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STABILIZATION OF NAVIER-STOKES EQUATION

by

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Abstract. — We survey here a few recent results and methods to stabilization of equilibrium solutions to Navier-Stokes in 2-D and 3-D.

1. The stabilization problem

Consider the Navier-Stokes equation in a domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$, with smooth boundary $\partial\mathcal{O}$,

$$
\begin{aligned}
&\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y = f_e + \nabla p && \text{in } \mathbb{R}^+ \times \mathcal{O},\\
&\nabla\cdot y = 0 && \text{in } \mathbb{R}^+ \times \mathcal{O},\\
&y = 0 && \text{on } \mathbb{R}^+ \times \partial\mathcal{O},\\
&y(0) = y_0 && \text{on } \mathbb{R}^+ \times \partial\mathcal{O},
\end{aligned}
\tag{1.1}
$$

where $f_e \in (L^2(\mathcal{O}))^d$, $\nabla\cdot f_e = 0$, $f_e\cdot n = 0$.

Here n is the unit normal and is directed toward the exterior of $\partial\mathcal{O}$.

Let $y_e \in (H^2(\mathcal{O}))^d$ be an equilibrium solution to (1.1), that is,

$$
\begin{aligned}
&-\nu\Delta y_e + (y_e\cdot\nabla)y_e = f_e + \nabla p_e \quad \text{in } \mathcal{O},\\
&\nabla\cdot y_e = 0 \ \text{ in } \mathcal{O}, \qquad y_e = 0 \quad \text{on } \partial\mathcal{O}.
\end{aligned}
\tag{1.2}
$$

1.1. Internal stabilization. — Let $\mathcal{O}_0 \subset \mathcal{O}$ be an open subdomain of $\mathcal{O}$ and consider the controlled system associated with (1.1)

$$\begin{aligned}
&\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y = f_e + \nabla p + \mathbb{1}_{\mathcal{O}_0}u \ \text{ in } \mathbb{R}^+\times\mathcal{O},\\
&\nabla\cdot y \ \text{ in } \mathbb{R}^+\times\mathcal{O}; \qquad y=0 \ \text{ on } \mathbb{R}^+\times\partial\mathcal{O},\\
&y(0)=y_0 \ \text{ in } \mathcal{O},
\end{aligned} \tag{1.3}$$

where the controller u is in $L^2(0,\infty;(L^2(\mathcal{O}))^d)$.

Problem 1.1. — *Find the controller u in feedback form, that is $u(t)=\phi(y(t)-y_e)$ such that the solution to the corresponding solution y to the closed loop system* (1.3) *satisfies for all y_0 in a neighborhood of y_e*

$$\|y(t)-y_e\|_{(L^2(\mathcal{O}))^d} \le Ce^{-\gamma t}\|y_0-y_e\|_{(L^2(\mathcal{O}))^d},\ \forall t\ge 0, \tag{1.4}$$

where $\gamma>0$.

If we set $y-y_e\to y$, Problem 1.1 reduces to find $u=\phi(y)$ such that the solution y to the equation

$$\begin{aligned}
&\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y + (y_e\cdot\nabla)y + (y\cdot\nabla)y_e = \nabla p + \mathbb{1}_{\mathcal{O}_0}u,\quad t\ge 0,\\
&\nabla\cdot y = 0 \ \text{ in } \mathbb{R}^+\times\mathcal{O},\\
&y=0 \ \text{ on } \mathbb{R}^+\times\partial\mathcal{O},\\
&y(0,x)=y_0(x)-y_e(x)=y^0(x),\ x\in\mathcal{O}.
\end{aligned} \tag{1.5}$$

satisfies

$$\|y(t)\|_{(L^2(\mathcal{O}))^d} \le Ce^{-\gamma t}\|y^0\|_{(L^2(\mathcal{O}))^d},\quad \forall t\ge 0. \tag{1.6}$$

We use the standard formalism to represent the Navier-Stokes equations as infinite-dimensional differential equations (see, e.g., **[9]**, **[20]**, **[21]**). That is we set

$$\begin{aligned}
H &= \{y\in (L^2(\mathcal{O}))^d; \nabla\cdot y = 0 \ \text{ in } \mathcal{O},\ y\cdot n = 0 \ \text{ on } \partial\mathcal{O}\},\\
Ay &= -P(\Delta y),\ \forall y\in D(A) = y\in H\cap (H^1_0(\mathcal{O}))^d\cap (H^2(\mathcal{O}))^d,\\
A_0 y &= P((y_e\cdot\nabla)y + (y\cdot\nabla)y_e), D(A_0) = H\cap (H^1_0(\mathcal{O}))^d,\\
By &= P((y\cdot\nabla)y),
\end{aligned}$$

where $P:(L^2(\mathcal{O}))^d\to H$ is the Leray projector.

We may rewrite (1.5) as

$$\frac{dy}{dt} + \nu Ay + A_0 y + By = P(\mathbb{1}_{\mathcal{O}_0}u),\ t\ge 0,\quad y(0)=y^0, \tag{1.7}$$

or, in a more compact form,

$$\frac{dy}{dt} + \mathcal{A}y + By = P(\mathbb{1}_{\mathcal{O}_0}u),\ t \geq 0, \quad y(0) = y^0, \tag{1.8}$$

where $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is the so called Oseen-Stokes operator

$$\mathcal{A} = \nu A + A_0, \quad D(\mathcal{A}) = D(A). \tag{1.9}$$

Then the *internal stabilization problem* reduces to find a feedback controller $u = \phi(y)$ such that the corresponding solution y to (1.8), that is,

$$\frac{dy}{dt} + \mathcal{A}y + By = P(\mathbb{1}_{\mathcal{O}_0}\phi(y)),\ \forall t \geq 0, \quad y(0) = y_0, \tag{1.10}$$

satisfies

$$|y\tau|_H \leq Ce^{-\gamma t}|y_0|_H,\ \forall t \geq 0, \tag{1.11}$$

for $\gamma > 0$ and all y_0 in a neighborhood of the origin. Here and everywhere in the following, $|\cdot|_H$ is the norm of the space H and $(\cdot,\cdot)_H$ is the corresponding scalar product.

1.2. Boundary stabilization. — Consider the boundary control system associated with (1.1)

$$\begin{aligned}
&\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y = f_e + \nabla p \ \text{ in } \mathbb{R}^+ \times \mathcal{O},\\
&\nabla\cdot y = 0 \qquad \text{on } \mathbb{R}^+ \times \mathcal{O}, \quad y = u \ \text{ on } \mathbb{R}^+ \times \partial\mathcal{O},\\
&y(0) = y_0 \qquad \text{in } \mathcal{O}.
\end{aligned} \tag{1.12}$$

Problem 1.2. — *Find a boundary controller u in the feedback form $u = \psi(y - y_e)$ such that the corresponding solution y to* (1.5) *satisfies* (1.4) *for all y_0 in a neighborhood of y_e.*

Equivalently, the solution y to

$$\begin{aligned}
&\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y + (y\cdot\nabla)y_e + (y_e\cdot\nabla)y = \nabla p \ \text{ in } \mathbb{R}^+\times\mathcal{O},\\
&y = u \ \text{ on } \mathbb{R}^+ \times \partial\mathcal{O}, \quad y(0) = y^0 - y_e,\\
&\nabla\cdot y = 0 \ \text{ on } \mathbb{R}^+ \times \mathcal{O},
\end{aligned} \tag{1.13}$$

where $u = \psi(y)$, satisfies (1.4).

If u is tangential, that is $u\cdot n = 0$ on $\mathbb{R}^+ \times \partial\mathcal{O}$, then the stabilization is said to be *tangential* while, if $u\cdot\tau = 0$ on $\mathbb{R}^+ \times \partial\mathcal{O}$ (where τ is the tangent vector to $\partial\mathcal{O}$), the stabilization is called *normal.*

Denote by $D : (L^2(\partial\mathcal{O}))^d \to H$ the Dirichlet map defined by

$$-\nu\Delta(Du) + (y_e\cdot\nabla)Du + (Du\cdot\nabla)y_e + kDu = \nabla p \ \text{ in } \mathcal{O}, \qquad Du = u \ \text{ on } \partial\mathcal{O}, \tag{1.14}$$

where $k > 0$ is sufficiently large but fixed.

It turns out that D is well defined on the space of all $u \in (L^2(\partial\mathcal{O}))^d$ such that $u \cdot n = 0$ on $\partial\mathcal{O}$ and that D is continuous from $(H^s(\partial\mathcal{O}))^d \to (H^{s+\frac{1}{2}}(\mathcal{O}))^d \cap H$ if $s \geq \frac{1}{2}$. (See Theorem A.2.1 in [**1**].) Then (1.13) reduces to

$$\frac{dy}{dt} + \mathcal{A}(y - Du) + By = kDu, \ t \geq 0, \quad y(0) = y^0. \tag{1.15}$$

If we denote by $\widetilde{\mathcal{A}}$ the extension, by transposition, $\widetilde{\mathcal{A}} : H \to (D(\mathcal{A}^*))'$ with respect to H as pivot space of the original operator $\mathcal{A}$, that is, $(\widetilde{\mathcal{A}}y, z) = (y, \mathcal{A}^* z)$, for all $z \in D(\mathcal{A})$, we can write (1.15) as

$$\frac{dy}{dt} + \widetilde{\mathcal{A}}y + By = kDu + \widetilde{\mathcal{A}}Du, \ t \geq 0, \quad y(0) = y^0, \tag{1.16}$$

and so, the tangential stabilization problem reduces to find a feedback controller $u = \psi(y)$ such that the solution y to (1.16) satisfies (1.4) for all y in a neighborhood of the origin.

It is obvious that the solution y to the Cauchy problem is taken here in a mild sense

$$y\tau = e^{-\mathcal{A}t} y^0 - \int_0^t e^{-\widetilde{\mathcal{A}}(t-s)} (By(s) + kDu + \widetilde{\mathcal{A}}Du(s)) ds, \quad t \geq 0. \tag{1.17}$$

Of course, if $\dfrac{d}{dt} Du \in L^2_{\rm loc}(0, \infty; H)$, we may rewrite (1.17) as

$$\begin{aligned} y\tau = {} & Du\tau + e^{-\mathcal{A}t}(y^0 - Du(0)) \\ & - \int_0^t e^{-\mathcal{A}(t-s)} \Big(By(s) + kDu(s) - \frac{d}{ds} Du(s) \Big) ds, \quad \forall t \geq 0. \end{aligned} \tag{1.18}$$

The functional representation of system (1.13) with normal boundary controller is a more delicate problem.

1.3. Main results

***Theorem 1.3* (Barbu & Triggiani 2004)**. — *There is a feedback controller*

$$u = \sum_{i=1}^{M} (R(y - y_e), \psi_i)_{(L^2(\mathcal{O}_0))^d} \psi_i, \quad R \in (L^2(\mathcal{O})), \tag{1.19}$$

which stabilizes exponentially y_e for

$$\|y_0 - y_e\|_W \leq \rho, \quad W = (H^{\frac{1}{2}}(\mathcal{O}))^d.$$

Here M^* is dependent of the multiplicity of eigenvalues λ_j of the Oseen-Stokes operator $\operatorname{Re} \lambda_j \leq 0$*, $j = 1, \dots, N$. The functions ψ_j are linear combinations of eigenfunctions φ_j^*.*