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LOCAL NORMAL FORMS FOR C-PROJECTIVELY EQUIVALENT METRICS AND PROOF OF THE YANO-OBATA CONJECTURE IN ARBITRARY SIGNATURE. PROOF OF THE PROJECTIVE LICHNEROWICZ CONJECTURE FOR LORENTZIAN METRICS

BY Alexey V. BOLSINOV, Vladimir S. MATVEEV and Stefan ROSEMANN

Abstract. – Two Kähler metrics on a complex manifold are called c-projectively equivalent if their J-planar curves coincide. These curves are defined by the property that the acceleration is complex proportional to the velocity. We give an explicit local description of all pairs of c-projectively equivalent Kähler metrics of arbitrary signature and use this description to prove the classical Yano-Obata conjecture: we show that on a closed connected Kähler manifold of arbitrary signature, any c-projective vector field is an affine vector field unless the manifold is $\mathbb{C}P^n$ with (a multiple of) the Fubini-Study metric. As a by-product, we prove the projective Lichnerowicz conjecture for metrics of Lorentzian signature: we show that on a closed connected Lorentzian manifold, any projective vector field is an affine vector field.

Résumé. – Deux métriques kählériennes sur une variété complexe sont appelées c-projectivement équivalentes si leurs courbes J-planaires coïncident. Ces courbes sont définies par la propriété que l'accélération est proportionnelle (au sens complexe) à la vitesse. Nous donnons une description locale de tous les paires de métriques kählériennes c-projectivement équivalentes de signature arbitraire et utilisons cette description pour prouver la conjecture classique de Yano-Obata: nous montrons que sur une variété kählérienne de signature arbitraire, connexe et fermée, tout champ de vecteurs c-projectif est un champ de vecteur affine sauf si la variété est $\mathbb{C}P^n$, munie de la métrique de Fubini-Study. En tant que sous-produit, nous prouvons la conjecture de Lichnerowicz pour les métriques de signature lorentzienne. Plus précisément, sur une variété lorentzienne connexe fermée tout champ de vecteurs projectif est un champ de vecteurs affine.

1. Introduction

1.1. Definitions and description of results

Let (M, g, J) be a Kähler manifold of arbitrary signature of real dimension $2n \geq 4$. We denote by ∇ the Levi-Civita connection of g and let $\omega = g(J\cdot, \cdot)$ denote the Kähler form. All objects under consideration are assumed to be sufficiently smooth.

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A regular curve $\gamma : \mathbb{R} \supseteq I \to M$ is called *J-planar* if there exist functions $\alpha, \beta : I \to \mathbb{R}$ such that

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = \alpha\dot\gamma(t) + \beta J(\dot\gamma(t)) \text{ for all } t \in I, \tag{1.1}$$

where $\dot\gamma = \frac{\mathrm{d}}{\mathrm{d}t}\gamma$.

From the definition we see immediately that the property of J-planarity is independent of the parameterisation of the curve, and that geodesics are J-planar curves. We also see that J-planar curves form a much bigger family than the family of geodesics; at every point and in every direction there exist infinitely many geometrically different J-planar curves.

Two metrics g and $\hat g$ of arbitrary signature that are Kähler w.r.t the same complex structure J are *c-projectively equivalent* if any J-planar curve of g is a J-planar curve of $\hat g$. Actually, the condition that the metrics are Kähler with respect to the same complex structure is not essential; it is an easy exercise to show that if any J-planar curve of a Kähler structure (g, J) is a $\hat J$-planar curve of another Kähler structure $(\hat g, \hat J)$, then $\hat J = \pm J$.

C-projective equivalence was introduced (under the name "h-projective equivalence") by Otsuki and Tashiro in [37, 43]. Their motivation was to generalize the notion of projective equivalence to the Kähler situation. Since the notion of projective equivalence plays an essential role in our paper let us recall it. Two metrics g and $\hat g$ of arbitrary signature are *projectively equivalent*, if each g-geodesic is, up to an appropriate reparameterisation, a $\hat g$-geodesic.

Otsuki and Tashiro have shown that projective equivalence is not interesting in the Kähler situation, since only simple examples are possible, and suggested c-projective equivalence as an interesting object of study instead. This suggestion appeared to be very fruitful and between the 1960s and the 1970s, the theory of c-projectively equivalent metrics and c-projective transformations was one of the main research topics in Japanese and Soviet (mostly Odessa and Kazan) differential geometry schools. For a collection of results of these times, see for example the survey [34] with more than 150 references. Moreover, two classical books [42, 46] contain chapters on c-projectively equivalent metrics and connections.

Relatively recently c-projective equivalence was re-introduced, under different names and because of different motivation. In fact, c-projectively equivalent metrics are essentially the same as Hamiltonian 2-forms, defined and investigated in Apostolov *et al.* [1, 2, 3, 4] for positive definite metrics, see also [17]. Though the definition of Hamiltonian 2-forms is visually different from that of c-projectively equivalent metrics, the defining equation [1, equation (12)] of a Hamiltonian 2-form is algebraically equivalent to a reformulation (see (1.4) below) of the condition "$\hat g$ is c-projectively equivalent to g" into the language of PDE. The motivation of Apostolov *et al.* to study Hamiltonian 2-forms is different from that of Otsuki and Tashiro. Roughly speaking, in [1, 2] Apostolov *et al.* observe that many interesting problems in Kähler geometry lead to Hamiltonian 2-forms and suggest studying them. The motivation is justified in [3, 4], where the authors indeed construct interesting and useful examples of Kähler manifolds. In dimension ≥ 6, c-projectively equivalent metrics are also essentially the same as Hermitian conformal Killing (or twistor) $(1, 1)$-forms studied in [35, 40, 41], see [1, Appendix A] or [33, §1.3] for details. Finally, such metrics are closely related to the so-called Kähler-Liouville integrable systems of type *A* introduced by Kiyohara in [24], see also [25].

We would also like to mention a recent review on c-projective geometry [16] that contains many new and old results in this area. Let us however make clear that our paper is totally independent of [16]. The work on these two papers was carried out more or less simultaneously and the second author of the present paper, being also one of the authors of [16], was quite careful about possible intersections between them. The only exception is Lemma 2.2 for which we suggest an alternative proof, for more details see discussion just after Lemma 2.2. It is worth noticing that the present paper and paper [16] represent two rather different approaches in c-projective geometry. Our approach is based on the reduction to the *real* projective setting, we explain it in §1.2, whereas [16] studies c-projectively equivalent metrics using ideas and methods of *parabolic* geometry.

Our paper contains three main results. The first result is a local description (near a generic point) of c-projectively equivalent Kähler metrics of arbitrary signature, see Example 5 and Theorem 1.6. If g is positive definite, such a description follows from the local description of Hamiltonian 2-forms due to Apostolov *et al.* [1]. Although the precise statements are slightly lengthy, we indeed provide an explicit description of the components of the metrics and of the Kähler form $\omega = g(J\cdot,\cdot)$. The parameters in this description are almost arbitrary numbers and functions of one variable and, in certain cases, almost arbitrary affinely equivalent Kähler metrics of smaller dimension (note that the description of affinely equivalent Kähler metrics was recently obtained by Boubel in [14]).

It is hard to overestimate the future role of a local description in the local and global theory of c-projectively equivalent metrics. Almost all known local results can easily be proved using it. Roughly speaking, using the local description, one can reduce any problem that can be stated using geometric PDEs (for example, any problem involving the curvature) to the analysis of a system of ODEs. As we mentioned above, in the positive definite case, the description of c-projectively equivalent metrics in the language of Hamiltonian 2-forms is due to Apostolov *et al.* [1], and with the help of such a description they did a lot. In particular they described possible topologies of closed manifolds admitting c-projectively equivalent Kähler metrics, described Bochner-flat Kähler metrics and constructed new examples of Einstein and extremal Kähler metrics on closed manifolds, see [1, 2, 3, 4]. We expect similar applications of our description and some have been already obtained, e.g., in [13] the local description of c-projectively equivalent metrics has been used to describe all Bochner-flat (pseudo-)Kähler metrics, generalizing results of [1] and [15] to the case of arbitrary signature. We plan to look for other applications and in particular to study the topology of c-projectively equivalent closed Kähler manifolds of arbitrary signature in further papers.

A demonstration of the importance of the local description is our second main result, which is a proof of the natural generalization of the Yano-Obata conjecture for Kähler manifolds of arbitrary signature. A vector field on a Kähler manifold is called *c-projective* if its local flow sends J-planar curves to J-planar curves, and *affine* if its local flow preserves the Levi-Civita connection.

THEOREM 1.1 (Yano-Obata conjecture). – *Let* (M, g, J) *be a closed connected Kähler manifold of arbitrary signature and of real dimension* $2n \geq 4$ *such that it admits a c-projective*