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## ON CARLOTTO–SCHOEN-TYPE SCALAR-CURVATURE GLUINGS

BY PIOTR T. CHRUSCIEL & ERWANN DELAY

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ABSTRACT. — We carry out a Carlotto–Schoen-type gluing with interpolating scalar curvature on cone-like sets, or deformations thereof, in the category of smooth Riemannian asymptotically Euclidean metrics.

RÉSUMÉ (*Sur les recollements de courbure scalaire de type Carlotto-Schoen*). — Nous effectuons des recollements de type Carlotto-Schoen en interpolant la courbure scalaire sur des ensembles de type cône ou des déformations de ceux-ci, dans la catégorie des métriques Riemanniennes lisses asymptotiquement Euclidiennes.

### 1. Introduction

In an outstanding paper [6], Carlotto and Schoen constructed non-trivial asymptotically Euclidean scalar-flat metrics that are Minkowskian outside of a

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solid cone. Their paper strongly suggests, although no explicit statements are made, that their construction can be generalised as follows:

1. Rather than gluing an asymptotically Euclidean metric to a flat one, any two asymptotically Euclidean metrics  $g_1$  and  $g_2$  can be glued together.
2. In the spirit of [12], the gluings at zero-scalar curvature can be replaced by gluings where the scalar curvature of the final metric equals

$$\chi R(g_1) + (1 - \chi)R(g_2)$$

with a cut-off function  $\chi$ .

3. The geometry of the gluing region can be somewhat more general than the interface between two rotationally symmetric cones.

The aim of this paper is to give detailed proofs of the above. While the overall structure of the argument is rather similar, the details are different. Both the proof in [6] and ours require a weighted Poincaré inequality, with nonstandard weights adapted to the geometry of the problem at hand. We provide a version of this inequality that is different from the one in [6], which might be of more general interest. We also provide a detailed proof of the gluing with exponential weights near the boundary, which is only sketched in [6]. Finally, we replace some of the tailor-made arguments in [6] by off-the-shelf results from [8].

A special case of our main Theorem 3.1 below provides the following variant of the Riemannian-geometry version of the main theorem of [6]:

**THEOREM 1.1.** — *Let  $n \geq 3$ . Consider two smooth  $n$ -dimensional asymptotically Euclidean Riemannian metrics  $g_1$  and  $g_2$  and two nested cones with non-zero opening angles and vertices displaced along the symmetry axis, see Figure 1.1. Simultaneously scaling-up the cones if necessary, or simultaneously shifting them to the asymptotic region, there exists a smooth asymptotically Euclidean metric  $g$ , which coincides with  $g_1$  outside of the larger cone and with  $g_2$  inside the smaller one, with the scalar curvature  $R(g)$  lying between  $R(g_1)$  and  $R(g_2)$  in the intermediate region.*

In particular, if  $R(g_1) = R(g_2) = 0$ , then  $R(g) = 0$ .

In Section 6, we show how Theorem 1.1 follows from our main Theorem 3.1 below. The key to the proof of this last theorem is a weighted Poincaré inequality involving radially scaled exponential weights, proved in Proposition 5.6 below. The rest of the proof is a verification of the hypotheses of the rather general results proved in [8].

As was already mentioned, Theorem 3.1 below allows more general cones than the ones in [6], as described at the beginning of Section 3. In fact, our arguments apply to a large class of deformed cones as well, e.g. “logarithmically rotated ones”; cf. Theorem 4.1 and the discussion in Section 4.

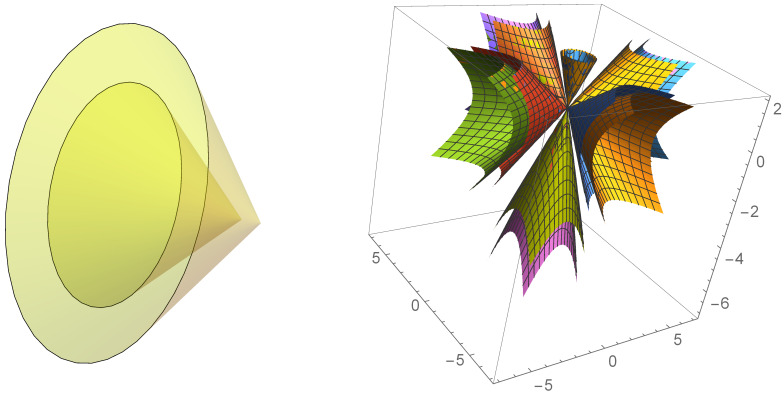


FIGURE 1.1. Left: If the second metric is flat and both metrics are scalar-flat, then the new metric is flat inside the smaller cone, scalar-flat everywhere, and coincides with the first one outside the larger cone. Both cones extend to infinity, and their tips are located very far in the asymptotically Euclidean region. Right: Iterating the construction one can embed any finite number of distinct initial data sets into Minkowskian data, or paste-in Minkowskian data inside several cones into a given data set.

## 2. Definitions, notations and conventions

We use the summation convention throughout; indices are lowered with  $g_{ij}$  and raised with its inverse  $g^{ij}$ .

We will have to control the asymptotic behaviour of the objects at hand frequently. Given a tensor field  $T$  and a function  $f$ , we will write

$$T = O(f),$$

when there exists a constant  $C$  such that the  $g$ -norm of  $T$  is dominated by  $Cf$ .

A metric  $g$  on a manifold  $M$  will be said to be asymptotically Euclidean (AE), if  $M$  contains a region, diffeomorphic to the complement of a ball in  $\mathbb{R}^n$ , on which the metric  $g$  approaches the Euclidean metric  $\delta$  as one recedes to infinity.

Let  $\phi$  and  $\psi$  be two smooth strictly positive functions on an  $n$ -dimensional manifold  $M$ . The function  $\psi$  will be used to control the growth of the fields involved near boundaries or in the asymptotic regions, while  $\phi$  will control how the growth is affected by derivations. For  $k \in \mathbb{N}$ , let  $H_{\phi, \psi}^k(g)$  be the space of