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FINITENESS OF SUPERELLIPTIC CURVES WITH CM JACOBIANS

BY Ke CHEN, Xin LU AND Kang ZUO

ABSTRACT. – This paper proves the Coleman conjecture for superelliptic curves: there are, up to isomorphism, at most finitely many superelliptic curves whose Jacobians are CM abelian varieties, as long as these curves are of genus at least 8. Here superelliptic curves are smooth projective curves over $\mathbb{C}$ admitting affine equations of the form $y^n = \zeta(x)$ with ζ a separable polynomial. The proof is reduced to the geometry of superelliptic Torelli locus $\mathcal{TS}_g$ in the Siegel modular variety $\mathcal{A}_g$: we establish the generic exclusion from $\mathcal{TS}_g$ of any special subvariety of dimension > 0 in $\mathcal{A}_g$ for $g \geq 8$, and the stability properties of Higgs bundles associated to surface fibrations play a crucial role in our study.

RÉSUMÉ. – Dans ce travail on montre la conjecture de Coleman pour les courbes super-elliptiques: l'ensemble des classes d'isomorphismes des courbes super-elliptiques dont les jacobiennes sont à multiplication complexe comme variétés abéliennes est au plus fini, lorsque ces courbes sont de genre au moins 8. Par courbes super-elliptiques on comprend les courbes projectives lisses sur $\mathbb{C}$ admettant une équation affine sous la forme $y^n = \zeta(x)$ où ζ est un polynôme séparable. La démonstration se réduit à la géométrie du lieu de Torelli super-elliptique $\mathcal{TS}_g$ dans la variété modulaire de Siegel $\mathcal{A}_g$: on montre qu'aucune sous-variété spéciale de dimension > 0 dans $\mathcal{A}_g$ n'est contenue génériquement dans $\mathcal{TS}_g$ pour $g \geq 8$, et un rôle crucial dans nos études est joué par les propriétés de stabilité des fibrés de Higgs associés aux fibrations des surfaces algébriques.

1. Introduction

This paper is dedicated to the Coleman conjecture for the superelliptic curves.

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1.1. Coleman conjecture

We start with the Coleman conjecture in its original form:

CONJECTURE 1.1 (Coleman). – *Up to isomorphism, there are at most finitely many smooth projective curves whose Jacobians are abelian varieties with complex multiplication, as long as these curves are of sufficiently high genus.*

The conjecture was made in the 1980s, cf. [7], and no precise bound on the genus was given. We can reformulate the conjecture in terms of geometry of moduli spaces. Recall that the Torelli morphism in genus $g \geq 2$ is

$$j : \mathcal{M}_g \to \mathcal{A}_g,\ [C] \mapsto [\mathrm{Jac}(C)],$$

where

(1) $\mathcal{A}_g$ is the moduli scheme of principally polarized abelian varieties of dimension g, with suitable level structure to insure the representability of the moduli functor;

(2) $\mathcal{M}_g$ is the moduli scheme of smooth projective curves of genus g, with similar constraints on the level structure induced from (1);

(3) j sends the isomorphism class of a curve $[C]$ to the isomorphism class of its Jacobian variety $[\mathrm{Jac}(C)]$, which is functorial because $\mathrm{Jac}(C)$ is nothing but the neutral component of the Picard scheme Pic°_C, and thus j is well-defined as a morphism between moduli schemes.

In this setting the Coleman conjecture amounts to the finiteness of CM points inside the schematic image $\mathrm{Im} j \subset \mathcal{A}_g$, where by CM points we mean points in $\mathcal{A}_g$ parametrizing abelian varieties with complex multiplication. It is well-known that $\mathrm{Im} j$ is a locally closed subscheme in $\mathcal{A}_g$ of dimension $3g-3$, and it is referred to as the open Torelli locus $\mathcal{T}_g^\circ$. Its closure is the Torelli locus $\mathcal{T}_g$. Since $\mathcal{T}_g^\circ$ is dense in $\mathcal{A}_g$ for $g = 2, 3$, it suffices to study Coleman's conjecture for $g \geq 4$.

The following equivalent form of Conjecture 1.1 is more convenient to work with:

CONJECTURE 1.2 (Coleman-Oort). – *When the integer g is sufficiently large, any special subvariety $S \subset \mathcal{A}_g$ of dimension > 0 is NOT* generically contained *in $\mathcal{T}_g$.*

Here a closed subvariety $S \subset \mathcal{A}_g$ is said to be *generically contained* in $\mathcal{T}_g$ if S is contained in $\mathcal{T}_g$ and that the intersection $S \cap \mathcal{T}_g^\circ$ is Zariski dense inside S; and special subvarieties in $\mathcal{A}_g$ are (geometrically) connected closed subvarieties parametrizing abelian varieties with "additional Hodge symmetry", see section 2 as well as [5] for details. Special subvarieties of dimension zero inside $\mathcal{A}_g$ are the same as CM points, and any special subvariety always contains a Zariski-dense subset of CM points. The equivalence of Conjecture 1.2 and Conjecture 1.1 is an immediate consequence of the following theorem, proved by Tsimerman in [37]:

THEOREM 1.3 (André-Oort conjecture for $\mathcal{A}_g$). – *Let Σ be an infinite subset of CM points in $\mathcal{A}_g$. Then the Zariski closure of Σ equals a finite union of special subvarieties.*

Roughly speaking, a special subvariety in $\mathcal{A}_g$ supports a universal family of abelian varieties with prescribed Hodge classes, and Conjecture 1.2 predicts that such a family cannot be deduced generically from a universal family of Jacobians (supported by a subvariety inside $\mathcal{T}_g^\circ$).

We refer to [26] (and the references therein) for a thorough discussion on the Coleman-Oort conjecture. The rich geometry of Shimura varieties has led to various results confirming the conjecture for many classes of special subvarieties, cf. [16, 19, 5] etc. Also, examples of special subvarieties generically contained in $\mathcal{T}_g$ are known up to genus 7, mainly constructed out of cyclic branched covers of $\mathbb{P}^1$, cf. [25].

1.2. Variant for superelliptic curves

Naturally one may formulate problems of Coleman-Oort type for moduli spaces of curves with additional data.

Definition 1.4. – For an integer $n > 1$, an n-superelliptic curve is a complex smooth projective algebraic curve of genus $g \geq 2$ (or simply superelliptic curve if n is clear from the text) which can be defined by an n-superelliptic equation, i.e., an affine equation of the form

$$y^n = \zeta(x),$$

with ζ a separable polynomial (i.e., admitting no multiple root). When $n = 2$ we obtain the usual notion of hyperelliptic curves. Note that the genus of such a curve can be computed explicitly in terms of n and $\deg \zeta$, cf. (3-4), and that for any fixed $g \geq 2$, there are finitely many possibilities of $(n, \deg \zeta)$ such that the curve defined above is of genus g.

Define $\mathcal{S}_{g,n}$ to be the moduli space of cyclic branched cover $C \to \mathbb{P}^1$ defined by an n-superelliptic equation as above, with C of fixed genus g. We have the evident morphism forgetting the cover

$$\mathcal{S}_{g,n} \to \mathcal{M}_g, \ \ (C \to \mathbb{P}^1) \mapsto C,$$

and we write $\mathcal{T}S_{g,n}^\circ$ for its image inside $\mathcal{A}_g$ under the Torelli morphism, referred to as the n-superelliptic open Torelli locus. Similar to the case of $\mathcal{T}_g^\circ$, it is locally closed in $\mathcal{A}_g$, and its closure $\mathcal{T}S_{g,n} = \overline{\mathcal{T}S_{g,n}^\circ}$ is called the n-superelliptic Torelli locus. Often the integer n is omitted when it is clear from the context. We can now state our main result:

Main Theorem. – *For $g \geq 8$, the superelliptic Torelli locus does not contain generically any special subvariety of $\mathcal{A}_g$ of positive dimension.*

Thanks to Theorem 1.3, our result is equivalent to the finiteness of superelliptic curves with CM Jacobians:

Corollary 1.5. – *For fixed genus $g \geq 8$, there exist, up to isomorphism, at most finitely many smooth superelliptic curves of genus g whose Jacobians are CM abelian varieties.*

Note that our result is sharp due to the counterexample with $g = 7$ given in [25] mentioned above. Precedent to our result, various cases of the superelliptic Coleman-Oort conjecture have been studied by Y. Zarhin in a series of works (see for example [40, 41], and the references therein), with emphasis on the endomorphism algebras of the Jacobians when the Galois