AUTOUR DES MOTIFS

École d'été franco-asiatique de géométrie algébrique et de théorie des nombres

Asian-French summer school on algebraic geometry and number theory

Volume II

M. Levine, J. Wildeshaus, B. Kahn

Panoramas et Synthèses

Numéro 41

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique

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Société mathématique de France 2013
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2000 *Mathematics Subject Classification.* — 11F, 11G, 11R39, 14F, 14G, 14K, 19E15.

Key words and phrases. — Motives, automorphic forms, Shimura varieties.

Mots-clé et phrases. — Motifs, formes automorphes, variétés de Shimura.

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J.-B. Bost et J.-M. Fontaine, éditeurs

Abstract. — This volume contains the second part of the lectures notes of the *Asian-French summer school on algebraic geometry and number theory*, which was held at the Institut des Hautes Études Scientifiques (Bures-sur-Yvette) and the université Paris-Sud XI (Orsay) in July 2006. This summer school was devoted to the theory of motives and its recent developments, and to related topics, notably Shimura varieties and automorphic representations.

The contributions in this second part are expanded versions of the series of lectures by M. Levine on triangulated categories of motives and motivic homotopy of schemes (*Six lectures on motives*), and of the additional lectures by J. Wildeshaus on boundary motives and their applications (*Boundary motives, relative motives and extensions of motives*) and by B. Kahn on a triangulated version of the conjectures of Tate and Beilinson on algebraic cycles over a finite field (*The full faithfullness conjectures in characteristic p*).

Résumé. — Ce volume contient la seconde partie des notes de cours de l'*École d'été franco-asiatique de géométrie algébrique et de théorie des nombres*, qui s'est tenue à l'Institut des Hautes Études Scientifiques (Bures-sur-Yvette) et à l'université Paris-Sud XI en juillet 2006. Cette école était consacrée à la théorie des motifs et à ses récents développements, ainsi qu'à des sujets voisins, comme la théorie des variétés de Shimura et des représentations automorphes.

Cette seconde partie est constituée de versions développées des cours de M. Levine consacrés aux catégories triangulées de motifs et à la théorie homotopique des schémas

(*Six lectures on motives*) et des leçons de J. Wildeshaus sur les motifs bords et leurs applications (*Boundary motives, relative motives and extensions of motives*) et de B. Kahn sur une version triangulée des conjectures de Tate et Beilinson sur les cycles algébriques sur les corps finis (*The full faithfullness conjectures in characteristic p*).

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FOREWORD

This second volume contains notes of the lectures by M. Levine and of parts of the seminars 6, 7, and 8 mentioned in the *Foreword* of the first volume of these proceedings. A third volume will contain the notes of the lectures by T. Saito and of parts of seminars 3, 6, 7, and 8.

Panoramas & Synthèses
41, 2013, p. 1–141

SIX LECTURES ON MOTIVES

by

Marc Levine

Preface

These lecture notes are taken from my lecture series in the Asian-French summer school on motives and related topics. My goal in my lectures was two-fold: to give first of all a sketch of Voevodsky's foundational construction of the triangulated category of motives and its basic properties, and then to give an idea of some of the applications and wider vistas this construction has made possible. In doing this, wanted also to point out some of the origins of this theory, coming from both the categorical side involving aspects of sheaf theory and triangulated categories, as well as the input from algebraic geometry, mainly through algebraic cycles. This latter aspect lead me to devote an entire lecture to so-called moving lemmas, as I felt this subject captured much of the geometric side of the theory. I also reviewed much of the necessary material about triangulated categories and sheaves on a Grothendieck site, with the intention of making the discussion as accessible as possible.

I chose mixed Tate motives for illustrating applications,. This subject touches on a broad range of subjects, including the theory of t-structures, Tannakian categories, rational homotopy theory, Grothendieck-Teichmüller theory, moduli of curves, polylogarithms and multiple zeta values. For this reason, I felt that an overview would be of interest to a fairly wide audience. I have also included two lectures on the extension of motives given by the motivic stable homotopy category of Morel-Voevodsky, giving a sketch of the construction as well as a discussion of the motivic Postnikov tower.

Many thanks are due to Joël Riou, whose careful reading and thoughtful comments allowed me to correct quite a few errors, as well as greatly improving the exposition.

The author thanks the IHÉS. and the Université Paris-Sud 11 at Orsay, as well as the NSF (grant DMS-DMS-0457195) and the Alexander von Humboldt Foundation, for their financial support.

Other than this, I have made only minor changes and additions to my original lectures in these notes; I hope this will transmit the informal nature of the lectures to the reader. The rewriting of these notes lets me recall how much I enjoyed the summer school at the I.H.E.S and gives me the opportunity of thanking most heartily the organizers of summer school, Jean-Marc Fontaine and Jean-Benoit Bost, for putting together a truly worthwhile conference.

Marc Levine
Essen, December 2006 and October 2010

LECTURE I
TRIANGULATED CATEGORIES OF MOTIVES

Our goal in this lecture is to construct a *category of motives* that should capture the fundamental properties and structures of a reasonable cohomology theory on smooth varieties over a field k. To guide the construction, we ask the rather vague question: what kind of structures does "cohomology" have? At the very least, one should have

1. Pull-back maps $f^* : H^*(Y) \to H^*(X)$ maps $f : X \to Y$.
2. Products $H^*(X) \times H^*(Y) \to H^*(X \times Y)$.
3. Some long exact sequences: for example, Mayer-Vietoris for (Zariski) open covers.
4. Some isomorphisms, for example $H^*(X) \cong H^*(X \times \mathbb{A}^1)$.

Next, what categorical constructions will lead to all these structures? First of all, there is an algebraic machinery for generating long exact sequences and imposing isomorphisms, namely the machinery of triangulated categories. This structure is a result of axiomatizing the basic example of the derived category of an abelian category. For example, if one considers the abelian category Shv_T of sheaves of abelian groups on a topological space T, then the sheaf cohomology $H^*(T, A)$ with coefficients in an abelian group A is given as the Ext-group

$$H^n(T, A) \cong \mathrm{Ext}^n_{\mathrm{Shv}_T}(\mathbb{Z}_T, A_T),$$

where $\mathbb{Z}_T$, A_T are the constant sheaves with value $\mathbb{Z}, A$. In the derived category $D(\mathrm{Shv}_T)$, one has the canonical isomorphism

$$\mathrm{Ext}^n(\mathbb{Z}_T, A_T) \cong \mathrm{Hom}_{D(\mathrm{Shv}_T)}(\mathbb{Z}_T, A_T[n]).$$

All the well-known long exact sequences for cohomology, such as the Mayer-Vietoris sequence, or the Bockstein sequence, arise from the long exact sequence machinery encoded in the triangulated category $D(\mathrm{Shv}_T)$. In a general triangulated category D, one can define the cohomology of an object X with values in another object A by

$$H^n(X, A) := \mathrm{Hom}_D(X, A[n]);$$

we shall see how the triangulated structure in D gives rise to lots of long exact sequence. This formal view of cohomology has proven extremely valuable in many areas of mathematics.

The product in cohomology comes from a tensor structure in the triangulated category D, namely a bi-functor

$$\otimes_D : D \times D \to D$$

with certain exactness properties. If our coefficient group A has a multiplication $A \otimes A \to A$ and our object X has a "diagonal" $\delta : X \to X \otimes X$, then our formal cohomology becomes a ring via

$$\begin{aligned}\mathrm{Hom}_D(X, A[n]) \otimes_{\mathbb{Z}} \mathrm{Hom}_D(X, A[m]) &\xrightarrow{\otimes_D} \mathrm{Hom}_D(X \otimes X, A[n+m]) \\ &\xrightarrow{\delta^*} \mathrm{Hom}_D(X, A[n+m]).\end{aligned}$$

We need some geometric input to feed this machine, coming from algebraic cycles. Finally, to understand what comes out of this construction, we need the homological algebra of sheaf theory. Schematically, we have the following picture:

$$\begin{tikzcd}
\text{Triangulated tensor categories} \arrow[dr] & \\
\text{Algebraic cycles} \arrow[r] & \text{Motives} \\
\text{Sheaf theory} \arrow[ur] &
\end{tikzcd}$$

1. Triangulated categories

1.1. Translations and triangles

Definition 1.1. – A *translation* on an additive category $\mathscr{A}$ is an equivalence $T : \mathscr{A} \to \mathscr{A}$. We write $X[1] := T(X)$. An additive functor $F : \mathscr{A} \to \mathscr{B}$ between additive categories with translation is *graded* if $F(X[1]) = (FX)[1]$ and similarly for morphisms.

Let $\mathscr{A}$ be an additive category with translation. A *triangle* (X, Y, Z, a, b, c) in $\mathscr{A}$ is a sequence of maps

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1].$$

A morphism of triangles

$$(f, g, h) : (X, Y, Z, a, b, c) \to (X', Y', Z', a', b', c')$$

is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[r, "b"] \arrow[d, "g"'] & Z \arrow[r, "c"] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"'] \\
X' \arrow[r, "a'"'] & Y' \arrow[r, "b'"'] & Z' \arrow[r, "c'"'] & X'[1].
\end{tikzcd}$$

1.2. Triangulated categories. – Verdier [**18**] has defined a *triangulated category* as an additive category $\mathscr{A}$ with translation, together with a collection $\mathscr{E}$ of triangles, called the *distinguished triangles* of $\mathscr{A}$, which satisfy

TR1 $\mathscr{E}$ is closed under isomorphism of triangles.

$A \xrightarrow{\mathrm{id}} A \to 0 \to A[1]$ is distinguished.

Each morphism $X \xrightarrow{u} Y$ extends to a distinguished triangle

$$X \xrightarrow{u} Y \to Z \to X[1]$$

TR2 $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished $\Leftrightarrow$ $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished.

TR3 Given a commutative diagram with distinguished rows

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & X[1] \\
X \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z \arrow[r, "w'"'] & X[1]
\end{tikzcd}$$

there exists a morphism $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"] \\
X \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z \arrow[r, "w'"'] & X][1]
\end{tikzcd}$$

TR4 If we have three distinguished triangles $(X, Y, Z', u, i, *)$, $(Y, Z, X', v, *, j)$, and $(X, Z, Y', w, *, *)$, with $w = v \circ u$, then there are morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that

– (id_X, v, f) is a morphism of triangles
– (u, id_Z, g) is a morphism of triangles
– $(Z', Y', X', f, g, i[1] \circ j)$ is a distinguished triangle.

A graded functor $F : \mathscr{A} \to \mathscr{B}$ of triangulated categories is called *exact* if F takes distinguished triangles in $\mathscr{A}$ to distinguished triangles in $\mathscr{B}$.

Remark 1.2. – Suppose $(\mathscr{A}, T, \mathscr{E})$ satisfies (TR1), (TR2) and (TR3). If (X, Y, Z, a, b, c) is in $\mathscr{E}$, and A is an object of $\mathscr{A}$, then the sequences

$$\cdots \xrightarrow{c[-1]_*} \mathrm{Hom}_{\mathscr{A}}(A, X) \xrightarrow{a_*} \mathrm{Hom}_{\mathscr{A}}(A, Y) \xrightarrow{b_*}$$
$$\mathrm{Hom}_{\mathscr{A}}(A, Z) \xrightarrow{c_*} \mathrm{Hom}_{\mathscr{A}}(A, X[1]) \xrightarrow{a[1]_*} \cdots$$

and

$$\cdots \xrightarrow{a[1]^*} \mathrm{Hom}_{\mathscr{A}}(X[1], A) \xrightarrow{c^*} \mathrm{Hom}_{\mathscr{A}}(Z, A) \xrightarrow{b^*}$$
$$\mathrm{Hom}_{\mathscr{A}}(Y, A) \xrightarrow{a^*} \mathrm{Hom}_{\mathscr{A}}(X, A) \xrightarrow{c[-1]^*} \cdots$$

are exact. This yields:

- (five-lemma): If (f,g,h) is a morphism of triangles in $\mathscr{E}$, and if two of f,g,h are isomorphisms, then so is the third.
- If (X,Y,Z,a,b,c) and (X,Y,Z',a,b',c') are two triangles in $\mathscr{E}$, there is an isomorphism $h : Z \to Z'$ such that
$$(\mathrm{id}_X, \mathrm{id}_Y, h) : (X,Y,Z,a,b,c) \to (X,Y,Z',a,b',c')$$
is an isomorphism of triangles.

If (TR4) holds as well, then one has "Mayer-Vietoris"-type distinguished triangles: Given a commutative diagram with distinguished rows

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, equal] & Y \arrow[r, "b"] \arrow[d, "f"'] & Z \arrow[r, "c"] & X[1] \arrow[d, equal] \\ X \arrow[r, "a'"'] & Y' \arrow[r, "b'"'] & Z' \arrow[r, "c'"'] & X[1] \end{tikzcd}$$

there exists a morphism $g : Z \to Z'$ so that

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, equal] & Y \arrow[r, "b"] \arrow[d, "f"'] & Z \arrow[r, "c"] \arrow[d, "g"'] & X[1] \arrow[d, equal] \\ X \arrow[r, "a'"'] & Y' \arrow[r, "b'"'] & Z' \arrow[r, "c'"'] & X[1] \end{tikzcd}$$

is a map of triangles and the sequence

$$Y \xrightarrow{(f,-b)} Y' \oplus Z \xrightarrow{b'+g} Z' \xrightarrow{a[1]\circ c'} Y[1]$$

is distinguished (see [**49**, Lemma 1.4.3]).

Shortly speaking: A triangulated category is a machine for generating natural long exact sequences.

1.3. An example: the homotopy category of an additive category. – Let $\mathscr{A}$ be an additive category, $C^?(\mathscr{A})$ the category of cohomological complexes (with boundedness condition $? = \varnothing, +, -, b$). Recall that a morphism of complexes $f : A \to B$ is a collection of maps $f^n : A^n \to B^n$ such that

$$d_B^n f^n = f^{n+1} d_A^n.$$

For a complex (A, d_A), let $A[1]$ be the complex

$$A[1]^n := A^{n+1}; \quad d^n_{A[1]} := -d_A^{n+1}.$$

For a map of complexes $f : A \to B$, we have the *cone sequence*

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} A[1]$$