}$ is $2^{\operatorname{card} E - 2k - \dim H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})}$. Supposing it is non-zero, we get the following estimate where $h = -\dim H_1(\hat{\Gamma}, \mathbb{Z}/2\mathbb{Z}) + \dim H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$.

$$\frac{C_{\gamma,p}(m,n)}{Z_p(\Sigma \times S^1)} = r^{-k} \frac{\operatorname{vol} P \cap \hat{V}_{\frac{n}{r}}}{\operatorname{vol} \hat{P}} 2^{k+h} + O(r^{-k-1}). \tag{9}$$

Let l be the number of connected components of Γ. The exact sequence of the pair $(\hat{\Gamma}, \Gamma)$ gives the formula $h + k - l + 1 = 0$ which replaces the power of 2 in Equation (9) with 2^{l-1}.

Consider now the problem of the non-vanishing of $C_{\gamma,p}(m,n)$. An element $c \in \Lambda \cap V_{m,n}$ satisfies $c(p_i) = m_j$, $c(q_i) = n_i$ and $c(e_i) + c(e_j) + c(e_k) = 1$ for all triple of edges e_i, e_j, e_k incident to a same vertex. Summing $c + 1$ twice over each edge of any component Γ_i of Γ gives modulo 2 the equality $l_i(c) = \operatorname{card} \partial\Gamma_i$ where $l_i(c) = \sum_{x \in \partial\Gamma_i} c(x)$. Summing over the connected components and observing that $m_j + n_j = b_j$ modulo 2, this gives the identity $\sum_j b_j = 0$.

Conversely, a simple argument involving the homology of $(\Gamma_i, \partial\Gamma_i)$ modulo 2 implies that $C_{\gamma,p}(m,n)$ is non-zero if and only if $l_i(m,n) = 0$ for all i. Hence, assuming that $\sum_j b_j$ is even, we get

$$\operatorname{ev}_{A_p^\sigma} \operatorname{tr}_p[\gamma, w] = \frac{1}{r^k} \sum_{\substack{n,\varepsilon \\ l_i(n+\varepsilon b, n) = 0}} e^{\frac{2i\pi\sigma}{p} \sum_j \varepsilon_j a_j n_j} 2^{l-1} \frac{\operatorname{vol} P \cap \hat{V}_{\frac{n}{r}}}{\operatorname{vol} \hat{P}} + O(\frac{1}{p}).$$

The parity conditions l_i should divide the sum by 2^l but the sum $\sum_{i=1}^{l} l_i(n + \varepsilon b, n) = \sum b_i$ vanishes modulo 2. Hence they divide the sum by 2^{l-1} and this factor cancels.

We recognize a Riemann sum: setting $f(x) = \frac{\operatorname{vol} \hat{P} \cap \hat{V}_x}{\operatorname{vol} \hat{P}}$ we get

$$\begin{aligned} \lim_{p\to\infty} \operatorname{ev}_{A_p} \operatorname{tr}_p[\gamma, w] &= \int_{\mathbb{R}} \sum_{\varepsilon} e^{2i\pi\sigma \sum_j \varepsilon_j a_j x_j} f(x) dx \\ &= \int_{\mathbb{R}} \prod_j 2\cos(2\pi\sigma a_j x_j) f(x) dx. \end{aligned}$$

This proves the proposition as $f(x)dx$ is equal to the push-forward $\Theta_* \frac{\nu}{\nu_g}$ and by construction $\rho(\gamma_j)$ is conjugate to $\begin{pmatrix} e^{i\pi\theta_j} & 0 \\ 0 & e^{-i\pi\theta_j} \end{pmatrix}$. □

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FLAG-APPROXIMABILITY OF CONVEX BODIES AND VOLUME GROWTH OF HILBERT GEOMETRIES

BY CONSTANTIN VERNICOS AND CORMAC WALSH

ABSTRACT. – We introduce the flag-approximability of a convex body to measure how easy it is to approximate by polytopes. We show that the flag-approximability is exactly half the volume entropy of the Hilbert geometry on the body, and that both quantities are maximized when the convex body is a Euclidean ball.

We also compute explicitly the asymptotic volume of a convex polytope, which allows us to prove that simplices have the least asymptotic volume.

RÉSUMÉ. – Nous introduisons l'approximabilité-drapeau d'un corps convexe qui mesure la difficulté de l'approcher par des polytopes convexes. Nous montrons que l'approximabilité-drapeau d'un corps convexe est égale à la moitié de l'entropie volumique de sa géométrie de Hilbert associée et que les deux invariants sont maximaux lorsque le corps convexe est une boule euclidienne.

Nous calculons également le volume asymptotique de la géométrie de Hilbert d'un polytope convexe, ce qui nous permet de démontrer que les simplexes ont le volume asymptotique minimal.

Introduction

An important problem with many practical applications is to approximate convex bodies with polytopes that are as simple as possible, in some sense. Various measures of complexity of a polytope have been considered in the literature. These include counting the number of vertices, the number of facets, or even the number of faces [3]. One could also use, however, the number of *maximal flags*. Recall that a maximal flag of a d-dimensional polytope is a finite sequence $(f_0, \dots, f_d)$ of faces of the polytope such that each face f_i has dimension i and is contained in the boundary of f_{i+1}.

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Suppose we wish to approximate a convex body Ω by a polytope within a Hausdorff distance $\varepsilon > 0$. Let $N_{\mathrm{f}}(\varepsilon, \Omega)$ be the least number of maximal flags over all polytopes satisfying this criterion. We define the *flag approximability* of Ω to be

$$a_{\mathrm{f}}(\Omega) := \liminf_{\varepsilon \to 0} \frac{\log N_{\mathrm{f}}(\varepsilon, \Omega)}{-\log \varepsilon}.$$

This is analogous to how Schneider and Wieacker [10] defined the (vertex) approximability, where the least number of vertices was used instead of the least number of maximal flags.

Facet and face approximabilities can also be defined in a similar fashion. It is not known if any equalities hold between the vertex, facet, face, and flag approximabilities. An advantage of using the flag approximability is that one can relate it to the *volume entropy* of the Hilbert metric on the body in the following way.

Choose a base point p in the interior of the convex body Ω, and for each $R > 0$ denote by $B_\Omega(p, R)$ the closed ball centered at p of radius R in the Hilbert geometry. Let Vol^H denote the Holmes-Thompson volume. The (lower) volume entropy of the Hilbert geometry on Ω is defined to be

$$\mathrm{Ent}(\Omega) := \liminf_{R \to \infty} \frac{\log \mathrm{Vol}^H\big(B_\Omega(p, R)\big)}{R}.$$

Observe that this does not depend on the base point p, and moreover does not change if one takes instead the Busemann volume. One can also define the upper flag approximability and the upper volume entropy by taking supremum limits instead of infimum ones. Although the two entropies do not generally coincide, as shown by the first author in [13], all our results and proofs hold when replacing lim inf with lim sup.

Theorem 1. – *Let $\Omega \subset \mathbb{R}^d$ be a convex body. Then,*

$$\mathrm{Ent}(\Omega) = 2a_{\mathrm{f}}(\Omega).$$

The same result concerning the vertex approximability was proved by the first author [13] in dimensions two and three. In higher dimension, it was shown only that the volume entropy is greater than or equal to twice the vertex approximability. The motivation was to try to prove the entropy upper bound conjecture, which states that the volume entropy of any convex body is no greater than $d - 1$. This would follow from equality of the two quantities just mentioned using the well-known result, proved by Fejes-Toth [7] in dimension two and by Bronshteyn-Ivanov [5] in the general case, that the vertex approximability of any convex body is no greater than $(d - 1)/2$.

We show, using a slight modification of the technique in Arya-da Fonseca-Mount [3], that the Bronshteyn-Ivanov bound also holds for the flag approximability.

Theorem 2. – *Let $\Omega \subset \mathbb{R}^d$ be a convex body. Then,*

$$a_{\mathrm{f}}(\Omega) \leq \frac{d-1}{2}.$$

This allows us to deduce the entropy upper bound conjecture. N. Tholozan has also proved this conjecture recently using a different method [11].

COROLLARY 3. – *Let $\Omega \subset \mathbb{R}^d$ be a convex body. Then,*

$$\mathrm{Ent}(\Omega) \leq d - 1.$$

For many Hilbert geometries, such as hyperbolic space, the volume of balls grows exponentially. However, for some Hilbert geometries, the volume grows only polynomially. In this case it is useful to make the following definition. Fix some notion of volume Vol. The *asymptotic volume* of the Hilbert geometry on a d-dimensional convex body Ω is defined to be

$$\mathrm{Asvol}(\Omega) := \liminf_{R\to\infty} \frac{\mathrm{Vol}(B_\Omega(p, R))}{R^d}.$$

Note that, unlike in the case of the volume entropy, the asymptotic volume depends on the choice of volume. The first author has shown in [12] that the asymptotic volume of a convex body is finite if and only if the body is a polytope.

In the next theorem, we again see a connection appearing between volume in Hilbert geometries and the number of maximal flags. We denote by $\mathrm{Flags}(\mathscr{P})$ the set of maximal flags of a polytope $\mathscr{P}$. Let Σ be a simplex of dimension d. Observe that $\mathrm{Flags}(\Sigma)$ consists of $(d + 1)!$ elements.

THEOREM 4. – *Let $\mathscr{P}$ be a polytope of dimension d, and fix some notion of volume* Vol. *Then,*

$$\mathrm{Asvol}(\mathscr{P}) = \frac{|\mathrm{Flags}(\mathscr{P})|}{(d + 1)!} \mathrm{Asvol}(\Sigma).$$

An immediate consequence is that the simplex has the smallest asymptotic volume among all convex bodies. This was conjectured by the first author in [12].

COROLLARY 5. – *Let $\Omega \subset \mathbb{R}^d$ be a convex body. Then,*

$$\mathrm{Asvol}(\Omega) \geq \mathrm{Asvol}(\Sigma),$$

with equality if and only if Ω is a simplex.

Another corollary is the following result, proved originally by Foertsch and Karlsson [8].

COROLLARY 6. – *If the Hilbert geometry on a convex body Ω is isometric to a finite-dimensional normed space, then Ω is a simplex.*

1. Preliminaries

A *proper* open set in $\mathbb{R}^d$ is an open set not containing a whole line. A non-empty proper open convex set will be called a *convex domain*. The closure of a bounded convex domain is called a *convex body*.

1.1. Hilbert geometries

A Hilbert geometry (Ω, d_Ω) is a convex domain Ω in $\mathbb{R}^d$ with the Hilbert distance d_Ω defined as follows. For any distinct points p and q in Ω, the line passing through p and q meets the boundary $\partial\Omega$ of Ω at two points a and b, labeled so that the line passes consecutively through a, p, q, and b. We define

$$d_\Omega(p,q) := \frac{1}{2}\log[a,p,q,b],$$

where $[a,p,q,b]$ is the cross ratio of (a,p,q,b), that is,

$$[a,p,q,b] := \frac{|qa|}{|pa|}\frac{|pb|}{|qb|} > 1,$$

with $|xy|$ denoting the Euclidean distance between x and y in $\mathbb{R}^d$. If either a or b is at infinity, the corresponding ratio is taken to be 1.

Note that the invariance of the cross ratio under projective maps implies the invariance of d_Ω under such maps. In particular, since any convex domain is projectively equivalent to a bounded convex domain, most of our proofs will reduce to that case without loss of generality.

1.2. The Holmes-Thompson and Busemann volumes

Each Hilbert geometry Ω is naturally endowed with a C^0 Finsler metric F_Ω as follows. Let p be a point in Ω, and let v be a non-zero vector in the tangent space $T_p\Omega$, which we identify with $\mathbb{R}^d$. The straight line passing through p in the direction v meets $\partial\Omega$ at two points p_Ω^+ and p_Ω^-. Let t^+ and t^- be two positive numbers such that $p+t^+v = p_\Omega^+$ and $p-t^-v = p_\Omega^-$. These numbers correspond to the time necessary from an affine point of view to reach the boundary starting at p with velocities v and $-v$, respectively. We define

$$F_\Omega(p,v) := \frac{1}{2}\left(\frac{1}{t^+}+\frac{1}{t^-}\right) \quad \text{and} \quad F_\Omega(p,0) := 0.$$

If p_Ω^+ or p_Ω^- is at infinity, the corresponding ratio should be taken to be 0.

The Hilbert distance d_Ω is the distance induced by F_Ω. We shall denote by $B_\Omega(p,r)$ the closed metric ball of radius r centered at the point $p \in \Omega$, and by $S_\Omega(p,r)$ the corresponding metric sphere.

From the Finsler metric, we can construct two important Borel measures on Ω. The first is called the *Busemann* volume and is denoted by Vol_Ω^B. It is actually the Hausdorff measure associated to the metric space (Ω, d_Ω); see Burago-Burago-Ivanov [6], Example 5.5.13. It is defined as follows. For any $p \in \Omega$, let

$$\beta_\Omega(p) := \{v \in \mathbb{R}^d \mid F_\Omega(p,v) < 1\}$$

be the open unit ball in $T_p\Omega = \mathbb{R}^d$ of the norm $F_\Omega(p,\cdot)$. Denote by ω_d the Euclidean volume of the open unit ball of the standard Euclidean space $\mathbb{R}^d$. Consider the (density) function $h_\Omega^B\colon \Omega \longrightarrow \mathbb{R}$ given by $h_\Omega^B(p) := \omega_d / \mathrm{Leb}\big[\beta_\Omega(p)\big]$, where Leb is the canonical Lebesgue measure of $\mathbb{R}^d$, equal to 1 on the unit "hypercube". Then for any Borel set A in Ω,

$$\mathrm{Vol}_\Omega^B(A) := \int_A h_\Omega^B(p)\,\mathrm{d}\,\mathrm{Leb}(p).$$

The second, called the *Holmes-Thompson* volume, will be denoted by Vol_Ω^H, and is defined as follows. Let $\beta_\Omega^*(p)$ be the polar dual of $\beta_\Omega(p)$, and let $h_\Omega^H\colon \Omega \longrightarrow \mathbb{R}$ be the density defined by $h_\Omega^H(p) := \mathrm{Leb}\big[\beta_\Omega^*(p)\big]/\omega_d$. Then Vol_Ω^H is the measure associated to this density.

In what follows, we will denote by Area_Ω^H and Area_Ω^B, respectively, the $d-1$-dimensional measures associated to the Holmes-Thompson and Busemann measures.

LEMMA 7 (Monotonicity of the Holmes-Thompson area). – *Let (Ω, d_Ω) be a Hilbert geometry in $\mathbb{R}^d$. The Holmes-Thompson area measure is monotonic on the set of convex bodies in Ω, that is, for any pair of convex bodies K_1 and K_2 in Ω, such that $K_1 \subset K_2$, one has*

$$\mathrm{Area}_\Omega^H(\partial K_1) \leq \mathrm{Area}_\Omega^H(\partial K_2). \tag{1}$$

Proof. – If $\partial\Omega$ is C^2 with everywhere positive Gaussian curvature, then the tangent unit spheres of the Finsler metric are quadratically convex. According to Álvarez Paiva and Fernandes [2, Theorem 1.1 and Remark 2], there exists a Crofton formula for the Holmes-Thompson area, from which inequality (1) follows. Such smooth convex bodies are dense in the set of all convex bodies in the Hausdorff topology. By approximation, it follows that inequality (1) is valid for any Ω. □

LEMMA 8 (Minimality of flats). – *Let (Ω, d_Ω) be a Hilbert geometry in $\mathbb{R}^d$. Then, the affine hypersurfaces contained in Ω are minimal with respect to the Holmes-Thompson area.*

Proof. – It was shown in [1, Theorem 5.5], that, in Finsler manifolds where the tangent unit spheres are quadratically convex, totally geodesic hypersurfaces are minimal with respect to the Holmes-Thompson area. This applies to Hilbert geometries when the boundary $\partial\Omega$ of the domain is C^2 with everywhere positive Gaussian curvature. In this case the totally geodesic hypersurfaces are affine. A simple approximation process like in the proof of Lemma 7 allows us to extend the result to all Hilbert geometries. □

The next result was essentially proved by Berck-Bernig-Vernicos in [4, Lemma 2.13].

LEMMA 9 (Co-area inequalities). – *Let (Ω, d_Ω) be a Hilbert geometry containing the origin o, and let L be a cone with apex o. Then, for some constant $C > 1$ depending only on the dimension d,*

$$\frac{1}{C}\,\mathrm{Area}_\Omega^B\big(S_\Omega(o,R)\cap L\big) \leq \frac{d}{dR}\,\mathrm{Vol}_\Omega^B\big(B_\Omega(o,R)\cap L\big) \leq C\,\mathrm{Area}_\Omega^B\big(S_\Omega(o,R)\cap L\big),$$

for all $R \geq 0$.

REMARK 10. – It is important here to point out that all the statements presented in this paper are actually independent of the definition of volume chosen, provided the volume has the following properties: continuity with respect to the Hausdorff pointed topology, monotony with respect to inclusion, and invariance under projective transformations. We also need, as a normalization, that the volume coincides with the standard one in the case of an ellipsoid (see Vernicos [13] for more details). In fact, the value of the volume entropy does not depend at all on the chosen volume, and so Theorem 1 and its corollaries are true whatever the choice. On the other hand, while the asymptotic volume *does* depend on the choice of volume, the dependence is such that Theorem 4 and its corollaries remain true whichever volume satisfying the above properties is chosen, as long as one is consistent in this choice.

1.3. Asymptotic balls

Let Ω be a bounded open convex set, and denote by $\operatorname{cl}\Omega$ its closure. For each $R \geq 0$ and $y \in \Omega$, we call the dilation of $\operatorname{cl}\Omega$ about y by a factor $1 - \exp(-2R)$ the *asymptotic ball* of radius R about y, and we denote it by

$$\mathrm{AsB}_\Omega(y, R) := y + (1 - e^{-2R})(\operatorname{cl}\Omega - y).$$

Some authors dilate by a factor $\tanh R$ instead, but there is very little difference when R is large. By convention, we take $\mathrm{AsB}_\Omega(y, R)$ to be empty if $R < 0$. When there is no ambiguity, we sometimes omit mention of Ω or y when denoting a ball or asymptotic ball.

The following lemma shows the close connection between asymptotic balls and the balls of the Hilbert geometry.

LEMMA 11. – *Let Ω be a bounded open convex set, containing a point y. Assume that Ω contains the Euclidean ball of radius $l > 0$ about y, and is contained in the Euclidean ball of radius $L > 0$ about y. Then for all $R > 0$ we have*

$$\mathrm{AsB}_\Omega\Big(y, R - \frac{1}{2}\log\big(1 + \frac{L}{l}\big)\Big) \subset B_\Omega(y, R) \subset \mathrm{AsB}_\Omega(y, R).$$

Proof. – Let $x \in \Omega$, and let w and z be the points in the boundary of Ω that are collinear with x and y, labeled so that w, x, y, and z lie in this order. Observe that $|xy| \leq L$ and $|yz| \geq l$. Therefore,

$$1 \leq \frac{|xz|}{|yz|} = 1 + \frac{|xy|}{|yz|} \leq 1 + \frac{L}{l}.$$

The point x is in the ball $B_\Omega(y, R)$ if and only if

$$\log \frac{|wy|}{|wx|}\frac{|xz|}{|yz|} \leq 2R,$$

and is in the asymptotic ball $\mathrm{AsB}_\Omega(y, R)$ if and only if

$$\log \frac{|wy|}{|wx|} \leq 2R.$$

The result follows easily. □

Recall that the Löwner-John ellipsoid of Ω is the unique ellipsoid of minimal volume containing Ω. By performing affine transformations, we may assume without loss of generality that the Löwner-John ellipsoid of Ω is the Euclidean unit ball $\mathcal{E}$. It is known that $(1/d)\mathcal{E}$ is then contained in Ω, that is,

$$\frac{1}{d}\mathcal{E} \subset \Omega \subset \mathcal{E}.$$

Thus, in this case the assumptions of Lemma 11 are satisfied with $L = 1$ and $l = 1/d$, taking y to be the origin. A convex body will be said to be in *canonical form* if its Löwner-John ellipsoid is the Euclidean unit ball.

2. Asymptotic volume and flags

In this section, we study the asymptotic volume of polytopes. By polytope, we mean the convex hull of a finite number of points. Our technique is to decompose the polytope into *flag simplices*. We show that the asymptotic volume of a flag simplex is independent of the shape of the polytope, and depends only on the dimension. Since there is one flag simplex for every maximal flag of the polytope, our formula follows.

2.1. Flags and flag simplices

Recall that to a closed convex set $K \subset \mathbb{R}^d$ we can associate an equivalence relation, where two points a and b are equivalent if they are equal or if there exists an open segment $(c, d) \subset K$ containing the closed segment $[a, b]$. The equivalence classes are called *faces*. A face is called a k*-face* if the dimension of its affine hull, that is, the smallest affine set containing it, is k. A 0-face is usually called an *extremal point*, or, in the case of polytopes, a *vertex*. A *facet* is the closure of a face of co-dimension 1.

Thus defined, each face is an open set in its affine hull. For instance, the segment $[a, b]$ in $\mathbb{R}$ admits three faces, namely $\{a\}$, $\{b\}$, and the open segment (a, b). Notice that if K has non-empty interior, then the interior is a d-dimensional face.

When a face f is in the relative boundary of another face F, we write $f < F$.

Definition 12 (Flag). – Let $\mathcal{P}$ be a d-dimensional polytope. A *maximal flag* of $\mathcal{P}$ is a $(d+1)$-tuple $(f_0, \dots, f_d)$ of faces of $\mathcal{P}$ such that each f_i has dimension i, and $f_0 < \cdots < f_d$.

We denote by $\mathrm{Flags}(\mathcal{P})$ the set of maximal flags of a polytope $\mathcal{P}$. We use $|\cdot|$ to denote the number of elements in a finite set. The following formula will be useful. Let $\{F_i\}$ be the set of facets of $\mathcal{P}$. So, each F_i is a polytope of dimension $d - 1$. We have that

$$|\,\mathrm{Flags}(\mathcal{P})| = \sum_i |\,\mathrm{Flags}(F_i)|. \tag{2}$$

In this paper, a *simplex* in $\mathbb{R}^d$ is the convex hull of $d + 1$ projectively independent points, that is, a triangle in $\mathbb{R}^2$, a tetrahedron in $\mathbb{R}^3$, and so forth. If Σ is a simplex of dimension d, then $\mathrm{Flags}(\Sigma)$ consists of $(d + 1)!$ elements.

Definition 13 (Flag simplex). – A simplex $\mathcal{S}$ is a *flag simplex* of a polytope $\mathcal{P}$ if there is a maximal flag $(f_0, \dots, f_d)$ of $\mathcal{P}$ such that each of the faces f_i contains exactly one vertex of $\mathcal{S}$.

Let $\mathcal{P}$ be a polytope. Suppose that for each face of $\mathcal{P}$ we are given a point in the face. Then, associated to each maximal flag there is a flag simplex of $\mathcal{P}$, obtained by taking the convex hull of the corresponding points. Moreover, these flag simplices form a simplicial complex, and their union is equal to $\mathcal{P}$. We call this a *flag decomposition* of $\mathcal{P}$. If each point is the barycenter of its respective face, then the resulting flag decomposition is just the well known *barycentric decomposition*.

2.2. Flag simplices of simplices

LEMMA 14. – *Let T and S be flag simplices of a d-dimensional simplex Σ. Then, there exists a projective linear map ϕ leaving Σ invariant, such that $\phi(T) \subset S$.*

Proof. – We use induction on the dimension. The induction hypothesis is that if T and S are flag simplices of a d-dimensional simplex Σ, and $\{p_i\}$ is a finite set of points in the interior of Σ, then there exists a projective linear map ϕ leaving Σ invariant, such that $\phi(T) \subset S$ and the points $\{\phi(p_i)\}$ are all contained in the interior of S.

The hypothesis is clearly true in dimension 1, since in this case Σ is a closed interval; the flag simplices are the closed segments in Σ having one endpoint that coincides with an endpoint of Σ and the other in the interior; and the group of projective linear maps is generated by a one-parameter family that acts transitively on the interior of Σ, along with reflection in the midpoint of the interval.

Assume the hypothesis is true in dimension d, let T and S be flag simplices of a $d + 1$-dimensional simplex Σ, and let $\{p_i\}$ be a finite subset of the interior of Σ. Since the group of projective linear maps acts transitively on the facets of Σ, we may assume that the flags associated to, respectively, T and S have, as their facets, the same facet F of Σ.

Let v be the vertex of Σ not contained in F, and let x be the vertex of T not contained in F. Project the points $\{p_i\}$ onto F along rays emanating from v, to get a set of points $\{q_i\}$. Project x in the same way to get a point y. By the induction hypothesis, there exists a projective linear map ϕ_0 leaving F invariant such that $\phi_0(T \cap F) \subset S \cap F$, and the point y and all the points $\{q_i\}$ are mapped by ϕ_0 into the relative interior of $S \cap F$. We can extend ϕ_0 to a projective linear map on the whole of Σ that fixes v. We denote this extended map again by ϕ_0.

There exists a 1-parameter family of projective linear maps that fix F and v. Among these maps, we can find one that maps x as close as we wish to y, and each of the points p_i as close as we wish to the corresponding point q_i. We choose such a map ϕ_1 so that the image of x and of each of the points $\{p_i\}$ is in the interior of $\phi_0^{-1}(S)$. So, the map $\phi := \phi_0 \circ \phi_1$ maps x and each of the points $\{p_j\}$ into the interior of S, and furthermore maps $T \cap F$ into $S \cap F$. Since T is the convex hull of x and $T \cap F$, we have that $\phi(T) \subset S$. This completes the induction step. □

LEMMA 15. – *Consider the Hilbert geometry on a d-dimensional simplex Σ. Let S be a flag simplex of Σ. Then for any z in Σ,*

$$\lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(\operatorname{AsB}(z, R) \cap S\big) = \frac{1}{(d+1)!} \operatorname{Asvol}(\Sigma).$$

Proof. – Because all simplices of the same dimension are affinely equivalent, we may assume that Σ is a regular simplex with the origin o as its barycenter.

Let T be a barycentric flag simplex of Σ.

A projective linear map leaving Σ invariant is an isometry of the Hilbert metric on Σ, and therefore preserves volume. Combining this with the fact that

$$B\big(x, R - d(x, y)\big) \subset B(y, R) \subset B\big(x, R + d(x, y)\big), \tag{3}$$

for any points $x, y \in \operatorname{int}\Sigma$ and $R > 0$, we get

$$\lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(B(o,R) \cap \phi(T)\big) = \lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(B(o,R) \cap T\big), \tag{4}$$

for any projective linear map ϕ leaving Σ invariant.

From Lemma 14, there exist projective linear maps ϕ_1 and ϕ_2 leaving Σ invariant, such that $\phi_1(T) \subset S \subset \phi_2(T)$. Combining this with (4), we get

$$\lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(B(o,R) \cap S\big) = \lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(B(o,R) \cap T\big).$$

Denote by Π the group of permutations of vertices of Σ. Observe that Π has $(d+1)!$ elements. The group Π acts on Σ, leaving the center o of Σ fixed. We have that the union of the sets $\{\phi(T)\}_{\phi\in\Pi}$ is Σ, and that the interiors of these sets are pairwise disjoint. So, by symmetry,

$$\lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(B(o,R) \cap T\big) = \frac{1}{(d+1)!} \operatorname{Asvol}(\Sigma).$$

The last step is to use (3) and Lemma 11 to get that

$$\lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(\operatorname{AsB}(z,R) \cap S\big) = \lim_{R\to\infty} \frac{1}{R^d} \operatorname{Vol}\big(B(o,R) \cap S\big). \qquad \square$$

2.3. Flag simplices of polytopes

LEMMA 16. – *Let $\mathscr{P}$ be a polytope, and let S be a flag simplex of $\mathscr{P}$. Then there exist simplices U and V satisfying $U \subset \mathscr{P} \subset V$ such that S is a flag simplex of both U and V.*

Proof. – We prove the existence of U by induction on the dimension. The one dimensional case is trivial, since here $\mathscr{P}$ is already a simplex. So, assume the result holds in dimension d, and let $\mathscr{P}$ be $d+1$-dimensional. Let p be the vertex of S that lies in the interior of $\mathscr{P}$. The remaining vertices of S form a flag simplex S' of a facet of $\mathscr{P}$. Applying the induction hypothesis, we get a simplex U' contained in this facet such that S' is a flag simplex of U'. It is not difficult to see that we may perturb p in such a way as to get a point $p' \in \mathscr{P}$ such that the simplex U formed from p' and U' contains p in its interior. It follows that $U \subset \mathscr{P}$, and that S is a flag simplex of U.

We also prove the existence of V by induction on the dimension. Again, the 1-dimensional case is trivial. As before, we assume the result holds in dimension d, and let $\mathscr{P}$ be $d+1$-dimensional. Recall that p is the vertex of S that lies in the interior of $\mathscr{P}$, and that the remaining vertices of S form a flag simplex S' of a facet F of $\mathscr{P}$. Applying the induction hypothesis, we get a simplex V' containing this facet such that S' is a flag simplex of V'. Denote by o the vertex of S that is also a vertex of $\mathscr{P}$. Without loss of generality we may assume that o is the origin of the vector space $\mathbb{R}^{d+1}$. Observe that if we multiply the vertices of V' by any scalar α greater than 1, then S' remains a flag simplex of $\alpha V'$. Choose $q \in \mathbb{R}^{d+1}$ and $\alpha > 1$ such that every vertex of $\mathscr{P}$ lies in the convex hull

$$V := \operatorname{conv}\{q, \alpha V'\}.$$

Then, $\mathscr{P} \subset V$ and S is a flag simplex of V. $\square$

Proof of Theorem 4. – Choose a flag decomposition of $\mathcal{P}$. Let x be the vertex that is common to all the flag simplices, which lies in the interior of $\mathcal{P}$.

Let S be any one of the flag simplices. By Lemma 16, there are simplices U and V satisfying $U \subset \mathcal{P} \subset V$ such that S is a flag simplex both of U and of V. Hence,

$$\mathrm{Vol}_U(X) \geq \mathrm{Vol}_{\mathcal{P}}(X) \geq \mathrm{Vol}_V(X), \tag{5}$$

for any measurable subset X of the interior of U. Observe that, for any $R > 0$,

$$\mathrm{AsB}_U(x, R) \cap S = \mathrm{AsB}_{\mathcal{P}}(x, R) \cap S = \mathrm{AsB}_V(x, R) \cap S. \tag{6}$$

Combining (5) and (6) with Lemma 15, we get

$$\lim_{R\to\infty} \frac{1}{R^d} \mathrm{Vol}_{\mathcal{P}}\big(\mathrm{AsB}_{\mathcal{P}}(x, R) \cap S\big) = \frac{1}{(d+1)!} \mathrm{Asvol}(\Sigma).$$

Using Lemma 11, we get from this that

$$\lim_{R\to\infty} \frac{1}{R^d} \mathrm{Vol}_{\mathcal{P}}\big(B_{\mathcal{P}}(x, R) \cap S\big) = \frac{1}{(d+1)!} \mathrm{Asvol}(\Sigma).$$

But this holds for any flag simplex of the decomposition, and summing over all the flags we get the result. □

Proof of Corollary 5. – The first author proved in [12] that the asymptotic volume of a convex body is finite if and only if it is a polytope. The result follows because the simplex has fewer flags than any other polytope of the same dimension. □

Proof of Corollary 6. – When one considers the Busemann volume, the asymptotic volume of every normed space of a fixed dimension d is the same, and is equal to $\mathrm{Asvol}(\Sigma)$ since the Hilbert geometry on a simplex is isometric to a normed space. Hence $\mathrm{Asvol}(\Omega) = \mathrm{Asvol}(\Sigma)$, and the result follows from Corollary 5. □

3. A general bound on the flag approximability

Here we prove Theorem 2, that is, that the flag approximability of a d-dimensional convex body is no greater than $(d-1)/2$.

Our technique is to modify the proof of the main result of Arya-da Fonseca-Mount [3]. In that paper, essentially the same result was proved for the *face-approximability*, which is defined analogously to the flag-approximability, but counting the least number of faces rather than the least number of flags.

Their proof uses the witness-collector method. Assume we have a set S of points in $\mathbb{R}^d$, a set $\mathcal{W}$ of regions called *witnesses*, and a set $\mathcal{C}$ of regions called *collectors*, satisfying the following properties.

i. each witness in $\mathcal{W}$ contains a point of S in its interior;

ii. any halfspace H of $\mathbb{R}^d$ either contains a witness $W \in \mathcal{W}$, or $H \cap S$ is contained in a collector $C \in \mathcal{C}$;

iii. each collector $C \in \mathcal{C}$ contains some constant number of points of S.

We strengthen Lemma 4.1 of Arya-da Fonseca-Mount [3]. In what follows, given a quantity D, any other quantity is said to be $O(D)$ if it is bounded from above by a multiple, depending only on the dimension of D.

LEMMA 17. – *Given a set of witnesses and collectors satisfying the above properties, the number of flags of the convex hull P of S is $O(|\mathcal{C}|)$.*

Proof. – Take any facet F of P, and let H be the half-space whose intersection with P is F. As in the original proof, H does not contain any witness, for otherwise, by property (i), it would contain a point of S in its interior. So, by (ii), the intersection of H and S is contained in some collector C. Therefore, by (iii), F has at most n vertices, where n is the number of points in each collector.

So, we see that each facet has at most 2^n faces, and so has at most $(2^n)^{d+1}$ flags, since each flag can be written as a sequence of $d+1$ faces. Also, the number of facets is at most $2^n|\mathcal{C}|$ since each facet has a different set of vertices, and this set is a subset of some collector. We deduce that the number of flags is at most $(2^n)^{d+2}|\mathcal{C}|$. □

We conclude that the main theorem of [3] holds when measuring complexity using flags instead of faces.

Proof of Theorem 2. – The proof follows that of the main result of [3], but using Lemma 17 above instead of Lemma 4.1 of that paper. □

4. Upper bound on the volume entropy

We show that the volume entropy of a convex body is no greater than twice the flag approximability.

4.1. A uniform upper bound on the volume of a ball

To prove the upper bound on the volume entropy, we will need to bound uniformly the volume of balls of any radius in a polytopal Hilbert geometry in terms of the number of flags of the polytope and the radius. The asymptotic result of Theorem 4 is insufficient because it says nothing about the rate of convergence, which might be slower for some polytopes than others. On the other hand, we will not be too concerned here with the exact dependence on the radius—showing that it is polynomial will be enough.

We use $B(R)$ to denote the ball in a Hilbert geometry of radius R and centered at o, and $S(R)$ to denote the boundary of this ball. We remind the reader that $\mathcal{E}$ stands for the Euclidean unit ball.

LEMMA 18. – *For each $d \in \mathbb{N}$ and $0 < l \leq 1$, there exists a polynomial $p_{d,l}$ of degree d such that the following holds. Let $\mathcal{P}$ be a d-dimensional polytope endowed with its Hilbert geometry, satisfying $l.\mathcal{E} \subset \mathcal{P} \subset \mathcal{E}$. Let F be a facet of $\mathcal{P}$, and let L be the cone with base F and apex o. Then,*

$$\mathrm{Vol}^H\big(B(R) \cap L\big) \leq p_{d,l}(R)|\mathrm{Flags}(F)|, \qquad \textit{for all } R \geq 0.$$

Proof. – We will use induction on the dimension d. When $d = 1$, there is only one Hilbert geometry, up to isometry. In this case, $\mathrm{Vol}^H\big(B(R) \cap L\big) = R$, and $|\mathrm{Flags}(F)| = 1$, and so the conclusion is evident.

Assume now that the conclusion is true when the dimension is $d-1$ and l is unchanged.

Using the co-area formula in Lemma 9, we get that

$$\frac{d}{dR}\,\mathrm{Vol}^B\big(B(R) \cap L\big) \le C\,\mathrm{Area}^B\big(S(R) \cap L\big),$$

for some constant C depending only on the dimension d. Since the Busemann and Holmes-Thompson areas agree up to a constant,

$$\mathrm{Area}^B\big(S(R) \cap L\big) \le C'\,\mathrm{Area}^H\big(S(R) \cap L\big),$$

with C' depending only on the dimension.

Denote the facets of F by $\{F_i\}_i$. So, each F_i is the closure of a face of $\mathscr{P}$ of co-dimension 2. By (2),

$$\sum_i |\mathrm{Flags}(F_i)| = |\mathrm{Flags}(F)|.$$

For each i, let L_i be the $d-1$ dimensional cone with base F_i and apex o.

Observe that, from Lemma 11, $B(R) \cap L \subset \mathrm{AsB}(R) \cap L$, for all $R \ge 0$. So, using the monotonicity of the Holmes-Thompson measure (Lemma 7), we get

$$\begin{aligned}\mathrm{Area}^H\big(S(R) \cap\ L\big) &+ \sum_i \mathrm{Area}^H\big(B(R) \cap L_i\big)\\ &\le \mathrm{Area}^H\big(\mathrm{AsS}(R) \cap L\big) + \sum_i \mathrm{Area}^H\big(\mathrm{AsB}(R) \cap L_i\big).\end{aligned}$$

Here $\mathrm{AsS}(R)$ is the boundary of the asymptotic ball of radius R about o. By the minimality of flats for the Holmes-Thompson volume (Lemma 8), we have that

$$\mathrm{Area}^H\big(\mathrm{AsS}(R) \cap L\big) \le \sum_i \mathrm{Area}^H\big(\mathrm{AsB}(R) \cap L_i\big).$$

From Lemma 11, we have that $\mathrm{AsB}(R) \subset B(R+c)$, where c depends only on l. Also, by the induction hypothesis, we have, for each i,

$$\mathrm{Area}^H\big(B(R+c) \cap L_i\big) \le p_{d-1,l}(R+c)|\mathrm{Flags}(F_i)|.$$

Putting all this together, we get that

$$\frac{d}{dR}\,\mathrm{Vol}^B(B(R) \cap L) \le 2CC'p_{d-1,l}(R+c)|\mathrm{Flags}(F)|.$$

The result follows upon integrating with respect to R, and using that the Holmes-Thompson volume is no greater than the Busemann volume. $\square$

The two- and three-dimensional cases of the following theorem follow from Theorem 10 in first author's paper [13].

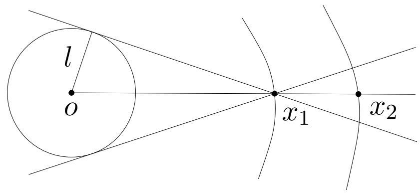

FIGURE 1. Diagram for the proof of Lemma 20.

THEOREM 19. – *For each $d \in \mathbb{N}$ and $0 < l \leq 1$, there is a polynomial $p_{d,l}$ of degree d such that, for any d-dimensional polytope $\mathscr{P}$ satisfying $l.\mathcal{E} \subset \mathscr{P} \subset \mathcal{E}$, we have*

$$\mathrm{Vol}^H(B(R)) \leq p_{d,l}(R)|\mathrm{Flags}(\mathscr{P})|, \qquad \textit{for all } R \geq 0.$$

The same result holds for the asymptotic balls.

Proof. – We will consider the metric balls; passing from these to the asymptotic balls can be accomplished using Lemma 11.

Let $p_{d,l}$ be the polynomial obtained from Lemma 18. According to that lemma, for each facet F of $\mathscr{P}$ and for each $R > 0$, we have

$$\mathrm{Vol}^H(B(R) \cap L) \leq p_{d,l}(R)|\mathrm{Flags}(F)|,$$

where L is the cone with base F and apex o. Using (2) and summing over all the facets of $\mathscr{P}$, we get the result. □

4.2. The upper bound on the volume entropy

LEMMA 20. – *Let Ω_1 and Ω_2 be convex bodies within a Hausdorff distance $\varepsilon > 0$ of each other, each containing the Euclidean ball $l.\mathcal{E}$ of radius $l > 0$ centered at the origin. Then, $(1/\lambda)\Omega_2 \subset \Omega_1$, with $\lambda := 1 + \varepsilon/l$.*

Proof. – Consider a ray emanating from the origin, and let x_1 and x_2 be the intersections of this ray with the boundaries of Ω_1 and Ω_2, respectively. Let l_1 and l_2 be the distances from the origin to x_1 and x_2, respectively, and suppose that $l_2 > l_1$. Define the cone

$$F := \{x_1 + \alpha(x_1 - z) \mid \alpha > 0 \text{ and } z \in l.\mathcal{E}\}.$$

See Figure 1. No point in the interior of the cone F can be in Ω_1. However, the distance from x_2 to Ω_1 is no greater than ε. This implies that the ball of radius ε around x_2 is not contained in the interior of F. Looking at the sine of the angle subtended by F at x_1, we see that $l/l_1 \leq \varepsilon/(l_2 - l_1)$. We deduce that

$$\frac{l_2}{l_1} = 1 + \frac{l_2 - l_1}{l_1} \leq 1 + \frac{\varepsilon}{l}.$$

The conclusion follows. □

LEMMA 21. – *Let Ω be a convex body in $\mathbb{R}^d$. The volume entropy of Ω is no greater than twice its flag approximability, that is,*

$$\mathrm{Ent}(\Omega) \leq 2a_{\mathrm{f}}(\Omega).$$

Proof. – Without loss of generality, we may assume that Ω is in canonical form. Let $R > 0$, and let $\varepsilon > 0$ be such that $-2R = \log \varepsilon$. Let P^* be a polytope approximating Ω within Hausdorff distance ε, having the least possible number $N_{\mathrm{f}}(\varepsilon, \Omega)$ of maximal flags. Write $P := (1/\lambda)P^*$, where $\lambda := 1 + 2d\varepsilon$. When ε is small enough, both Ω and P^* contain $(1/2d)\mathcal{E}$, and so, by Lemma 20,

$$(1/\lambda^2)\Omega \subset P \subset \Omega. \tag{7}$$

We will henceforth assume that ε is small enough for this to be the case, and for P to contain $(1/4d)\mathcal{E}$. Since Ω is in canonical form, this implies that P satisfies the assumptions of Theorem 19, with $l = 1/4d$. Therefore, there exists a polynomial p_d of degree d, depending only on the dimension d, such that

$$\mathrm{Vol}_P^H\big(\mathrm{AsB}_P(o, R)\big) \leq N_{\mathrm{f}}(\varepsilon, \Omega)\, p_d(R).$$

From (7),

$$\mathrm{Vol}_\Omega^H(\cdot) \leq \mathrm{Vol}_P^H(\cdot).$$

Observe that $((1-\varepsilon)/\lambda^2)\Omega$ is the asymptotic ball of Ω of radius R', where $-2R' = \log \varepsilon'$, with $1-\varepsilon' = (1-\varepsilon)/\lambda^2$. Also, the asymptotic ball of P of radius R is $(1-\varepsilon)P$. So, according to (7),

$$\mathrm{AsB}_\Omega(o, R') \subset \mathrm{AsB}_P(o, R).$$

Finally, Lemma 11 gives that $B_\Omega(o, R') \subset \mathrm{AsB}_\Omega(o, R')$.

Putting all this together, we conclude that

$$\frac{1}{R'} \log \mathrm{Vol}_\Omega^H\big(B_\Omega(o, R')\big) \leq 2\frac{\log\big(N_{\mathrm{f}}(\varepsilon, \Omega)\, p_d(R)\big)}{-\log \varepsilon'}.$$

We now take the limit infimum as R tends to infinity, in which case R' also tends to infinity, and ε and ε' tend to zero. A simple calculation shows that, in this limit, the ratio ε'/ε converges to $4d + 1$. The result follows. ☐

5. Lower bound on the volume entropy

We show that the volume entropy of a convex body is no less than twice the flag approximability.

LEMMA 22. – *Let Ω be a convex body in $\mathbb{R}^d$. Then, $2a_{\mathrm{f}}(\Omega) \leq \mathrm{Ent}(\Omega)$.*

Our proof will be a modification of the method used in Arya-da Fonseca-Mount [3]. We start with a lemma concerning the centroid of a convex body, otherwise known as its barycenter or center of mass.

LEMMA 23. – *Let D be a convex body in $\mathbb{R}^d$. Let $p \in \partial D$ and $q \in D$ be such that the centroid x of D lies on the line segment $[pq]$. Then, $|px| \geq |pq|/(d + 1)$.*

Proof. – Let h be a hyperplane tangent to D at p. The ratio $|px|/|pq|$ is minimized when D is a simplex with a vertex at q and all the other vertices on h. ☐

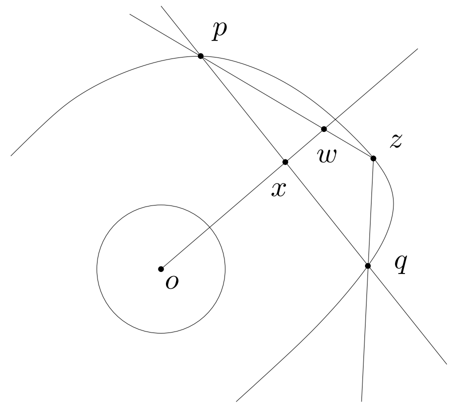

FIGURE 2. Diagram for the proof of Lemma 24.

Recall the following definitions. A *cap* C of a convex body Ω is a non-empty intersection of Ω with a closed halfspace H. The *base* of the cap C is the intersection of Ω with the hyperplane h that bounds the halfspace. An *apex* of C is a point of C of maximum distance from h. Thus, the apexes of C all lie in a hyperplane tangent to Ω and parallel to h. The width of the cap is the distance from any apex to h.

Let Ω be a convex body containing the origin o in its interior. Consider the ray emanating from o and passing through another point x. We define the *ray-distance* $\mathrm{ray}(x)$ to be the distance from x to the point where this ray intersects $\partial\Omega$.

LEMMA 24. – *Let $\Omega \subset \mathbb{R}^d$ be a convex body in canonical form. Let x be the centroid of the base of a cap of width ε of Ω, with $\varepsilon < 1/3d$. Then, the ray-distance $\mathrm{ray}(x)$ is greater than $C'\varepsilon$, for some constant $C' > 0$ depending only on the dimension d.*

Proof. – Let C be a cap of width ε, and let x be the centroid of its base D. Let z be an apex of C. So, z is at distance ε from h, the hyperplane defining the cap.

Consider the 2-plane Π containing the points o, x, and z. (If these points are collinear, then take Π to be any 2-plane containing them.)

The intersection of D with Π is a line segment. Let p and q be the endpoints of this line segment. Label them in such a way that the ray ox intersects the line segment pz at a point w. See Figure 2. Think of D as a convex body in h. We get from Lemma 23 that $|px| \geq |pq|/d$, since x is the centroid of D.

We consider separately the cases where the angle $\angle pzq$ is acute and where it is not.

Case $\angle pzq \leq \pi/2$. Observe that the cap C does not intersect the Euclidean ball $(1/3d)\mathcal{E}$. Moreover, the intersection of this ball with the plane Π lies in the infinite wedge-shaped region of Π defined by the angle $\angle pzq$. Since z is at distance at most 1 from the origin, we deduce that the magnitude of $\angle pzq$ must be at least $A := 2\arcsin(1/3d)$. In the present case, this implies that $\sin\angle pzq$ is at least $\sin A$. Note that $|zq| \geq \varepsilon$. Two applications of the sine rule give

$$|xw| = |zq|\frac{|px|}{|pq|}\frac{\sin\angle pzq}{\sin\angle pwx}.$$

We deduce that $|xw| \geq \varepsilon \sin(A)/d$

Case $\angle pzq \geq \pi/2$. In this case there is a point y between p and q such that $\angle pzy = \pi/2$. Drop the perpendicular from x to the line pz to get a point w' such that $\angle pw'x = \pi/2$. Using similarity of triangles, we get

$$|xw| \geq |xw'| = \frac{|px||yz|}{|py|} \geq \frac{|px||yz|}{|pq|} \geq \frac{\varepsilon}{d}.$$

In both cases we have shown that $|xw|$ is at least ε times some constant depending on the dimension. The conclusion follows since $\mathrm{ray}(x) \geq |xw|$. □

The following is part of Theorem 2 of [12].

LEMMA 25. – *For each dimension d, there is a constant $c > 0$ such that*

$$\mathrm{Vol}^H_\Omega\big(B_\Omega(x, R)\big) \geq cR^d,$$

for each convex body Ω, point $x \in \operatorname{int}\Omega$, and radius $R > 0$.

Let Ω be a convex body containing a point x in its interior. The *Macbeath region* about x is defined to be

$$M'(x) := x + \Big(\frac{1}{5}(\Omega - x) \cap \frac{1}{5}(x - \Omega)\Big).$$

Macbeath regions were introduced by A. M. Macbeath in [9]. They are related to balls of the Hilbert geometry as follows.

LEMMA 26. – *The Macbeath region $M'(x)$ about any point x satisfies*

$$B\Big(x, \frac{1}{2}\log\frac{6}{5}\Big) \subset M'(x) \subset B\Big(x, \frac{1}{2}\log\frac{3}{2}\Big).$$

Proof. – Recall that the Funk distance between two points p and q is defined to be

$$d_F(p, q) := \log\frac{|pb|}{|qb|},$$

where b is as in the definition of the Hilbert metric in Section 1. The Funk metric is not actually a metric since it is not symmetric. Its symmetrisation is the Hilbert metric: $d_\Omega(p, q) = (d_F(p, q) + d_F(q, p))/2$.

One can show that a point y is in $M'(x)$ if and only if both $d_F(x, y) \leq \log(5/4)$ and $d_F(y, x) \leq \log(6/5)$. The conclusion follows. □

The following is a modification of Lemma 3.2 of [3]. The assumptions are similar; the main difference is the bound on the number of caps. The original bound was $O(1/\delta^{(d-1)/2})$.

LEMMA 27. – *Let $\Omega \subset \mathbb{R}^d$ be a convex body in canonical form. Let $0 < \delta < 1/3d$, and let $\mathcal{C}$ be a set of caps each of width δ, such that the Macbeath regions $M'(x)$ centered at the centroids x of the bases of these caps are disjoint. Then,*

$$|\mathcal{C}| = O\Big(\mathrm{Vol}^H\big(\mathrm{AsB}(o, R)\big)\Big),$$

where $2R := -\log C\delta$, and C is a constant depending only on the dimension.

Proof. – Let x be the centroid of the base of one of the caps in $\mathcal{C}$. By Lemma 24, the ray-distance satisfies $\mathrm{ray}(x) \geq C'\delta$, where C' is the constant appearing in that lemma. Since Ω is contained in the unit ball, this implies that $x \in \mathrm{AsB}(o, R')$, where $2R' = -\log C'\delta$.

So, from Lemma 11, we get that x lies within the Hilbert metric ball $B(o, R' + C'')$, where $C'' := (1/2)\log(1 + d)$. Applying Lemma 26 then gives that the whole Macbeath region $M'(x)$ is contained in $B(o, R)$, with $R := R' + C'' + C'''$ and $C''' := (1/2)\log(3/2)$. Using Lemma 11 again, we get that the Macbeath region is contained within $\mathrm{AsB}(o, R)$.

Combining Lemmas 25 and 26, we get that there is a constant C_1 such that the Macbeath region $M'(x)$ has volume at least C_1.

A volume argument now gives that $|\mathcal{C}|C_1 \leq \mathrm{Vol}^H(\mathrm{AsB}(o, R))$. □

We can now prove the lower bound on the volume entropy.

Proof of Lemma 22. – We may assume without loss of generality that Ω is in canonical form.

We follow the method of [3], but using the bound in Lemma 27 on the number of non-intersecting Macbeath regions, rather than the bound given in Lemma 3.2 of [3]. Given an $\varepsilon > 0$, this method produces a set of points S, a set $\mathcal{W}$ of witnesses, and a set $\mathcal{C}$ of collectors satisfying the assumptions in Section 3, such that the convex hull of S is an ε-approximation of Ω. Furthermore, Lemma 27 leads to the following bound on the number of collectors:

$$|\mathcal{C}| \leq \frac{1}{C_1} \mathrm{Vol}^H\big(\mathrm{AsB}(o, R)\big),$$

where $2R := -\log C\delta$ and $\delta := C_2\varepsilon/\log(1/\varepsilon)$, with the constants C, C_1, and C_2 depending only on the dimension.

Since we are concerned with the flag-approximability, we must, just as in the proof of Theorem 2, use Lemma 17 from Section 3 instead of Lemma 4.1 of [3]. We get that the number $N_{\mathrm{f}}(\varepsilon, \Omega)$ of flags of the approximating polytope is at most a fixed multiple $C_3|\mathcal{C}|$ of $|\mathcal{C}|$.

Now let ε tend to zero. Observe that $\log\delta/\log\varepsilon$ converges to 1. So,

$$\begin{aligned} a_{\mathrm{f}}(\Omega) &= \liminf_{\varepsilon\to 0} \frac{\log N_{\mathrm{f}}(\varepsilon, \Omega)}{-\log\varepsilon} \\ &\leq \liminf_{R\to\infty} \frac{\log\big((C_3/C_1)\,\mathrm{Vol}^H(\mathrm{AsB}(o, R))\big)}{2R + \log C} \\ &= \frac{1}{2}\,\mathrm{Ent}(\Omega). \end{aligned}$$

□

The proof of the main result of the paper is now complete.

Proof of Theorem 1. – We combine Lemmas 21 and 22. □

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GLOBAL WELL-POSEDNESS FOR THE 2D STABLE MUSKAT PROBLEM IN $H^{3/2}$

BY DIEGO CÓRDOBA AND OMAR LAZAR

ABSTRACT. – We prove a global existence result of a unique strong solution in $H^{5/2}$ with small $\dot{H}^{3/2}$ semi-norm for the 2D Muskat problem. Hence, allowing the interface to have arbitrary large finite slopes and finite energy (thanks to the L^2 maximum principle). The proof is based on the use of a new formulation of the Muskat equation that involves oscillatory terms. Then, a careful use of interpolation inequalities in homogeneneous Besov spaces allows us to close the a priori estimates.

RÉSUMÉ. – Nous prouvons un résultat d'existence globale d'une unique solution forte dans $H^{5/2}$ en supposant que la semi-norme $\dot{H}^{3/2}$ soit petite pour le problème de Muskat 2D. Ceci permet donc d'avoir des interfaces dont la pente peut être arbitrairement grande et finie (grâce au principe du maximum L^2). La preuve est basée sur l'introduction d'une nouvelle formulation de l'équation de Muskat en termes d'intégrales oscillantes. Ensuite, une utilisation minutieuse d'inégalités d'interpolations dans des espaces de Besov homogènes permet de clore les estimations *a priori*.

1. Introduction

In this paper, we are interested in the Muskat problem which was introduced in [27] by Morris Muskat in order to describe the dynamics of water and oil in sand. The Muskat problem models the motion of an interface separating two incompressible fluids in a porous medium. One can imagine the plane $\mathbb{R}^2$ split into two regions, say $\Gamma_1(t)$, $\Gamma_2(t)$ that evolve with time. We assume that the first region $\Gamma_1(t)$ is occupied by an incompressible fluid with density ρ_1 and the second region $\Gamma_2(t)$ is occupied by another fluid with density ρ_2. We further assume that both fluids are immiscible. The non mixture condition allows one to consider the interface between these two fluids. This interface corresponds to their common boundary $\partial\Gamma_1(t)$ and $\partial\Gamma_2(t)$. The velocity in each region $\Gamma_i(t)$ ($i = 1$ or 2) is governed by the so-called Darcy's law [18], which states that the velocity depends on the gradient pressure,

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the gravity and the density of the fluid (which is transported by the flow) via the following relation,

$$\frac{\mu}{\kappa}u(x,t) = -\nabla p - (0, g\rho), \tag{1.1}$$

where μ is the constant viscosity, κ is the permeability of the porous media and g is the gravity. For the sake of simplicity, we may, without loss of generality, assume that all those constants are equal to 1. The system is then driven by the following transport equation

$$\partial_t \rho + u \cdot \nabla \rho = 0. \tag{1.2}$$

Since the fluids are incompressible we also have

$$\nabla \cdot u = 0. \tag{1.3}$$

Equations (1.1), (1.2) and (1.3) give rise to the so-called incompressible porous media system (IPM). Saffman and Taylor [31] pointed out that in 2D the Muskat problem is similar to the evolution of an interface in a vertical Hele-Shaw cell.

For the Muskat problem we can rewrite the IPM system in terms of the dynamics of the interface in between both fluids (see [1] and [14]). If we denote the interface by a planar curve $z(\alpha,t)$ and if we neglect surface tension, then the interface satisfies

$$\partial_t z(\alpha,t) = \frac{\rho_2-\rho_1}{2\pi}\int \frac{z_1(\alpha,t)-z_1(\beta,t)}{|z(\alpha,t)-z(\beta,t)|^2}(\partial_\alpha z(\alpha,t)-\partial_\beta z(\beta,t))d\beta,$$

where the curve z is asymptotically flat at infinity *i.e.,* $(z(\alpha,t)-(\alpha,0)) \to 0$ as $|\alpha| \to \infty$. The point $(0,\infty)$ belongs to $\Gamma_1(t)$, whereas the point $(0,-\infty)$ belongs to $\Gamma_2(t)$. From elementary potential theory, we can derive explicit formulas for the velocity field u and the pressure p from the curve z.

A convenient way of studying the evolution of the interface is to consider this latter as a parametrized graph of a function. When the interface is a graph of a function $z(x,t) = (x, f(x,t))$, then this characterization is preserved locally in time by the system and f satisfies the contour equation

$$f_t(x,t) = -\rho(\Lambda f + T(f)), \tag{1.4}$$

where ρ is equal to $\frac{\rho_2-\rho_1}{2}$, and the operator Λ^γ, $0 < \gamma < 2$, denotes the usual fractional Laplacian operator of order γ and is defined as

$$\Lambda^\gamma f = (-\Delta)^{\gamma/2} f = C_\gamma P.V. \int_{\mathbb{R}} \frac{f(x)-f(x-y)}{|y|^{1+\gamma}}dy,$$

where $C_\gamma > 0$ is a positive constant. In particular, when $\gamma = 1$, the constant is equal to $\frac{1}{\pi}$.

The operator $\mathcal{H}$ denotes the Hilbert transform operator which is defined by

$$\mathcal{H} f = \frac{1}{\pi} P.V. \int \frac{f(x-\alpha)-f(x)}{\alpha}\, d\alpha.$$

In particular, one may easily check that $\partial_x \mathcal{H} = \Lambda$.

As for T, which is the nonlinear term, it is defined by

$$T(f) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\partial_x f(x)-\partial_x f(x-\alpha)}{\alpha} \frac{\left(\frac{f(x)-f(x-\alpha)}{\alpha}\right)^2}{1+\left(\frac{f(x)-f(x-\alpha)}{\alpha}\right)^2} d\alpha. \tag{1.5}$$

Equivalently, the Muskat equation can be written as

$$(\mathcal{M}) \ : \ \begin{cases} \partial_t f = \dfrac{\rho}{\pi} P.V.\, \partial_x \displaystyle\int \arctan \Delta_\alpha f \ d\alpha \\ f(x,0) = f_0(x), \end{cases}$$

where $\Delta_\alpha f \equiv \frac{f(x,t)-f(x-\alpha,t)}{\alpha} = \frac{\delta_\alpha f(x,t)}{\alpha}$.

Indeed, it is well known that linearizing $\mathcal{M}$ around the flat solution gives rise to the fractional heat equation $f_t = \rho\Lambda f$ (see e.g., [1] and [14]). The equation is linearly stable if and only if the heavier fluid is below the interface (that is $\rho_2 > \rho_1$), otherwise we say that the curve is in the unstable regime. This is known as the Rayleigh-Taylor condition and is determined by the normal component of the pressure gradient jump at the interface having a distinguished sign (also called the Saffman-Taylor condition).

This equation has attracted the attention of the mathematical community in the past several years and we shall briefly sum up the results known regarding the Cauchy problem for $(\mathcal{M})$ in the stable regime ($\rho > 0$). First of all, let us recall that this equation has a maximum principle for $\|f(\cdot,t)\|_{L^\infty}$ and $\|f(\cdot,t)\|_{L^2}$. Indeed, it is shown in [15] that

$$\|f(t)\|_{L^\infty(\mathbb{R})} \leq \frac{\|f_0\|_{L^\infty(\mathbb{R})}}{1+t}.$$

Moreover, the authors showed in [15] that if $\|\partial_x f_0\|_{L^\infty} < 1$, then $\|\partial_x f(\cdot,t)\|_{L^\infty} < \|\partial_x f_0\|_{L^\infty}$ for all $t > 0$. On the other hand, there is also an L^2 maximum principle (see [12]). More precisely we have

$$\|f(T)\|^2_{L^2(\mathbb{R})} + \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} \log\left(1+\left(\frac{f(\alpha,s)-f(\beta,s)}{\alpha-\beta}\right)^2\right) d\alpha d\beta ds = \|f_0\|^2_{L^2(\mathbb{R})}.$$

This does not imply, for large initial data, a gain of regularity in the system. However, it was observed in [22] that a gain of regularity is possible even if we start with L^2 initial data, under the condition that the slope is initially less than 1 (see [22]). Recall that the Muskat equation has a scaling: if f is a solution associated to the initial data f_0, then the function $\lambda^{-1} f(\lambda x, \lambda t)$, $\lambda > 0$ is also a solution for the corresponding initial data $\lambda^{-1} f_0(\lambda x)$. In particular, the Lipschitz norm $\dot{W}^{1,\infty}$ is critical as well as the homogeneous Sobolev norm $\dot{H}^{3/2}$. More generally, the whole family of homogeneous Besov spaces $\dot{B}^{1+p^{-1}}_{p,q}$ with $(p,q) \in [1,+\infty]^2$ is critical with respect to the scaling of $\mathcal{M}$.

As far as the local well-posedness results are concerned, in [14], the authors proved local existence in H^3. The authors of [10] were able to lower the local theory to H^2. Recently, in [13], Constantin, Gancedo, Shvydkoy and Vicol have proved that the equation is locally well-posed in $\dot{W}^{2,p}$ with $p > 1$. There is another result by Matioc [26] where local existence is obtained in H^s, $s \in (3/2, 2]$. Instant analyticity is obtained in [9] from any initial data in H^4 (see also [26]).

If the heavier fluid is above the interface, *i.e.*, $\rho < 0$, then the equation $(\mathcal{M})$ is ill-posed in Sobolev spaces (see [14] and [3]). However, there exists weak solutions to the (IPM) system starting with an initial data with a jump of densities in the unstable regime. These solutions create a zone around the initial interface where the two fluids mix. This zone grows over time, for more details see [28], [33], [7] and [21].

As for the global well-posedness results, all the available theorems were obtained in the case of data having small slopes $\|\partial_x f_0\|_{L^\infty} < 1$. These theorems are usually proved by taking advantage of the parabolic nature of ($\mathscr{M}$) for very small initial data. The first global well-posedness result for small initial data is established in [14], which was inspired by the proof in [32] for a different setting (the fluids have same densities but different viscosities). In the critical framework, the authors of [12] and [11] obtained a global existence result of a unique strong solution in H^3, if initially the Wiener norm (*i.e.,* those f verifying that the Fourier transform of Λf is integrable) is smaller than 1/3 (which, in particular, implies a slope smaller than 1). As well, the authors of [12] were able to show the global propagation of the Lipschitz regularity provided the initial data has a Lipschitz norm less than 1. Recently, the authors of [13] improved the continuation criteria for solutions from the previously known $C^{2,\alpha}$ bound to instead uniformly continuous bounded slope. In [26], the authors obtained the same kind of result in the space $H^{3/2+\epsilon}$ (for $\epsilon > 0$) with smallness in the same space. Furthermore, in [13], the authors obtained a regularity criteria, that as long as the slope has a uniform modulus of continuity there is existence. Recently, by taking advantage of this regularity criteria, it was proved in [6] a global existence of smooth solutions under a smallness assumption of some critical quantity : the product $\sup f_0'(x) \times \sup -f_0'(y)$ has to be smaller than 1. In [29] they prove optimal decay estimates for higher derivatives in the Wiener norm. By adapting the proof of [12], Deng, Lei, and Lin [20] have been able to prove the existence of a global solution for arbitrarily large monotonic initial data. The key ingredient in the proof is that, under a monotonicity assumption, the slope of the interface still satisfies the L^∞ maximum principle. Although, these solutions fail to be in L^2.

There are also several results about the development of singularities for the Muskat equation ($\mathscr{M}$) in the stable regime. Indeed, it has been shown in [9] that there exists a family of initial data (with very large slope) where the interface reaches a regime in finite time in which it is no longer a graph. Therefore there exists a time T^* where the slope of the solution of ($\mathscr{M}$) blows up: $\|f_x\|_{L^\infty}(T^*) = \infty$. In [8], they proved that there exists a class of analytic initial data in the stable regime for the Muskat problem such that the solution turns to the unstable regime and no longer belongs to C^4. Another recent physical scenario was studied in [17] where the authors proved that some solutions can pass from the stable to the unstable regime and return back to the stable regime before the solution breaks down. This shift of stability phenomena illustrates the unpredictability of the solutions to the Muskat equation even starting in the stable regime. Moreover, there is numerical evidence of initial data $||\partial_x f_0||_{L^\infty} = 50$ that develops an infinite slope in finite time (see [16]). Similar results in a confined porous media have been obtained; for global existence see [24] and for shift of stability see [23].

In this paper, we develop a $\dot{H}^{\frac{3}{2}}$ critical theory under an arbitrary bounded slope assumption. The approach in this paper is completely new and is based on a reformulation of the usual Muskat equation ($\mathscr{M}$). This new formulation allows one to take advantage of the oscillations which are crucial in this problem. There are many ways of measuring smoothness while trying to do a priori estimates in critical spaces. Contrarily to (almost) all previous works in the Muskat equation we shall never split the study into high/low frequencies or small/big increment in the finite difference operator. On the contrary, we shall consider the interaction between both and the Besov spaces techniques will be the main tool to achieve this. It is worth saying that the new formulation of the problem turns out to be crucial to prove the main theorems of this paper since it gives new features that are very difficult to see in the original formulation ($\mathcal{M}$). If one tries to do $\dot{H}^{3/2}$ estimates using Besov spaces techniques by using the classical formulation of the problem as stated in the introduction ($\mathcal{M}$), then one would quickly notice that there is a big issue to control the higher order terms. Therefore, a direct use of Besov space estimates would not give a satisfactory result.

This article is a significant step in understanding the theory of global well-posedness of large solutions in the Lipschitz space. Indeed, the main result of this paper is the global well-posedness of strong solution in $H^{5/2}$ under a smallness assumption on the $\dot{H}^{3/2}$ norm of the initial data. It would not be possible to prove a global result for all data in the Lipschitz space since the authors of [17] have shown that there are solutions with initial data having a (relatively) high slope that become singular in finite time showing the instability of the Cauchy problem associated to initial data in critical spaces.

The strategy of the proof is classical. We first establish the a priori estimates and then we construct a solution *via* classical compactness arguments that allow us to pass to the weak limit in a parabolic regularized Muskat equation.

The outline of this paper is as follows. In the next section, we state the main results. In the third section, we give the definitions of the spaces along with some harmonic analysis tools that we shall use throughout the article. The fourth section is devoted to the new formulation of the problem. The fifth section is the proof of the a priori estimates in $\dot{H}^{3/2}$. In the sixth section, we give the a priori estimates in $\dot{H}^{5/2}$. The last section is the proof of the main results.

2. Main results

The main result of this article is the following global existence theorem of a unique strong solution for small data in the critical Sobolev space.

THEOREM 2.1. – *Assume $f_0 \in H^{5/2}$ with $\|f_0\|_{\dot{H}^{3/2}} < \frac{C}{1+\|\partial_x f_0\|^2_{L^\infty}}$ where C is a fixed constant, then, there exists a unique strong solution f to equation ($\mathcal{M}$) which verifies $f \in L^\infty([0,T], H^{5/2}) \cap L^2([0,T], \dot{H}^3)$, for all $T > 0$.*

REMARK 1. – By using the L^2 maximum principle, it suffices to prove a priori estimates in $\dot{H}^{5/2}$.

REMARK 2. – It would be possible to lower the initial regularity by considering a data in H^s with $3/2 < s \leq 5/2$. This would lead to tedious computations, the present article does only treat the case of regular enough data with finite energy for the sake of simplicity.

In order to prove the main result, we shall prove a regularity criterion in terms of the control of the slopes that gives a condition for a weak solution to be strong.

THEOREM 2.2. – *Let $f_0 \in \dot{W}^{1,\infty} \cap \dot{H}^{3/2}$ and let $T > 0$. Assume there is a solution f on $[0,T]$ with slope $\|\partial_x f\|_{L^\infty}$ and $\|f_0\|_{\dot{H}^{3/2}} < \frac{C}{1+\|\partial_x f_0\|^2_{L^\infty}}$, where C is a fixed constant, then there exists a unique weak solution f to equation ($\mathcal{M}$) which verifies $f \in L^\infty([0,T], (\dot{H}^{3/2} \cap \dot{W}^{1,\infty})) \cap L^2([0,T], \dot{H}^2)$ and in particular, $\|f(T)\|_{\dot{H}^{3/2}} \leq \|f_0\|_{\dot{H}^{3/2}}$.*

REMARK 3. – The definition of weak solutions is easy to get. Indeed, we say that f is a weak solution to the Muskat problem if, for all $\phi \in \mathscr{D}([0,T]\times\mathbb{R})$, we have the following equality

$$\int_0^T\int \phi_s(x,s)f(x,s)\,ds\,dx + \int \phi(x,0)f_0(x)\,dx \\ = \int_0^T\int \phi_x(x,s)\iint_0^\infty \delta^{-1}e^{-\delta}\sin(\delta\Delta_\alpha f)\,d\delta\,d\alpha\,dx\,ds.$$

3. Functional setting

We shall use the homogeneous Sobolev space $\dot{H}^s$, which is endowed with the semi-norm

$$\|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2}.$$

We recall the definition of the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R})$ (see [5], [30]). Let $(p,q,s) \in [1,\infty]^2\times\mathbb{R}$, we say that a tempered distribution f (which is such that its Fourier transform is integrable near 0) belongs to the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$ if the following quantity (which is a semi-norm), is finite, that is

$$\|f\|_{\dot{B}^s_{p,q}} = \left\| \frac{\|1_{]0,1[}(s)\delta_y f + 1_{[1,2[}(s)(\delta_y f + \bar{\delta}_y f)\|_{L^p}}{|y|^s}\right\|_{L^q(\mathbb{R},|y|^{-1}dy)} < \infty,$$

where $\delta_y f(x) = f(x) - f(x-y)$ and $\bar{\delta}_y f(x) = f(x) - f(x+y)$.

Let us recall some classical embeddings (see e.g., [4], [2]).

We have, for all $(p_1,p_2,r) \in [1,\infty]^3$

$$\dot{B}^{s_1}_{p_1,r}(\mathbb{R}) \hookrightarrow \dot{B}^{s_2}_{p_2,r}(\mathbb{R}),$$

where $s_1 + \frac{1}{p_2} = s_2 + \frac{1}{p_1}$ and $p_1 < p_2$. We also have for all $(p_1,s_1) \in [2,\infty]\times\mathbb{R}$,

$$\dot{B}^{s_1}_{p_1,r_1}(\mathbb{R}) \hookrightarrow \dot{B}^{s_1}_{p_1,r_2}(\mathbb{R}),$$

for all $(r_1,r_2) \in]1,\infty]^2$ such that $r_1 \le r_2$.

Let $(s_1,s_2) \in \mathbb{R}^2$ so that $s_1 < s_2$. Then, for all $\theta \in]0,1[$ and $p \in [1,\infty]$, we have the following real interpolation inequality

$$\|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \le \frac{C}{s_2-s_1}\left(\frac{1}{\theta}+\frac{1}{1-\theta}\right)\|f\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}}\|f\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}. \tag{3.1}$$

We shall use the following generalized Calderón commutator type estimate which was proved by Dawson, Macghan, and Ponce in [19]. Let $\Phi \in \dot{W}^{k+l,\infty}$ and let us consider the commutator

$$T_\Phi f = [\mathscr{H},\Phi]\,f,$$

then, for all $p \in]1,\infty[$, $(k,l) \in \mathbb{N}$ and for all $f \in L^p$ the following estimate holds,

$$\left\|T_\Phi \partial_x^k f\right\|_{\dot{W}^{l,p}} \le C_{k,l,p}\|\Phi\|_{\dot{W}^{k+l,\infty}}\|f\|_{L^p}. \tag{3.2}$$

Throughout the article, we shall use the notation $A \lesssim B$ if there exists a fixed constant $C > 0$ such that $A \le CB$.

4. A new formulation of the stable Muskat equation

In this section, we give another formulation of the Muskat equation in terms of oscillatory integrals that will be useful when doing estimates (especially to control high regularity terms) in Besov spaces.

PROPOSITION 4.1. – *Assume that f solves $(\mathscr{M})$ then f is a solution of $(\tilde{\mathscr{M}})$ that is*

$$(\tilde{\mathscr{M}}) \ : \begin{cases} f_t(t,x) = \dfrac{\rho}{\pi} P.V. \displaystyle\int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f)\, d\delta\, d\alpha \\ f(0,x) = f_0(x). \end{cases} \tag{4.1}$$

Reciprocally, if f solves $(\tilde{\mathscr{M}})$ then f solves $(\mathscr{M})$. That is,

$$(\tilde{\mathscr{M}}) \iff (\mathscr{M}).$$

Proof of Proposition 4.1. – Consider the following integrable function

$$\mu(x) \ = \frac{1}{2} \exp(-|x|). \tag{4.2}$$

Its Fourier transform is well defined and since μ is even, we have,

$$\widehat{\mu}(\xi) = \int_0^\infty e^{-\delta} \cos(\delta\xi)\, d\delta.$$

By denoting $I \ = \int_0^\infty e^{-\delta} \cos(\delta\xi)\, d\delta$ and by integrating by parts twice, it is not difficult to check that $I = -\xi^2 I + 1$. Therefore,

$$\int_0^\infty e^{-\delta} \cos(\delta\xi)\, d\delta = \frac{1}{1+\xi^2}. \tag{4.3}$$

Then, by considering the restriction of this Fourier transform onto $\Delta_\alpha f$ one readily arrives to $(\tilde{\mathscr{M}})$. Conversely, if f is a solution of $(\tilde{\mathscr{M}})$ then it is obviously also a solution of $(\mathscr{M})$ by using once again the identity (4.3) applied to $\xi = \Delta_\alpha f$. □

We shall assume, without loss of generality, that $\rho = \pi$. The purpose of the next section is to prove that one has nice a priori estimates in $\dot{H}^{3/2}$.

5. A priori estimates in $\dot{H}^{3/2}$

We start this section by proving some useful identities that we shall use in the a priori estimates. Introduce the operator $\bar{\Delta}_\alpha f \ = \ \frac{f(x,t)-f(x+\alpha,t)}{\alpha}$ and consider the difference $D = \Delta_\alpha f - \bar{\Delta}_\alpha f$. We have the following identity for D.

$$D = \frac{f(x+\alpha) - f(x-\alpha)}{\alpha} = \frac{1}{\alpha}\int_0^\alpha f_x(x+s) + f_x(x-s) - 2f_x(x)\, ds + 2f_x(x). \tag{5.1}$$

We shall also need an expression for $\partial_\alpha D$.

$$\partial_\alpha D = \frac{f_x(x+\alpha) - f_x(x-\alpha) - 2f_x(x)}{\alpha} - \frac{\int_0^\alpha (f_x(x-s) + f_x(x+s) - 2f_x(x))\, ds}{\alpha^2}. \tag{5.2}$$

Set $S = \Delta_\alpha f - \bar{\Delta}_\alpha f$. Analogously, we shall need to get some nice expression of S and $\partial_\alpha S$. More precisely, we have the following identities:

$$S = \Delta_\alpha f + \bar{\Delta}_\alpha f = -\frac{(f(x+\alpha) + f(x-\alpha) - 2f(x))}{\alpha}. \tag{5.3}$$

For $\partial_\alpha S$, we have that,

$$\partial_\alpha S = \bar{\Delta}_\alpha f_x - \Delta_\alpha f_x + \frac{f(x+\alpha) + f(x-\alpha) - 2f(x)}{\alpha^2}. \tag{5.4}$$

Note that those identities will be useful since they make appear the more favorable second finite order differences. We can now state the main result of this section which is the following lemma.

Lemma 5.1. – *Let $T > 0$, assume that $f_0 \in \dot{H}^{3/2} \cap \dot{W}^{1,\infty}$ and that the $L^\infty([0,T], \dot{W}^{1,\infty})$ norm remains bounded. Then, if we set $K = \sup_{(x,t)\in\mathbb{R}\times[0,T]} |f_x(x,t)|$, we have*

$$\begin{aligned}\|f\|^2_{\dot{H}^{3/2}}(T) + \frac{\pi}{1+K^2}\int_0^T \|f\|^2_{\dot{H}^2}\,ds \\ \lesssim \|f_0\|_{\dot{H}^{3/2}} + \left(\|f\|_{L^\infty([0,T],\dot{H}^{3/2})} + \|f\|^2_{L^\infty([0,T],\dot{H}^{3/2})}\right)\int_0^T \|f\|^2_{\dot{H}^2}\,ds.\end{aligned}$$

Proof of Lemma 5.1. – We multiply $\Lambda^{3/2} f$ against $\Lambda^{3/2} f_t$ and integrate with respect to the space variable, we obtain

$$\begin{aligned}\frac{1}{2}\partial_t\|f\|^2_{\dot{H}^{3/2}} &= \int \mathcal{H} f_{xx} \int \partial_{xx}\Delta_\alpha f \int_0^\infty e^{-\delta}\cos(\delta\Delta_\alpha f(x))\,d\delta\,d\alpha\,dx \\ &\quad - \int \mathcal{H} f_{xx} \int (\partial_x\Delta_\alpha f)^2 \int_0^\infty \delta e^{-\delta}\sin(\delta\Delta_\alpha f(x))\,d\delta\,d\alpha\,dx \\ &= I_1 + I_2.\end{aligned}$$

□

Obviously, the more singular term is I_1. The estimation of such a term requires a long splitting into several terms that will be nicely estimated. This is the aim of the next subsection.

5.1. Decomposition of the term I_1

The aim of this subsection is to decompose I_1 into a sum of more easily controllable terms. The idea is to symmetrize the terms in order to make appear second finite order differences. Indeed, having a second finite order difference would allow us to share the powers of α in a more balanced way. Besides, estimating terms rigorously in homogeneous Besov spaces with high regularity (that is $s \in (1,2)$) requires to have second finite order differences. The main strategy will be to use the convenient identity

$$\partial_x(f(x+\alpha) - f(x-\alpha)) = \partial_\alpha(f(x+\alpha) + f(x-\alpha) - 2f(x)).$$

This identity shows that $f(x+\alpha) - f(x-\alpha) = \Delta_\alpha f - \bar{\Delta}_\alpha f$ contains a hidden second finite order difference. To estimate the term I_1 we shall try to force the appearance of such symmetric terms using this idea.

$$\begin{aligned}
I_1 = &\int \mathcal{H} f_{xx} \int (\partial_{xx}\Delta_\alpha f - \partial_{xx}\bar{\Delta}_\alpha f) \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f(x))\, d\delta\, d\alpha\, dx \\
&- \int \mathcal{H} f_{xx} \int \partial_{xx}\Delta_\alpha f \int_0^\infty e^{-\delta} \cos(\delta \bar{\Delta}_\alpha f)\, d\delta\, d\alpha\, dx \\
= &\int \mathcal{H} f_{xx} \int (\partial_{xx}\Delta_\alpha f - \partial_{xx}\bar{\Delta}_\alpha f) \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f(x))\, d\delta\, d\alpha\, dx \\
&+ \int \mathcal{H} f_{xx} \int \partial_{xx}\Delta_\alpha f \int_0^\infty e^{-\delta} \big(\cos(\delta \Delta_\alpha f) - \cos(\delta \bar{\Delta}_\alpha f)\big)\, d\delta\, d\alpha\, dx \\
&- \int \mathcal{H} f_{xx} \int \partial_{xx}\Delta_\alpha f \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f)\, d\delta\, d\alpha\, dx.
\end{aligned}$$

Hence, by symmetry (note that $\partial_{xx}\Delta_\alpha f - \partial_{xx}\bar{\Delta}_\alpha f$ is invariant by $\alpha \to -\alpha$), one obtains

$$\begin{aligned}
I_1 = &\frac{1}{4}\int \mathcal{H} f_{xx} \int (\partial_{xx}\Delta_\alpha f - \partial_{xx}\bar{\Delta}_\alpha f) \int_0^\infty e^{-\delta} (\cos(\delta \Delta_\alpha f(x)) + \cos(\delta \bar{\Delta}_\alpha f(x))\, d\delta\, d\alpha\, dx \\
&+ \frac{1}{2}\int \mathcal{H} f_{xx} \int \partial_{xx}\Delta_\alpha f \int_0^\infty e^{-\delta} \big(\cos(\delta \Delta_\alpha f) - \cos(\delta \bar{\Delta}_\alpha f)\big)\, d\delta\, d\alpha\, dx.
\end{aligned}$$

By using classical trigonometry formulas, I_1 may be rewritten as

$$\begin{aligned}
I_1 = &\frac{1}{2}\int \mathcal{H} f_{xx} \int (\partial_{xx}\Delta_\alpha f - \partial_{xx}\bar{\Delta}_\alpha f) \\
&\times \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&- \int \mathcal{H} f_{xx} \int \partial_{xx}\Delta_\alpha f \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
I_1 = &-\int \mathcal{H} f_{xx} \int (\partial_{xx}\Delta_\alpha f - \partial_{xx}\bar{\Delta}_\alpha f) \\
&\times \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin^2(\frac{\delta}{4}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&+ \frac{1}{2}\int \mathcal{H} f_{xx} \int (\partial_{xx}\Delta_\alpha f - \partial_{xx}\bar{\Delta}_\alpha f) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&- \int \mathcal{H} f_{xx} \int \partial_{xx}\Delta_\alpha f \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
= &I_{1,1} + I_{1,2} + I_{1,3}.
\end{aligned}$$

The aim will be to estimate the $I_{1,j}$ with $j = 1, 2, 3$. It is important to note that the gain of regularity will come from the term $I_{1,2}$. Indeed, since $\cos(x) = 1 - 2\sin^2(x/2)$, we immediately notice that $I_{1,2} =$ dissipation $+$ remainder. The main task in this term will be to estimate the remainder. As a matter of fact, this remainder will be the sum of a term that will be nicely estimated plus the elliptic component (see (5.9)).

5.1.1. *Estimates of* $I_{1,1}$. – To control $I_{1,1}$, we use the continuity of the Hilbert transform on L^2 along with the embedding $\dot{H}^{3/2} \hookrightarrow \dot{B}^1_{\infty,2}$, then one gets

$$\begin{aligned}
I_{1,1} &= -\int \mathcal{H} f_{xx} \int (\partial_{xx}\Delta_\alpha f - \partial_{xx}\bar{\Delta}_\alpha f) \\
&\quad \times \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin^2(\frac{\delta}{4}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\leq \frac{\Gamma(3)}{4} \|f\|^2_{\dot{H}^2} \int \frac{\|\delta_\alpha f + \bar{\delta}_\alpha f\|^2_{L^\infty}}{|\alpha|^3}\, d\alpha \\
&\lesssim \frac{1}{2}\|f\|^2_{\dot{H}^2}\|f\|^2_{\dot{B}^1_{\infty,2}} \\
&\lesssim \frac{1}{2}\|f\|^2_{\dot{H}^2}\|f\|^2_{\dot{H}^{3/2}}.
\end{aligned}$$

5.1.2. *Estimates of* $I_{1,2}$. – Recall that $I_{1,2}$ is given by

$$\begin{aligned}
I_{1,2} &= \frac{1}{2}\int \mathcal{H} f_{xx} \int (\partial_{xx}\Delta_\alpha f - \partial_{xx}\bar{\Delta}_\alpha f) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&= \frac{1}{2}\int \mathcal{H} f_{xx} \int \frac{1}{\alpha}\partial_\alpha\{\delta_\alpha f_x + \bar{\delta}_\alpha f_x\} \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx,
\end{aligned}$$

therefore, an integration by parts (with respect to α) gives

$$\begin{aligned}
I_{1,2} &= \frac{1}{2}\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha^2} \\
&\quad \times \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\quad + \frac{1}{4}\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha} \\
&\quad \times \int_0^\infty \delta e^{-\delta} \partial_\alpha\{\Delta_\alpha f - \bar{\Delta}_\alpha f\} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx.
\end{aligned}$$

Then, in order to force the appearance of a second finite order difference, we use the identity (5.2). Therefore, we get

$$\begin{aligned}
I_{1,2} &= \frac{1}{2}\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha^2} \\
&\quad \times \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\quad + \frac{1}{4}\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha} \\
&\quad \times \int_0^\infty \delta e^{-\delta} \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\quad - \frac{1}{4}\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha^3} \int_0^\infty \delta e^{-\delta} \\
&\quad \times \int_0^\alpha f_x(x-s) + f_x(x+s) - 2f_x(x)\, ds \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx.
\end{aligned}$$

Then, importantly, one has to remark that the first term of $I_{1,2}$, that is

$$\frac{1}{2}\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx,$$

contains a dissipative term. Indeed, using the trigonometry formula $\cos(\theta)=1-2\sin^2(\frac{1}{2}\theta)$ and the well-know identity

$$\int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2}\, d\alpha = -2\pi\Lambda f_x = -2\pi\,\mathcal{H} f_{xx}, \tag{5.5}$$

one deduces that,

$$\begin{aligned}
&\frac{1}{2}\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta}\cos(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx\\
&\qquad = -\pi\int |\mathcal{H} f_{xx}|^2\, dx - \int \mathcal{H} f_{xx}\int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2} \qquad (5.6)\\
&\qquad\quad \times \int_0^\infty e^{-\delta}\sin^2(\frac{\delta}{4}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx\\
&\qquad = -\pi\|f\|_{\dot{H}^2}^2 - \int \mathcal{H} f_{xx}\int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2} \qquad (5.7)\\
&\qquad\quad \times \int_0^\infty e^{-\delta}\sin^2(\frac{\delta}{4}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx.
\end{aligned}$$

As we shall see, the remaining nonlinear term, namely

$$-\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta}\sin^2(\frac{\delta}{4}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx,$$

will be split into two terms. One term will be nicely estimated and the other term will be the elliptic component. Hence, using (5.6), we finally have

(5.8)

$$\begin{aligned}
I_{1,2} &= -\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2}\\
&\quad \times \int_0^\infty e^{-\delta}\sin^2(\frac{\delta}{4}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx\\
&\quad + \frac{1}{4}\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha}\\
&\quad \times \int_0^\infty \delta e^{-\delta}\frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha}\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx\\
&\quad - \frac{1}{4}\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^3} \int_0^\infty \delta e^{-\delta}\\
&\quad \times \int_0^\alpha f_x(x-s)+f_x(x+s)-2f_x(x)\, ds\,\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx - \pi\|f\|_{\dot{H}^2}^2\\
&= I_{1,2,1}+I_{1,2,2}+I_{1,2,3}+I_{1,2,4}.
\end{aligned}$$

We shall estimate the $I_{1,2,j}$, $j=1,2,3$, in the next subsection.

5.1.2.1. *Estimate of the* $I_{1,2,1}$. – Recall that $I_{1,2,1}$ is given by

$$I_{1,2,1} = -\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2} \times \int_0^\infty e^{-\delta} \sin^2\Big(\frac{\delta}{4}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\Big)\, d\delta\, d\alpha\, dx. \tag{5.9}$$

Since we have

$$\int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2}\, d\alpha = -2\pi\Lambda f_x = -2\pi \mathcal{H} f_{xx}.$$

Therefore, we notice that if we neglect the effect of the oscillatory integral we find,

$$-\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2}\, d\alpha\, dx = 2\pi \int |\mathcal{H} f_{xx}|^2\, dx. \tag{5.10}$$

This is a bad term since it cancels the gain of regularity obtained in the term $I_{1,2,4}$ (see (5.8)). So it is crucial to use the oscillations. The phase of the oscillatory integrals is not regular enough to be controlled in L^∞. Indeed, estimating $\Delta_\alpha f - \bar{\Delta}_\alpha f$ in L^∞ would imply a smallness on the slope. The idea is to find a second order finite difference in $\Delta_\alpha f - \bar{\Delta}_\alpha f$. By using the formula (5.1), we know that $\Delta_\alpha f - \bar{\Delta}_\alpha f$ can be written as a second finite difference plus a correction term which is $2f_x(x)$. The idea will be therefore to force the appearance of this correction term in the oscillatory integral. The correct quantity to substract and add in order to force the appearance of $2f_x(x)$ is $\sin^2(\frac{\delta}{2} f_x)$. Hopefully, this quantity is independent of α. This will allow us to remove the α-dependence in the δ-integral which is crucial in order to break the regularity issue. The remaining integral will be the desired elliptic component which will be of the form $\frac{1}{2}\frac{f_x^2}{1+f_x^2}$ as we shall see below. More precisely, we write

$$\begin{aligned} I_{1,2,1} &= -\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2} \\ &\quad \times \int_0^\infty e^{-\delta} \left(\sin^2\Big(\frac{\delta}{4}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\Big) - \sin^2\Big(\frac{\delta}{2} f_x\Big)\right) d\delta\, d\alpha\, dx \\ &\quad -\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta} \sin^2\Big(\frac{\delta}{2} f_x\Big)\, d\delta\, d\alpha\, dx \\ &= I_{1,2,1,1} + I_{1,2,1,2}. \end{aligned}$$

In order to estimate $I_{1,2,1,1}$ we use the following useful identity

$$\begin{aligned} &\sin^2\Big(\frac{\delta}{4}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\Big) - \sin^2\Big(\frac{\delta}{2} f_x\Big) \\ &= 4\sin\Big(\frac{\delta}{4}(\Delta_\alpha f - \bar{\Delta}_\alpha f - 2f_x(x))\Big)\sin\Big(\frac{\delta}{4}(\Delta_\alpha f - \bar{\Delta}_\alpha f + 2f_x(x))\Big) \\ &\quad \times \cos\Big(\frac{\delta}{4}(\Delta_\alpha f - \bar{\Delta}_\alpha f - 2f_x(x))\Big)\cos\Big(\frac{\delta}{4}(\Delta_\alpha f - \bar{\Delta}_\alpha f + 2f_x(x))\Big). \end{aligned} \tag{5.11}$$

So that, by using the classical estimate $|\sin\theta| \le |\theta|$ we get

$$\left|\sin^2\Big(\frac{\delta}{4}(\Delta_\alpha f - \bar{\Delta}_\alpha f)\Big) - \sin^2\Big(\frac{\delta}{2} f_x\Big)\right| \le \delta\, \big|\Delta_\alpha f - \bar{\Delta}_\alpha f + 2f_x(x)\big|. \tag{5.12}$$

Hence, by using the identity (5.1) one gets

$$|I_{1,2,1,1}| \lesssim \|f\|_{\dot H^2} \int \frac{\|f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)\|_{L^\infty}}{|\alpha|^3} \times \int_0^\alpha \frac{\|f_x(x+s)+f_x(x-s)-2f_x(x)\|_{L^2}}{s^q}\, s^q\, ds.$$

Let $q \in [1,\infty)$ be a real number, using Hölder inequality in s (with $r^{-1}+\bar r^{-1}=1$) we find

$$\begin{aligned}
|I_{1,2,1,1}| &\lesssim \|f\|_{\dot H^2} \int \frac{\|f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)\|_{L^\infty}}{|\alpha|^3} \\
&\quad\times \left(\int_0^\alpha \frac{\|f_x(x+s)+f_x(x-s)-2f_x(x)\|_{L^2}^r}{s^{qr}}\, ds\right)^{1/r} \left(\int_0^\alpha s^{q\bar r}\, ds\right)^{1/\bar r} d\alpha \\
&\lesssim \|f\|_{\dot H^2} \int \frac{\|f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)\|_{L^\infty}}{|\alpha|^{3-q-\frac{1}{\bar r}}} \\
&\quad\times \left(\int_0^\alpha \frac{\|f_x(x+s)+f_x(x-s)-2f_x(x)\|_{L^2}^r}{s^{qr}}\, ds\right)^{1/r} d\alpha \\
&\le \|f\|_{\dot H^2} \|f_x\|_{\dot B^{2-q-\frac{1}{\bar r}}_{\infty,1}} \|f_x\|_{\dot B^{q-\frac{1}{r}}_{2,r}}.
\end{aligned}$$

Then, by real interpolation and by choosing $q=9/8$, $r=\bar r=2$ one obtains

$$|I_{1,2,1,1}| \le \|f\|_{\dot H^2} \|f\|^{1/4}_{\dot B^1_{\infty,\infty}} \|f\|^{3/4}_{\dot B^{3/2}_{\infty,\infty}} \|f_x\|_{\dot B^{5/8}_{2,2}}.$$

Using the interpolation inequality, we get

$$\|f\|_{\dot B^{13/8}_{2,2}} \le \|f\|^{3/4}_{\dot H^{3/2}} \|f\|^{1/4}_{\dot H^2},$$

along with the fact that $\dot H^{1/2+\eta} \hookrightarrow \dot B^\eta_{\infty,\infty}$ for $\eta = 3/2$ and $\eta = 1$, one finally gets

$$|I_{1,2,1,1}| \lesssim \|f\|^2_{\dot H^2} \|f\|_{\dot H^{3/2}}.$$

Then, it remains to estimate $I_{1,2,1,2}$, we write

$$I_{1,2,1,2} = -\int \mathscr{H} f_{xx} \int \frac{f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)}{\alpha^2} \times \int_0^\infty e^{-\delta}\, \sin^2\!\left(\frac{\delta}{2} f_x(x)\right) d\delta\, d\alpha\, dx. \tag{5.13}$$

Importantly, we may explicity compute independently the α-integral and the δ-integral. Indeed, the α-integral gives $-2\pi\, \mathscr{H} f_{xx}$ (see Equation (5.10) above). As for the δ-integral, one may easily check that it is equal to

$$\int_0^\infty e^{-\delta}\, \sin^2\!\left(\frac{\delta}{2} f_x(x)\right) d\delta = \frac{1}{2}\frac{f_x^2}{1+f_x^2}. \tag{5.14}$$

Indeed, to prove equality (5.14) it suffices to use equality (4.3) applied to $\xi = f_x(x)$. One gets

$$\int_0^\infty e^{-\delta} \cos(\delta f_x(x))\, d\delta = \frac{1}{1+f_x(x)^2}.$$

Using the trigonometric identity $\cos(\theta) = 1 - 2\sin^2(\frac{1}{2}\theta)$ applied to $\theta = \delta f_x(x)$ gives

$$\int_0^\infty e^{-\delta}\left(1 - 2\sin^2(\frac{\delta}{2} f_x(x))\right) d\delta = \frac{1}{1 + f_x(x)^2}.$$

Therefore, using that $\int_0^\infty e^{-\delta} d\delta = 1$ one gets the desired identity, that is

$$\int_0^\infty e^{-\delta}\sin^2(\frac{\delta}{2} f_x(x))\, d\delta = \frac{1}{2}\frac{f_x(x)^2}{1 + f_x(x)^2}.$$

Hence, combining the two identities (5.10) and (5.14), we get that

$$I_{1,2,1,5} = -\int \mathscr{H} f_{xx} \int \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha^2} \int_0^\infty e^{-\delta}\ \sin^2(\frac{\delta}{2} f_x(x))\, d\delta\, d\alpha\, dx$$

$$= \pi \int \frac{f_x^2}{1 + f_x^2} |\mathscr{H} f_{xx}|^2\, dx. \tag{5.15}$$

Set $K = \sup_{(x,t)\in\mathbb{R}\times[0,T]} |f_x(x,t)|$. Using the fact that $x \mapsto \frac{x^2}{1+x^2}$ is an increasing function on $\mathbb{R}^+$, we get that,

$$I_{1,2,1,2} \leq \pi \frac{K^2}{1 + K^2} \|f\|_{\dot{H}^2}^2. \tag{5.16}$$

Therefore, we find that

$$|I_{1,2,1,1} + I_{1,2,1,2}| \lesssim \|f\|_{\dot{H}^2}^2 \left(\|f\|_{\dot{H}^{3/2}} + \|f\|_{\dot{H}^{3/2}}^2\right) + \pi \frac{K^2}{1 + K^2} \|f\|_{\dot{H}^2}^2. \tag{5.17}$$

5.1.2.2. *Estimate of* $I_{1,2,2}$. – To estimate $I_{1,2,2}$, we observe that

$$\begin{aligned}
I_{1,2,2} &= \frac{1}{4}\int \mathscr{H} f_{xx} \int \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha} \\
&\quad\times \int_0^\infty \delta e^{-\delta} \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\lesssim \|f\|_{\dot{H}^2} \left(\int \frac{\|f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)\|_{L^2}^2}{\alpha^2}\, d\alpha\right)^{1/2} \\
&\quad\times \left(\int \frac{\|f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)\|_{L^\infty}^2}{\alpha^2}\, d\alpha\right)^{1/2} \\
&\lesssim \|f\|_{\dot{H}^2} \|f_x\|_{\dot{B}^{1/2}_{\infty,2}} \|f\|_{\dot{H}^{3/2}} \\
&\lesssim \|f\|_{\dot{H}^2}^2 \|f\|_{\dot{H}^{3/2}}.
\end{aligned}$$

To estimate $I_{1,2,3}$ we write

$$\begin{aligned}
I_{1,2,3} &= -\frac{1}{4}\int \mathscr{H} f_{xx} \int \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha^3} \\
&\quad\times \int_0^\infty \delta e^{-\delta} \int_0^\alpha f_x(x-s) + f_x(x+s) - 2f_x(x)\, ds \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx.
\end{aligned}$$

Hence,

$$I_{1,2,3} \leq \frac{1}{4}\|f\|_{\dot{H}^2} \iint_0^\infty \delta e^{-\delta} \frac{\|f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)\|_{L^\infty}}{|\alpha|^3}$$

$$
\begin{aligned}
&\times \int_0^\alpha \|f_x(x-s)-f_x(x)\|_{L^2}+\|f_x(x+s)-f_x(x)\|_{L^2}\,ds\,d\delta\,d\alpha\\
&\lesssim \|f\|_{\dot H^2}\left(\int \frac{\|f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)\|_{L^\infty}}{|\alpha|^{3-q-\frac{1}{\bar r}}}\,d\alpha\right)\\
&\times\left[\left(\int \frac{\|f_x(x-s)-f_x(x)\|_{L^2}^r}{|s|^{qr}}\,ds\right)^{1/r}+\left(\int \frac{\|f_x(x+s)-f_x(x)\|_{L^2}^r}{|s|^{qr}}\,ds\right)^{1/r}\right]\\
&\lesssim \|f\|_{H^2}\|f_x\|_{\dot B^{2-q-\frac{1}{\bar r}}_{\infty,1}}\|f_x\|_{\dot B^{q-\frac{1}{r}}_{2,r}}.
\end{aligned}
$$

Then, by choosing $q=9/8$, $r=\bar r=2$, and real interpolation (to get a control of $\dot B^{11/8}_{\infty,1}$) along with classical homogeneous Besov embeddings one gets

$$
\begin{aligned}
|I_{1,2,3}| &\le \frac{1}{2}\|f\|_{\dot H^2}\|f\|_{\dot B^{13/8}_{2,2}}\|f\|_{\dot B^{11/8}_{\infty,1}}\\
&\le \frac{1}{2}\|f\|_{\dot H^2}\|f\|^{1/4}_{\dot B^1_{\infty,1}}\|f\|^{3/4}_{\dot B^{3/2}_{\infty,1}}\|f\|^{1/4}_{\dot B^2_{2,2}}\|f\|^{3/4}_{\dot B^2_{2,2}}\\
&\le \frac{1}{2}\|f\|^2_{\dot H^2}\|f\|_{\dot H^{3/2}}.
\end{aligned}
$$

Recalling that (5.8) is a dissipative term, and that (5.15) is the elliptic component, we have proved that

$$
|I_{1,2}|\lesssim \|f\|^2_{H^2}(\|f\|^2_{\dot H^{3/2}}+\|f\|_{\dot H^{3/2}})-\pi\|f\|^2_{\dot H^2}+\pi\frac{K^2}{1+K^2}\|f\|^2_{\dot H^2},
$$

where $K=\sup_{(x,t)\in\mathbb{R}\times[0,T]}|f_x(x,t)|$. Finally,

$$
|I_{1,2}|\lesssim \|f\|^2_{H^2}(\|f\|^2_{\dot H^{3/2}}+\|f\|_{\dot H^{3/2}})-\frac{\pi}{1+K^2}\|f\|^2_{\dot H^2}. \tag{5.18}
$$

5.1.3. *Estimates of $I_{1,3}$.* – Let us estimate $I_{1,3}$. Let us recall that

$$
I_{1,3}=-\int \mathscr{H}f_{xx}\int \partial_{xx}\Delta_\alpha f\int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\,d\delta\,d\alpha\,dx.
$$

The estimate of $I_{1,3}$ is not immediate. Indeed, if we do a direct estimate of $I_{1,3}$ we get that

$$
I_{1,3}\lesssim \|f\|^2_{\dot H^2}\int \frac{\|\delta_\alpha f+\bar\delta_\alpha f\|_{L^\infty}}{\alpha^2}d\alpha=\|f\|^2_{\dot H^2}\|f\|_{\dot B^1_{\infty,1}}.
$$

Using the interpolation inequality (3.1), for $p=\infty$ and $\theta s_1+(1-\theta)s_2=1$ where $s_1<s_2$ one obtains

$$
\|f\|_{\dot B^1_{\infty,1}}\lesssim \|f\|^\theta_{\dot B^{s_1}_{\infty,\infty}}\|f\|^{1-\theta}_{\dot B^{s_2}_{\infty,\infty}}. \tag{5.19}
$$

The conditions $\theta s_1+(1-\theta)s_2=1$ and $s_1<s_2$ necessarily imply that $s_2>1$. Therefore, this would give a control of $\|f\|_{\dot B^1_{\infty,1}}$ by a quantity of the type $\|f\|_{\dot B^{1+\epsilon}_{\infty,\infty}}$ which is a subcritical norm with respect to the scaling of the equation (recall that the critical space in that scale is $\dot B^1_{\infty,\infty}$).

Since there is a regularity issue, the idea is to try to balance the derivatives. The idea is to write that $f_{xx}(x-\alpha)-f_{xx}(x)=\partial_\alpha(f_x(x)-f_x(x-\alpha))-f_{xx}(x)$. This kind of splitting is usually not very helpful because the finite difference is always more regular than each term written separately. However, the presence of $\mathcal{H} f_{xx}$ in front of $f_{xx}(x)$ creates a new regularizing effect since it has a nice commutator structure. More precisely, we write

$$\begin{aligned}
I_{1,3} &= -\int \mathcal{H} f_{xx} \int \frac{f_{xx}(x) - f_{xx}(x-\alpha)}{\alpha} \\
&\quad \times \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&= \int \mathcal{H} f_{xx} \int \frac{f_{xx}(x-\alpha) - f_{xx}(x)}{\alpha} \\
&\quad \times \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&= \int \mathcal{H} f_{xx} \int \frac{\partial_\alpha \{f_x(x) - f_x(x-\alpha)\}}{\alpha} \\
&\quad \times \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\quad \underbrace{- \int \mathcal{H} f_{xx} \int \frac{f_{xx}(x)}{\alpha} \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx}_{\text{commutator structure}}.
\end{aligned}$$

Integrating by parts (in α) and using the notations $D = \Delta_\alpha f - \bar{\Delta}_\alpha f$ and $S = \Delta_\alpha f + \bar{\Delta}_\alpha f$, one finds

$$\begin{aligned}
I_{1,3} &= \int \mathcal{H} f_{xx} \int \frac{f_x(x) - f_x(x-\alpha)}{\alpha^2} && (5.20) \\
&\quad \times \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx && (5.21) \\
&\quad - \frac{1}{2} \int \mathcal{H} f_{xx} \int \frac{f_x(x) - f_x(x-\alpha)}{\alpha} \\
&\quad \times \int_0^\infty \delta e^{-\delta} \partial_\alpha D \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx && (5.22) \\
&\quad - \frac{1}{2} \int \mathcal{H} f_{xx} \int \frac{f_x(x) - f_x(x-\alpha)}{\alpha} \\
&\quad \times \int_0^\infty \delta e^{-\delta} \partial_\alpha S \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx && (5.23) \\
&\quad - \int \mathcal{H} f_{xx} \int \frac{f_{xx}(x)}{\alpha} \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&= \sum_{i=1}^4 I_{1,3,i}.
\end{aligned}$$

In the next subsection, we shall estimate the $I_{1,3,i}$ for $i = 1, \ldots, 4$.

5.1.3.1. *Estimates of* $I_{1,3,1}$. – This term is estimated as follows, by observing that $\dot H^2 \hookrightarrow \dot B^{3/2}_{\infty,2}$, we may for instance write

$$
\begin{aligned}
|I_{1,3,1}| &\leq \|f\|_{\dot H^2} \int_0^\infty \delta e^{-\delta} \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^2}\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}}{|\alpha|^3}\, d\delta\, d\alpha\\
&\leq \Gamma(2)\|f\|_{\dot H^2}\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^2}^2}{|\alpha|^2}\, d\alpha \int \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}^2}{|\alpha|^4}\, d\alpha\right)^{1/2}\\
&\lesssim \|f\|_{\dot H^2}\|f_x\|_{\dot B^{1/2}_{2,2}}\|f\|_{\dot B^{3/2}_{\infty,2}}\\
&\lesssim \|f\|_{\dot H^2}^2\|f\|_{\dot H^{3/2}}.
\end{aligned}
$$

5.1.3.2. *Estimates of* $I_{1,3,2}$. – Recall that,

$$
\begin{aligned}
I_{1,3,2} = -\frac12 \int \mathcal{H} f_{xx} \int \frac{f_x(x)-f_x(x-\alpha)}{\alpha}\\
\times \int_0^\infty \delta e^{-\delta}\partial_\alpha D \cos(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\, d\delta\, d\alpha\, dx.
\end{aligned}
$$

Hence, by using the identity (5.2), one gets and we infer that,

$$
\begin{aligned}
I_{1,3,2} = &-\frac12 \int \mathcal{H} f_{xx} \int \frac{f_x(x)-f_x(x-\alpha)}{\alpha} \int_0^\infty \delta e^{-\delta}\frac{f_x(x+\alpha)+f_x(x-\alpha)-2f_x(x)}{\alpha}\\
&\times \cos(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\, d\delta\, d\alpha\, dx\\
&+\frac12 \int \mathcal{H} f_{xx} \int \frac{f_x(x)-f_x(x-\alpha)}{\alpha} \int_0^\infty \delta e^{-\delta}\frac{\int_0^\alpha (f_x(x-s)+f_x(x+s)-2f_x(x))\, ds}{\alpha^2}\\
&\times \cos(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\, d\delta\, d\alpha\, dx
\end{aligned}
\tag{5.24}
$$

$$
= I_{1,3,2,1} + I_{1,3,2,2}.
\tag{5.25}
$$

We may estimate those two terms as follows. For the first one, it suffices to use the embedding $\dot H^1 \hookrightarrow \dot B^{1/2}_{\infty,2}$, indeed we have

$$
\begin{aligned}
|I_{1,3,2,1}| &\leq \frac12\|f\|_{\dot H^2}\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^2}}{|\alpha|}\ \frac{\|f_x(x+\alpha)+f_x(x-\alpha)-2f_x(x)\|_{L^\infty}}{|\alpha|}\, d\alpha\\
&\lesssim \|f\|_{\dot H^2}\|f_x\|_{\dot B^{1/2}_{2,2}}\|f_x\|_{\dot B^{1/2}_{\infty,2}}\\
&\lesssim \|f\|_{\dot H^2}^2\|f\|_{\dot H^{3/2}}.
\end{aligned}
$$

For $I_{1,3,2,2}$, for some q, r and $\bar r$ (so that $1/r+1/\bar r=1$) that will be chosen later, we write

$$
\begin{aligned}
|I_{1,3,2,2}| &\leq \frac12\|f\|_{\dot H^2}\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^\infty}}{|\alpha|^3}\int_0^\infty \delta e^{-\delta}|\alpha|^{q+\frac{1}{\bar r}}\\
&\quad\times\left(\int_0^\alpha \frac{\|f_x(x-s)+f_x(x+s)-2f_x(x)\|_{L^2}^r}{s^{qr}}\, ds\right)^{1/r}\frac{\|\delta_\alpha f+\bar\delta_\alpha f\|_{L^\infty}}{|\alpha|}\, d\delta\, d\alpha\\
&\leq \frac{\Gamma(2)}{2}\|f\|_{\dot H^2}\|f_x\|_{\dot B^{q-\frac1r}_{2,r}}\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^\infty}}{|\alpha|^{3-q-\frac{1}{\bar r}}}\ \frac{\|\delta_\alpha f+\bar\delta_\alpha f\|_{L^\infty}}{|\alpha|}\, d\alpha
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|f\|_{\dot H^2}\|f\|_{\dot B^{q+1-\frac{1}{r}}_{2,r}}\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|^2_{L^\infty}}{|\alpha|^{4-2q-\frac{2}{r}}}\,d\alpha\int \frac{\|\delta_\alpha f+\bar\delta_\alpha f\|^2_{L^\infty}}{\alpha^4}\,d\alpha\right)^{1/2}\\
&\lesssim \|f\|_{\dot H^2}\|f\|_{\dot B^{q+1-\frac{1}{r}}_{2,r}}\|f\|_{\dot B^{\frac{5}{2}-q-\frac{1}{r}}_{\infty,2}}\|f\|_{\dot B^{3/2}_{\infty,2}}\\
&\lesssim \|f\|_{\dot H^2}\|f\|_{\dot B^{q+\frac{1}{2}}_{2,r}}\|f\|_{\dot B^{2-q}_{\infty,2}}\|f\|_{\dot B^{3/2}_{\infty,2}}.
\end{aligned}
$$

Then, by choosing $\bar r = r = 2$ and $q = 1$, we obtain

$$|I_{1,3,2,2}| \lesssim \|f\|^2_{\dot H^2}\|f\|^2_{\dot H^{3/2}}.$$

Finally, we have shown that

$$|I_{1,3,2}| \lesssim \|f\|^2_{\dot H^2}\|f\|^2_{\dot H^{3/2}}.$$

5.1.3.3. *Estimates of* $I_{1,3,3}$. – Recall that

$$
\begin{aligned}
I_{1,3,3} = -\frac{1}{2}\int \mathcal{H} f_{xx}\int \frac{f_x(x)-f_x(x-\alpha)}{\alpha}\\
\times \int_0^\infty \delta e^{-\delta}\sin\big(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f)\big)\partial_\alpha S\cos\big(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f)\big)d\delta\, d\alpha\, dx.
\end{aligned}
$$

By using the identity (5.4), one finds

$$
\begin{aligned}
I_{1,3,3} &= -\frac{1}{2}\int \mathcal{H} f_{xx}\int \frac{f_x(x)-f_x(x-\alpha)}{\alpha}\\
&\quad\times \int_0^\infty \delta e^{-\delta}\ \sin\big(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f)\big)\bar\Delta_\alpha f_x\ \cos\big(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f)\big)d\delta\, d\alpha\, dx\\
&\quad+\frac{1}{2}\int \mathcal{H} f_{xx}\int \frac{f_x(x)-f_x(x-\alpha)}{\alpha}\\
&\quad\times \int_0^\infty \delta e^{-\delta}\ \sin\big(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f)\big)\Delta_\alpha f_x\ \cos\big(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f)\big)d\delta\, d\alpha\, dx\\
&\quad-\frac{1}{2}\int \mathcal{H} f_{xx}\ \int \frac{f_x(x)-f_x(x-\alpha)}{\alpha}\int_0^\infty \delta e^{-\delta}\sin\big(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f)\big)\\
&\quad\times \frac{f(x+\alpha)+f(x-\alpha)-2f(x)}{\alpha^2}\cos\big(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f)\big)\, d\delta\, d\alpha\, dx\\
&= \sum_{i=1}^{3} I_{1,3,3,i}.
\end{aligned}
$$

The control of $I_{1,3,3,1}$ is relatively easy, indeed, it suffices to write

$$
\begin{aligned}
|I_{1,3,3,1}| &\le \frac{\Gamma(2)}{2}\|f\|_{\dot H^2}\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|^2_{L^4}}{|\alpha|^2}\,d\alpha\int \frac{\|f_x(x)-f_x(x+\alpha)\|^2_{L^4}}{|\alpha|^2}\,d\alpha\right)^{1/2}\\
&\lesssim \|f\|_{\dot H^2}\|f_x\|^2_{\dot B^{1/2}_{4,2}}\\
&\lesssim \|f\|_{\dot H^2}\|f\|^2_{\dot B^{7/4}_{2,2}}\\
&\lesssim \|f\|^2_{\dot H^2}\|f\|_{\dot H^{3/2}}.
\end{aligned}
$$

As well, one may easily estimate $I_{1,3,3,2}$ by writing

$$\begin{aligned}
|I_{1,3,3,2}| &\leq \frac{\Gamma(2)}{2}\|f\|_{\dot{H}^2}\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^4}^2}{|\alpha|^2}\,d\alpha\\
&\lesssim \|f\|_{\dot{H}^2}\|f_x\|_{\dot{B}^{1/2}_{4,2}}^2\\
&\lesssim \|f\|_{\dot{H}^2}\|f\|_{\dot{B}^{7/4}_{2,2}}^2\\
&\lesssim \|f\|_{\dot{H}^2}^2\|f\|_{\dot{H}^{3/2}}.
\end{aligned}$$

For $I_{1,3,3,3}$, it suffices to see that

$$\begin{aligned}
|I_{1,3,3,3}| \leq \frac{\Gamma(2)}{2}&\|f\|_{\dot{H}^2}\\
&\times\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^4}^2}{|\alpha|^2}\,d\alpha\int \frac{\|f(x+\alpha)+f(x-\alpha)-2f(x)\|_{L^4}^2}{|\alpha|^4}\,d\alpha\right)^{1/2}\\
&\lesssim \|f\|_{\dot{H}^2}\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^4}^2}{|\alpha|^2}d\alpha\int \frac{\|f(x+\alpha)+f(x-\alpha)-2f(x)\|_{L^4}^2}{|\alpha|^4}\,d\alpha\right)^{1/2}\\
&\lesssim \|f\|_{\dot{H}^2}\|f_x\|_{\dot{B}^{1/2}_{4,2}}\|f\|_{\dot{B}^{3/2}_{4,2}}\\
&\lesssim \|f\|_{\dot{H}^2}\|f_x\|_{\dot{H}^{3/4}}\|f\|_{\dot{H}^{7/4}}\\
&\lesssim \|f\|_{\dot{H}^2}^2\|f\|_{\dot{H}^{3/2}}.
\end{aligned}$$

Therefore,

$$|I_{1,3,3}| \lesssim \|f\|_{\dot{H}^2}^2\|f\|_{\dot{H}^{3/2}}. \tag{5.26}$$

5.1.3.4. *Estimates of* $I_{1,3,4}$. – It remains to estimate $I_{1,3,4}$, to this end, we shall use the following decomposition in terms of nice controlled commutators. One may indeed rewrite $I_{1,3,4}$, using the anti-symmetry of $\mathscr{H}$, as follows

$$\begin{aligned}
I_{1,3,4} = -&\int \mathscr{H} f_{xx}\int \frac{f_{xx}(x)}{\alpha}\\
&\times\int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar{\Delta}_\alpha f))\,d\delta\,d\alpha\,dx\\
=&\frac{1}{2}\int f_{xx}\int_0^\infty\int e^{-\delta}\frac{1}{\alpha}\left[\mathscr{H},\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar{\Delta}_\alpha f))\right]f_{xx}\,d\delta\,d\alpha\,dx.
\end{aligned}$$

Hence, we may rewrite this term as a sum of two commutators, namely

$$\begin{aligned}
I_{1,3,4} = &\frac{1}{2}\iint \frac{f_{xx}(x)-f_{xx}(x-\alpha)}{\alpha}\\
&\quad\times\int_0^\infty e^{-\delta}\left[\mathscr{H},\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar{\Delta}_\alpha f))\right]f_{xx}d\delta\,d\alpha\,dx\\
&+\frac{1}{2}\iint \frac{f_{xx}(x-\alpha)}{\alpha}\\
&\quad\times\int_0^\infty e^{-\delta}\left[\mathscr{H},\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar{\Delta}_\alpha f))\right]f_{xx}\,d\delta\,d\alpha\,dx
\end{aligned}$$

$$= \frac{1}{2}\iint \frac{f_{xx}(x) - f_{xx}(x-\alpha)}{\alpha}$$
$$\times \int_0^\infty e^{-\delta}\left[\mathcal{H}, \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\right] f_{xx} d\delta\, d\alpha\, dx$$
$$-\frac{1}{2}\iint \frac{\partial_\alpha(f_x(x-\alpha) - f_x(x))}{\alpha}$$
$$\times \int_0^\infty e^{-\delta}\left[\mathcal{H}, \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\right] f_{xx} d\delta\, d\alpha\, dx.$$

Finally, by integrating by parts with respect to x in the first integral and with respect to α in the last integral, one finds,

$$I_{1,3,4} = -\frac{1}{2}\iint \frac{f_x(x) - f_x(x-\alpha)}{\alpha}$$
$$\times \int_0^\infty e^{-\delta}\partial_x\left[\mathcal{H}, \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\right] f_{xx} d\delta\, d\alpha\, dx$$
$$-\frac{1}{2}\iint \frac{f_x(x-\alpha) - f_x(x)}{\alpha^2}$$
$$\times \int_0^\infty e^{-\delta}\left[\mathcal{H}, \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\right] f_{xx} d\delta\, d\alpha\, dx$$
$$+\frac{1}{2}\iint \frac{f_x(x-\alpha) - f_x(x)}{\alpha}$$
$$\times \int_0^\infty e^{-\delta}\partial_\alpha\left[\mathcal{H}, \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\right] f_{xx} d\delta\, d\alpha\, dx$$
$$= I_{1,3,4,1} + I_{1,3,4,2} + I_{1,3,4,3}. \tag{5.27}$$

Let us estimate $I_{1,3,4,1}$.

$$|I_{1,3,4,1}| \leq \frac{1}{2}\int \frac{\|f_x(x) - f_x(x-\alpha)\|_{L^2}}{|\alpha|}\int_0^\infty e^{-\delta}$$
$$\times \left\|\partial_x\left[\mathcal{H}, \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\right] f_{xx}\right\|_{L^\infty} d\delta\, d\alpha\, dx.$$

We then use the commutator estimate (3.2) in the case $l = 1$ and $k = 0$. To do so, we first notice that, by using $|\sin(\frac{\delta}{2}(\Delta_\alpha f_x + \bar{\Delta}_\alpha f_x))| \leq \frac{\delta}{2}|\Delta_\alpha f_x + \bar{\Delta}_\alpha f_x|$ and that cos and sin functions are bounded by 1, we have for any $\delta \geq 0$

$$\left\|\partial_x\left(\sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\right)\right\|_{L^\infty}$$
$$\leq \frac{\delta^2}{4}\|\Delta_\alpha f_x - \bar{\Delta}_\alpha f_x\|_{L^\infty}\|\Delta_\alpha f + \bar{\Delta}_\alpha f\|_{L^\infty} + \frac{\delta}{2}\|\Delta_\alpha f_x + \bar{\Delta}_\alpha f_x\|_{L^\infty}$$
$$\leq \frac{\delta^2}{4}\left(\|\Delta_\alpha f_x\|_{L^\infty} + \|\bar{\Delta}_\alpha f_x\|_{L^\infty}\right)\|\Delta_\alpha f + \bar{\Delta}_\alpha f\|_{L^\infty} + \frac{\delta}{2}\|\Delta_\alpha f_x + \bar{\Delta}_\alpha f_x\|_{L^\infty}.$$

Then, we find,

$$|I_{1,3,4,1}| \lesssim \|f\|_{\dot{H}^2}\int \frac{\|f_x(x) - f_x(x-\alpha)\|_{L^2}}{|\alpha|}\frac{\|f_x(x) - f_x(x-\alpha)\|_{L^\infty}}{|\alpha|}$$

$$
\begin{aligned}
&\times \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}}{|\alpha|}\,d\alpha\\
&+\|f\|_{\dot H^2}\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^2}}{|\alpha|}\frac{\|f_x(x)-f_x(x+\alpha)\|_{L^\infty}}{|\alpha|}\\
&\times \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}}{|\alpha|}\,d\alpha\\
&+\|f\|_{\dot H^2}\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^2}}{|\alpha|}\frac{\|f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)\|_{L^\infty}}{|\alpha|}\,d\alpha\\
&\lesssim \|f\|_{\dot H^2}\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^2}^2}{|\alpha|^2}\,d\alpha\right)^{1/2}\\
&\times\left(\int \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}^4}{|\alpha|^5}\,d\alpha\right)^{1/4}\\
&\times\left[\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^\infty}^4}{|\alpha|^3}\,d\alpha\right)^{1/4}+\left(\int \frac{\|f_x(x)-f_x(x+\alpha)\|_{L^\infty}^4}{|\alpha|^3}\,d\alpha\right)^{1/4}\right]\\
&+\ \|f\|_{\dot H^2}\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^2}^2}{|\alpha|^2}\,d\alpha\right)^{1/2}\\
&\times\left(\int \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}^2}{|\alpha|^2}\,d\alpha\right)^{1/2}\\
&\lesssim \|f\|_{\dot H^2}\|f\|_{\dot H^{3/2}}\|f\|_{\dot B^1_{\infty,4}}\|f_x\|_{\dot B^{1/2}_{\infty,4}}+\|f\|_{\dot H^2}\|f\|_{\dot H^{3/2}}\|f\|_{\dot B^1_{\infty,2}}\\
&\lesssim \|f\|_{\dot H^2}^2\|f\|_{\dot H^{3/2}}^2,
\end{aligned}
$$

where we used the following embeddings $\dot H^{3/2}\hookrightarrow \dot B^1_{\infty,4}$, $\dot H^2\hookrightarrow \dot B^{3/2}_{\infty,4}$, and $\dot H^{3/2}\hookrightarrow \dot B^1_{\infty,2}$.

Then, we estimate $I_{1,3,4,2}$, we see that this term may be written as

$$
\begin{aligned}
(5.28)\quad I_{1,3,4,2} &= \frac{1}{2}\int \mathcal{H}f_{xx}\frac{f_x(x-\alpha)-f_x(x)}{\alpha^2}\\
&\quad\times\int_0^\infty\int e^{-\delta}\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\,d\delta\,d\alpha\,dx\\
(5.29)\quad &+\frac{1}{2}\int f_{xx}\frac{\mathcal{H}f_x(x-\alpha)-\mathcal{H}f_x(x)}{\alpha^2}\\
&\quad\times\int_0^\infty\int e^{-\delta}\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\,d\delta\,d\alpha\,dx\\
(5.30)\quad &= I_{1,3,4,2,1}+I_{1,3,4,2,2}.
\end{aligned}
$$

To estimate $I_{1,3,4,2,1}$, we write

$$
\begin{aligned}
|I_{1,3,4,2,1}| &\le \frac{1}{2}\|f\|_{\dot H^2}\int \frac{\|f_x(x-\alpha)-f_x(x)\|_{L^2}}{\alpha^2}\int_0^\infty e^{-\delta}\frac{\|\delta_\alpha f+\bar\delta_\alpha f\|_{L^\infty}}{|\alpha|}\,d\delta\,d\alpha\\
&\lesssim \|f\|_{\dot H^2}\left(\int \frac{\|f_x(x-\alpha)-f_x(x)\|_{L^2}^2}{\alpha^2}\,d\alpha\right)^{1/2}\left(\int \frac{\|\delta_\alpha f+\bar\delta_\alpha f\|_{L^\infty}^2}{\alpha^4}\,d\alpha\right)^{1/2}\\
&\lesssim \|f\|_{\dot H^2}\|f\|_{\dot H^{3/2}}\|f\|_{\dot B^{3/2}_{\infty,2}}
\end{aligned}
$$

$$\lesssim \|f\|_{\dot{H}^2}^2 \|f\|_{\dot{H}^{3/2}}.$$

Since $\mathcal{H}$ is continuous on L^2, one gets the same control for $I_{1,3,4,2,2}$ so that one finally obtains

$$|I_{1,3,4,2}| \lesssim \|f\|_{\dot{H}^2}^2 \|f\|_{\dot{H}^{3/2}}.$$

It therefore remains to estimate $I_{1,3,4,3}$. To do so, we open the commutator and use the anti-symmetry of $\mathcal{H}$. Then, we explicitely compute the derivative in α. Hence, we obtain the following decomposition

(5.31)
$$I_{1,3,4,3} = -\frac{1}{4}\int f_{xx} \frac{\mathcal{H} f_x(x-\alpha) - \mathcal{H} f_x(x)}{\alpha} \int_0^\infty \int \delta e^{-\delta} \partial_\alpha(\Delta_\alpha f - \bar{\Delta}_\alpha f)$$
$$\times \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx$$

(5.32)
$$-\frac{1}{4}\int \mathcal{H} f_{xx} \frac{f_x(x-\alpha) - f_x(x)}{\alpha}$$
$$\times \int_0^\infty \int \delta e^{-\delta} \partial_\alpha(\Delta_\alpha f - \bar{\Delta}_\alpha f) \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f) \, d\delta \, d\alpha \, dx$$

(5.33)
$$-\frac{1}{4}\int f_{xx} \int \frac{\mathcal{H} f_x(x-\alpha) - \mathcal{H} f_x(x)}{\alpha}$$
$$\times \int_0^\infty \delta e^{-\delta} \partial_\alpha(\Delta_\alpha f + \bar{\Delta}_\alpha f) \ \cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f) \ \, d\delta \, d\alpha \, dx$$

(5.34)
$$-\frac{1}{4}\int \mathcal{H} f_{xx} \int \frac{f_x(x-\alpha) - f_x(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_\alpha(\Delta_\alpha f + \bar{\Delta}_\alpha f) \ \cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))$$
$$\times \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f) \ \, d\delta \, d\alpha \, dx$$

(5.35)
$$= \sum_{i=1}^4 I_{1,3,4,3,i}.$$

Since we shall do L^p estimates, $p \in (1,\infty)$ on the terms involving $\mathcal{H}$ one observes that these terms have the same regularity as the terms $I_{1,3,j}$ for $j = 2, 3$ (see (5.24)). It is therefore not difficult to see that,

$$|I_{1,3,4,3,1} + I_{1,3,4,3,2}| \lesssim \|f\|_{\dot{H}^2}^2 \|f\|_{\dot{H}^{3/2}}^2,$$

and,

$$|I_{1,3,4,3,3} + I_{1,3,4,3,4}| \lesssim \|f\|_{\dot{H}^2}^2 \|f\|_{\dot{H}^{3/2}}.$$

Therefore, we have obtained

$$|I_{1,3}| \lesssim \|f\|_{\dot{H}^2}^2 \left(\|f\|_{\dot{H}^{3/2}}^2 + \|f\|_{\dot{H}^{3/2}}\right). \tag{5.36}$$

5.2. Estimate of I_2

To estimate

$$I_2 = -\int \mathcal{H} f_{xx} \int (\partial_x \Delta_\alpha f)^2 \int_0^\infty \delta e^{-\delta} \sin(\delta \Delta_\alpha f(x))\, d\delta\, d\alpha\, dx,$$

it suffices to write that,

$$|I_2| \leq \Gamma(2) \|f\|_{\dot{H}^2} \left(\int \frac{\|f_x(x) - f_x(x-\alpha)\|_{L^2}^2}{\alpha^2}\, d\alpha \right)^{1/2} \left(\int \frac{\|f_x(x) - f_x(x-\alpha)\|_{L^\infty}^2}{\alpha^2}\, d\alpha \right)^{1/2}.$$

Therefore, we obtain

$$\begin{aligned} |I_2| &\lesssim \|f\|_{\dot{H}^2} \|f_x\|_{\dot{B}^{1/2}_{\infty,2}} \|f\|_{\dot{H}^{3/2}} \\ &\lesssim \|f\|_{\dot{H}^2}^2 \|f\|_{\dot{H}^{3/2}}. \end{aligned} \tag{5.37}$$

Finally, combining all the estimates, we get that

$$\frac{1}{2}\partial_t \|f\|_{\dot{H}^{3/2}}^2 + \frac{\pi}{1+K^2} \|f\|_{\dot{H}^2}^2 \lesssim \|f\|_{\dot{H}^2}^2 \left(\|f\|_{\dot{H}^{3/2}}^2 + \|f\|_{\dot{H}^{3/2}} \right). \tag{5.38}$$

And then integrating in time $s \in [0,T]$ one gets the desired energy inequality of Lemma 5.1. Therefore, if $\|f_0\|_{\dot{H}^{3/2}}$ is smaller than some $C(K)$ that depends only on K, then the solution is in $L^\infty([0,T], L^2) \cap L^2([0,T], \dot{H}^2)$. This concludes the $\dot{H}^{3/2}$-estimates and therefore Lemma 5.1 is proved.

6. A priori estimates in $\dot{H}^{5/2}$ estimates

In this section we shall prove the following lemma.

LEMMA 6.1. – *Let $T > 0$ and $f_0 \in \dot{H}^{5/2} \cap \dot{H}^{3/2}$ so that $\|f_0\|_{\dot{H}^{3/2}} < C(\|\partial_x f_0\|_{L^\infty})$, then we have*

$$\begin{aligned} &\|f\|_{\dot{H}^{5/2}}^2(T) + \frac{\pi}{1+K^2} \int_0^T \|f\|_{\dot{H}^3}^2\, ds \\ &\qquad \lesssim \|f_0\|_{\dot{H}^{5/2}} + \left(\|f\|_{L^\infty([0,T],\dot{H}^{3/2})} + \|f\|_{L^\infty([0,T],\dot{H}^{3/2})}^2 \right) \int_0^T \|f\|_{\dot{H}^3}^2\, ds, \end{aligned}$$

where $K = \sup_{(x,t)\in\mathbb{R}\times[0,T]} |f_x(x,t)|$.

Proof of Lemma 6.1. – We now do an a priori estimate in $\dot{H}^{5/2}$. Using the anti-symmetry property of the Hilbert transform, one finds

$$\begin{aligned} \frac{1}{2}\partial_t \|f\|_{\dot{H}^{5/2}}^2 &= \int \mathcal{H}\partial_x^3 f \int \partial_x^3 \Delta_\alpha f \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f)\, dx\, d\alpha\, d\delta \\ &\quad - 3 \int \mathcal{H}\partial_x^3 f \int \partial_x \Delta_\alpha f\, \partial_{xx} \Delta_\alpha f \int_0^\infty \delta e^{-\delta} \sin(\delta \Delta_\alpha f)\, dx\, d\alpha\, d\delta \\ &\quad - \int \mathcal{H}\partial_x^3 f \int (\partial_x \Delta_\alpha f)^3 \int_0^\infty \delta^2 e^{-\delta} \cos(\delta \Delta_\alpha f)\, dx\, d\alpha\, d\delta \\ &= T_1 + T_2 + T_3. \end{aligned}$$

Analogously to the $\dot{H}^{3/2}$ a priori estimate, we decompose the first term as follows. By noticing that I_1 is analogous to T_1, we find that

$$\begin{aligned}
T_1 &= -\int \mathcal{H}\partial_x^3 f \int (\partial_x^3 \Delta_\alpha f - \partial_x^3 \bar{\Delta}_\alpha f)\\
&\quad \times \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin^2(\frac{\delta}{4}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx\\
&\quad + \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int (\partial_x^3 \Delta_\alpha f - \partial_x^3 \bar{\Delta}_\alpha f) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx\\
&\quad + \int \mathcal{H}\partial_x^3 f \int \frac{\partial_x^3 f(x) - \partial_x^3 f(x-\alpha)}{\alpha}\\
&\quad \times \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx\\
&= T_{1,1} + T_{1,2} + T_{1,3}.
\end{aligned}$$

Note that $T_{1,1}$, $T_{1,2}$, and $T_{1,3}$ are respectively analogous to $I_{1,1}$, $I_{1,2}$ and $I_{1,3}$. Using (5.20), we immedialy infer that

$$\begin{aligned}
T_1 &= -\int \mathcal{H}\partial_x^3 f \int (\partial_x^3 \Delta_\alpha f - \partial_x^3 \bar{\Delta}_\alpha f)\\
&\quad \times \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin^2(\frac{\delta}{4}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx\\
&\quad + \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int (\partial_x^3 \Delta_\alpha f - \partial_x^3 \bar{\Delta}_\alpha f) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx\\
&\quad + \int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x-\alpha)}{\alpha^2}\\
&\quad \times \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx\\
&\quad - \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x-\alpha)}{\alpha}\\
&\quad \times \int_0^\infty \delta e^{-\delta} \partial_\alpha D\, \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) d\delta\, d\alpha\, dx\\
&\quad - \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x-\alpha)}{\alpha}\\
&\quad \times \int_0^\infty \delta e^{-\delta}\, \partial_\alpha S\, \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, \cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) d\delta\, d\alpha\, dx\\
&\quad - \int \mathcal{H}\partial_x^3 f \int \partial_x^3 f(x) \int_0^\infty e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) \frac{1}{\alpha}\, d\delta\, d\alpha\, dx\\
&= \sum_{i=1}^{6} T_{1,j}. \qquad \square
\end{aligned}$$

We then estimate the $T_{1,j}$, $j = 1, \ldots, 6$.

6.1. Estimate of $T_{1,1}$

To control $T_{1,1}$, we use the continuity of the Hilbert transform on L^2 along with the embedding $\dot{H}^{3/2} \hookrightarrow \dot{B}^1_{\infty,2}$, then one gets,

$$\begin{aligned}
T_{1,1} &= -\int \mathcal{H}\partial_x^3 f \int (\partial_x^3 \Delta_\alpha f - \partial_x^3 \bar{\Delta}_\alpha f) \\
&\quad \times \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin^2(\frac{\delta}{4}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\le \frac{\Gamma(3)}{4} \|f\|^2_{\dot{H}^3} \int \frac{\|\delta_\alpha f + \bar{\delta}_\alpha f\|^2_{L^\infty}}{|\alpha|^3}\, d\alpha \\
&\lesssim \|f\|^2_{\dot{H}^3} \|f\|^2_{\dot{B}^1_{\infty,2}} \\
&\lesssim \|f\|^2_{\dot{H}^3} \|f\|^2_{\dot{H}^{3/2}}.
\end{aligned}$$

6.2. Estimate of $T_{1,2}$

We first rewrite $T_{1,2}$ as follows, by integrating by parts, one finds

$$\begin{aligned}
T_{1,2} &= \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int (\partial_x^3 \Delta_\alpha f - \partial_x^3 \bar{\Delta}_\alpha f) \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&= \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int \frac{1}{\alpha}\partial_\alpha\{\delta_\alpha f_{xx} + \bar{\delta}_\alpha f_{xx}\} \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&= \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)}{\alpha^2} \\
&\quad \times \int_0^\infty e^{-\delta} \cos(\frac{\delta}{2}(\bar{\Delta}_\alpha f - \Delta_\alpha f))\, d\delta\, d\alpha\, dx \\
&\quad + \frac{1}{4}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)}{\alpha} \\
&\quad \times \int_0^\infty \delta e^{-\delta} \partial_\alpha\{\Delta_\alpha f - \bar{\Delta}_\alpha f\} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx.
\end{aligned}$$

Then, we obtain that

$$\begin{aligned}
T_{1,2} &= -\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)}{\alpha^2} \\
&\quad \times \int_0^\infty e^{-\delta} \sin^2(\frac{\delta}{4}(\bar{\Delta}_\alpha f - \Delta_\alpha f))\, d\delta\, d\alpha\, dx \\
&\quad + \frac{1}{4}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)}{\alpha} \\
&\quad \times \int_0^\infty \delta e^{-\delta} \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\quad - \frac{1}{4}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)}{\alpha^3} \\
&\quad \times \int_0^\infty \delta e^{-\delta} \int_0^\alpha f_x(x-s) + f_x(x+s) - 2f_x(x)\, ds \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\quad - \pi \int \mathcal{H}\partial_x^3 f \Lambda f_{xx}\, dx
\end{aligned}$$

$$= T_{1,2,1} + T_{1,2,2} + T_{1,2,3} + T_{1,2,4}.$$

In order to estimate $T_{1,2}$ we shall estimate the $T_{1,2,j}$, for $j = 1,2,3$. Note that $T_{1,2,4}$ is a dissipative term. We start with $T_{1,2,1}$, and observe that this term is analogous to $I_{1,2,1}$ (see (5.8)). More precisely, we write

$$\begin{aligned}
T_{1,2,1} = &-\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha)+f_{xx}(x+\alpha)-2f_{xx}(x)}{\alpha^2} \\
&\times \int_0^\infty e^{-\delta} \sin^2(\frac{\delta}{4}(\bar{\Delta}_\alpha f - \Delta_\alpha f))\, d\delta\, d\alpha\, dx \\
= &-\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha)+f_{xx}(x+\alpha)-2f_{xx}(x)}{\alpha^2} \\
&\times \int_0^\infty e^{-\delta} \left(\sin^2(\frac{\delta}{4}(\bar{\Delta}_\alpha f - \Delta_\alpha f)) - \sin^2(\frac{\delta}{2} f_x(x))\right) d\delta\, d\alpha\, dx \\
&+\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha)+f_{xx}(x+\alpha)-2f_{xx}(x)}{\alpha^2} \int_0^\infty e^{-\delta} \sin^2(\frac{\delta}{2} f_x(x))\, d\delta\, d\alpha\, dx \\
= &\, T_{1,2,1,1} + T_{1,2,1,2}.
\end{aligned}$$

In order to estimate $T_{1,2,1,1}$ we use the estimate (5.12) together with the identity (5.1). So that by using Hölder inequality one finds that

$$\begin{aligned}
|T_{1,2,1,1}| \lesssim \|f\|_{\dot{H}^3} \int & \frac{\|f_{xx}(x-\alpha)+f_{xx}(x+\alpha)-2f_{xx}(x)\|_{L^\infty}}{|\alpha|^3} \\
&\times \int_0^\alpha \|f_x(x-s)-f_x(x)\|_{L^2}\, ds\, d\alpha.
\end{aligned}$$

Then, by using Hölder inequality in the s-integral (where r and $\bar{r}$ are conjugate) and for some real number $q \in [0,2]$ that will be chosen later, we have

$$\begin{aligned}
|T_{1,2,1,1}| &\lesssim \|f\|_{\dot{H}^3} \int \frac{\|f_{xx}(x-\alpha)-f_{xx}(x)\|_{L^\infty}}{|\alpha|^3} |\alpha|^{q+\frac{1}{\bar{r}}} \left(\int \frac{\|f_x(x-s)-f_x(x)\|_{L^2}^r}{|s|^{qr}}\, ds\right)^{1/r} d\alpha \\
&\lesssim \|f\|_{\dot{H}^3} \|f_{xx}\|_{\dot{B}^{2-q-\frac{1}{\bar{r}}}_{\infty,1}} \|f_x\|_{\dot{B}^{q-\frac{1}{r}}_{2,r}}.
\end{aligned}$$

By choosing $q = 9/8$, $r = \bar{r} = 2$, we find

$$|T_{1,2,1,1}| \lesssim \|f\|_{\dot{H}^3} \|f_{xx}\|_{\dot{B}^{3/8}_{\infty,1}} \|f_x\|_{\dot{B}^{5/8}_{2,2}}.$$

Then, by interpolating in the space $\dot{B}^{19/8}_{\infty,1}$, we find,

$$|T_{1,2,1,1}| \lesssim \|f\|_{\dot{H}^3} \|f\|_{\dot{B}^{13/8}_{2,2}} \|f\|^{1/12}_{\dot{B}^1_{\infty,\infty}} \|f\|^{11/12}_{\dot{B}^{5/2}_{\infty,\infty}}.$$

Since,

$$\|f\|_{\dot{B}^{13/8}_{2,2}} \leq \|f\|^{1/12}_{\dot{H}^3} \|f\|^{11/12}_{\dot{H}^{3/2}},$$

then, by using classical Besov embeddings, one finally gets

$$|T_{1,2,1,1}| \lesssim \|f\|^2_{\dot{H}^3} \|f\|_{\dot{H}^{3/2}}.$$

As for the term $T_{1,2,1,2}$, we use the same idea as the estimate of the term $I_{1,2,1,2}$ (see (5.16)). We have that,

$$T_{1,2,1,2} = -\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha)+f_{xx}(x+\alpha)-2f_{xx}(x)}{\alpha^2}$$

$$\times \int_0^\infty e^{-\delta} \ \sin^2(\frac{\delta}{2} f_x(x))\, d\delta\, d\alpha\, dx$$

$$= 2\pi \int (\mathscr{H}\partial_x^3 f)^2 \int_0^\infty e^{-\delta} \ \sin^2(\frac{\delta}{2} f_x(x))\, d\delta \ dx \leq \pi \frac{K^2}{1+K^2} \|f\|^2_{\dot{H}^3}, \tag{6.1}$$

where $K = \sup\limits_{(x,t)\in\mathbb{R}\times[0,T]} |f_x(x,t)|$. Therefore, we find that for some fixed constant $C > 0$,

$$T_{1,2,1,1} + T_{1,2,1,2} \leq C\|f\|^2_{\dot{H}^3}\|f\|_{\dot{H}^{3/2}} + \pi \frac{K^2}{1+K^2}\|f\|^2_{\dot{H}^3}. \tag{6.2}$$

To estimate $T_{1,2,2}$, it suffices to write that

$$\begin{aligned}
T_{1,2,2} &= -\int \mathscr{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)}{\alpha} \\
&\quad \times \int_0^\infty \delta e^{-\delta} \frac{f_x(x-\alpha) + f_x(x+\alpha) - 2f_x(x)}{\alpha} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\lesssim \|f\|^2_{\dot{H}^3} \left(\int \frac{\|f_{xx}(x-\alpha)+f_{xx}(x+\alpha)-2f_{xx}(x)\|^2_{L^\infty}}{\alpha^2} \frac{\|f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)\|_{L^2}}{\alpha^2}\, d\alpha \right)^{1/2} \\
&\lesssim \|f\|_{\dot{H}^3}\|f\|_{\dot{B}^{5/2}_{\infty,2}}\|f_x\|_{\dot{B}^{1/2}_{2,2}} \\
&\lesssim \|f\|^2_{\dot{H}^3}\|f\|_{\dot{H}^{3/2}}.
\end{aligned}$$

We now estimate $T_{1,2,3}$. Let $q \in [0,2]$ and let $r, \bar{r}$ so that $1/r + 1/\bar{r} = 1$, we write

$$\begin{aligned}
T_{1,2,3} &= -\frac{1}{4}\int \mathscr{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)}{\alpha^3} \int_0^\infty \delta e^{-\delta} \\
&\quad \times \int_0^\alpha f_x(x-s) + f_x(x+s) - 2f_x(x)\, ds \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\leq \frac{1}{4}\|f\|_{\dot{H}^3} \int_0^\infty \delta e^{-\delta} \frac{\|f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_x(xx)\|_{L^\infty}}{|\alpha|^3} \\
&\quad \times \int_0^\alpha \|f_x(x-s) - f_x(x)\|_{L^2} + \|f_x(x+s) - f_x(x)\|_{L^2}\, ds\, d\delta\, d\alpha \\
&\lesssim \|f\|_{\dot{H}^3} \left(\int \frac{\|f_{xx}(x-\alpha) + f_{xx}(x+\alpha) - 2f_{xx}(x)\|_{L^\infty}}{|\alpha|^{3-q-\frac{1}{\bar{r}}}}\, d\alpha \right) \\
&\quad \times \left[\left(\int \frac{\|f_x(x-s) - f_x(x)\|^r_{L^2}}{|s|^{qr}}\, ds \right)^{1/r} + \left(\int \frac{\|f_x(x+s) - f_x(x)\|^r_{L^2}}{|s|^{qr}}\, ds \right)^{1/r} \right] \\
&\lesssim \|f\|_{H^3}\|f_{xx}\|_{\dot{B}^{2-q-\frac{1}{\bar{r}}}_{\infty,1}}\|f_x\|_{\dot{B}^{q-\frac{1}{r}}_{2,r}}.
\end{aligned}$$

Then, by choosing $q = 2$, $r = \bar{r} = 2$, we get

$$|T_{1,2,3}| \lesssim \|f\|_{H^3}\|f\|_{\dot{B}^{3/2}_{\infty,1}}\|f\|_{\dot{B}^{5/2}_{2,2}}.$$

Then, using the interpolations $\dot{H}^{5/2} = \left[\dot{H}^{3/2}, \dot{H}^3\right]_{\frac{1}{3},\frac{2}{3}}$ and $\dot{B}^{3/2}_{\infty,1} = \left[\dot{B}^1_{\infty,\infty}, , \dot{B}^{5/2}_{\infty,\infty}\right]_{\frac{2}{3},\frac{1}{3}}$, we find

$$\begin{aligned}
|T_{1,2,3}| &\lesssim \|f\|_{H^3}\|f\|^{2/3}_{\dot{H}^3}\|f\|^{1/3}_{\dot{H}^{3/2}}\|f\|^{1/3}_{\dot{B}^{5/2}_{\infty,\infty}}\|f\|^{2/3}_{\dot{B}^1_{\infty,\infty}} \\
&\lesssim \|f\|^2_{H^3}\|f\|_{\dot{H}^{3/2}}.
\end{aligned}$$

6.3. Estimate of $T_{1,3}$

To estimate $T_{1,3}$, it suffices to write that

$$
\begin{aligned}
|T_{1,3}| &\leq \|f\|_{\dot H^3}\int_0^\infty \delta e^{-\delta}\frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|_{L^\infty}\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^2}}{|\alpha|^3}\,d\delta\,d\alpha\\
&\leq \Gamma(2)\|f\|_{\dot H^3}\left(\int \frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|^2_{L^\infty}}{|\alpha|^2}\,d\alpha\int\frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|^2_{L^2}}{|\alpha|^4}\,d\alpha\right)^{1/2}\\
&\lesssim \|f\|_{\dot H^3}\|f_{xx}\|_{\dot B^{1/2}_{\infty,2}}\|f\|_{\dot B^{3/2}_{2,2}}\\
&\lesssim \|f\|^2_{\dot H^3}\|f\|_{\dot H^{3/2}}.
\end{aligned}
$$

6.4. Estimate of $T_{1,4}$

The control of $T_{1,4}$ is done thanks to the following decomposition which is analogous to the decomposition of $I_{1,3,2}$ (see (5.24)), namely

$$
\begin{aligned}
T_{1,4} &= -\frac{1}{2}\int \mathcal{H}\partial_x^3 f\int\frac{f_{xx}(x)-f_{xx}(x-\alpha)}{\alpha}\int_0^\infty \delta e^{-\delta}\frac{f_x(x+\alpha)+f_x(x-\alpha)-2f_x(x)}{\alpha}\\
&\qquad\times\cos(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\,d\delta\,d\alpha\,dx\\
&\quad+\frac{1}{2}\int\mathcal{H}\partial_x^3 f\int\frac{f_{xx}(x)-f_{xx}(x-\alpha)}{\alpha}\int_0^\infty\delta e^{-\delta}\frac{\int_0^\alpha f_x(x-s)+f_x(x+s)-2f_x(x)\,ds}{\alpha^2}\\
&\qquad\times\cos(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\,d\delta\,d\alpha\,dx\\
&= T_{1,4,1}+T_{1,4,2}.
\end{aligned}
$$

We have

$$
\begin{aligned}
|T_{1,4,1}| &\lesssim \|f\|_{\dot H^3}\int\frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|_{L^\infty}}{|\alpha|}\,\frac{\|f_x(x+\alpha)+f_x(x-\alpha)-2f_x(x)\|_{L^2}}{|\alpha|}\,d\alpha\\
&\lesssim \|f\|_{\dot H^3}\|f\|_{\dot B^{5/2}_{\infty,2}}\|f\|_{\dot B^{3/2}_{2,2}}\\
&\lesssim \|f\|^2_{\dot H^3}\|f\|_{\dot H^{3/2}}.
\end{aligned}
$$

For $T_{1,4,2}$, we write

$$
\begin{aligned}
|T_{1,4,2}| &\leq \frac{1}{2}\|f\|_{\dot H^3}\int\frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|_{L^\infty}}{|\alpha|^3}\int_0^\infty\delta e^{-\delta}|\alpha|^{q+\frac{1}{r}}\\
&\qquad\times\left(\int_0^\alpha\frac{\|f_x(x-s)+f_x(x+s)-2f_x(x)\|^r_{L^2}}{|s|^{qr}}\,ds\right)^{1/r}\frac{\|\delta_\alpha f+\bar\delta_\alpha f\|_{L^\infty}}{|\alpha|}\,d\delta\,d\alpha\\
&\leq\frac{\Gamma(3)}{2}\|f\|_{\dot H^3}\|f_x\|_{\dot B^{q-\frac{1}{r}}_{2,r}}\int\frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|_{L^\infty}}{|\alpha|^{2-q-\frac{1}{r}}}\,\frac{\|\delta_\alpha f+\bar\delta_\alpha f\|_{L^\infty}}{\alpha^2}\,d\alpha\\
&\lesssim\|f\|_{\dot H^3}\|f_x\|_{\dot B^{q-\frac{1}{r}}_{2,r}}\left(\int\frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|^2_{L^\infty}}{|\alpha|^{4-2q-\frac{2}{r}}}\,d\alpha\int\frac{\|\delta_\alpha f+\bar\delta_\alpha f\|^2_{L^\infty}}{\alpha^4}\,d\alpha\right)^{1/2}\\
&\lesssim\|f\|_{\dot H^3}\|f\|_{\dot B^{q+1-\frac{1}{r}}_{2,r}}\|f\|_{\dot B^{7/2-q-\frac{1}{r}}_{\infty,2}}\|f\|_{\dot B^{3/2}_{\infty,2}}\\
&\lesssim\|f\|_{\dot H^3}\|f\|_{\dot H^2}\|f\|_{\dot B^{q+\frac{1}{2}}_{2,2}}\|f\|_{\dot B^{3-q}_{\infty,2}},
\end{aligned}
$$

where we have chosen $\bar{r} = r = 2$ and then by choosing $q = 1$ we obtain

$$|T_{1,4,2}| \lesssim \|f\|_{\dot{H}^3}\|f\|_{\dot{H}^{3/2}}\|f\|_{\dot{H}^{5/2}}\|f\|_{\dot{H}^2}$$
$$\lesssim \|f\|_{\dot{H}^3}^2\|f\|_{\dot{H}^{3/2}}^2.$$

Therefore,

$$|T_{1,4}| \lesssim \|f\|_{\dot{H}^3}^2(\|f\|_{\dot{H}^{3/2}} + \|f\|_{\dot{H}^{3/2}}^2).$$

6.5. Estimate of $T_{1,5}$

We now estimate $T_{1,5}$, analogously to I_{133} (see (5.24)) we use the identity (5.4). So that,

$$\begin{aligned}
T_{1,5} &= -\frac{1}{2}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x-\alpha)}{\alpha} \\
&\quad \times \int_0^\infty \delta e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\partial_\alpha S \cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&= \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x-\alpha)}{\alpha} \\
&\quad \times \int_0^\infty \delta e^{-\delta}\ \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\bar{\Delta}_\alpha f_x\ \cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\quad + \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x-\alpha)}{\alpha} \\
&\quad \times \int_0^\infty \delta e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f))\Delta_\alpha f_x\ \cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&\quad - \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x) - f_{xx}(x-\alpha)}{\alpha}\int_0^\infty \delta e^{-\delta} \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \\
&\quad \times \frac{f(x+\alpha) + f(x-\alpha) - 2f(x)}{\alpha^2}\cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f))\, d\delta\, d\alpha\, dx \\
&= \sum_{j=1}^{3} T_{1,5,j}.
\end{aligned}$$

The estimate of $T_{1,5,1}$ is relatively easy, indeed it suffices to write

$$\begin{aligned}
|T_{1,5,1}| &\leq \frac{\Gamma(2)}{2}\|f\|_{\dot{H}^3}\left(\int \frac{\|f_{xx}(x) - f_{xx}(x-\alpha)\|_{L^\infty}^2}{|\alpha|^2}\, d\alpha \int \frac{\|f_x(x) - f_x(x+\alpha)\|_{L^2}^2}{|\alpha|^2}\, d\alpha\right)^{1/2} \\
&\lesssim \|f\|_{\dot{H}^3}\|f\|_{\dot{B}^{5/2}_{\infty,2}}\|f\|_{\dot{B}^{3/2}_{2,2}} \\
&\lesssim \|f\|_{\dot{H}^3}^2\|f\|_{\dot{H}^{3/2}}.
\end{aligned}$$

As well, one may easily estimate $T_{1,5,2}$ by writing

$$\begin{aligned}
|T_{1,5,2}| &\leq \frac{\Gamma(2)}{2}\|f\|_{\dot{H}^3}\left(\int \frac{\|f_{xx}(x) - f_{xx}(x-\alpha)\|_{L^\infty}^2}{|\alpha|^2}\, d\alpha \int \frac{\|f_x(x) - f_x(x+\alpha)\|_{L^2}^2}{|\alpha|^2}\, d\alpha\right)^{1/2} \\
&\lesssim \|f\|_{\dot{H}^3}\|f\|_{\dot{B}^{5/2}_{\infty,2}}\|f\|_{\dot{B}^{3/2}_{2,2}} \\
&\lesssim \|f\|_{\dot{H}^3}^2\|f\|_{\dot{H}^{3/2}}.
\end{aligned}$$

For $T_{1,5,3}$, it suffices to write

$$\begin{aligned}|T_{1,5,3}| &\leq \frac{\Gamma(2)}{2}\|f\|_{\dot H^3}\left(\int \frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|^2_{L^\infty}}{|\alpha|^2}\,d\alpha \int \frac{\|f(x+\alpha)+f(x-\alpha)-2f(x)\|^2_{L^2}}{|\alpha|^4}\,d\alpha\right)^{1/2}\\
&\lesssim \|f\|_{\dot H^3}\left(\int \frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|^2_{L^\infty}}{|\alpha|^2}\,d\alpha \int \frac{\|f(x+\alpha)+f(x-\alpha)-2f(x)\|^2_{L^2}}{|\alpha|^4}\,d\alpha\right)^{1/2}\\
&\lesssim \|f\|_{\dot H^3}\|f_{xx}\|_{\dot B^{1/2}_{\infty,2}}\|f\|_{\dot B^{3/2}_{2,2}}\\
&\lesssim \|f\|^2_{\dot H^3}\|f\|_{\dot H^{3/2}}.\end{aligned}$$

Therefore,

$$|T_{1,5}| \lesssim \|f\|^2_{\dot H^3}\|f\|_{\dot H^{3/2}}. \tag{6.3}$$

It remains to estimate $T_{1,6}$, this is the purpose of the next subsection.

6.6. Estimate of $T_{1,6}$

As we did for $I_{1,3,4}$ (see equality (5.27)), we first rewrite $T_{1,6}$ in term of controlled commutators.

$$\begin{aligned}T_{1,6} &= -\int \mathscr{H}\partial_x^3 f_{xx}\int \frac{\partial_x^3 f(x)}{\alpha}\int_0^\infty e^{-\delta}\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\,d\delta\,d\alpha\,dx\\
&= \frac12\int \partial_x^3 f\int_0^\infty e^{-\delta}\int\frac{1}{\alpha}\left[\mathscr{H},\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\right]\partial_x^3 f\,d\delta\,d\alpha\,dx\\
&= \frac12\iint \frac{\partial_x^3 f-\partial_x^3 f(x-\alpha)}{\alpha}\\
&\quad\times\int_0^\infty e^{-\delta}\left[\mathscr{H},\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\right]\partial_x^3 f\,d\delta\,d\alpha\,dx\\
&\quad+\frac12\iint\frac{\partial_x^3 f(x-\alpha)}{\alpha}\\
&\quad\times\int_0^\infty e^{-\delta}\left[\mathscr{H},\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\right]\partial_x^3 f\,d\delta\,d\alpha\,dx.\end{aligned}$$

Finally, by integrating by parts one obtains

$$\begin{aligned}T_{1,6,1} &= -\frac12\iint\frac{f_{xx}(x)-f_{xx}(x-\alpha)}{\alpha}\\
&\quad\times\int_0^\infty e^{-\delta}\partial_x\left[\mathscr{H},\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\right]\partial_x^3 f\,d\delta\,d\alpha\,dx\\
&\quad-\frac12\iint\frac{f_{xx}(x-\alpha)-f_{xx}(x)}{\alpha^2}\\
&\quad\times\int_0^\infty e^{-\delta}\left[\mathscr{H},\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\right]\partial_x^3 f\,d\delta\,d\alpha\,dx\\
&\quad+\frac12\iint\frac{f_{xx}(x-\alpha)-f_{xx}(x)}{\alpha}\\
&\quad\times\int_0^\infty e^{-\delta}\partial_\alpha\left[\mathscr{H},\sin(\frac{\delta}{2}(\Delta_\alpha f-\bar\Delta_\alpha f))\sin(\frac{\delta}{2}(\Delta_\alpha f+\bar\Delta_\alpha f))\right]\partial_x^3 f\,d\delta\,d\alpha\,dx\end{aligned}$$

$$= T_{1,6,1} + T_{1,6,2} + T_{1,6,3}.$$

By using the generalized Calderón commutator estimate (3.2) along with some classical Besov embeddings, we may control $T_{1,6,1}$ as follows:

$$\begin{aligned}
|T_{1,6,1}| &\lesssim \|f\|_{\dot H^3} \int \frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|_{L^2}}{|\alpha|} \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^\infty}}{|\alpha|} \\
&\quad \times \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}}{|\alpha|}\, d\alpha \\
&\quad + \|f\|_{\dot H^3} \int \frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|_{L^2}}{|\alpha|} \frac{\|f_x(x)-f_x(x+\alpha)\|_{L^\infty}}{|\alpha|} \\
&\quad \times \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}}{|\alpha|}\, d\alpha \\
&\quad + \|f\|_{\dot H^3} \int \frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|_{L^2}}{|\alpha|} \frac{\|f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)\|_{L^\infty}}{|\alpha|}\, d\alpha \\
&\lesssim \|f\|_{\dot H^3} \left(\int \frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|_{L^2}^2}{\alpha^2}\, d\alpha\right)^{1/2} \\
&\quad \times \left(\int \frac{\|f(x-\alpha)+f(x+\alpha)-2f(x)\|_{L^\infty}^4}{|\alpha|^5}\, d\alpha\right)^{1/4} \\
&\quad \times \left[\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|_{L^\infty}^4}{|\alpha|^3}\, d\alpha\right)^{1/4} + \left(\int \frac{\|f_x(x)-f_x(x+\alpha)\|_{L^\infty}^4}{|\alpha|^3}\, d\alpha\right)^{1/4}\right] \\
&\quad + \|f\|_{\dot H^3} \left(\int \frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|_{L^2}^2}{\alpha^2}\, d\alpha\right)^{1/2} \\
&\quad \times \left(\int \frac{\|f_x(x-\alpha)+f_x(x+\alpha)-2f_x(x)\|_{L^\infty}^2}{\alpha^2}\, d\alpha\right)^{1/2} \\
&\lesssim \|f\|_{\dot H^3}\|f\|_{\dot H^{5/2}} \left(\|f\|_{\dot B^1_{\infty,4}}\|f\|_{\dot B^{3/2}_{\infty,4}} + \|f\|_{\dot B^{3/2}_{\infty,2}}\right) \\
&\lesssim \|f\|_{\dot H^3}\|f\|_{\dot H^{5/2}} \left(\|f\|_{\dot B^1_{\infty,4}}\|f\|_{\dot B^{3/2}_{\infty,4}} + \|f\|_{\dot B^{3/2}_{\infty,2}}\right) \\
&\lesssim \|f\|_{\dot H^3}\|f\|_{\dot H^{3/2}}^{1/3}\|f\|_{\dot H^3}^{2/3} \left(\|f\|_{\dot H^{3/2}}\|f\|_{\dot B^2_{2,2}} + \|f\|_{\dot H^2}\right) \\
&\lesssim \|f\|_{\dot H^3}^2\|f\|_{\dot H^{3/2}}^2 + \|f\|_{\dot H^3}^2\|f\|_{\dot H^{3/2}}.
\end{aligned}$$

Then, we estimate $T_{1,6,2}$. Analogously to $I_{1,3,4,2}$ (see (5.28)) we see that this term may be rewritten as

$$\begin{aligned}
T_{1,6,2} &= \frac{1}{2}\int \mathcal{H}\partial_x^3 f \int \frac{f_{xx}(x-\alpha)-f_{xx}(x)}{\alpha^2} \\
&\qquad \times \int_0^\infty e^{-\delta} \sin\Big(\frac{\delta}{2}(\Delta_\alpha f - \bar\Delta_\alpha f)\Big) \sin\Big(\frac{\delta}{2}(\Delta_\alpha f + \bar\Delta_\alpha f)\Big)\, d\delta\, d\alpha\, dx \\
&\quad + \frac{1}{2}\int \partial_x^3 f \int \frac{\mathcal{H} f_{xx}(x-\alpha) - \mathcal{H} f_{xx}(x)}{\alpha^2} \\
&\qquad \times \int_0^\infty e^{-\delta} \sin\Big(\frac{\delta}{2}(\Delta_\alpha f - \bar\Delta_\alpha f)\Big) \sin\Big(\frac{\delta}{2}(\Delta_\alpha f + \bar\Delta_\alpha f)\Big)\, d\delta\, d\alpha\, dx
\end{aligned}$$

$$= T_{1,6,2,1} + T_{1,6,2,2}.$$

To estimate $T_{1,6,2,1}$, we use the embedding $\dot{H}^2 \hookrightarrow \dot{B}^{3/2}_{\infty,2}$ along with classical interpolation inequalites to get

$$\begin{aligned}
|T_{1,6,2,1}| &\lesssim \|f\|_{\dot{H}^3} \int \frac{\|f_{xx}(x-\alpha) - f_{xx}(x)\|_{L^2}}{\alpha^2} \int_0^\infty \delta e^{-\delta} \frac{\|\delta_\alpha f + \bar{\delta}_\alpha f\|_{L^\infty}}{|\alpha|} \, d\delta \, d\alpha \\
&\lesssim \|f\|_{\dot{H}^3} \left(\int \frac{\|f_{xx}(x-\alpha) - f_{xx}(x)\|^2_{L^2}}{\alpha^2} \, d\alpha \right)^{1/2} \left(\int \frac{\|\delta_\alpha f + \bar{\delta}_\alpha f\|^2_{L^\infty}}{\alpha^4} \, d\alpha \right)^{1/2} \\
&\lesssim \|f\|_{\dot{H}^3} \|f\|_{\dot{H}^{5/2}} \|f\|_{\dot{H}^2} \\
&\lesssim \|f\|^2_{\dot{H}^3} \|f\|_{\dot{H}^{3/2}}.
\end{aligned}$$

Since $\mathcal{H}$ is continuous on L^2, one gets the same control for $T_{1,6,2,2}$. Therefore,

$$|T_{1,6,2}| \lesssim \|f\|^2_{\dot{H}^3} \|f\|_{\dot{H}^{3/2}}.$$

It remains to estimate $T_{1,6,3}$, to do so, we use the following decomposition (which is analogous to equality (5.31)).

$$\begin{aligned}
T_{1,6,3} = &-\frac{1}{4} \int \partial_x^3 f \int \frac{\mathcal{H} f_{xx}(x-\alpha) - \mathcal{H} f_{xx}(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_\alpha (\Delta_\alpha f - \bar{\Delta}_\alpha f) \\
&\times \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) \, d\delta \, d\alpha \, dx \\
&- \frac{1}{4} \int \mathcal{H} \partial_x^3 f \int \frac{f_{xx}(x-\alpha) - f_{xx}(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_\alpha (\Delta_\alpha f - \bar{\Delta}_\alpha f) \\
&\times \cos(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f) \, d\delta \, d\alpha \, dx \\
&- \frac{1}{4} \int \partial_x^3 f \int \frac{\mathcal{H} f_{xx}(x-\alpha) - \mathcal{H} f_{xx}(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_\alpha (\Delta_\alpha f + \bar{\Delta}_\alpha f) \\
&\times \cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f) \, d\delta \, d\alpha \, dx \\
&- \frac{1}{4} \int \partial_x^3 f \int \frac{f_{xx}(x-\alpha) - f_{xx}(x)}{\alpha} \int_0^\infty \delta e^{-\delta} \partial_\alpha (\Delta_\alpha f + \bar{\Delta}_\alpha f) \\
&\times \cos(\frac{\delta}{2}(\Delta_\alpha f + \bar{\Delta}_\alpha f)) \sin(\frac{\delta}{2}(\Delta_\alpha f - \bar{\Delta}_\alpha f) \, d\delta \, d\alpha \, dx.
\end{aligned}$$

Since we shall do L^p estimates, $p \in (1, \infty)$ on the terms involving $\mathcal{H}$. We observe that these terms have the same regularity as the terms $I_{1,3,4,3}$ (see (5.31)) and therefore one analogously infers that,

$$|T_{1,6,3}| \lesssim \|f\|^2_{\dot{H}^3} \|f\|^2_{\dot{H}^{3/2}} + \|f\|^2_{\dot{H}^3} \|f\|_{\dot{H}^{3/2}},$$

hence,

$$|T_{1,6}| \lesssim \|f\|^2_{\dot{H}^3} (\|f\|_{\dot{H}^{3/2}} + \|f\|^2_{\dot{H}^{3/2}}).$$

Therefore, we get that

$$|T_1| \lesssim \|f\|^2_{\dot{H}^3} (\|f\|^2_{\dot{H}^{3/2}} + \|f\|_{\dot{H}^{3/2}}) - \pi \|f\|^2_{\dot{H}^3} + \pi \frac{K^2}{1+K^2} \|f\|^2_{\dot{H}^3}.$$

Finally, we obtain that

$$|T_1| \lesssim \|f\|^2_{\dot H^3}(\|f\|^2_{\dot H^{3/2}} + \|f\|_{\dot H^{3/2}}) - \frac{\pi}{1+K^2}\|f\|^2_{\dot H^3}. \tag{6.4}$$

6.7. Estimate of T_2

Recall that

$$T_2 = -3\int \mathscr{H}\partial_x^3 f \int \partial_{xx}\Delta_\alpha f\ \partial_x\Delta_\alpha f \int_0^\infty \delta e^{-\delta}\sin(\delta\Delta_\alpha f(x))\,d\delta\,d\alpha\,dx.$$

To estimate this term, it suffices to observe that

$$|T_2| \lesssim \|f\|_{\dot H^3}\left(\int \frac{\|f_{xx}(x)-f_{xx}(x-\alpha)\|^2_{L^2}}{\alpha^2}\,d\alpha\right)^{1/2}\left(\int \frac{\|f_x(x)-f_x(x-\alpha)\|^2_{L^\infty}}{\alpha^2}\,d\alpha\right)^{1/2}.$$

Therefore,

$$|T_2| \lesssim \|f\|_{\dot H^3}\|f\|_{\dot H^{5/2}}\|f_x\|_{\dot B^{1/2}_{\infty,2}}.$$

Hence, using the embedding $\dot H^1 \hookrightarrow \dot B^{1/2}_{\infty,2}$ along with Sobolev interpolations $\dot H^2 = \left[\dot H^{3/2},\dot H^3\right]_{\frac23,\frac13}$ and $\dot H^{5/2} = \left[\dot H^{3/2},\dot H^3\right]_{\frac13,\frac23}$, one finds

$$|T_2| \lesssim \|f\|^2_{\dot H^3}\|f\|_{\dot H^{3/2}}. \tag{6.5}$$

6.8. Estimate of T_3

It suffices to observe for instance that $\dot B^{5/3}_{6,3} \hookleftarrow \dot H^2 = \left[\dot H^{3/2},\dot H^3\right]_{\frac23,\frac13}$, so that

$$\begin{aligned} T_3 &= -2\int \mathscr{H}\partial_x^3 f \int (\partial_x\Delta_\alpha f)^3 \int_0^\infty \delta^2 e^{-\delta}\ \cos(\delta\Delta_\alpha f)\,dx\,d\alpha\,d\delta \\ &\lesssim \|f\|_{\dot H^3}\|f\|^3_{\dot B^{5/3}_{6,3}} \\ &\lesssim \|f\|^2_{\dot H^3}\|f\|^2_{\dot H^{3/2}}. \end{aligned} \tag{6.6}$$

Finally, by (6.4), (6.5) and (6.6) and by integrating in time $s \in [0,T]$ we find that for all $T>0$,

$$\|f\|^2_{\dot H^{5/2}}(T) + \frac{\pi}{1+K^2}\int_0^T \|f\|^2_{\dot H^3}\,ds \lesssim \|f_0\|_{\dot H^{5/2}} + P(f)\int_0^T \|f\|^2_{\dot H^3}\,ds,$$

where $K = \sup_{(x,t)\in\mathbb{R}\times[0,T]} |f_x(x,t)|$ and $P(f) = \|f\|_{L^\infty([0,T],\dot H^{3/2})} + \|f\|^2_{L^\infty([0,T],\dot H^{3/2})}$. This ends the proof of Lemma 6.1. □

7. Proofs of Theorems 2.1 and 2.2

Consider the following approximated system,

$$(7.1) \qquad (\tilde{\mathscr{M}}_\epsilon) \ : \begin{cases} \partial_t f_\epsilon(t,x) - \dfrac{\rho}{\pi} \displaystyle\int \partial_x \Delta_\alpha f_\epsilon \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f_\epsilon) \, d\delta \, d\alpha - \epsilon \Delta f_\epsilon = 0 \\ f_{0,\epsilon}(x) = f_0(x) * \phi_\epsilon(x), \end{cases}$$

where ϕ_ϵ is a classical mollifier that is $\phi_\epsilon(x) = \epsilon^{-1}\phi(x\epsilon^{-1})$, $\phi \in \mathscr{D}(\mathbb{R})$ so that ϕ is nonnegative and $\int \phi(x)dx = 1$. If the Lipschitz norm remains bounded on a time interval $[0, T_\epsilon)$ and if $\|f_\epsilon\|_{L^\infty \dot{H}^{3/2}}$ is smaller than a constant that depends only on the $L^\infty([0,T_\epsilon), \dot{W}^{1,\infty})$ norm then we may locally solve the equation. Using the same a priori estimate as proved in Section 5 for the regularized equation $(\tilde{\mathscr{M}}_\epsilon)$ we may show that actually $T_\epsilon = +\infty$. One has (uniformly in ϵ) that

$$\|f_{0,\epsilon}\|_{H^{3/2}} \lesssim \|f_0\|_{H^{3/2}}$$

as well as

$$\|f_{0,\epsilon}\|_{\dot{W}^{1,\infty}} \lesssim \|f_0\|_{\dot{W}^{1,\infty}}.$$

Therefore, the regularized initial data converges strongly in $H^{3/2} \cap \dot{W}^{1,\infty}$. Let $\psi \in \mathscr{D}([0,T] \times \mathbb{R})$ be nonnegative, from the a priori estimates we know that ψf_ϵ is bounded in $L^\infty([0,T]; H^{3/2} \cap \dot{W}^{1,\infty}) \cap L^2([0,T]; H^2)$. Since those spaces are separable Banach spaces, thanks to the Banach-Alaoglu theorem, we may extract from these solutions f_ϵ a subsequence $\{f_{\epsilon_k}\}_{k \geq 0}$ that converges weakly to a solution $f \in L^2([0,T]; H^2)$ and *-weakly in $L^\infty([0,T]; H^{3/2})$. In order to obtain the convergence in the sense of distribution, one needs to prove a strong convergence in $(L^2L^2)_{\mathrm{loc}}$. To do so, we need for instance to get a nice bound on $\partial_t f_\epsilon$ locally in space and time, namely on

$$-\epsilon \Lambda^2 f_\epsilon - \Lambda f_\epsilon - 2 \int \partial_x \Delta_\alpha f_\epsilon \int_0^\infty \delta e^{-\delta} \sin^2(\frac{\delta}{2} \Delta_\alpha f_\epsilon) \, d\delta \, d\alpha.$$

Since f_ϵ is bounded in $L^2([0,T]; \dot{H}^2)$ then $\partial_x f_\epsilon$ is bounded in $L^2([0,T]; \dot{H}^1)$. It is not difficult to see that the contribution coming from the nonlinear part of $\partial_t f_\epsilon$ is a locally bounded sequence in $L^2H^{-1/4}$. Indeed, by using the dual form of the Sobolev embedding it suffices to have a bound on the $L^2L^{4/3}$ norm of the nonlinearity. By controling the product, we find that if $f_\epsilon \in \dot{B}^{3/2}_{2,2} \cap \dot{B}^{3/4}_{8,2}$ then the nonlinear part of $\partial_t f$ is bounded in $L^2\dot{H}^{-1/4}$. We know that f_ϵ is locally bounded in $L^2\dot{H}^{3/2} \cap L^2\dot{H}^{9/8}$. Since this space embeds into $\dot{B}^{3/2}_{2,2} \cap \dot{B}^{3/4}_{8,2}$, therefore, we get that the nonlinear term of $\partial_t f_\epsilon$ is a bounded sequence in $L^2\dot{H}^{-1/4}$. Since the linear part is locally bounded in $L^2\dot{H}^1$, we may use the Rellich compactness theorem (see e.g., [25]) to get the strong convergence of a subsequence in $(L^2L^2)_{\mathrm{loc}}$. Consequently, the nonlinear term converges in $\mathscr{D}\tilde{\mathrm{O}}$. It is then classical to prove that the limit is a solution of the equation. For higher regularity data (that is $f_0 \in H^{5/2}$, with f_0 small enough in $\dot{H}^{3/2}$), using the a priori estimates proved in Section 6 we know that up to an extraction, f_ϵ is a bounded sequence in $L^\infty([0,T], H^{5/2})$ (in particular the semi-norm $\sup_{t>0} \|f_x\|_{L^\infty}$ will remain bounded). Hence, we can pass to the weak limit as well since Rellich gives also the strong compactness in $(L^2L^2)_{\mathrm{loc}}$. This concludes the result.

By doing the same a priori estimates for the difference of two solutions in $H^{3/2}$ (resp. $H^{5/2}$), one observes that the uniqueness follows easily from the regularizing effect together with the fact that the $L^\infty_t H^{3/2}$ norm decays (resp $L^\infty_t H^{5/2}$). Hence, Grönwall's inequality gives the uniqueness in the usual way and we therefore omit the details. □

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ERRATUM: "PRIME NUMBER RACES FOR ELLIPTIC CURVES OVER FUNCTION FIELDS"

BY Byungchul CHA, Daniel FIORILLI AND Florent JOUVE

Abstract. – The paper mentioned in the title contains a mistake in Proposition 3.1. The expression for the L-function of the elliptic curve $E/\mathbb{F}_q(t)$ is wrong by a small uniformly bounded number of linear factors in $\mathbb{Z}[T]$. In this note we fix the problem and its minor consequences on other results in the same paper.

Résumé. – L'article auquel le titre fait référence contient une erreur dans la proposition 3.1. L'expression donnée pour la fonction L de la courbe elliptique $E/\mathbb{F}_q(t)$ diffère de la valeur correcte par un nombre uniformément borné de facteurs linéaires dans $\mathbb{Z}[T]$. Le but de cette note est de corriger cette erreur ainsi que les conséquences mineures qu'elle a entrainées sur d'autres résultats du même article.

1. The L-function of elliptic curves in Ulmer's family

First recall some notation used in [1, §3]. Let $\mathbb{F}_q(t)$ be the rational function field over a finite field $\mathbb{F}_q$ of characteristic $p \geq 3$. Following [2], fix $d \in \mathbb{Z}_{>0}$ and define $E_d/\mathbb{F}_q(t)$ to be the elliptic curve over $\mathbb{F}_q(t)$ given by the Weierstrass equation

$$E_d\colon y^2 + xy = x^3 - t^d \,.$$

The following explicit description of the Hasse-Weil L-function of $E_d/\mathbb{F}_q(t)$ is essential to the analysis of Chebyshev's bias for Ulmer's family performed in [1]. This corrects the flawed expression for $L(E_d/\mathbb{F}_q(t), T)$ given in [1, Prop. 3.1].

Proposition 1.1. – *Suppose that d divides $p^n + 1$ for some n, and let $L(E_d/\mathbb{F}_q(t), T)$ be the Hasse-Weil L-function of E_d over $\mathbb{F}_q(t)$. Then,*

$$L(E_d/\mathbb{F}_q(t), T) = (1 - qT)^{\epsilon_d}(1 + qT)^{\eta_d} \prod_{\substack{e \mid d \\ e \nmid 6}} \left(1 - (qT)^{o_e(q)}\right)^{\phi(e)/o_e(q)} . \tag{1}$$

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Here, $\phi(e) = \#(\mathbb{Z}/e\mathbb{Z})^*$ is the Euler-phi function and $o_e(q)$ is the (multiplicative) order of q in $(\mathbb{Z}/e\mathbb{Z})^*$. Further, ϵ_d and η_d are defined as

$$\epsilon_d := \begin{cases} 0 & \textit{if } 2 \nmid d \textit{ or } 4 \nmid q-1 \\ 1 & \textit{if } 2 \mid d \textit{ and } 4 \mid q-1 \end{cases} + \begin{cases} 0 & \textit{if } 3 \nmid d \\ 1 & \textit{if } 3 \mid d \textit{ and } 3 \nmid q-1 \\ 2 & \textit{if } 3 \mid d \textit{ and } 3 \mid q-1; \end{cases}$$

$$\eta_d := \begin{cases} 0 & \textit{if } 2 \nmid d \textit{ or } 4 \mid q-1 \\ 1 & \textit{if } 2 \mid d \textit{ and } 4 \nmid q-1 \end{cases} + \begin{cases} 0 & \textit{if } 3 \nmid d \textit{ or } 3 \mid q-1 \\ 1 & \textit{if } 3 \mid d \textit{ and } 3 \nmid q-1. \end{cases}$$

Note that Proposition 1.1 only differs from its original version [1, Prop. 3.1] by the factor $(1+qT)^{\eta_d}$ appearing in (1). In particular the assumptions as well as the statement about the rank of $E_d/\mathbb{F}_q(t)$ in [1, Prop. 3.1] are unchanged.

Proof of Proposition 1.1. – We combine three arguments in order to obtain the expression stated in the proposition for $f_d(T) := L(E_d/\mathbb{F}_q(t), T)$ as an element of $\mathbb{Z}[T]$.

(i) We first compute the degree of $f_d(T)$ using the conductor-degree formula.

(ii) We use our knowledge of $\deg f_d(T)$ and the work of Ulmer ([2, Cor. 7.7, Prop. 8.1 and Th. 9.2]) to obtain the following factorization of $f_d(T)$ in $\mathbb{Z}[T]$:

$$f_d(T) = (1-qT)^{\epsilon_d} g_d(T) P_2(T),$$

where P_2 is the product over divisors e of d not dividing 6 appearing in (1), and $g_d(T) \in \mathbb{Z}[T]$ has degree η_d.

(iii) We use the geometric construction described in [2, §5] explaining that the difference between $P_2(T)$ and $f_d(T)$ is the result of blowing up some relevant quotient F_d/Γ of a Fermat surface at points that are either defined over $\mathbb{F}_q$ or over a quadratic extension of $\mathbb{F}_q$ (these points are cube roots or fourth roots of 1).

In the rest of the proof we let $k = \mathbb{F}_q$. For (i) we use [3, §3.1.7] and the reduction data [2, §2] for $E_d/k(T)$ to deduce that

$$\deg f_d = -4 + \left(1 + d + \begin{cases} 0 & \text{if } 6 \mid d \\ 2 & \text{if } 6 \nmid d \end{cases}\right),$$

where the first summand -4 on the right-hand side comes from the fact that the base curve is $\mathbb{P}^1/k$ and the three remaining summands correspond to the contributions of the bad reduction places above t, $1 - 2^4 3^3 t^d$, ∞, respectively. Overall

$$\deg f_d = \begin{cases} d-3 & \text{if } 6 \mid d\,, \\ d-1 & \text{if } 6 \nmid d\,. \end{cases} \tag{2}$$

As expected, the geometric invariant $\deg f_d$ does not depend on k, but only on d.

Step (ii) merely consists in extracting information from Ulmer's work [2]. Since we assume that $d \mid p^n + 1$ for some n, we deduce from [2, Cor. 7.7, Prop. 8.1] that $L(E/k, T)$ is divisible in $\mathbb{Z}[T]$ by

$$P_2(T) := \prod_{\substack{e \mid d \\ e \nmid 6}} \left(1 - (qT)^{o_e(q)}\right)^{\phi(e)/o_e(q)}.$$

Note that this factor depends a priori on q since making a field extension k'/k will result in replacing q by $|k'|$ each time it occurs in the expression for P_2. Moreover, invoking [2, Th. 9.2], we obtain an extra factor (a power of $1-qT$) for $L(E/k(t),T)$ so that overall we deduce that in $\mathbb{Z}[T]$, the polynomial f_d is a multiple of

$$h_d(T) := (1-qT)^{\epsilon_d} \prod_{\substack{e|d \\ e\nmid 6}} \left(1-(qT)^{o_e(q)}\right)^{\phi(e)/o_e(q)} . \tag{3}$$

Again note that ϵ_d depends on d *and* on k; precisely its value is affected by the presence of cube roots or fourth roots of 1 in k. In particular as soon as we work over a field extension k'/k containing the cube and fourth roots of 1, the parameter ϵ_d becomes independent of any further base extension.

Let $g_d := \frac{f_d}{h_d} \in \mathbb{Z}[T]$ and let $\eta_d = \deg g_d$. From (2) and (3) we deduce the formula for η_d stated in the proposition. In particular, the expression for η_d shows that $g_d = 1$ when k contains both the groups of cube roots and fourth roots of 1, and that in any case $\eta_d = \deg g_d \leq 2$.

We finally turn to (iii). From [4, (6.3)] we know precisely how the zeta function of $\mathcal{E}_d$ relates to $L(E_d/k(T),T)$ (here the notation is as in [2, §3]: $\mathcal{E}_d/k$ is the elliptic surface which is regular, proper and relatively minimal when seen as fibered over $\mathbb{P}^1$, and which has generic fiber $E_d/k(T)$). Also $\mathcal{E}_d$ is constructed (see [2, §5]) from some quotient F_d/Γ of the diagonal Fermat surface F_d/k by a sequence of blow-ups at k-points of μ_3 and μ_4 (the groups of cube roots and fourth roots of 1 in $\bar{k}$, respectively), as explained in [2, §5.6].

By [2, Cor. 7.7] the polynomial P_2 is the characteristic polynomial of the Frobenius acting on the middle étale cohomology of F_d/Γ. The "missing" factor g_d thus comes as the arithmetic translation of the sequence of blow-ups leading from F_d/Γ to $\mathcal{E}_d$. Let x_0 be a k'-rational point of F_d/Γ which is blown up in the process of constructing $\mathcal{E}_d$. As already mentioned, x_0 corresponds to an element of $\mu_3 \cup \mu_4$ seen as a subset of $\bar{k}$. In particular, k' either equals k or is a quadratic extension of k. In any case we can choose k' to be a quadratic extension of k such that x_0 is defined over k'. Then if $Y \to F_d/\Gamma$ is the result of blowing up x_0 we have by "multiplicativity of zeta functions" that

$$Z(Y/k',T) = \frac{Z((F_d/\Gamma)/k',T)}{1-q^2T} .$$

(Here we use the standard fact asserting that if X/k is a variety and if Y is a closed subvariety of X, then $Z(X,T) = Z(Y,T) \cdot Z(U,T)$ where U is the complement $U := X \setminus Y$. This is readily obtained from the definition of the zeta function of a variety over a finite field as an exponential generating series.) Also one has the following base change formula:

$$Z(Y/k',T^2) = Z(Y/k,T) \times Z(Y/k,-T) .$$

(Again this is a standard fact obtained by coming back to the definition of the zeta function of a variety over a finite field X/k and by exploiting elementary properties of r-th roots of 1 in $\mathbb{C}$, to show that if k_r/k is an extension of degree r, then one has $Z(X \times_k \operatorname{Spec} k_r, T^r) = \prod_{i=1}^{r} Z(X,\xi^i T)$, where $\xi \in \mathbb{C}$ is a primitive r-th root of 1.) Combining these facts on zeta functions, we deduce that

$$Z(Y/k,T) \times Z(Y/k,-T) = \frac{Z((F_d/\Gamma)/k', T^2)}{(1-qT)(1+qT)}.$$

One possibly has to iterate the above process several times (depending on the number of blow-ups that are necessary to construct $\mathcal{E}_d$ from F_d/Γ). However each time a point is blown up, the zeta function for the resulting variety only differs from that of the initial variety by a factor $1-qT$ or $1+qT$. Of course a factor $1-qT$ affects the rank of $E_d/k(T)$, but the rank is known by [2, Th. 9.2]. We deduce that g_d has to be a power of the polynomial $1+qT$ and that the exponent is necessarily η_d by (ii) above. □

We deduce the following corrected version of [1, Prop. 3.2].

PROPOSITION 1.2. – *Let $c_\pm(X)$ be defined by*

$$c_\pm(X) := \begin{cases} q/(q-1) & \text{for even } X, \\ \sqrt{q}/(q-1) & \text{for odd } X. \end{cases}$$

Then, denoting by $T_d(X)$ the quantity $T_{E_d}(X)$ defined by [1, (1)], *we have*

$$T_d(X) = -c_\pm(X) + \frac{\epsilon_d}{1-q^{-\frac{1}{2}}} + \frac{\eta_d e^{i\pi X}}{1+q^{-\frac{1}{2}}} + \sum_{\substack{e|d \\ e\nmid 6}} \phi(e) \frac{q^{-(X \bmod o_e(q))/2}}{1-q^{-o_e(q)/2}} + o_{X\to\infty}(1) \tag{4}$$

for X large enough, where $0 \le (X \bmod \ell) \le \ell - 1$ is the remainder in the Euclidean division of X by ℓ.

REMARK 1.3. – Note that the only difference between (4) and the expression for $T_d(X)$ stated in [1, Prop. 3.2] that it corrects, is the summand $\eta_d e^{i\pi X}/(1+q^{-\frac{1}{2}})$ on the right hand side. The hypotheses of Proposition 1.2 are the same as in the original [1, Prop. 3.2].

Proof of Proposition 1.2. – We combine [1, Cor. 2.10] and Proposition 1.1 to obtain that (5)

$$T_d(X) = -c_\pm(X) + \frac{\epsilon_d}{1-q^{-\frac{1}{2}}} + \frac{\eta_d e^{i\pi X}}{1+q^{-\frac{1}{2}}} + \sum_{\substack{e|d \\ e\nmid 6}} \frac{\phi(e)}{o_e(q)} \sum_{k=0}^{o_e(q)-1} \frac{e^{2\pi i k X/o_e(q)}}{1-q^{-\frac{1}{2}} e^{-2\pi i k/o_e(q)}} + o_{X\to\infty}(1).$$

The end of the proof of [1, Prop. 3.2] remains valid. □

The definition of the "periodic part" of $T_d(X)$ has to be corrected accordingly. We set (compare with [1, (44)])

$$T_d^{\mathrm{per}}(X) := -c_\pm(X) + \frac{\epsilon_d}{1-q^{-\frac{1}{2}}} + \frac{\eta_d e^{i\pi X}}{1+q^{-\frac{1}{2}}} + \sum_{\substack{e|d \\ e\nmid 6}} \phi(e) \frac{q^{-(X \bmod o_e(q))/2}}{1-q^{-o_e(q)/2}}.$$

The statement of [1, Cor. 3.4] remains valid (in the proof, one has to replace the flawed expression for $T_d(X)$ by (4) but this has no impact on the statement of [1, Cor. 3.4] since $\eta_d \ge 0$).

2. Corrected versions of [1, Th. 1.5] and of some related statements

The fact that the expression for $T_d(X)$ was incorrect in [1] has consequences on some of the results of [1] on Chebyshev's bias in Ulmer's family $(E_d/\mathbb{F}_q(t))$. In this section we state and prove the following corrected version of [1, Th. 1.5].

THEOREM 2.1. – *For the family* $\{E_d/\mathbb{F}_q(t)\}$ *(where we recall that the integer* d *and the characteristic* p *of* $\mathbb{F}_q$ *are linked by the relation* $d \mid p^n + 1$ *for some* $n \geq 1$*), one has the following cases of extreme bias.*

(i) *Suppose that* $3 \mid d$ *and* $3 \mid q-1$ *and that either* d *is odd or* $4 \mid q-1$*. Then,* $T_d(X) > 0$ *for all large enough* X*, and thus* $\underline{\delta}(E_d) = \overline{\delta}(E_d) = 1$.

(ii) *If* $q = p^k$ *with* p *large enough and* $d = p^n + 1$ *for some* $1 \leq n \leq \mathrm{e}^{q^{\frac{1}{2}}/6}$ *with* $n \equiv 0 \bmod k$*, then* $T_d(X) > 0$ *for all large enough* X*, and thus* $\underline{\delta}(E_d) = \overline{\delta}(E_d) = 1$.

(iii) *Fix* $\epsilon > 0$*. There exists primes* $d \geq 3$ *and* p *such that* p *is a primitive root modulo* d*, and such that if we pick* $q = p^{\frac{d+1}{2}}$*, then the associated curve* E_d *has analytic rank* 1 *(resp.* 2*) if* $(d-1)/2$ *is even (resp. odd) and*

$$0 < \underline{\delta}(E_d) \leq \overline{\delta}(E_d) < \epsilon\,.$$

REMARK 2.2. – Let us emphasize the differences between the statement of Theorem 2.1 and the statement that it corrects: [1, Th. 1.5]. In (i) we originally assumed that either d is a multiple of 3, or that d is even and $4 \mid q-1$. We see now that the proof requires the assumption $3 \mid d$ as well as the congruence condition $q \equiv 1 \bmod 3$. Statement (ii) only differs from the original one by the exponent in the upper bound on the range of n: we need to assume that it is $\sqrt{q}/6$ as opposed to the original exponent $\sqrt{q}/2$.

Finally, note that the statement (iii) is unchanged, compared with [1, Th. 1.5(iii)], but since its proof has to be amended, we chose, for completeness, to give a full corrected statement for [1, Th. 1.5].

Proof of Theorem 2.1(i). – If d and q satisfy the stated assumptions, then $\epsilon_d \in \{2,3\}$ and $\eta_d = 0$. Then, the statement easily follows because

$$\begin{aligned}-c_\pm(X) + \frac{\epsilon_d}{1-q^{-\frac{1}{2}}} + \frac{\eta_d \mathrm{e}^{\mathrm{i}\pi X}}{1+q^{-\frac{1}{2}}} &\geq -\frac{1}{1-q^{-1}} + \frac{\epsilon_d}{1-q^{-\frac{1}{2}}} - \frac{\eta_d}{1+q^{-\frac{1}{2}}} \\ &= \frac{-1+\epsilon_d - \eta_d + q^{-\frac{1}{2}}(\epsilon_d+\eta_d)}{1-q^{-1}} > 0,\end{aligned}$$

thus using Proposition 1.2 we see that $T_d(X) > 0$ for all large enough X. □

Proof of Theorem 2.1(ii). – By Proposition 1.2 and by positivity, we have that (recall that p is large enough, and therefore so is d)

$$\begin{aligned}T_d(X) &= -c_\pm(X) + \frac{\epsilon_d}{1-q^{-\frac{1}{2}}} + \frac{\eta_d \mathrm{e}^{\mathrm{i}\pi X}}{1+q^{-\frac{1}{2}}} + \sum_{\substack{e\mid d \\ e\nmid 6}} \phi(e)\frac{q^{-(X \bmod o_e(q))/2}}{1-q^{-o_e(q)/2}} + o_{X\to\infty}(1) \\ &\geq \phi(d) q^{-(X \bmod o_d(q))/2} - 1 - \eta_d + o_{p\to\infty}(1) + o_{X\to\infty}(1) \\ &\geq \phi(d) q^{-(o_d(q)-1)/2} - 3 + o_{p\to\infty}(1) + o_{X\to\infty}(1).\end{aligned}$$

However, we have that $q^{2n/k} \equiv 1 \bmod d$, that is $o_d(q) \mid 2n/k$. We conclude that

$$\begin{aligned} T_d(X) &\geq \phi(d) q^{\frac{1}{2}} q^{-\frac{n}{k}} - 3 + o_{p\to\infty}(1) + o_{X\to\infty}(1) \\ &= \phi(d) q^{\frac{1}{2}} (d-1)^{-1} - 3 + o_{p\to\infty}(1) + o_{X\to\infty}(1). \end{aligned}$$

This quantity is positive for large enough X since, for d large enough, we have that $\phi(d)/(d-1) \geq (\mathrm{e}^{-\gamma} + o(1))/\log\log d$ and the condition on n implies that

$$\log\log(p^n+1) \leq q^{\frac{1}{2}}/6 + \log\log p + 1.$$

Since $6\exp(-\gamma) > 3$ the proof is complete. □

The proof of [1, Th. 1.5(iii)] uses [1, Prop. 3.5] which remains valid although parts of its proof require some corrections that we now explain.

Proof of Proposition 3.5 *in* [1]. – The argument given in [1] to prove [1, Prop. 3.5(i)] remains valid since the hypotheses on the parameters imply that $\eta_d = 0$.

We turn to the proof of [1, Prop. 3.5(ii)]. The hypotheses imply $\epsilon_d = 0$ and $\eta_d = 1$. Thus, by Proposition 1.2, one has that

$$T_d(X) = -c_{\pm}(X) + \frac{\mathrm{e}^{\mathrm{i}\pi X}}{1+q^{-\frac{1}{2}}} + \phi(\ell)\frac{q^{-(X \bmod o_\ell(q))/2}}{1-q^{-o_\ell(q)/2}} + \phi(2\ell)\frac{q^{-(X \bmod o_{2\ell}(q))/2}}{1-q^{-o_{2\ell}(q)/2}} + o_{X\to\infty}(1).$$

We have $q^n \equiv (-p^{-1})^n \equiv -1 \bmod d$. We claim that n is the least positive integer such that this congruence holds. Indeed this minimality condition holds by definition for the congruence $p^n \equiv -1 \bmod d$. Now $q \equiv -p^{-1} \bmod d$ and n is even, thus the claim follows. Since q is odd the property $\ell \mid q^k - 1$ is equivalent to $2\ell \mid q^k - 1$, for all $k \in \mathbb{Z}_{\geq 1}$. In particular we have $o_d(q) = o_{2\ell}(q) = o_\ell(q) = 2n$.

Now, let j be an integer such that $4 \leq j \leq 2n-1$. We have that

$$\phi(\ell)\frac{q^{-(j \bmod o_\ell(q))/2}}{1-q^{-o_\ell(q)/2}} + \phi(2\ell)\frac{q^{-(j \bmod o_{2\ell}(q))/2}}{1-q^{-o_{2\ell}(q)/2}} \ll \ell q^{-4/2} \ll p^{n-2(n-1)} = p^{2-n}.$$

If j is odd, we have that $c_{\pm}(j) \gg q^{-\frac{1}{2}} = p^{(1-n)/2}$ and $\mathrm{e}^{\mathrm{i}\pi j} = -1$. We deduce that for p and X large enough we have $T_d(X) < 0$ (recall $n \geq 4$) as soon as $X \equiv j \bmod 2n$.

If j is even, we have that

$$T_d(X) = -c_{\pm}(j) + \frac{1}{1+q^{-\frac{1}{2}}} + O(p^{2-n}) + o_{X\to\infty}(1) = -\frac{p^{-(n-1)/2}}{1-q^{-1}} + O(p^{2-n}) + o_{X\to\infty}(1),$$

which is < 0 for large enough p and X. Hence we deduce that

$$0 \leq \underline{\delta}(E_d) \leq \overline{\delta}(E_d) \leq \frac{2}{n}.$$

As for the lower bound, it is given by [1, Cor. 3.4]. □

Proof of Theorem 2.1(iii). – This is a direct consequence of [1, Prop. 3.5(i)] and [1, Cor. 3.7] (the argument given in the proof of [1, Th. 1.5(iii)] can be applied without modifications). □

3. Corrected version of [1, Th. 1.7] and of some related results

The corrected version of [1, Th. 1.7] is the following statement.

THEOREM 3.1. – *For the family $\{E_d/\mathbb{F}_q(t)\}$, one has the following cases where $T_d(X)$ is completely unbiased. Fix $p \equiv 3 \bmod 4$ and let $d \geq 5$ be an* odd *divisor of p^2+1. Pick $q = p^{4k+1}$ with $k \geq 1$. Then the analytic rank of E_d is $(d-1)/4$ and we have*

$$\underline{\delta}(E_d) = \overline{\delta}(E_d) = \frac{1}{2}\,.$$

REMARK 3.2. – Here the only difference with the original statement [1, Th. 1.7] is that we now assume that d is odd. In Remark 3.3 below we prove that this hypothesis is in fact necessary for Theorem 3.1 to hold. Note that d being an odd divisor of a sum of two coprime squares implies that the stated value $(d-1)/4$ of the analytic rank is indeed an integer.

In [1], the proof of [1, Th. 1.7] follows another statement about moderate bias, namely [1, Prop. 3.8]. The latter remains valid although the proof requires some corrections that we now explain.

Proof of Proposition 3.8 in [1]. – Under the stated assumptions $q \equiv p \bmod 4$ thus $4 \nmid q-1$ and of course $2 \mid d$. Also since n is even, $3 \nmid d$. We deduce $\epsilon_d = 0$ and $\eta_d = 1$ and, from (4), we obtain the formula

$$T_d(X) = -c_{\pm}(X) + \frac{\mathrm{e}^{\mathrm{i}\pi X}}{1+q^{-\frac{1}{2}}} + \sum_{\substack{e \mid d \\ e \nmid 6}} \phi(e) \frac{q^{-(X \bmod o_e(q))/2}}{1-q^{-o_e(q)/2}} + o_{X\to\infty}(1).$$

As in the proof of [1, Prop. 3.8], one shows that, for each $e \mid d$ with $e \nmid 6$, we have that $o_e(q) \geq 3$.

Using this fact, we have that if $X \equiv 3 \bmod 2n$ (recall $n \geq 2$ so that 3 is an admissible remainder for the Euclidean division by $2n$), then

$$T_d(X) = -c_{\pm}(X) - \frac{1}{1+q^{-\frac{1}{2}}} + \sum_{\substack{e \mid d \\ e \nmid 6}} \phi(e) \frac{q^{-3/2}}{1-q^{-o_e(q)/2}} + o_{X\to\infty}(1).$$

This last quantity is negative for large enough $X \equiv 3 \bmod 2n$. Indeed $(1+q^{-\frac{1}{2}})^{-1} = 1 + o_{p\to\infty}(1)$, and since X is odd we have $c_{\pm}(X) = o_{p\to\infty}(1)$. Also,

$$\sum_{\substack{e \mid d \\ e \nmid 6}} \phi(e) \frac{q^{-3/2}}{1-q^{-o_e(q)/2}} \ll q^{-\frac{3}{2}} d \ll p^{n-3(kn+1)/2} \ll p^{-\frac{5}{2}},$$

which is $o_{p\to\infty}(1)$. We conclude by invoking [1, Cor. 3.4]. □

Proof of Theorem 3.1. – We argue as in [1, Proof of Th. 1.7] to see that $\epsilon_d = 0$. Moreover, since we assume d is odd and $d \mid p^2+1 \equiv 2 \bmod 3$, we have $\eta_d = 0$ and $d \equiv 1 \bmod 4$ (indeed d is an odd divisor of a sum of two coprime squares). Note also that if $e \mid d$ with $e \notin \{1,2\}$ (*i.e.*, $e \neq 1$ since d is odd), then $q^2 \equiv p^2 \equiv -1 \bmod e$, hence $o_e(q) = 4$.

The rank of E_d is then easily computed with the help of [1, Prop. 3.1]. Moreover, we deduce from (4) that

$$T_d(X) = -c_{\pm}(X) + \sum_{\substack{e|d \\ e\neq 1}} \phi(e)\frac{q^{-(X \bmod 4)/2}}{1-q^{-2}} + o_{X\to\infty}(1),$$

hence $T_d^{\mathrm{per}}(X)$ is 4-periodic.

If $X \equiv 0 \bmod 4$, then

$$T_d(X) = -\frac{q}{q-1} + \sum_{\substack{e|d \\ e\neq 1}} \frac{\phi(e)}{1-q^{-2}} + o_{X\to\infty}(1) \geq -2 + \frac{d-1}{1-q^{-2}} + o_{X\to\infty}(1),$$

which is positive for X large enough.

If $X \equiv 1 \bmod 4$, then

$$T_d(X) = -\frac{q^{\frac12}}{q-1} + \sum_{\substack{e|d \\ e\neq 1}} \frac{\phi(e)q^{-\frac12}}{1-q^{-2}} + o_{X\to\infty}(1) = q^{-\frac12}\left(\frac{d-2-q^{-1}}{1-q^{-2}}\right) + o_{X\to\infty}(1),$$

which is again positive for X large enough.

The analysis of cases $X \equiv 2, 3 \bmod 4$ is unchanged compared with [1, Proof of Th. 1.7], except for their index set in the sums $\sum_{e|d, e\neq 1,2}$ that have to be replaced by $\sum_{e|d, e\neq 1}$. □

REMARK 3.3. – In [1, Th. 1.7] the constraint "d odd" does not appear. However taking into account the correcting factor $(1+qT)^{\eta_d}$ in the expression for $L(E_d/\mathbb{F}_q(t), T)$ (in Proposition 1.1) we now see that this is a necessary restriction. Let us investigate the changes to the above proof implied in case d is *even*. Since $3 \nmid d$ and $4 \nmid q-1$ we still have $\epsilon_d = 0$; however if d is even, we have $\eta_d = 1$. The argument according to which any divisor $e \neq 1, 2$ of d satisfies $o_e(q) = 4$ is still valid and thus Proposition 1.2 gives

$$T_d(X) = -c_{\pm}(X) + \frac{(-1)^X}{1+q^{-\frac12}} + \sum_{\substack{e|d \\ e\neq 1,2}} \phi(e)\frac{q^{-(X \bmod 4)/2}}{1-q^{-2}} + o_{X\to\infty}(1),$$

hence with notation as before $T_d^{\mathrm{per}}(X)$ is again 4-periodic.

The formula for the rank in [1, Prop. 3.1] provides the explicit value $(d-2)/4$ for the analytic rank of $E_d/\mathbb{F}_q(t)$ (note that d is an even divisor of p^2+1 and thus $d \equiv 2 \bmod 4$).

Let us determine the sign of $T_d(X)$ depending on X mod 4. If $X \equiv 0 \bmod 4$, then

$$\begin{aligned} T_d(X) &= -\frac{q}{q-1} + \frac{1}{1+q^{-\frac12}} + \sum_{\substack{e|d \\ e\neq 1,2}} \frac{\phi(e)}{1-q^{-2}} + o_{X\to\infty}(1) \\ &\geq -2 + \frac{1}{1+q^{-\frac12}} + \frac{d-2}{1-q^{-2}} + o_{X\to\infty}(1), \end{aligned}$$

which is positive for X large enough.

Likewise for $X \equiv 3 \bmod 4$ the contribution is negative in case d is odd (as seen in the proof above). For even d the factor $(1+qT)^{\eta_d}$ of the L-function produces an extra negative contribution:

$$T_d(X) = -\frac{q^{\frac{1}{2}}}{q-1} - \frac{1}{1+q^{-\frac{1}{2}}} + \sum_{\substack{e|d \\ e\neq 1,2}} \frac{\phi(e)q^{-\frac{3}{2}}}{1-q^{-2}} + o_{X\to\infty}(1),$$

thus $T_d(X)$ is a fortiori negative for X large enough.

For $X \equiv 2 \bmod 4$, the contribution remains negative. Indeed we have that

$$\begin{aligned} T_d(X) &= -\frac{q}{q-1} + \frac{1}{1+q^{-\frac{1}{2}}} + \sum_{\substack{e|d \\ e\neq 1,2}} \frac{\phi(e)q^{-1}}{1-q^{-2}} + o_{X\to\infty}(1) \\ &= -\frac{q^{-\frac{1}{2}}}{1-q^{-1}} + \frac{(d-2)q^{-1}}{1-q^{-2}} + o_{X\to\infty}(1) \\ &= \frac{-q^{-\frac{1}{2}} + (d-2)q^{-1} - q^{-\frac{3}{2}}}{1-q^{-2}} + o_{X\to\infty}(1), \end{aligned}$$

which is negative for X large enough since $(d-2)q^{-\frac{1}{2}} \leq p^{2-2k-\frac{1}{2}} \leq p^{-\frac{1}{2}}$.

Finally if $X \equiv 1 \bmod 4$, the contribution is negative (contrary to the case d odd). Indeed we have that

$$\begin{aligned} T_d(X) &= -\frac{q^{\frac{1}{2}}}{q-1} - \frac{1}{1+q^{-\frac{1}{2}}} + \sum_{\substack{e|d \\ e\neq 1,2}} \frac{\phi(e)q^{-\frac{1}{2}}}{1-q^{-2}} + o_{X\to\infty}(1) \\ &= \frac{-1}{1-q^{-1}} + \frac{q^{-\frac{1}{2}}(d-2)}{1-q^{-2}} + o_{X\to\infty}(1) = \frac{-1-q^{-1}+q^{-\frac{1}{2}}(d-2)}{1-q^{-2}} + o_{X\to\infty}(1). \end{aligned}$$

As seen above $(d-2)q^{-\frac{1}{2}} \leq p^{-\frac{1}{2}}$ and thus $T_d(X)$ is negative for large enough X.

As a conclusion, we obtain that under the assumptions of Theorem 3.1 but fixing this time an *even* divisor d of p^2+1, we have $\delta(E_d) = 1/4$.

4. Corrected proof of [1, Th. 1.8] and of some related results

The statement of [1, Th. 1.8] remains valid, however its proof relies on [1, Prop. 3.10], the statement of which remains valid as well, although its proof requires some fixing. We now explain the details.

Proof of Proposition 3.10 *in* [1]. – We first see that the hypotheses on the parameters imply that $\epsilon_d = \eta_d = 0$. Indeed, as in [1, Proof of Prop. 3.10] we show that $d \equiv n \equiv 1 \bmod 2$ and that $d \equiv n \bmod 3$.

Since n is prime, we deduce that $(6,d) = 1$ as soon as $n > 3$. Now if $n = 3$, the assumption $3 \nmid p+1$ and the fact that $p^3+1 \equiv p+1 \bmod 3$ imply that $d \equiv 1 \bmod 3$ and again we conclude $(6,d) = 1$. Therefore under the assumptions stated in [1, Prop. 3.10], we have $\epsilon_d = \eta_d = 0$.

The rest of the proof of [1, Prop. 3.10] is unchanged. □

The way [1, Th. 1.8] is deduced from [1, Prop. 3.10] is unchanged.

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