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DISCONTINUITY OF THE PHASE TRANSITION FOR THE PLANAR RANDOM-CLUSTER AND POTTS MODELS WITH $q > 4$

BY Hugo DUMINIL-COPIN, Maxime GAGNEBIN, Matan HAREL, Ioan MANOLESCU AND Vincent TASSION

ABSTRACT. – We prove that the q-state Potts model and the random-cluster model with cluster weight $q > 4$ undergo a discontinuous phase transition on the square lattice. More precisely, we show (1) Existence of multiple infinite-volume measures for the critical Potts and random-cluster models, (2) Ordering for the measures with monochromatic (resp. wired) boundary conditions for the critical Potts model (resp. random-cluster model), and (3) Exponential decay of correlations for the measure with free boundary conditions for both the critical Potts and random-cluster models. The proof is based on a rigorous computation of the Perron-Frobenius eigenvalues of the diagonal blocks of the transfer matrix of the six-vertex model, whose ratios are then related to the correlation length of the random-cluster model.

As a byproduct, we rigorously compute the correlation lengths of the critical random-cluster and Potts models, and show that they behave as $\exp(\pi^2/\sqrt{q-4})$ as q tends to 4.

RÉSUMÉ. – Nous démontrons que la transition de phase du modèle de Potts à q états et de la percolation FK avec $q > 4$ est du premier ordre. Plus précisément, nous montrons: (1) l'existence de plusieurs mesures en volume infini pour ces modèles au point critique, (2) l'émergence d'une structure ordonnée pour les mesures avec conditions au bord monochromatiques (resp. liées) pour le modèle de Potts critique (resp. pour la percolation FK), et (3) la décroissance exponentielle des corrélations pour les mesures libres des deux modèles au point critique. La preuve repose sur un calcul rigoureux des valeurs propres de Perron Frobenius associées aux blocs diagonaux de la matrice de transfert du modèle "six-vertex", qui peuvent être directement reliées à la longueur de corrélation de la percolation FK. Notamment, cette approche nous donne un calcul rigoureux des longueurs de corrélation critiques pour la percolation FK et le modèle de Potts au point critique. Nous en déduisons un comportement asymptotique de la forme $\exp(\pi^2/\sqrt{q-4})$ lorsque le paramètre q tend vers 4.

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1. Introduction

1.1. Motivation

Lattice spin models were introduced to describe specific experiments; they were later found to be illustrative of a large variety of physical phenomena. Depending on a parameter (most commonly temperature), they exhibit different macroscopic behaviors (also called phases), and phase transitions between them. Phase transitions may be continuous or discontinuous, and determining their type is one of the first steps towards a deeper understanding of the model.

In recent years, the Potts and random-cluster models have been the object of revived interest after new rigorous results were proved. In [3], the critical points of the models were determined for any $q \geq 1$. In [12], the models were proved to undergo a continuous phase transition for $1 \leq q \leq 4$, thus proving half of a famous prediction by Baxter. The object of this paper is to prove the second half of his prediction - namely, that the phase transition is discontinuous when $q > 4$.

1.2. Results for the Potts model

The Potts model was introduced by Potts [21] following a suggestion of his adviser Domb. While the model received little attention early on, it became the object of great interest in the last 50 years. Since then, mathematicians and physicists have been studying it intensively, and much is known about its rich behavior, especially in two dimensions. For a review of the physics results, see [24].

In this paper, we will focus on the case of the square lattice $\mathbb{Z}^2$ composed of vertices $x = (x_1, x_2) \in \mathbb{Z}^2$, and edges between nearest neighbors. In the q-state ferromagnetic Potts model (where q is a positive integer larger than or equal to 2), each vertex of a graph receives a *spin* taking value in $\{1, \dots, q\}$. The energy of a configuration is then proportional to the number of neighboring vertices of the graph having different spins. Formally, the Potts measure on a finite subgraph $G = (V, E)$ of the square lattice, at inverse temperature $\beta > 0$ and boundary conditions $i \in \{0, 1, \dots, q\}$, is defined for every $\sigma \in \{1, \dots, q\}^V$ by the formula

$$\mu^i_{G,\beta}[\sigma] := \frac{\exp[-\beta \mathbf{H}^i_G(\sigma)]}{\sum\limits_{\sigma' \in \{1,\dots,q\}^V} \exp[-\beta \mathbf{H}^i_G(\sigma')]}, \tag{1.1}$$

where

$$\mathbf{H}^i_G(\sigma) := -\sum_{\{x,y\} \in E} \mathbf{1}[\sigma_x = \sigma_y] - \sum_{x \in \partial V} \mathbf{1}[\sigma_x = i].$$

Above, $\mathbf{1}[\cdot]$ denotes the indicator function and ∂V is the set of vertices of G with at least one neighbor (in $\mathbb{Z}^2$) outside of G. Note that when $i = 0$, the second sum is zero for all σ.

For any boundary conditions i, the family of measures $\mu^i_{G,\beta}$ converges as G tends to the whole square lattice. The resulting measure μ^i_β defined on the square lattice is called the Gibbs measure with *free* boundary conditions if $i = 0$ (respectively, *monochromatic* boundary conditions equal to i if $i \in \{1, \dots, q\}$).

The Potts model undergoes an order/disorder phase transition, meaning that there exists a *critical inverse temperature* $\beta_c = \beta_c(q) \in (0, \infty))$ such that:

— For $\beta < \beta_c$, the measures μ^i_β, $i = 0, \dots, q$, are all equal.
— For $\beta > \beta_c$, the measures μ^i_β, $i = 0, \dots, q$, are all distinct.

Baxter [1] conjectured that the phase transition is continuous if $q \le 4$ and discontinuous if $q > 4$, meaning that all the measures $\mu^i_{\beta_c}$ with $i = 0, \dots, q$ are equal if and only if $q \le 4$. It was shown in [3] that $\beta_c = \log(1 + \sqrt{q})$; moreover, when $q \le 4$, it was proved in [12] that the phase transition is indeed continuous, along with more detailed properties of the unique critical measure μ_{β_c}. The goal of this article is to complete the proof of Baxter's conjecture by proving the following theorem. Below, x_n denotes the site of $\mathbb{Z}^2$ with both coordinates equal to $\lfloor n/2 \rfloor$.

THEOREM 1.1. – *Consider the q-state Potts model on the square lattice with $q > 4$. Then,*

1. *all the measures $\mu^i_{\beta_c}$ for $i = 0, \dots, q$ are distinct and ergodic (in particular, $\mu^0_{\beta_c}$ is not equal to the average of the $\mu^i_{\beta_c}$ with $i \in \{1, \dots, q\}$);*
2. *for any $i \in \{1, \dots, q\}$, $\mu^i_{\beta_c}[\sigma_0 = i] > \frac{1}{q}$.*
3. *Let $\lambda > 0$ satisfy $\cosh(\lambda) = \sqrt{q}/2$. Then*

$$\lim_{n \to \infty} -\tfrac{1}{n} \log\left(\mu^0_{\beta_c}[\sigma_0 = \sigma_{x_n}] - \tfrac{1}{q}\right) = \lambda + 2 \sum_{k=1}^{\infty} \tfrac{(-1)^k}{k} \tanh(k\lambda).$$

Furthermore, the quantity above is strictly positive.

The limit computed in the final item above is the inverse correlation length of the critical Potts model in the diagonal direction. This theorem follows directly from Theorem 1.2 below via the standard coupling between the Potts and random-cluster models (see Section 3.4 for details).

1.3. Results for the random-cluster model

The random-cluster model (also called Fortuin-Kasteleyn percolation) was introduced by Fortuin and Kasteleyn around 1970 (see [14] and [15]) as a class of models satisfying specific series and parallel laws. It is related to many other models of statistical mechanics, including the Potts model. For background on the random-cluster model and the results mentioned below, we direct the reader to the monographs [18] and [7].

Consider a finite subgraph $G = (V, E)$ of the square lattice. A percolation configuration ω is an element of $\{0, 1\}^E$. An edge e is said to be *open* (in ω) if $\omega(e) = 1$, otherwise it is *closed*. A configuration ω can be seen as a subgraph of G with vertex set V and edge-set $\{e \in E : \omega(e) = 1\}$. When speaking of connections in ω, we view ω as a graph. A *cluster* is a connected component of ω (it may just be an isolated vertex). Let $o(\omega)$ and $c(\omega)$ denote the number of open edges and closed edges in ω respectively. Let $k_0(\omega)$ denote the number of clusters of ω, and $k_1(\omega)$ the number of clusters of ω when all clusters intersecting ∂V are counted as a single one – as before, ∂V is the set of vertices of G adjacent to a vertex of $\mathbb{Z}^2$ not contained in G.

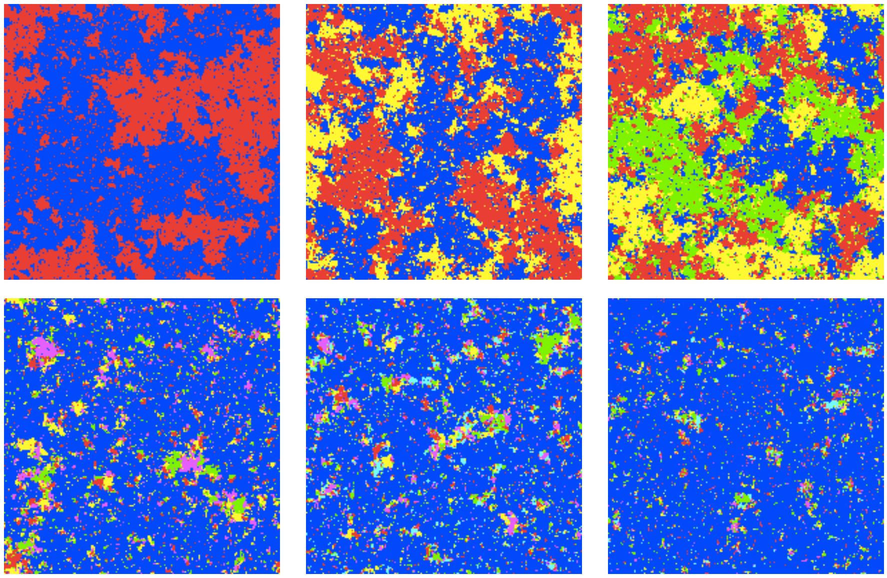

FIGURE 1. Simulations (courtesy of Vincent Beffara) of the critical planar Potts model $\mu^1_{\beta_c}$ (the spin 1 is depicted in blue) with q equal to 2, 3, 4, 5, 6 and 9 respectively. The behavior for $q \le 4$ is clearly different from the behavior for $q > 4$. In the first three pictures, each spin seems to play the same role, while in the last three, the blue spin dominates the other ones.

For $i \in \{0,1\}$, the random-cluster measure with parameters $p \in [0,1]$, $q > 0$ and boundary conditions i is given by

$$\phi^i_{G,p,q}(\omega) = \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k_i(\omega)}}{Z^i(G,p,q)},$$

where $Z^i(G,p,q)$ is a normalizing constant called the partition function. When $i = 0$ and $i = 1$, we speak of free and wired boundary conditions respectively.

The family of measures $\phi^i_{G,p,q}$ converges weakly as G tends to the whole square lattice. The limiting measures are denoted by $\phi^i_{\mathbb{Z}^2,p,q}$ and are called *infinite-volume* random-cluster measures with free and wired boundary conditions (for i equal to 0 and 1 respectively).

For $q \ge 1$, the random-cluster model undergoes a phase transition at the critical parameter $p_c = p_c(q) = \sqrt{q}/(1+\sqrt{q})$ (see [3] or [9, 10, 11] for alternative proofs), in the following sense:

— if $p > p_c(q)$, $\phi^0_{\mathbb{Z}^2,p,q} = \phi^1_{\mathbb{Z}^2,p,q}$ and the probability of having an infinite cluster in ω is 1.

— if $p < p_c(q)$, $\phi^0_{\mathbb{Z}^2,p,q} = \phi^1_{\mathbb{Z}^2,p,q}$ and the probability of having an infinite cluster in ω is 0.

As before, one may ask whether the phase transition is continuous or not; this comes down to whether there exists a single critical measure or multiple ones. In [12], it was proved that for $1 \le q \le 4$, $\phi^0_{\mathbb{Z}^2,p_c,q} = \phi^1_{\mathbb{Z}^2,p_c,q}$ and the probability of having an infinite cluster under this measure is 0. In this article, we complement this result by proving the following theorem.

Recall that, in this model, q is not necessarily an integer. Also recall that x_n is the site with both coordinates equal to $\lfloor n/2 \rfloor$.

THEOREM 1.2. – *Consider the random-cluster model on the square lattice with $q > 4$. Then*

1. $\phi^1_{\mathbb{Z}^2,p_c,q} \neq \phi^0_{\mathbb{Z}^2,p_c,q}$;
2. $\phi^1_{\mathbb{Z}^2,p_c,q}$[*there exists an infinite cluster*] $= 1$;
3. *if* $\lambda > 0$ *satisfies* $\cosh(\lambda) = \sqrt{q}/2$, *then*

$$\lim_{n\to\infty} -\tfrac{1}{n} \log \phi^0_{\mathbb{Z}^2,p_c,q}[0 \text{ and } x_n \text{ are in the same cluster}] = \lambda + 2\sum_{k=1}^{\infty} \tfrac{(-1)^k}{k} \tanh(k\lambda). \tag{1.2}$$

Furthermore, the quantity on the right-hand side is positive and as $q \searrow 4$,

$$\lambda + 2\sum_{k=1}^{\infty} \tfrac{(-1)^k}{k} \tanh(k\lambda) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\sinh\left(\frac{\pi^2(2k+1)}{2\lambda}\right)} \sim 8\exp\left(-\frac{\pi^2}{\sqrt{q-4}}\right). \tag{1.3}$$

As in the Potts model, the quantity on the left-hand side of (1.2) corresponds to the inverse correlation length in the diagonal direction. Note that it directly implies exponential tails for the radius of the cluster.

The proof of this theorem relies on the connection between the random-cluster model and the six-vertex model defined below. At the level of partition functions, this connection was made explicit by Temperley and Lieb in [23]. Here, we will further explore the connection to derive the inverse correlation length; see Section 3.3 for more details.

1.4. Results for the six-vertex model

The six-vertex model was initially proposed by Pauling in 1931 in order to study the thermodynamic properties of ice. While we are mainly interested in it for its connection to the previously discussed models, the six-vertex model is a major object of study on its own right. We do not attempt to give an overview of the six-vertex model here; instead, we refer to [22] and Chapter 8 of [2] (and references therein) for a bibliography on the subject and to the companion paper [8] for details specifically used below.

Fix two even numbers N and M, and consider the torus $\mathbb{T}_{N,M} := \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ as a graph with edge-set denoted $E(\mathbb{T}_{N,M})$. An *arrow configuration* $\vec{\omega}$ is a map attributing to each edge $e = \{x, y\} \in E(\mathbb{T}_{N,M})$ one of the two oriented edges (x, y) and (y, x). We say that an arrow configuration satisfies the *ice rule* if each vertex of $\mathbb{T}_{N,M}$ is incident to two edges pointing towards it (and therefore to two edges pointing outwards from it). The ice rule leaves six possible configurations at each vertex, depicted in Fig. 2, whence the name of the model. Each arrow configuration $\vec{\omega}$ receives a weight

$$w(\vec{\omega}) := \begin{cases} a^{n_1+n_2} \cdot b^{n_3+n_4} \cdot c^{n_5+n_6} & \text{if } \vec{\omega} \text{ satisfies the ice rule,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

where a, b, c are three positive numbers, and n_i denotes the number of vertices with configuration $i \in \{1, \dots, 6\}$ in $\vec{\omega}$. In this article, we will focus on the case $a = b = 1$ and $c > 2$, and will therefore only consider such weights from now on. This choice of parameters is such that the six-vertex model is related to the critical random-cluster model with cluster weight $q > 4$ on a tilted square lattice, as explained in Section 3.3.

Our choice of parameters corresponds to $\Delta := \frac{a^2+b^2-c^2}{2ab} < -1$, called the anti-ferroelectric phase. The regime $\Delta \in [-1,1)$, also called disordered, is also of interest and is related to the random-cluster model with $q \le 4$; see [2]. The regime $\Delta > 1$ (which requires $a \neq b$), called the ferroelectric phase, has also been studied under the name of stochastic six-vertex model and is related to interacting particle systems and random-matrix theory; see the recent paper [5] and references therein.

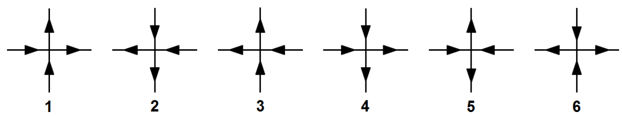

FIGURE 2. The 6 possibilities for vertices in the six-vertex model. Each possibility comes with a weight a, b or c.

In the context of this paper, the utility of the six-vertex model stems from its solvability using the transfer-matrix formalism. More precisely, the partition function of a toroidal six-vertex model may be expressed as the trace of the M-th power of a matrix V (depending on N) called the *transfer matrix*, which we define next. For more details, see [8].

Set $\vec{x} = (x_1, \dots, x_n)$ to be a set of ordered integers (called *entries*) $1 \le x_1 < \cdots < x_n \le N$ with $0 \le n \le N$. Let $\Omega = \{-1,1\}^{\otimes N}$ be the 2^N-dimensional real vector space spanned by the vectors $\Psi_{\vec{x}} \in \{\pm 1\}^N$ given by $\Psi_{\vec{x}}(i) = 1$ if $i \in \{x_1, \dots, x_n\}$, and -1 otherwise. The matrix V is defined by the formula

$$V(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = \begin{cases} 2 & \text{if } \Psi_{\vec{x}} = \Psi_{\vec{y}}, \\ c^{|\{i : \Psi_{\vec{x}}(i) \neq \Psi_{\vec{y}}(i)\}|} & \text{if } \Psi_{\vec{x}} \neq \Psi_{\vec{y}} \text{ and } \Psi_{\vec{x}} \text{ and } \Psi_{\vec{y}} \text{ are interlaced,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.5}$$

where $\vec{x}$ and $\vec{y}$ are *interlaced* if they have the same numbers of entries n and $x_1 \le y_1 \le x_2 \le \cdots \le x_n \le y_n$ or $y_1 \le x_1 \le y_2 \le \cdots \le y_n \le x_n$. It is immediate that V is a symmetric matrix; in particular, all its eigenvalues are real. Furthermore, it is made up of diagonal-blocks $V^{[n]}$ corresponding to its action on the vector spaces

$$\Omega_n := \mathrm{Vect}(\Psi_{\vec{x}} : \vec{x} \text{ has } n \text{ entries}) \qquad 0 \le n \le N.$$

As discussed in [8], each block $V^{[n]}$ satisfies the assumption of the Perron-Frobenius theorem [(1)], and thus has one dominant, positive, simple eigenvalue. For an integer $0 \le r \le N/2$, let $\Lambda_r(N)$ be the Perron-Frobenius eigenvalue of the block $V^{[N/2-r]}$, where we emphasize the dependence of Λ_r on N (recall that N is even). The main result dealing with the six-vertex model is the following asymptotic for the aforementioned eigenvalues.

[(1)] More precisely, the entries of $V^{[n]}$ are non-negative and there exists an integer k such that $(V^{[n]})^k$ has only positive entries. We will henceforth call a matrix with these properties a Perron-Frobenius matrix.

THEOREM 1.3. – *For* $c > 2$ *and* $r > 0$ *integer, fix* $\lambda > 0$ *to satisfy* $\cosh(\lambda) = \frac{c^2-2}{2}$. *Then,*

$$\lim_{\substack{N\to\infty \\ N\in 4\mathbb{N}}} \frac{1}{N} \log \Lambda_0(N) = \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{e^{-k\lambda}\tanh(k\lambda)}{k} \tag{1.6}$$

$$\lim_{\substack{N\to\infty \\ N\in 4\mathbb{N}}} \frac{\Lambda_r(N)}{\Lambda_0(N)} = \exp\Big[-r\Big(\lambda + 2\sum_{k=1}^{\infty} \tfrac{(-1)^k}{k}\tanh(k\lambda)\Big)\Big]. \tag{1.7}$$

The limit of $\frac{\Lambda_1(N)}{\Lambda_0(N)}$ is sometimes interpreted as twice the *surface tension* of the six-vertex model, and the second equation is effectively a computation of this quantity. The limit of $\frac{\Lambda_r(N)}{\Lambda_0(N)}$ does not have an immediate interpretation but will come in useful when transferring the result to the random-cluster model (see Remark 3.18). The first identity may be reformulated in terms of the free energy, which defines the asymptotic behavior of the partition function, as described below.

COROLLARY 1.4. – *Fix* $c > 2$ *and* $\lambda > 0$ *such that* $\cosh(\lambda) = \frac{c^2-2}{2}$. *Then the free-energy* $f(1,1,c)$ *of the six-vertex model satisfies*

$$f(1,1,c) := \lim_{N,M\to\infty} \frac{1}{NM} \log\Big(\sum_{\vec{\omega}\ on\ \mathbb{T}_{N,M}} w(\vec{\omega})\Big) = \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{e^{-k\lambda}\tanh(k\lambda)}{k}.$$

The previous corollary follows trivially from Theorem 1.3 once observed that the free energy does exist, and that the leading eigenvalue of V is the Perron-Frobenius eigenvalue of $V^{[N/2]}$ (see Section 3.2 for details).

Theorem 1.3 above will be obtained by applying the coordinate Bethe Ansatz to the blocks $V^{[n]}$ of the transfer matrix. This Ansatz, aimed at finding eigenvalues of certain types of matrices, was introduced by Bethe [4] in 1931 for the Hamiltonian of the XXZ model. It has since been widely studied and developed, with applications in various circumstances, such as the one at hand. Its formulation for the six-vertex model is described in detail in [8]. For completeness, let us briefly discuss this technique again.

The idea is to try to express the eigenvalues of $V_N^{[n]}$ as explicit functions (see Theorem 3.1 below) of an n-uplet $\mathbf{p} = (p_1, \dots, p_n) \in (-\pi,\pi)^n$ satisfying the n equations

$$Np_j = 2\pi I_j - \sum_{k=1}^{n} \Theta(p_j, p_k) \qquad \forall j \in \{1,\dots,n\}, \tag{BE$_\Delta$}$$

where the I_j are integers or half-integers (depending on whether n is odd or even) between $-N/2$ and $N/2$, and $\Theta : \mathbb{R}^2 \to \mathbb{R}$ is the unique continuous function [(2)] satisfying $\Theta(0,0) = 0$ and

$$\exp(-i\Theta(x,y)) = e^{i(x-y)} \cdot \frac{e^{-ix} + e^{iy} - 2\Delta}{e^{ix} + e^{-iy} - 2\Delta}, \tag{1.8}$$

where recall that $\Delta = (2-c^2)/2$. This parameterization of the six-vertex model will be used throughout the paper. We refer to BE_Δ as the Bethe equations. Depending on the choice of the I_j, the eigenvalue obtained may be different. It is also a priori unclear whether all eigenvalues of $V_N^{[n]}$ can be obtained via this procedure.

(2) The fact that Θ is well-defined, real-valued and analytic can be checked easily.

The asymptotic behavior of $\Lambda_0(N)$ was computed in [25] using the coordinate Bethe Ansatz. The argument of [25] assumed that $\Lambda_0(N)$ is produced by a solution $\mathbf{p}(N) = (p_1, \dots, p_{N/2})$ to (BE_Δ) with $n = N/2$ and the special choice $I_j = j - (n+1)/2$. An asymptotic analysis of the distribution of $p_1, \dots, p_{N/2}$ on $[-\pi, \pi]$ was then used to derive the asymptotic behavior of $\Lambda_0(N)$. To our best understanding, certain gaps prevent this derivation from being completely justified in this first paper. Among them are the existence of solutions to (BE_Δ), the fact that the associated eigenvector constructed by the Bethe Ansatz is non-zero, and the justification of the weak convergence of the point measure of $\mathbf{p}$ to an explicit continuous distribution.

The more refined asymptotic (1.7) requires further justification. For $r = 1$ (or equivalently -1), the limit was derived in [2] and [6]. Baxter's result [2] is based on computations involving a more sophisticated version of the Bethe Ansatz and the eight-vertex model, which generalizes the six-vertex model. The paper [6] relies on completeness of the six-vertex and Potts representations of the Bethe Ansatz. To our best understanding, both computations require assumptions which are difficult to rigorously justify. We are not aware of any computation of (1.7) for $|r| \geq 2$. Similar results were obtained rigorously in [20, 17] for related models (see the discussion before Theorem 2.3 and Remark 3.5 for more details).

In light of this, we chose to write a fully rigorous, self-contained derivation of both (1.6) (which matches Baxter's computation) and (1.7). Moreover, we only use elementary tools, so as to render it accessible to a more diverse audience, less accustomed to the mathematical physics literature. The computations of the two limits in Theorem 1.3 will be used in a crucial way in the proof of Theorem 1.2.

1.5. Organization of the paper

1.5.1. *Section 2: Study of the Bethe equations.* – This step consists in the study of (BE_Δ) with the choice

$$I_j := j - \frac{n+1}{2} \qquad \text{for } j \in \{1, \dots, n\}. \tag{1.9}$$

This section does not involve any reference to the Bethe Ansatz or the six-vertex model. It is divided in three steps:

1. We first study two functional equations that we call the *continuous Bethe Equation* and the *continuous Offset Equation*, respectively, via Fourier analysis.
2. We then construct solutions to (BE_Δ) with prescribed properties. This approach proves the existence of solutions to the Bethe equations and, more importantly, provides good control of the increments $p_{j+1} - p_j$ of the solution. This will be crucial when analyzing the asymptotic of $\Lambda_r(N)/\Lambda_0(N)$. It also provides tools for proving that the eigenvectors built via the Bethe Ansatz are non-zero and correspond to Perron-Frobenius eigenvalues.
3. Finally, we study the asymptotic behavior of the solutions of the discrete Bethe equations using the continuous Bethe Equation. Furthermore, we compare solutions with different values of n using the continuous Offset Equation.

1.5.2. *Section 3: From the Bethe equations to the different models.* – This part contains the proofs of the main theorems. It is divided in two steps.

1. We use the Bethe Ansatz to relate the Bethe equations to the eigenvalues of the transfer matrix of the six-vertex model. We then study the asymptotic behavior of the Perron-Frobenius eigenvalues of the different blocks of the transfer matrix using the asymptotic behavior of the solutions to the continuous Bethe Equation derived in the previous section (see the proof of Theorem 1.3).
2. We relate the six-vertex model to the random-cluster and Potts models via classical couplings. These relations, together with new results on the random-cluster model, enable us to prove Theorems 1.2 and 1.1.

1.5.3. *Section 4: Fourier computations.* – The study will require certain computations using Fourier decompositions. While these computations are elementary, they may be lengthy, and would break the pace of the proofs. We therefore defer all of them to Section 4.

1.5.4. *Notation.* – Most functions hereafter depend on the parameter $\Delta = \frac{2-c^2}{2} < -1$. For ease of notation, we will generally drop the dependency in Δ, and recall it only when it is relevant. We write ∂_i for the partial derivative in the i^{th} coordinate.

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2. Study of the Bethe Equation

2.1. The continuous Bethe and Offset Equations

This section studies the following continuous functional equations for $\Delta < -1$:

$$(\mathrm{cBE}_\Delta) \qquad 2\pi\rho(x) = 1 + \int_{-\pi}^{\pi} \partial_1\Theta(x,y)\rho(y)dy \qquad \forall x \in [-\pi,\pi],$$

$$(\mathrm{cOE}_\Delta) \qquad 2\pi\tau(x) = \frac{\Theta(x,-\pi)+\Theta(x,\pi)}{2} - \int_{-\pi}^{\pi} \partial_2\Theta(x,y)\tau(y)dy \qquad \forall x \in [-\pi,\pi].$$

The first equation naturally arises as a continuous version of the Bethe Equations (BE_Δ), while the second one will be useful when studying the displacement between solutions of the Bethe equations for different values of n.

The main object of the section is the following proposition. For $\Delta < -1$, let k be the unique continuous function [(3)] from $[-\pi,\pi]$ to itself satisfying

$$e^{ik(\alpha)} = \frac{e^\lambda - e^{-i\alpha}}{e^{\lambda - i\alpha} - 1},$$

where $\lambda > 0$ is such that $\cosh(\lambda) = -\Delta$.

(3) The existence of k follows by taking the complex logarithm and fixing $k(\pm\pi) = \pm\pi$. Furthermore, k is invertible. Also notice that $z \mapsto (e^\lambda - z)/(e^\lambda z - 1)$ is a Mobiüs transformation mapping the unit circle to itself and -1 to -1.

PROPOSITION 2.1. – *For $\Delta < -1$, let $x = k(\alpha)$. The functions ρ and τ defined by* [(4)]

$$\rho(x) := \frac{1}{4\lambda k'(\alpha)} \sum_{j \in \mathbb{Z}} \frac{1}{\cosh[\pi(2\pi j + \alpha)/(2\lambda)]}, \tag{2.1}$$

$$\tau(x) := \sum_{m>0} \frac{(-1)^m}{\pi m} \tanh(\lambda m) \sin(m\alpha)$$

are the only solutions in $L^2([-\pi, \pi])$ of (cBE_Δ) *and* (cOE_Δ) *respectively. In particular, the function $\rho : (\Delta, x) \mapsto \rho(x)$ is strictly positive and analytic in $\Delta < -1$ and $x \in [-\pi, \pi]$.*

We prove this result by making a change of variables $x = k(\alpha)$ to obtain equations involving a convolution operator, and then using Fourier analysis to compute ρ and τ (and therefore deduce their uniqueness).

Proof. – In this proof, we fix $\Delta < -1$ and drop it from the notation. Set $R(\alpha) = 2\pi\rho(k(\alpha))k'(\alpha)$ and $T(\alpha) = 2\pi\tau(k(\alpha))$. The change of variables $x = k(\alpha)$ transforms (cBE_Δ) and (cOE_Δ) into [(5)]

$$R(\alpha) = \Xi_\lambda(\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \Xi_{2\lambda}(\alpha - \beta) R(\beta) d\beta \qquad \forall \alpha \in [-\pi, \pi], \tag{cBE'_Δ}$$

$$T(\alpha) = \Psi(\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \Xi_{2\lambda}(\alpha - \beta) T(\beta) d\beta \qquad \forall \alpha \in [-\pi, \pi], \tag{cOE'_Δ}$$

where, for $\mu \in \mathbb{R}$ and $\alpha \in [-\pi, \pi]$,

$$\Xi_\mu(\alpha) := \frac{\sinh(\mu)}{\cosh(\mu) - \cos(\alpha)} \qquad \text{and} \qquad \Psi(\alpha) := \frac{\Theta(k(\alpha), -\pi) + \Theta(k(\alpha), \pi)}{2}.$$

For any function $f \in L^2([-\pi, \pi])$, denote by $(\hat{f}(m))_{m \in \mathbb{Z}}$ its Fourier coefficients defined as $\hat{f}(m) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-im\alpha} f(\alpha) d\alpha$. Then, ($\mathrm{cBE}'_\Delta$) and ($\mathrm{cOE}'_\Delta$) may be rewritten as

$$\hat{R}(m) = \hat{\Xi}_\lambda(m) - \hat{\Xi}_{2\lambda}(m)\hat{R}(m) \qquad \text{and} \qquad \hat{T}(m) = \hat{\Psi}(m) - \hat{\Xi}_{2\lambda}(m)\hat{T}(m) \qquad \forall m \in \mathbb{Z}. \tag{2.2}$$

The end of the proof is a simple computation which we resume next; details are given in Section 4. The residue theorem shows that $\hat{\Xi}_\mu(m) = \exp(-\mu|m|)$. In addition, a simple computation implies that $\hat{\Psi}(m) = \frac{(-1)^m}{im}\left(1 - \hat{\Xi}_{2\lambda}(m)\right)$ for $m \neq 0$ and $\hat{\Psi}(0) = 0$. Substituting these in (2.2), we deduce that, for all $m \in \mathbb{Z}$ ($m \neq 0$ for the second equality),

$$\hat{R}(m) = \frac{\hat{\Xi}_\lambda(m)}{1 + \hat{\Xi}_{2\lambda}(m)} = \frac{1}{2\cosh(\lambda m)} \quad \text{and} \quad \hat{T}(m) = \frac{\hat{\Psi}(m)}{1 + \hat{\Xi}_{2\lambda}(m)} = \frac{(-1)^m}{im} \tanh(\lambda|m|).$$

The conclusion follows by checking that functions given in (2.1) have the Fourier coefficients above; details are given in Section 4. The properties of positivity and analyticity of ρ follow directly from its explicit expression (observe that the terms of the sum in (2.1) are positive and converge exponentially fast to 0). □

(4) In the definition of τ, the series is not absolutely summable; it stands for the limit of the partial sums.
(5) A series of algebraic manipulations is necessary for this step. The key is to observe that $k'(\alpha) = \Xi_\lambda(\alpha)$ and $\Xi_{2\lambda}(\alpha - \beta) = -\frac{d}{d\alpha}\Theta(k(\alpha), k(\beta)) = \frac{d}{d\beta}\Theta(k(\alpha), k(\beta))$.

Before turning to the discrete equations, let us provide an alternative proof of the uniqueness of the solution to (cBE$_\Delta$) based on a fixed-point theorem. While this proof does not give an explicit formula for ρ (a formula which will be useful later on), it highlights the importance of a particular norm which will play a central role in the next section. The goal is to prove that the map T_c defined below is contractive, a fact which immediately implies that (cBE$_\Delta$) has a unique solution.

Fix $\Delta < -1$. Consider the map T_c from the set $\mathscr{H}$ of bounded functions $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ with $\int_{-\pi}^{\pi} f(x)dx = 1/2$ to itself[(6)] defined by

$$2\pi \mathsf{T}_c(f)(x) = 1 + \int_{-\pi}^{\pi} \partial_1 \Theta(x, y) f(y) dy \qquad \forall x \in [-\pi, \pi].$$

We claim that this map is contractive for the norm defined by

(2.3)
$$\|f\| := \sup\left\{ \left|k'(k^{-1}(x)) f(x)\right|, \, x \in [-\pi, \pi]\right\} = \sup\left\{ \left|k'(\alpha) f(k(\alpha))\right|, \, \alpha \in [-\pi, \pi]\right\}.$$

Note that k' is bounded away from 0 and infinity, so that the norm above is equivalent to the supremum norm $\|\cdot\|_\infty$ (with constants depending on $\Delta < -1$).

Indeed, let f and g be two functions in $\mathscr{H}$. Set $F = k' \cdot (f \circ k)$, $G = k' \cdot (g \circ k)$, $\tilde{F} = k' \cdot (\mathsf{T}_c(f) \circ k)$ and $\tilde{G} = k' \cdot (\mathsf{T}_c(g) \circ k)$, and notice that all these functions integrate to $1/2$ on $[-\pi, \pi]$. Letting $m_\Xi = \min\{\Xi_{2\lambda}(x) : x \in [-\pi, \pi]\} > 0$, we find that, for any $\alpha \in [-\pi, \pi]$,

$$\begin{aligned}
|\tilde{F}(\alpha) - \tilde{G}(\alpha)| &= \tfrac{1}{2\pi} \Big| \int_{-\pi}^{\pi} \Xi_{2\lambda}(\alpha - \beta)(F(\beta) - G(\beta)) d\beta \Big| \\
&= \tfrac{1}{2\pi} \Big| \int_{-\pi}^{\pi} \big(\Xi_{2\lambda}(\alpha - \beta) - m_\Xi\big)(F(\beta) - G(\beta)) d\beta \Big| \\
&\leq \tfrac{1}{2\pi} \|F - G\|_\infty \int_{-\pi}^{\pi} \big(\Xi_{2\lambda}(\beta) - m_\Xi\big) d\beta \\
&\leq (1 - m_\Xi) \|F - G\|_\infty, \qquad (2.4)
\end{aligned}$$

where we used the fact that $\Xi_{2\lambda}(\beta)$ integrates to 2π (since $\hat{\Xi}_{2\lambda}(0) = 1$) in the final line. Observing that $\|F - G\|_\infty = \|f - g\|$ and $\|\tilde{F} - \tilde{G}\|_\infty = \|\mathsf{T}_c(f) - \mathsf{T}_c(g)\|$, we conclude that T_c is contracting.

2.2. The discrete Bethe equations

The main object of this section is to prove the existence and regularity of solutions to the Bethe equations recalled below:

$$(\mathrm{BE}_\Delta) \qquad N p_j = 2\pi I_j - \sum_{k=1}^{n} \Theta(p_j, p_k), \qquad \forall j \in \{1, \ldots, n\},$$

with the choice (1.9) for the I_j, namely $I_j = j - \frac{n+1}{2}$ for $1 \leq j \leq n$. We will be looking for solutions $\mathbf{p}$ with additional symmetry (which takes into account the symmetry of the I_j). More precisely, we will be looking for solutions in

$$\mathscr{S}_n := \Big\{ \mathbf{p} = (p_1, \ldots, p_n) \, : -\pi < p_1 < p_2 < \cdots < p_n < \pi \text{ and } p_{n+1-j} = -p_j, \forall j \Big\}.$$

(6) That $\mathsf{T}_c(\mathscr{H}) \subset \mathscr{H}$ follows from Fubini's theorem and the fact that $\Theta(\pi, y) - \Theta(-\pi, y) = -2\pi$ for all $y \in [-\pi, \pi]$.

For any vector $\mathbf{p} = (p_1, \dots, p_n)$, we set $p_0 = p_n - 2\pi$ and $p_{n+1} = p_1 + 2\pi$. Hereafter, N will always denote an even integer.

Maybe the most natural approach to proving the existence of solutions to (BE_Δ) (for fixed Δ, N and n) is to apply the Brouwer Fixed-Point Theorem [(7)] to the map $\mathsf{T} : \mathcal{S}_n \to \mathcal{S}_n$ defined by

$$\mathsf{T}(p_1, \dots, p_n) = \Big(\frac{2\pi I_j}{N} - \frac{1}{N}\sum_{k=1}^{n} \Theta(p_j, p_k)\Big)_{1 \le j \le n}.$$

Indeed, $\mathbf{p}$ being a fixed point of T is equivalent to it satisfying (BE_Δ). The fact that T maps $\mathcal{S}_n$ to itself follows directly from the monotonicity and anti-symmetry of Θ, and from the fact that

$$-2\pi \le \Theta(x, y) + \Theta(x, -y) \le 2\pi \qquad \forall x, y \in [-\pi, \pi].$$

The Brouwer Fixed-Point Theorem indeed applies to T, and solutions to (BE_Δ) may thus be shown to exist for any $\Delta < -1$. Having said that, it will be important that the solutions vary continuously as functions of Δ, which does not follow from such arguments. Such a continuity statement was proved by Karol Kozlowksi in [20] and Pedro Goldbaum in [17] for the 1D Hubbard model. The argument used in the latter paper generalizes the earlier work of Yang and Yang [25], using an Index theorem on a well-chosen field, and thus deducing that the solutions form families of continuous curves, proving that there exists a continuous curve of solutions to (BE_Δ) in the set $[-\infty, -1) \times [-\pi, \pi]^n$, extending over the whole range of Δ.

However, we wish to prove a stronger statement: we would like the solutions to have some regularity, in that they should be close to ρ, the solution we explicitly computed in (2.1), in some appropriately-chosen sense. This will be important when comparing solutions for different values n to compute the limit of $\Lambda_r(N)/\Lambda_0(N)$.

We therefore choose another path to prove the existence of solutions, based on the Implicit Function Theorem. Our approach has the further advantage of being fairly short and elementary, and of proving that the obtained solution is close to the continuous one (which renders the asymptotic analysis of $\Lambda_0(N)$ essentially trivial). Furthermore, we will also prove that the map $\Delta \mapsto \mathbf{p}_\Delta$ is not only continuous but analytic, a fact which will be useful in proving that the eigenvalue associated with $\mathbf{p}_\Delta$ is the Perron-Frobenius one (see Section 3.1). The downside is that it only yields a solution on an interval $[-\infty, \Delta_N]$ with $\Delta_N < -1$, tending to -1 as N tends to infinity (which will be sufficient for the application we have in mind).

Before stating the theorem, let us explain how we will compare a solution $\mathbf{p}$ of (BE_Δ) to the continuous solution ρ of (cBE_Δ). For $\mathbf{p} \in \mathcal{S}_n$, introduce the step function $\rho_{\mathbf{p}} : [-\pi, \pi] \to \mathbb{R}$ defined by

$$\rho_{\mathbf{p}}(t) = \frac{I_{j+1} - I_j}{N(p_{j+1} - p_j)} \qquad \text{if } t \in [p_j, p_{j+1}), \tag{2.5}$$

(7) For sufficiently small values of Δ, the Brouwer Fixed-Point Theorem is not even necessary, since one may show that T is contractive for the ℓ^1 norm; this is not true for Δ close to -1.

where I_{n+1} and I_0 are defined by $I_{n+1} - I_1 = I_n - I_0 = N - n$. We measure the distance from $\mathbf{p}$ to the continuous solution using $\|\rho_{\mathbf{p}} - \rho\|$, where $\|\cdot\|$ is the norm introduced in (2.3). This norm appears naturally in this context since the map T_c – which may be viewed as a continuous version of T – is contractive for $\|\cdot\|$.

REMARK 2.2. – We chose to write $I_{j+1} - I_j$ in the numerators, since this would be the natural quantity would the I_j take arbitrary values. In our case, $I_{j+1} - I_j$ is equal to 1 for any $1 \le j < n$, and to $2r + 1$ for $j = 0$ and n (recall that $n = N/2 - r$).

We are now in a position to state the main theorem of this section.

THEOREM 2.3. – *Fix $r \ge 0$ and $\Delta_0 < -1$. There exist $K > 0$ and N_0 such that, for any $N \ge N_0$, there exists a family of solutions $(\mathbf{p}_\Delta)_{\Delta \le \Delta_0}$ to the Bethe Equations* (BE_Δ) *with $n = N/2 - r$ satisfying*

(i) $\Delta \mapsto \mathbf{p}_\Delta$ *is analytic on* $[-\infty, \Delta_0)$ [8],

(ii) $\|\rho_{\mathbf{p}_\Delta} - \rho\| \le \frac{K}{N}$ *for all* $\Delta \le \Delta_0$.

Property (ii) should be understood as a regularity statement. It implies in particular that, for all $0 \le j \le n$,

$$p_{j+1} - p_j - \frac{I_{j+1} - I_j}{\rho(p_j)N} = O\Big(\frac{1}{N^2}\Big), \tag{2.6}$$

where $O(\cdot)$ depends on Δ_0 only [9]. As an important consequence for us, the previous expression implies that, for $N > N_0$ large enough and $0 \le j \le n$,

$$p_{j+1} - p_j \le \frac{2(I_{j+1} - I_j)}{m_\rho N}, \tag{2.7}$$

where $m_\rho > 0$ is the infimum of ρ over $x \in [-\pi, \pi]$ and $\Delta \le \Delta_0$. It will be crucial to us that the bound (2.7) above does not depend on the quantity K of Theorem 2.3 (even though N_0 may depend on K). Also notice that (ii) implicitly shows that $\mathbf{p}_\Delta$ is in the interior of $\mathcal{S}_n$ for all $\Delta \le \Delta_0$, provided N is large enough.

The rest of this section is dedicated to proving Theorem 2.3.

As we already mentioned, our strategy is based on the Implicit Function Theorem, which will be applied to $\mathbb{I} - \mathsf{T}$ (seen as a function of Δ and $\mathbf{p}$), where $\mathbb{I}$ denotes the identity function:

$$\mathbb{I}(\Delta, \mathbf{p}) = \mathbf{p}, \qquad \forall \mathbf{p} \in \mathcal{S}_n \text{ and } \Delta < -1.$$

There is no a priori reason that allows us to apply the Implicit Function Theorem at any zero of $\mathbb{I} - \mathsf{T}$, as the differential is not guaranteed to be invertible. Nonetheless, we will show that we may construct a family of such zeros that remains close to the continuous solution, and that this ensures that the differential of $\mathbb{I} - \mathsf{T}$ is invertible. The key of this argument is the following stability lemma.

[8] Here, f is analytic at $-\infty$ if there exists (a_n) such that $f(x) = \sum_{n=0}^{\infty} a_n x^{-n}$ for all x sufficiently small.
[9] The notation $O(N^\alpha)$ is used to indicate a quantity that is bounded by CN^α for all N, where C is a constant. We say $O(\cdot)$ is uniform in certain parameters, to mean that C may be chosen independently of those parameters.

LEMMA 2.4. – *Fix $r \geq 0$ and $\Delta_0 < -1$. Then, for $K > 0$ and N_0 large enough, for any $\Delta \leq \Delta_0$ and $N \geq N_0$, there exists no solution $\mathbf{p} \in \mathcal{S}_n$ of* (BE$_\Delta$) *with $n = N/2 - r$ and*

$$\frac{K}{2N} \leq \|\rho_{\mathbf{p}} - \rho\| \leq \frac{K}{N}.$$

This lemma should not appear as a surprise. Indeed, as mentioned above, T is, in some sense, a discrete version of T_c (which is contractive and has fixed point ρ), at least in a vicinity of ρ, and could therefore be expected to be contractive for N large enough. We did not manage to prove this fact, but the lemma above is sufficient for our use.

Let us assume this lemma for now. Write $\mathbb{R}^n_{\text{sym}}$ for the $\lfloor n/2 \rfloor$-dimensional subspace of $\mathbb{R}^n$ of symmetric vectors:

$$\mathbb{R}^n_{\text{sym}} = \{(q_1, \dots, q_n) \in \mathbb{R}^n : q_j = -q_{n+1-j}, \forall j\}.$$

The map $\mathbb{I} - \mathsf{T}$ leaves this space stable (as can be seen by the symmetry properties of Θ). Therefore, we may apply the Implicit Function Theorem to $\mathbb{I} - \mathsf{T}$ as a function from $[-\infty, \Delta_0] \times \mathcal{S}_n$ to $\mathbb{R}^n_{\text{sym}}$ (recall that $\mathcal{S}_n \subset \mathbb{R}^n_{\text{sym}}$). Write $d(\mathbb{I} - \mathsf{T})$ for (the restriction of) the differential of $\mathbb{I} - \mathsf{T}$ in $\mathbf{p}$ as an automorphism of $\mathbb{R}^n_{\text{sym}}$. To apply the Implicit Function Theorem at some point $(\Delta, \mathbf{p})$ one needs to ensure that $d(\mathbb{I} - \mathsf{T})(\Delta, \mathbf{p})$ is invertible. This is done via the lemma below; its proof is deferred to the end of the section.

LEMMA 2.5. – *Fix $r \geq 0$, $\Delta_0 < 1$ and $K > 0$. Then there exists N_0 such that, for any $\Delta \leq \Delta_0$ and $N \geq N_0$, $d(\mathbb{I} - \mathsf{T})(\Delta, \mathbf{p})$ is invertible for any solution $\mathbf{p} \in \mathcal{S}_n$ of* (BE$_\Delta$) *with $n = N/2 - r$ such that $\|\rho_{\mathbf{p}} - \rho\| \leq K/N$.*

Theorem 2.3. – Fix $r \geq 0$ and $\Delta_0 < -1$; $K \geq 2r$ and N will be assumed large enough for Lemmas 2.4 and 2.5 to apply, further conditions on N will appear in the proof.

For $\Delta = -\infty$ we have $\Theta(x, y) = y - x$, and the Bethe equations have a unique solution $\mathbf{p}_{-\infty}$ with $p_j = 2\pi I_j/(N - n)$ for $1 \leq j \leq n$. This solution satisfies $\mathbf{p} \in \mathcal{S}_n$ and $\|\rho_{\mathbf{p}} - \rho\| \leq K/N$ (ρ is the constant function $1/(4\pi)$ when $\Delta = -\infty$ and we assumed $K \geq 2r$).

Due to Lemma 2.5, the Implicit Function Theorem may be repeatedly applied to extend the solution from $\mathbf{p}_{-\infty}$ to an analytic function $\Delta \mapsto \mathbf{p}_\Delta$, as long as $\|\rho_{\mathbf{p}_\Delta} - \rho\| \leq K/N$ and $\mathbf{p}_\Delta \in \mathcal{S}_n$. The latter condition is implied by the former when N is large enough; we may therefore ignore it. Lemma 2.4 shows that $\mathbf{p}_\Delta$, being continuous in Δ, may never exit the ball of radius $K/(2N)$ around ρ for the $\|\cdot\|$-norm. Thus, the map $\Delta \mapsto \mathbf{p}_\Delta$ is defined for all $\Delta \leq \Delta_0$, analytic and such that $\|\rho_{\mathbf{p}_\Delta} - \rho\| \leq K/N$ for all $\Delta \leq \Delta_0$. □

To close the section, we prove Lemmas 2.4 and 2.5.

Lemma 2.4. – Let $r \geq 0$ and $\Delta_0 < -1$; bounds on K and N_0 will appear throughout the proof. Consider $\Delta \leq \Delta_0$, $N \geq N_0$ and $\mathbf{p} = (p_1, \dots, p_n)$ a solution of (BE$_\Delta$) with $n = N/2 - r$ and $\|\rho_{\mathbf{p}} - \rho\| \leq K/N$.

In this proof, $O(\cdot)$ is uniform in K and $j = 1, \dots, n$ (but may depend on r). In particular, by (2.7), we may write that $p_{j+1} - p_j = O(1/N)$, provided N_0 is large enough. For further reference, note that the derivatives of the functions ρ, k, Θ, etc, are all bounded uniformly in $\Delta < \Delta_0$.

Let $f_{\mathbf{p}} : \mathbb{R} \to \mathbb{R}$ be the smooth function defined by

$$f_{\mathbf{p}}(x) := \frac{1}{2\pi}\Big(x + \frac{1}{N}\sum_{k=1}^{n} \Theta(x, p_k)\Big). \tag{2.8}$$

For any $t \in [p_j, p_{j+1})$, apply the Mean Value Theorem to construct $\xi_j \in (p_j, p_{j+1})$ such that

$$\rho_{\mathbf{p}}(t) \overset{(2.5)}{=} \frac{I_{j+1} - I_j}{N(p_{j+1} - p_j)} = \frac{f_{\mathbf{p}}(p_{j+1}) - f_{\mathbf{p}}(p_j)}{p_{j+1} - p_j} = f'_{\mathbf{p}}(\xi_j). \tag{2.9}$$

In the second identity, we used that $\mathbf{p}$ is a fixed point for T and therefore satisfies $f_{\mathbf{p}}(p_j) = I_j/N$ for any $j \in \{1, \ldots, n-1\}$. In fact, this relation also holds for $j = 0$ and n; to see this, we recall that $\Theta(x + 2\pi, y) - \Theta(x, y) = -2\pi$, and therefore

$$f_{\mathbf{p}}(x \pm 2\pi) = f_{\mathbf{p}}(x) \pm \frac{N-n}{N}.$$

Thus,

$$f(p_1) - f(p_0) = \frac{I_1 - (I_n - N + n)}{N} = \frac{I_1 - I_0}{N}.$$

The argument is identical for $j = n$. Since $p_{j+1} - p_j = O(1/N)$, for any $t \in [p_j, p_{j+1})$, we may approximate $\rho(t)$ by $\rho(\xi_j)$ and $k'(k^{-1}(t))$ by $k'(k^{-1}(\xi_j))$ to deduce that

$$\begin{aligned} k'(k^{-1}(t))|\rho_{\mathbf{p}}(t) - \rho(t)| &\le (1 + O(\tfrac{1}{N}))\, k'(k^{-1}(\xi_j))|f'_{\mathbf{p}}(\xi_j) - \rho(\xi_j)| + O(\tfrac{1}{N}) \\ &\le (1 + O(\tfrac{1}{N}))\|f'_{\mathbf{p}} - \rho\| + O(\tfrac{1}{N}). \end{aligned}$$

Therefore, the lemma follows readily from the following inequality, which we prove below:

$$\|f'_{\mathbf{p}} - \rho\| \le (1 - m_\Xi)\|\rho_{\mathbf{p}} - \rho\| + O(\tfrac{1}{N}), \tag{2.10}$$

where $m_\Xi = \inf\{\Xi_{2\lambda}(x) : x \in [-\pi, \pi] \text{ and } \Delta \le \Delta_0\} > 0$. Indeed, assuming (2.10) holds, the previous computation shows that

$$\|\rho_{\mathbf{p}} - \rho\| \le (1 - m_\Xi)\|\rho_{\mathbf{p}} - \rho\| + O(\tfrac{1}{N}),$$

which implies the result for K large enough (recall that the constant in $O(1/N)$ above does not depend on K).

Hence, we only need to prove (2.10) to finish the proof of the lemma. Set $R_{\mathbf{p}}(\alpha) := \rho_{\mathbf{p}}(k(\alpha))k'(\alpha)$. Fix $x = k(\alpha)$. With this definition, the change of variable explained in the previous section implies that

$$\begin{aligned} 2\pi f'_{\mathbf{p}}(x) = 1 + \frac{1}{N}\sum_{k=1}^{n} \partial_1\Theta(x, p_k) &= 1 + \int_{-\pi}^{\pi} \partial_1\Theta(x, y)\rho_{\mathbf{p}}(y)dy \;+\; O(\tfrac{1}{N}) \\ &= 1 + \frac{1}{k'(\alpha)}\int_{-\pi}^{\pi} \Xi_{2\lambda}(\alpha - \beta)R_{\mathbf{p}}(\beta)d\beta \;+\; O(\tfrac{1}{N}), \end{aligned}$$

where we used again that $\max\{p_{j+1} - p_j\} = O(\frac{1}{N})$ and that $\partial_2\partial_1\Theta$ is bounded uniformly to approximate $\partial_1\Theta(x, p_k)$ by $\partial_1\Theta(x, k(\beta))$. Thus,

$$\begin{aligned} k'(k^{-1}(x))|f'_{\mathbf{p}}(x) - \rho(x)| &= \left|\tfrac{1}{2\pi}\int_{-\pi}^{\pi} \Xi_{2\lambda}(\alpha - \beta)(R_{\mathbf{p}}(\beta) - R(\beta))d\beta\right| \;+\; O(\tfrac{1}{N}) \\ &\overset{(2.4)}{\le} (1 - m_\Xi)\|\rho_{\mathbf{p}} - \rho\| \;+\; O(\tfrac{1}{N}), \end{aligned}$$

where in the last inequality, we can apply (2.4) since $\int_{-\pi}^{\pi} R_{\mathbf{p}}(\alpha)d\alpha = \int_{-\pi}^{\pi} \rho_{\mathbf{p}}(x)dx = \frac{1}{2}$. □

Lemma 2.5. – Let r, Δ_0 and K be as in the statement of the lemma; N_0 will be chosen later in the proof. Fix $\Delta \le \Delta_0$, $N \ge N_0$ and $\mathbf{p} \in \mathcal{S}_n$ satisfying (BE$_\Delta$) with $n = N/2 - r$ and such that $\|\rho_{\mathbf{p}} - \rho\| \le K/N$.

Note that for $\Delta = -\infty$, T is equal to $\mathbb{I}/2$, and the result is trivial. We may therefore assume that $\Delta \in (-\infty, \Delta_0]$.

Write A for $d(\mathbb{I}-\mathsf{T})(\Delta,\mathbf{p})$, the differential of $\mathbb{I}-\mathsf{T}$ in $\mathbf{p}$ at the point $(\Delta,\mathbf{p})$ fixed above. Recall that we see A as an automorphism of $\mathbb{R}^n_{\mathrm{sym}}$. We will regard it as a square matrix of size $\lfloor n/2 \rfloor$, when written in the basis $(e_j - e_{n+1-j})_{1\le j\le \lfloor n/2\rfloor}$ of $\mathbb{R}^n_{\mathrm{sym}}$, where $(e_j)_{1\le j\le n}$ is the canonical basis of $\mathbb{R}^n$. We may write A explicitly:

$$A_{jk} = \frac{\partial\big[(\mathbb{I}-\mathsf{T})(\Delta,\mathbf{p})\big]_j}{\partial p_k} - \frac{\partial\big[(\mathbb{I}-\mathsf{T})(\Delta,\mathbf{p})\big]_j}{\partial p_{n+1-k}}$$
$$= \begin{cases} 1 + \frac{1}{N}\sum\limits_{\ell\ne j} \partial_1\Theta(p_j,p_\ell) - \frac{1}{N}\partial_2\Theta(p_j,-p_j) & \text{if } j = k, \\ \frac{1}{N}\big[\partial_2\Theta(p_j,p_k) - \partial_2\Theta(p_j,-p_k)\big] & \text{if } j \ne k, \end{cases}$$

for $1 \le j,k \le n/2$. For the second equality, we have used $p_{n+1-k} = -p_k$.

Also, write B for the diagonal matrix of size $\lfloor n/2\rfloor$, with entries $N(p_{j+1}-p_j) = \rho_{\mathbf{p}}(p_j)^{-1}$ on the diagonal. Rather than proving that A is invertible, we will prove that $\tilde{A} = AB$ is invertible, by showing that it is diagonally dominated – i.e., $\tilde{A}_{ii} > \sum_{j\ne i} \tilde{A}_{ij}$ for every i.

Below, the notation $O(\cdot)$ is considered uniform in $\Delta < \Delta_0$ and j, but may depend on the fixed constants K and r. Due to the condition $\|\rho_{\mathbf{p}} - \rho\| \le K/N$, we may write $p_{j+1} - p_j = O(1/N)$. Finally, we will use that the functions ρ and Θ and their derivatives are uniformly bounded for $\Delta \le \Delta_0$ (provided that N is large enough).

The diagonal terms of $\tilde{A}$ are

(2.11)
$$\tilde{A}_{jj} = \frac{1}{\rho_{\mathbf{p}}(p_j)}\Big(1 + \frac{1}{N}\sum_{k\ne j}\partial_1\Theta(p_j,p_k)\Big) + O\big(\tfrac{1}{N}\big)$$
$$= \frac{1}{\rho_{\mathbf{p}}(p_j)}\Big(1 + \int_{-\pi}^{\pi}\partial_1\Theta(x,y)\rho(y)\Big) + O\big(\tfrac{1}{N}\big) \overset{(\mathrm{cBE}_\Delta)}{=} \frac{2\pi\rho(p_j)}{\rho_{\mathbf{p}}(p_j)} + O\big(\tfrac{1}{N}\big) = 2\pi + O\big(\tfrac{1}{N}\big).$$

For the second equality [(10)], we used $\|\rho_{\mathbf{p}} - \rho\| \le K/N$. We further note that the final equality follows thanks to the fact that $m_\rho > 0$.

We now compute the off-diagonal terms of $\tilde{A}$. For $x, y \in [-\pi,\pi]$, write $G(x,y) := \Theta(x,y) - \Theta(-x,y)$. A direct computation shows that $G(x,y)$ is increasing in y when both x and y are in $[-\pi,0]$. For $1 \le j \ne k \le n/2$, since $\Theta(x,-y) = -\Theta(-x,y)$, we have

$$\tilde{A}_{jk} = (p_{k+1}-p_k)\big[\partial_2\Theta(p_j,p_k) - \partial_2\Theta(-p_j,p_k)\big] = (p_{k+1}-p_k)\partial_2 G(p_j,p_k) \ge 0.$$

(10) The equality is obtained by a simple computation similar to that of the proof of Theorem 2.6 below. We omit the details here.

Therefore, for any fixed $1 \le j \le n/2$,

$$\begin{aligned}\sum_{k\neq j}|\tilde{A}_{jk}| = \sum_{k\neq j}\tilde{A}_{jk} &= \sum_{k=1}^{\lfloor n/2\rfloor}(p_{k+1}-p_k)\partial_2 G(p_j,p_k) + O\big(\tfrac{1}{N}\big)\\ &= G(p_j,0) - G(p_j,-\pi) + O\big(\tfrac{1}{N}\big).\end{aligned} \tag{2.12}$$

A straightforward calculus exercise can show that, for any $\Delta < \Delta_0$, the function $G(x,0) - G(x,-\pi)$ satisfies

$$G(x,0) - G(x,-\pi) \le 4\arctan\Big(\frac{1}{2|\Delta_0|\sqrt{\Delta_0^2-1}}\Big) < 2\pi, \qquad \forall x \in [-\pi,0].$$

In conclusion, (2.11) and (2.12) show that for N large enough (depending on Δ_0, r and K only), $\tilde{A}$ is diagonal dominant and therefore invertible. □

2.3. The asymptotic behavior of the solutions to the Bethe equations

This section is devoted to two results that control the asymptotic behavior of solutions to the Bethe equations when $\rho_{\mathbf{p}}$ is close to ρ. The first deals with the "first order" asymptotics of solutions to (BE_Δ) with $n = N/2 - r$, for fixed r.

THEOREM 2.6. – *Fix $\Delta < -1$ and $r \ge 0$. Consider a family of $\mathbf{p}(N) \in \mathcal{S}_{N/2-r}$ for N even large enough satisfying $\|\rho_{\mathbf{p}(N)} - \rho\| \longrightarrow 0$. Then, $\mu_N := \frac{1}{N}\sum_{i=1}^n \delta_{p_i(N)}$ converges weakly to $\rho(x)dx$, where dx is Lebesgue's measure on $[-\pi,\pi]$.*

Proof. – Fix $\Delta < -1$, $r \ge 0$ and set $n = N/2 - r$. For any continuous function g on $[-\pi,\pi]$ and $N \ge 2r$, define $g_{\mathbf{p}(N)} : [-\pi,\pi] \to \mathbb{R}$ by $g_{\mathbf{p}(N)}(t) := g(p_j)$ if $t \in [p_j, p_{j+1})$ for some $0 \le j \le n$ (where we extend g periodically whenever needed). Then

$$\int_{-\pi}^{\pi} g(x)d\mu_N(x) = \frac{1}{N}\sum_{j=1}^n g(p_j) = \int_{-\pi}^{\pi} g_{\mathbf{p}(N)}(x)\rho_{\mathbf{p}(N)}(x)dx + \frac{g(p_n)-g(p_1)}{2N},$$

and we find

$$\begin{aligned}&\int_{-\pi}^{\pi} g(x)\rho(x)dx - \int_{-\pi}^{\pi} g(x)d\mu_N(x)\\ &= \int_{-\pi}^{\pi} g(x)\big[\rho(x)-\rho_{\mathbf{p}(N)}(x)\big]dx + \int_{-\pi}^{\pi}\big[g(x)-g_{\mathbf{p}(N)}(x)\big]\rho_{\mathbf{p}(N)}(x)dx + \frac{g(p_n)-g(p_1)}{2N},\end{aligned}$$

Then, (2.6) and (2.7) imply that each integral above converges to 0, and the result follows. □

The second result deals with the displacement of the solutions to the Bethe equations with N and $n = N/2 - r$ with respect to the solution with N and $n = N/2$. Fix $r > 0$ and write henceforth $n = N/2 - r$. For $\mathbf{p} = (p_1, \ldots, p_{N/2}) \in \mathcal{S}_{N/2}$ and $\tilde{\mathbf{p}} = (\tilde{p}_1, \ldots, \tilde{p}_n) \in \mathcal{S}_n$, introduce the *offset displacement* $\varepsilon = \varepsilon(\mathbf{p}, \tilde{\mathbf{p}}) \in \mathbb{R}^n$ defined for $1 \le j \le n$ by

$$\varepsilon_j = \begin{cases} N\big(\tilde{p}_j - p_{j+r/2}\big) & \text{if } r \text{ is even},\\ N\Big(\tilde{p}_j - \dfrac{p_{j-(r-1)/2} + p_{j-(r+1)/2}}{2}\Big) & \text{if } r \text{ is odd}\end{cases} \tag{2.13}$$

and the *offset function* $f_{\mathbf{p},\tilde{\mathbf{p}}}(t) := \varepsilon_j$ if $t \in [\tilde{p}_j, \tilde{p}_{j+1})$ for some $0 \le j \le n$.

REMARK 2.7. – The difference of index in (2.13) between $\tilde{\mathbf{p}}$ and $\mathbf{p}$ is made in such a way that the indices coincide when "starting from the middle of the interval $[-\pi, \pi]$".

REMARK 2.8. – Consider $\mathbf{p}$ and $\tilde{\mathbf{p}}$ given by Theorem 2.3 for r and $r+1$. Then, the solutions may be proved to be interlaced [11], in the sense that $p_j < \tilde{p}_j < p_{j+1}$ for any $1 \leq j < n$. We will not use this property later, but this may be useful in subsequent works.

While the asymptotic behavior of individual solutions $\mathbf{p}$ is described by the continuous Bethe Equation, that of the offset displacement is governed by the Offset Equation, as shown in the next theorem.

THEOREM 2.9. – *Fix $\Delta < -1$ and $r \geq 0$. Consider two families $\mathbf{p}(N) \in \mathcal{S}_{N/2}$ and $\tilde{\mathbf{p}}(N) \in \mathcal{S}_{N/2-r}$ of solutions to the Bethe equations with parameters Δ and N even sufficiently large. If $\|\rho_{\mathbf{p}(N)} - \rho\| = O(\frac{1}{N})$ and $\|\rho_{\tilde{\mathbf{p}}(N)} - \rho\| = O(\frac{1}{N})$, then*

1. $\rho \cdot f_{\mathbf{p}(N),\tilde{\mathbf{p}}(N)}$ *converges uniformly on $[-\pi, \pi]$ to $r \cdot \tau$.*
2. *There exists $C > 0$ such that $|f_{\mathbf{p}(N),\tilde{\mathbf{p}}(N)}(x)| \leq C|x| + O(1/N)$ for all N and $x \in [-\pi, \pi]$.*

The second property is slightly technical but will be useful when integrating functions against the empirical measure of the $\tilde{\mathbf{p}}(N)$ (see Section 3.2).

Proof. – We drop N and $n = N/2 - r$ from the notation in the computations, except that we set $f_N = f_{\mathbf{p}(N),\tilde{\mathbf{p}}(N)}$. We treat the case r even and odd separately. Below, all quantities $O(\cdot)$ may depend on Δ and r but are uniform in $j = 1, \ldots, n$ and $x \in [-\pi, \pi]$.

2.3.1. *Case r even.* – First, we bound the increments of f_N and show that f_N is almost equal to 0 at the origin, so as to prove the second property. Equation (2.6) and the bound (2.7) on the increments of $\mathbf{p}$ and $\tilde{\mathbf{p}}$ (both valid due to our assumptions) imply that

$$|\varepsilon_{j+1} - \varepsilon_j| \overset{(2.6)}{\leq} \left|\frac{1}{\rho(p_{j+r/2+1})} - \frac{1}{\rho(p_{j+r/2})}\right| + \left|\frac{1}{\rho(\tilde{p}_{j+1})} - \frac{1}{\rho(\tilde{p}_j)}\right| + O\left(\tfrac{1}{N}\right) \overset{(2.7)}{=} O\left(\tfrac{1}{N}\right). \tag{2.14}$$

Now, by symmetry, $p_{N/4} = -p_{N/4+1}$ (recall that $p_{N/4}$ and $p_{N/4+1}$ are the two elements of $\mathbf{p}$ closest to the origin) and $\tilde{p}_{n/2} = -\tilde{p}_{n/2+1}$ so

$$\varepsilon_{n/2} = -\varepsilon_{n/2+1} = O\left(\tfrac{1}{N}\right). \tag{2.15}$$

Finally, observe that $\rho_{\tilde{p}}$ is bounded uniformly in N (since it converges to ρ in the norm $\|\cdot\|$, it also does in the uniform norm), and therefore $\tilde{p}_{j+1} - \tilde{p}_j > c/N$ for all N and j, where $c > 0$ is some constant independent of N and j. This implies the existence of $C > 0$, independent of N and j, such that

$$|f_N(x)| \leq C|x| + O\left(\tfrac{1}{N}\right) \qquad \text{for all } x \in [-\pi, \pi]. \tag{2.16}$$

[11] The strategy is to show that the property of being interlaced is true for $\Delta = -\infty$ (this is a straightforward computation) and that this property does not cease to be true when increasing Δ continuously. Namely, one can prove that for any $\Delta < -1$, it is not possible that $p_j \leq \tilde{p}_j \leq p_{j+1}$ for every $1 \leq j < n$ and $\tilde{p}_k$ be equal to p_k or p_{k+1} for some $1 \leq k < n$. This is based on the fact that $\Theta(x, 0) \in (-\pi, \pi)$ for any $x \in (-\pi, \pi)$, and that $G(x, y) = \Theta(x, y) - \Theta(-x, y)$ defined on $[-\pi, 0]^2$ is decreasing in the first variable and increasing in the second one. The continuity of $\Delta \mapsto \mathbf{p}, \tilde{\mathbf{p}}$ is then used to conclude. We leave the details of the computation to the reader.

Let us now prove the first statement - that is, the convergence of ρf_N. In light of (2.14), we may apply the Arzela-Ascoli theorem to the sequence (f_N) to extract a sub-sequential limit f. It suffices to show that $\rho f = r \cdot \tau$ to conclude.

For N and $1 \le j \le n$, the Bethe equations applied to $p_{j+r/2}$ and $\tilde{p}_j$ imply

$$\varepsilon_j = \sum_{k=1}^{N/2} \Theta(p_{j+r/2}, p_k) - \sum_{k=1}^{n} \Theta(\tilde{p}_j, \tilde{p}_k).$$

In the first sum, we Taylor expand the terms $\Theta(p_{j+r/2}, p_{k+r/2})$ at $(\tilde{p}_j, \tilde{p}_k)$ for any $1 \le k \le n$ (while leaving the remaining terms as they are). This gives

$$\begin{aligned} \varepsilon_j = &\underbrace{\sum_{k=1}^{r/2} \Theta(p_{j+r/2}, p_k) + \Theta(p_{j+r/2}, p_{n+1-k})}_{(1)} \\ &- \underbrace{\tfrac{1}{N} \sum_{k=1}^{n} \partial_1 \Theta(\tilde{p}_j, \tilde{p}_k)\varepsilon_j}_{(2)} - \underbrace{\tfrac{1}{N} \sum_{k=1}^{n} \partial_2 \Theta(\tilde{p}_j, \tilde{p}_k)\varepsilon_k}_{(3)} + O\big(\tfrac{1}{N}\big). \end{aligned} \tag{2.17}$$

The final term is due to the second order errors in the Taylor expansion; it is indeed $O\big(\frac{1}{N}\big)$, since it contains $O(N)$ terms of order $O\big(\frac{1}{N^2}\big)$.

Fix $x \in [-\pi, \pi]$ and for each N (along the subsequence for which f_N tends to f) pick $\tilde{p}_j$ so that $x \in [\tilde{p}_j, \tilde{p}_{j+1})$. Then the equation displayed above offers an expression for $f_N(x)$. Taking N to infinity, we find that (1) converges to $\frac{r}{2}(\Theta(x, -\pi) + \Theta(x, \pi))$, and (2) and (3) converge to $(1 - 2\pi\rho(x)) f(x)$ and $\int_{-\pi}^{\pi} \partial_2\Theta(x, y) f(y)\rho(y)dy$, respectively, by the definition of f and the weak convergence of μ_N (defined in statement of Theorem 2.6). Thus,

$$2\pi f(x)\rho(x) = \tfrac{r}{2}\big(\Theta(x, -\pi) + \Theta(x, \pi)\big) - \int_{-\pi}^{\pi} \partial_2\Theta(x, y) f(y)\rho(y)dy.$$

It follows that $\frac{1}{r} f(x)\rho(x) = \tau(x)$ by the uniqueness of the solution to the Offset Equation (cOE$_\Delta$).

2.3.2. *Case r odd.* – The reasoning is similar. Equation (2.14) may be obtained in the same way and (2.15) may be replaced by $\varepsilon_{(n+1)/2} = 0$, which results from the symmetry of $\mathbf{p}$ and $\tilde{\mathbf{p}}$. One then expands around $(\tilde{p}_i, \tilde{p}_k)$ the expression

$$\sum_{k=1}^{n} \Theta(\tilde{p}_j, \tilde{p}_k) - \tfrac{1}{2}\big[\Theta(p_{j+(r-1)/2}, p_k) + \Theta(p_{j+(r+1)/2}, p_k)\big]$$

to obtain the same result. □

3. Proofs of the theorems

3.1. Perron-Frobenius eigenvalues of six-vertex model via Bethe Ansatz

The goal of this section is to show that the Perron-Frobenius eigenvalue of $V^{[n]}$ is given by the Bethe Ansatz from the solution $\mathbf{p}$ of (BE$_\Delta$) given by Theorem 2.3 (recall the choice $I_j = j - \frac{n+1}{2}$ for $1 \le j \le n$ in the theorem). We start by recalling the Bethe Ansatz for the transfer matrix of the six-vertex model. A more detailed discussion (with references) and an expository proof may be found in the companion paper [8].

Recall that $\Delta = (2-c^2)/2$ and that the function Θ depends implicitly on Δ. For $z \neq 1$, define

$$L(z) := 1 + \frac{c^2 z}{1-z} \qquad \text{and} \qquad M(z) := 1 - \frac{c^2}{1-z}\,. \tag{3.1}$$

THEOREM 3.1 (Bethe Ansatz for V). – *Fix $n \leq N/2$. Let $(p_1, p_2, \ldots, p_n) \in (-\pi,\pi)^n$ be distinct and satisfy the equations*

$$\exp\left(iNp_j\right) = (-1)^{n-1} \exp\left(-i \sum_{k=1}^{n} \Theta(p_j, p_k)\right) \quad \forall j \in \{1,2,\ldots,n\}. \tag{BE}$$

Then, $\psi = \sum_{|\vec{x}|=n} \psi(\vec{x})\, \Psi_{\vec{x}}$, where $\psi(\vec{x})$ is given by

$$\psi(\vec{x}) := \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{k=1}^{n} \exp\left(ip_{\sigma(k)}x_k\right)$$

$$\text{where } A_\sigma := \varepsilon(\sigma) \prod_{1\leq k<\ell\leq n} e^{ip_{\sigma(k)}}\,(e^{-ip_{\sigma(k)}} + e^{ip_{\sigma(\ell)}} - 2\Delta),$$

(for σ an element of the symmetry group $\mathfrak{S}_n$) satisfies the equation $V\psi = \Lambda\psi$, where

$$\Lambda = \Lambda(\mathbf{p}) := \begin{cases} \displaystyle\prod_{j=1}^{n} L(e^{ip_j}) + \prod_{j=1}^{n} M(e^{ip_j}) & \text{if } p_1,\ldots,p_n \neq 0,\\ \displaystyle\Big[2 + c^2(N-1) + c^2 \sum_{j\neq\ell} \partial_1\Theta(0,p_j)\Big] \cdot \prod_{j\neq\ell} M(e^{ip_j}) & \text{if } p_\ell = 0 \text{ for some } \ell. \end{cases}$$

It is a priori unclear whether ψ is non-zero, so that the previous theorem does not trivially imply that $\Lambda(\mathbf{p})$ is an eigenvalue of V. It is also unclear whether solutions of (BE) exist. Nonetheless, any solutions of (BE_Δ) do also satisfy (BE). In particular, Theorem 2.3 provides us with a family of solutions to (BE_Δ), and our goal is to prove that the corresponding value Λ given by the theorem above is the Perron-Frobenius eigenvalue of $V^{[n]}$.

Below, we will view $V^{[n]}$ as a function of Δ, hence we write it $V^{[n]}_\Delta$. We begin by computing the asymptotic of the Perron-Frobenius eigenvalue of $V^{[n]}_\Delta$ when Δ tends to $-\infty$.

LEMMA 3.2. – *Fix $r \geq 0$ and $N > 2r$ an even integer. Set $n = N/2 - r$. Then the largest eigenvalue λ of the matrix*

$$V^{[n]}_\infty := \lim_{\Delta\to-\infty} \frac{V^{[n]}_\Delta}{(-2\Delta)^n}$$

is simple and satisfies

$$\lambda \leq 2^r \prod_{j=0}^{r-1} \Big[1 + \cos\Big(\frac{\pi(2j+1)}{n+2r}\Big)\Big], \tag{3.2}$$

where the empty product is set to 1.

REMARK 3.3. – The matrix $V_\infty^{[n]}$ is symmetric and thus all its eigenvalues are real; its largest eigenvalue is therefore well-defined. It is not a Perron-Frobenius matrix, and thus we cannot be sure a priori that the largest eigenvalue is simple and largest in absolute value. We further note that the largest eigenvalue of $V_\infty^{[n]}$ is actually equal to the RHS of (3.2), as will be shown in the proof of Corollary 3.4 below.

Proof. – Fix $N \ge 2r$ and $n = N/2 - r$. For two distinct configurations $\Psi_{\vec{x}}$ and $\Psi_{\vec{y}}$ in Ω_n (12), recall that $V_\Delta^{[n]}(\Psi_{\vec{x}}, \Psi_{\vec{y}})$ is non-zero only when $\Psi_{\vec{x}}$ and $\Psi_{\vec{x}}$ are interlacing, and in this case it is equal to

$$c^{|\{i:\Psi_{\vec{x}}(i)\neq\Psi_{\vec{y}}(i)\}|} = (2-2\Delta)^{\frac{1}{2}|\{i:\Psi_{\vec{x}}(i)\neq\Psi_{\vec{y}}(i)\}|}.$$

Since $\Psi_{\vec{x}}, \Psi_{\vec{y}} \in \Omega_n$, the number $P(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = |\{i : \Psi_{\vec{x}}(i) \neq \Psi_{\vec{y}}(i)\}|$ is at most $2n$. The normalization $(-2\Delta)^n$ is chosen to ensure that, for any pair of configurations $\vec{x}$ and $\vec{y}$ as above,

$$V_\infty^{[n]}(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = \begin{cases} 1 & \text{if } P(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

If $\vec{x}$ and $\vec{y}$ are configurations as above with $V_\infty^{[n]}(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = 1$, then $\Psi_{\vec{x}}$ has no consecutive up-arrows (and by symmetry neither does $\Psi_{\vec{y}}$). Indeed, if we suppose that $\Psi_{\vec{x}}$ has at least two consecutive up-arrows, then interlacement requires $\Psi_{\vec{y}}$ to have an up-arrow above at least one of the consecutive up-arrows of $\vec{x}$, which induces $P(\Psi_{\vec{x}}, \Psi_{\vec{y}}) < 2n$ and therefore $V_\infty^{[n]}(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = 0$. Thus, to study $V_\infty^{[n]}$, we may study its restriction to the set of configurations with no consecutive up-arrows.

In the case $n = N/2$, there is only one pair of such configurations: the completely staggered configurations – i.e., those with alternating up and down arrows. Hence, $V_\infty^{[N/2]}$ breaks down into a block-diagonal structure: a 2×2 block of the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and a $\left[\binom{N}{N/2} - 2\right]$-dimensional block of 0's. The spectral structure of this matrix is very straightforward - there are simple eigenvalues at ± 1, and all other eigenvalues are 0, as required.

For $n = N/2 - r$, the situation is more complicated, and a direct computation of the spectrum of $V_\infty^{[n]}$ is best avoided. However, we do have the tools to bound the dominant eigenvalue.

The set of configurations with no consecutive up-arrows can be parameterized by the location of the $2r$ "defects" – i.e., coordinates i with a down arrow preceded by another down arrow. By periodicity, we say that $\vec{x}$ has a defect at 1 if $\vec{x}$ has a down arrow at 1 and at N.

It is straightforward to show that a configuration with n up-arrows has no consecutive up-arrows if and only if there are exactly $2r$ defects whose parities alternate. Moreover, $\vec{x}$ and $\vec{y}$ are such that $V_\infty^{[n]}(\Psi_{\vec{x}}, \Psi_{\vec{y}}) = 1$ if and only if $\Psi_{\vec{x}} \neq \Psi_{\vec{y}}$ and the locations of the defects of $\vec{y}$ may be obtained from those of $\vec{x}$, by moving each defect by precisely one unit on the left or on the right (taken toroidally). Since the parity of the defects alternates in both states, no two defects can exchange positions between $\vec{x}$ and $\vec{y}$. See Fig. 3 for an example.

Write $\tilde{\Omega}_n$ for the subspace of Ω_n generated by basis vectors $\Psi_{\vec{x}}$ with no two consecutive up arrows. Then, $V_\infty^{[n]}$ leaves this space stable, and we may consider its restriction to $\tilde{\Omega}_n$.

(12) Recall that Ω_n is the vector space generated by the $\Psi_{\vec{x}}$, where $\vec{x}$ has n entries.

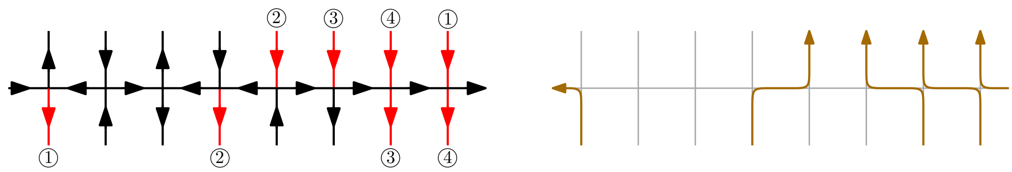

FIGURE 3. *Left*: Two configurations $\vec{x}$ and $\vec{y}$ with $N = 8$ and $r = 2$. The defects are marked by red arrows and are numbered. Notice that each defect has moved by one unit when going from $\Psi_{\vec{x}}$ (below) to $\Psi_{\vec{y}}$ (above), but none have exchanged places. *Right*: The $2r$ paths corresponding to the evolutions of the defects.

A straightforward computation shows that this matrix is irreducible in the sense that, for any $\Psi_{\vec{x}}, \Psi_{\vec{y}} \in \tilde{\Omega}_n$, there exists K such that $[V_\infty^{[n]}]^K(\Psi_{\vec{x}}, \Psi_{\vec{y}}) > 0$. As any symmetric irreducible matrix, it is either aperiodic or of period 2; a more precise analysis can show that the latter occurs in the case of $V_\infty^{[n]}$. Thus, the Perron-Frobenius theorem for irreducible (but not aperiodic) matrices guarantees that the largest eigenvalue is simple and maximizes the absolute value; the *smallest* eigenvalue actually has the same absolute value as the largest, unlike for true Perron-Frobenius matrices.

To determine λ, the largest eigenvalue, consider the following related construction. Let M be an even integer and $(a_1, \dots, a_{2r})$ be an ordered set of integers between 1 and N of alternating parity. Consider families of $2r$ paths on $\mathbb{Z}/N\mathbb{Z}$ denoted

$$\{X_j(t) : 0 \le t \le M;\ j = 1, \dots, 2r\},$$

such that, for each j, $X_j(0) = X_j(M) = a_j$ and $|X_j(t+1) - X_j(t)| = 1$ for $1 \le t < M$. Additionally, impose that the paths $X_1, \dots, X_n$ are *non-intersecting*, in the sense that no pair of adjacent paths ever exchange position. Let $Z(M; a_1, \dots, a_{2r})$ be the number of such paths, and $Z(M)$ the sum of $Z(M; a_1, \dots, a_{2r})$ over all admissible $(a_1, \dots, a_{2r})$. The discussion above indicates that

$$Z(M) = \mathrm{Tr}\left([V_\infty^{[n]}]^M\right),$$

which in turn implies that the largest eigenvalue (in absolute value) of $V_\infty^{[n]}$ is given by

$$\lambda = \lim_{M\to\infty} Z(M)^{1/M}.$$

Families of non-intersecting paths as those appearing in the definition of $Z(M)$ have been studied before, in particular in the work of Fulmek [16], which enables us to compute the asymptotic of $Z(M)$ directly. Fulmek enumerates the number of vertex-avoiding paths (i.e., families of paths as above, but such that no two ever hit the same vertex, rather than not intersecting). Luckily, the two are closely related: consider the transformation of a set of paths $\{X_j(t) : t, j\}$ as above to the set of paths $\{\tilde{X}_j(t) : t, j\}$ on $\mathbb{Z}/(N+2r)\mathbb{Z}$, where

$$\tilde{X}_j(t) = X_j(t) + j, \qquad \forall 1 \le t \le M,\ 1 \le j \le 2r.$$

One may check that this transformation induces a bijection between the set of non-intersecting paths starting and ending at $(a_1, \dots, a_{2r})$ on $\mathbb{Z}/N\mathbb{Z}$ and that of vertex-avoiding paths starting and ending at $(a_1+1,\dots,a_{2r}+2r)$ on $\mathbb{Z}/(N+2r)\mathbb{Z}$. Note that, while vertex-avoiding paths are generally allowed to intersect, the parity constraints of the starting positions prevent them from doing so in this case (more precisely, observe that $\tilde{X}_{j+1}(t)-\tilde{X}_j(t)$ is even for all t and j).

Since we may get from any admissible starting position (that is, with even spacing between the starting points) to the position $(2,4,\dots,4r)$ in at most N steps, the limit of interest to us may be computed as

$$\lim_{M\to\infty} Z(M)^{1/M} = \lim_{M\to\infty} Z(M;2,4,\dots,2r)^{1/M}.$$

We now state Corollary 7 of [16], which provides an exact expression of $Z(M;2,4,\dots,2r)$ as the determinant of a matrix of size $2r$:

$$Z(M;2,4,\dots,2r) = (N+2r)^{-2r}\det\left(\xi^{i-j}\sum_{\ell=0}^{N+2r-1}\xi^{2(i-j)\ell}\left[2\cos\left(\frac{\pi(2\ell+1)}{N+2r}\right)\right]^M\right)_{1\le i,j\le 2r},$$

where we set $\xi = e^{i\frac{2\pi}{N+2r}}$. Since we are only interested in $\lim_{M\to\infty} Z(M;2,4,\dots,2r)^{1/M}$, we can simply study the dominating terms (as $M\to\infty$) in the Leibniz formula for the determinant on the right-hand side. However, the apparently maximal terms cancel out in the computation of the determinant, and some care is needed.

To start, observe that the entries of the matrix above may be rewritten by grouping the terms ℓ and $\ell+N/2+r$ (which are equal) as a sum with only half the terms:

$$2\xi^{i-j}\sum_{\ell=0}^{n+2r-1}\xi^{2(i-j)\ell}\Big[2\cos\Big(\frac{\pi(2\ell+1)}{N+2r}\Big)\Big]^M.$$

Then, we write the determinant out as

$$(N+2r)^{2r}2^{-2r(M+1)}Z(M;2,4,\dots,2r) = \sum_{\substack{\sigma\in\mathfrak{S}_{2r}\\ 0\le\ell_1,\dots,\ell_{2r}<n+2r}}\varepsilon(\sigma)\prod_{i=1}^{2r}\xi^{(i-\sigma(i))(2\ell_i+1)}\Big[\cos\Big(\frac{\pi(2\ell_i+1)}{N+2r}\Big)\Big]^M.$$

In the above, note that the term taken to the power M does not depend on σ. We conclude that

$$\lim_{M\to\infty} Z(M)^{1/M} = 2^{2r}\prod_{i=1}^{2r}\Big|\cos\Big(\frac{\pi(2\ell_i+1)}{N+2r}\Big)\Big|, \tag{3.3}$$

where $\ell_1,\dots,\ell_{2r}\in\{0,\dots,n+2r-1\}$ maximize the product above and are such that

$$\sum_{\sigma\in\mathfrak{S}_{2r}}\varepsilon(\sigma)\prod_{i=1}^{2r}\xi^{(i-\sigma(i))(2\ell_i+1)}\neq 0. \tag{3.4}$$

Consider $\ell_1,\dots,\ell_{2r}$ as above with $\ell_j=\ell_{j'}$ for some $j\neq j'$ and a permutation σ. Write $\tau_{j,j'}$ for the transposition of j and j'. The sum of the terms corresponding to $\ell_1,\dots,\ell_{2r}$ with σ and $\sigma\circ\tau_{j,j'}$ sum up to 0, and we find that the term in (3.4) is zero.

Thus, we may limit ourselves to terms with $\ell_1,\dots,\ell_{2r}$ all distinct. Among such sets, one maximizing the product in (3.3) is given by $\ell_i=i-1$ for $i\le r$ and $\ell_i=n+2r-i$ for $i>r$.

For this set, we find that the term in (3.3) is equal to

$$2^{2r}\prod_{i=1}^{r}\left[\cos\left(\frac{\pi(2\ell_i+1)}{N+2r}\right)\right]^2=2^r\prod_{j=0}^{r-1}\left[1+\cos\left(\frac{\pi(2j+1)}{n+2r}\right)\right].$$

This does not prove that $\lim_{M\to\infty} Z(M)^{1/M}$ is equal to the above, since (3.4) may not be satisfied. It does, however, show the claimed inequality. □

COROLLARY 3.4. – *Fix $\Delta_0 < -1$ and $r \geq 0$. Then, for N large enough, the Perron-Frobenius eigenvalue of $V_N^{[N/2-r]}$ for Δ_0 is given by $\Lambda(\mathbf{p}_{\Delta_0})$, where $(\mathbf{p}_\Delta)_{\Delta\leq\Delta_0}$ is the family given by Theorem 2.3 applied to Δ_0 and r.*

Proof. – Fix $\Delta_0 < -1$, $r \geq 0$ and let N be large enough for Theorem 2.3 to apply. Write $n = N/2 - r$. Since N is fixed, we drop it from the notation.

The dependency of the Perron-Frobenius eigenvalue of $V_\Delta^{[n]}$ on Δ will be important, and we therefore denote it by $\Lambda_r(\Delta)$. Also, write $\psi(\mathbf{p}_\Delta)$ for the vector given by Theorem 3.1 for the solution $\mathbf{p}_\Delta$ to (BE_Δ). We wish to prove that $\Lambda_r(\Delta) = \Lambda(\mathbf{p}_\Delta)$ for $\Delta = \Delta_0$. We will prove more generally that this is true for all $\Delta \leq \Delta_0$.

First, observe that the Perron-Frobenius eigenvalue of a family of irreducible symmetric matrices varying analytically in a parameter (here Δ) varies analytically in this parameter as well (since it is a simple zero of the characteristic polynomial). Therefore, $\Lambda_r(\Delta)$ is an analytic function. Since $\Delta \mapsto \mathbf{p}_\Delta$ is analytic, we deduce that $\Delta \mapsto \Lambda(\mathbf{p}_\Delta)$ also is, so that it is sufficient to show that $\Lambda(\mathbf{p}_\Delta) = \Lambda_r(\Delta)$ for Δ small enough in order to conclude that the two are equal for all $\Delta \leq \Delta_0$. To do this, we shall prove two facts, namely that

- $\psi(\mathbf{p}_\Delta)$ is non-zero for Δ small enough (which implies that $\Lambda(\mathbf{p}_\Delta)$ is an eigenvalue of $V_\Delta^{[n]}$ for the corresponding values of Δ),
- $\lim_{\Delta\to-\infty} \frac{1}{(-2\Delta)^n}\Lambda(\mathbf{p}_\Delta)$ is the largest eigenvalue of $V_\infty^{[n]}$ (defined in Lemma 3.2).

These two facts indeed prove the result: since the largest eigenvalue of $V_\infty^{[n]}$ is simple, by continuity of $\Delta \mapsto \Lambda(\mathbf{p}_\Delta)$ and $\Delta \mapsto V_\Delta^{[n]}$, we deduce that $\Lambda(\mathbf{p}_\Delta)$ is the largest eigenvalue of $V_\Delta^{[n]}$ for Δ small enough. However, for finite Δ, $V_\Delta^{[n]}$ is a Perron Frobenius matrix, and $\Lambda(\mathbf{p}_\Delta)$ is then its Perron Frobenius eigenvalue. The observation of the previous paragraph is then sufficient to conclude.

The rest of the proof is dedicated to the two facts listed above. Recall that, at $\Delta = -\infty$, we have a simple formula for $\mathbf{p}$, namely

$$p_j = \frac{2\pi I_j}{N-n} \qquad \text{for all } 1 \leq j \leq n.$$

For the rest of the proof, write $\zeta = e^{2\pi i/(N-n)}$.

We start with the study of $\psi(\mathbf{p}_\Delta)$. Set $\psi_\infty := \lim_{\Delta\to-\infty}(-2\Delta)^{-\frac{n(n-1)}{2}}\psi(\mathbf{p}_\Delta)$. It suffices then to prove that ψ_∞ has at least one non-zero coordinate, and we shall do so for the coordinate $\psi_\infty(2, 4, \dots, 2n)$. First, we need to study the asymptotics of the coefficients A_σ appearing in the definition of ψ. For $\sigma \in \mathfrak{S}_n$, as $\Delta \to -\infty$,

$$A_\sigma = \varepsilon(\sigma)\prod_{1\leq j<k\leq n}\left[-2\Delta\zeta^{\sigma(j)-\frac{n+1}{2}}\right] + o(\Delta^{\frac{n(n-1)}{2}}).$$

By injecting this into the definition of ψ, we find that,

$$\begin{aligned}\psi_\infty(2,\dots,2n) &= \sum_{\sigma\in\mathfrak{S}_n} A_\sigma \prod_{k=1}^{n} \exp(i p_{\sigma(k)}\cdot 2k)\\ &= \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\Big(\prod_{1\le j<k\le n} \zeta^{\sigma(j)-\frac{n+1}{2}}\Big)\times\prod_{k=1}^{n}\zeta^{2(\sigma(k)-\frac{n+1}{2})k}\\ &= \zeta^{-\frac14(n+1)^2 n}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\zeta^{\sum_{j=1}^n \sigma(j)j}.\end{aligned}$$

In the sum above, we recognize the determinant of the matrix $\left(\zeta^{j\cdot k}\right)_{1\le j,k\le n}$. This is the Vandermonde matrix corresponding to the values $\zeta,\zeta^2,\dots,\zeta^n$, which are all distinct (since $2n\le N$). Thus,

$$\psi_\infty(2,\dots,2n) = \lim_{\Delta\to-\infty}(-2\Delta)^{-\frac{n(n-1)}{2}}\psi_\Delta(2,\dots,2n)\neq 0.$$

We now turn to the study of $\lim_{n\to\infty}(-2\Delta)^{-n}\Lambda(\mathbf{p}_\Delta)$ (we show below that this limit exists). Before starting, we mention that, since $\psi_\infty\neq 0$, the above limit is an eigenvalue of $V_\infty^{[n]}$. With this and Lemma 3.2 in mind, it suffices to prove that it is equal to the RHS of (3.2) to deduce that it is the largest eigenvalue of $V_\infty^{[n]}$. We do this below.

The functions L and M defined in (3.1) depend on Δ and degenerate when $\Delta\to-\infty$. However, we have

$$\frac{1}{-2\Delta}L(z)\xrightarrow[\Delta\to-\infty]{}\frac{z}{1-z}\quad\text{and}\quad\frac{1}{-2\Delta}M(z)\xrightarrow[\Delta\to-\infty]{}\frac{-1}{1-z}\qquad\forall z\in[-\pi,\pi]\setminus\{0\}.$$

Therefore, we find that

$$\frac{\Lambda(\mathbf{p}_\Delta)}{(-2\Delta)^n}\xrightarrow[\Delta\to-\infty]{}\begin{cases}\frac{2}{\prod_{j=1}^n(1-\zeta^{j-(n+1)/2})} & \text{if } n \text{ is even},\\ (N-n)\times\prod_{j=1,j\neq(n+1)/2}^n\left(\frac{1}{1-\zeta^{j-(n+1)/2}}\right) & \text{if } n \text{ is odd}.\end{cases}\tag{3.5}$$

(Recall that $\Theta_{-\infty}(x,y)=y-x$, c^2 behaves like -2Δ, and $\zeta^{N-n}=1$.) When n is an even number, the decomposition of the polynomial $x^{N-n}-1$ reads

$$x^{N-n}-1=\prod_{j=1}^{N-n}\left(x-\zeta^{j-n/2}\right).$$

Thus, if we multiply the numerator and denominator in (3.5) by the terms corresponding to $j=n+1$ to $N-n$ and apply the above to $x=\zeta^{1/2}$, we find that

$$\begin{aligned}\frac{2}{\prod_{j=1}^n(1-\zeta^{j-(n+1)/2})} &= \frac{2\times\prod_{j=n+1}^{N-n}\left(1-\zeta^{j-(n+1)/2}\right)}{\zeta^{-(N-n)/2}\left(\zeta^{(N-n)/2}-1\right)}\\ &= \prod_{j=n+1}^{N-n}\left(1-\zeta^{j-(n+1)/2}\right)\\ &= \prod_{j=0}^{2r-1}\left(1-\zeta^{j+(n+1)/2}\right)\end{aligned}$$

$$= \prod_{j=0}^{r-1} \left[2 - 2\cos\left(\frac{\pi(2j+n+1)}{N-n}\right)\right]$$
$$= 2^r \prod_{j=0}^{r-1} \left[1 + \cos\left(\frac{\pi(2j+1)}{N-n}\right)\right],$$

where in the second equality we used that $\zeta^{(N-n)/2} = -1$, in the third we changed j to $N-n-j$ and used that $N-n = n+2r$, in the fourth we grouped the $j = k$ and $j = 2r-1-k$ terms together. The last equality follows again from the fact that $N-n = n+2r$ and changing j to $r-1-j$. This matches the expression in (3.2), as required.

We use a similar strategy when n is odd. Noting that

$$\prod_{\substack{1\le j\le N-n\\ j\neq (n+1)/2}} \left(1-\zeta^{j-(n+1)/2}\right) = \lim_{x\to 1} \frac{x^{N-n}-1}{x-1} = N-n,$$

we may perform a similar computation to obtain again

$$(N-n)\times \prod_{j=1, j\neq (n+1)/2}^{n} \left(\frac{1}{1-\zeta^{j-(n+1)/2}}\right) = 2^r \prod_{j=0}^{r-1} \left[1+\cos\left(\frac{\pi(2j+1)}{N-n}\right)\right]. \qquad \square$$

REMARK 3.5. – The analyticity of $\Delta \mapsto \mathbf{p}_\Delta$ allows us to avoid using a highly non-trivial fact (which would be necessary would we have continuity only), namely that for each Δ, N and n, the vector obtained by the Bethe Ansatz from the solution $\mathbf{p}_\Delta$ to (BE_Δ) is non-zero. This is necessary to deduce that the associated value $\Lambda(\mathbf{p}_\Delta)$ is indeed an eigenvalue of the transfer-matrix. Let us mention that Goldbaum proves that the vector obtained by the Bethe Ansatz for the 1D Hubbard model is indeed non-zero for every Δ. The proof relies on a symmetry of the model which is not satisfied by the six-vertex model. Kozlowski claims a similar result for the XXZ chain in [20].

3.2. From the Bethe Equation to the six-vertex model: proof of Theorem 1.3

The goal of this section is the proofs of Theorem 1.3 and Corollary 1.4.

Theorem 1.3. – We divide the proof in three steps. We first treat relation (1.6). We then focus on (1.7) with $r > 0$, even, and finally treat the case of (1.7) with $r > 0$, odd. Note that (1.7) with $r < 0$ follows directly from $r > 0$ since the transfer matrix V is invariant under global arrow flip, and therefore, the spectrums of V on Ω_n and Ω_{N-n} are identical.

Fix $c > 2$ and recall that $\Delta = \frac{2-c^2}{2} < -1$. Generically, in this proof $\mathbf{p} = \mathbf{p}_\Delta(N)$ and $\tilde{\mathbf{p}} = \tilde{\mathbf{p}}_\Delta(N)$ are given by Theorem 2.3 applied to $\Delta_0 = \Delta$ and $n = N/2$ and $N/2 - r$ respectively. We will always assume N to be a multiple of 4 (in particular $N/2$ is even). For clarity, we will drop N and Δ from the notation and write $n = N/2 - r$.

3.2.1. *Proof of* (1.6). – The Bethe Ansatz and Corollary 3.4 imply that

$$\Lambda_0(N) := 2\prod_{j=1}^{n} |M(e^{ip_j})|,$$

where we used above that $\mathbf{p}$ is symmetric with respect to the origin and that $L(z) = M(\bar{z})$ for $|z| = 1$ to deduce both products in the expression in Theorem 3.1 are equal to the product of the $|M|$. By Theorem 2.6, we deduce[13] that

$$\lim_{N\to\infty} \frac{1}{N} \log \Lambda_0(N) = \int_{-\pi}^{\pi} \log \left|M\left(e^{ix}\right)\right| \rho(x)dx. \tag{3.6}$$

The explicit form of ρ enables us to compute this integral explicitly via Fourier analysis (see Section 4 for details) to obtain the result.

3.2.2. *Proof of* (1.7)*, case $r > 0$ even.* – In this case, both $N/2$ and n are even, so that the Bethe Ansatz together with Corollary 3.4 imply that

$$\frac{\Lambda_r(N)}{\Lambda_0(N)} = \underbrace{\prod_{j=1}^{n} \frac{|M\left(e^{i\tilde{p}_j}\right)|}{|M\left(e^{ip_{j+r/2}}\right)|}}_{(1)} \cdot \underbrace{\Big(\prod_{j=1}^{r/2} |M\left(e^{ip_j}\right)|\Big)^{-2}}_{(2)}, \tag{3.7}$$

where again we used that $\mathbf{p}$ is symmetric with respect to the origin to group the two products into a single one.

We study the two terms separately. The term (2) converges to $|\Delta|^{-r}$, since M is continuous, $M(-1) = \Delta$ and the first $r/2$ coordinates of $\mathbf{p}$ converge to $-\pi$ as $N \to \infty$. As for the first term, by taking the logarithm and using that μ_N converges weakly (by Theorem 2.6) and $f_{\mathbf{p}(N),\tilde{\mathbf{p}}(N)}$ converges uniformly (by Theorem 2.9), we deduce that (1) converges to

$$\exp\Big(r\int_{-\pi}^{\pi} \ell'(x)\tau(x)dx\Big), \tag{3.8}$$

where $\ell(x) := \log|M(e^{ix})|$. Note that $\ell'(x)$ behaves like $1/|x|$ near the origin. Nonetheless, this does not raise any issue here since by Theorem 2.9, $f_{\mathbf{p}(N),\tilde{\mathbf{p}}(N)}(x) \le C|x|$ uniformly in N; thus, $\ell'(x)\tau(x)$ is uniformly bounded, and the weak convergence applies.

The explicit forms of τ and ℓ lead to the expression in the statement of Theorem 1.3, thus concluding the proof. The relevant computation is based on Fourier analysis and is deferred to Section 4.

(13) One should be wary of the log singularity at 0 of $\log|M|$. However, since $\log|M|$ is in $L^1[(-\pi,\pi)]$, standard truncation techniques are sufficient to show the convergence of the sum to the integral above. In particular one uses that the p_j's are well-separated – that is that $p_{j+1} - p_j \ge \pi/N$ for all sufficiently large values of N, which follows from (BE$_\Delta$) and the monotonicity of Θ – to ensure that there are not too many p_j's near the origin.

3.2.3. *Proof of* (1.7)*, case* $r > 0$ *odd.* – In this case $N/2$ is even and n is odd. The Bethe Ansatz and Corollary 3.4 imply that

$$\frac{\Lambda_r(N)}{\Lambda_0(N)} = \underbrace{\frac{2 + c^2(N-1) + c^2 \displaystyle\sum_{j \neq (n+1)/2} \partial_1 \Theta(0, \tilde{p}_j)}{2}}_{(A)} \underbrace{\frac{\displaystyle\prod_{j=1: j \neq (n+1)/2}^{n} |M\left(e^{i\tilde{p}_j}\right)|}{\displaystyle\prod_{j=1}^{N/2} |M\left(e^{ip_j}\right)|}}_{(B)}.$$

(The 2 in the denominator of the first fraction comes from the fact that $\Lambda_0(N)$ involves two products, whereas $\Lambda_r(N)$ only contains one.) First, observe that the weak convergence of $\tilde{p}_j$ and (cBE$_\Delta$) implies that

$$(A) = \frac{Nc^2}{2}\Big(1 + \int_{-\pi}^{\pi} \partial_1 \Theta(0, x)\rho(x)dx + o(1)\Big) = \frac{c^2}{2} 2\pi\rho(0)N + o(N).$$

We now focus on (B) and divide it into four terms

$$(B) = \underbrace{\prod_{\substack{j=1 \\ j \neq (n+1)/2}}^{n} \frac{|M\left(e^{i\tilde{p}_j}\right)|}{|M\left(e^{i\hat{p}_j}\right)|}}_{(1)} \cdot \underbrace{\Big(|M(e^{ip_{(r+1)/2}})| \prod_{j=1}^{(r-1)/2} |M\left(e^{ip_j}\right)|^2\Big)^{-1}}_{(2)} \cdot \underbrace{|M(e^{ip_{N/2}})|^{-1}}_{(3)}$$

$$\cdot \underbrace{\prod_{j=1}^{n} \frac{|M\left(e^{i\hat{p}_j}\right)|}{\sqrt{|M\left(e^{ip_{j+(r-1)/2}}\right)||M\left(e^{ip_{j+(r+1)/2}}\right)|}}}_{(4)},$$

where $\hat{p}_j = \frac{1}{2}(p_{j+(r-1)/2} + p_{j+(r+1)/2})$. To obtain the terms (2) and (3), we have used that $p_{N/2+1-j} = -p_j$. The same arguments as in the previous case imply that (1) converges to $\exp(r \int_{-\pi}^{\pi} \ell'(x)\tau(x)dx)$ and (2) to $|\Delta|^{-r}$. Furthermore, symmetry and (2.6) imply [14] that for each fixed k,

$$p_{N/2+k} = \frac{k - 1/2}{\rho(0)N} + O\Big(\frac{k^2}{N^2}\Big),$$

where $O(\cdot)$ is uniform in k and N. Since $|M(e^{ip})| = c^2/|p| + o(1)$ for p close to 0, we deduce that

$$(3) = \frac{1}{2\rho(0)c^2 N} + o\Big(\frac{1}{N}\Big),$$

$$(4) = \prod_{k=N/2+1}^{\infty} \frac{4p_k p_{k+1}}{(p_k + p_{k+1})^2} + o(1) = \prod_{k=1}^{\infty}\Big(1 - \frac{1}{4k^2}\Big) + o(1) = \frac{2}{\pi} + o(1).$$

(In approximating (4), we used (2.6) to control $p_{N/2+k+1} - p_{N/2+k}$ for $k \geq N^{1/2}$.) Combining the estimates above and appealing to the computation of $\int_{-\pi}^{\pi} \ell'(x)\tau(x)dx$ in Section 4, we obtain the expected result. □

[14] We used that $p_{N/2+1} = \frac{1}{2}(p_{N/2+1} - p_{N/2}) = \frac{1}{2\rho(0)N} + O\Big(\frac{1}{N^2}\Big)$ and $p_{N/2+k+1} - p_{N/2+k} = \frac{1}{\rho(0)N} + O\Big(\frac{k}{N^2}\Big)$.

We now prove Corollary 1.4. The proof consists in two steps. We first prove that the free energy exists and that it is related to the sum of weighted configurations that are "balanced". We then relate the latter to the rate of growth of the Perron-Frobenius eigenvalue of $V_N^{[N/2]}$.

The proof of the existence of the free energy is slightly tedious due to the fact that the six-vertex model does not enjoy the finite-energy property. Nevertheless, it is close in spirit to corresponding proofs for other models. The next section enables us to deduce the existence of the limit along N and M even using the connection to the random-cluster model. Nonetheless, we believe that a direct proof is of value.

The proof below is not connected to other arguments in this paper except through its result. We encourage the reader mostly interested in Theorems 1.1 and 1.2 to skip this proof.

Corollary 1.4. – Step 1: Existence of free energy. – For $N, M \in \mathbb{N}$, let $\mathsf{R}_{N,M}$ be the subgraph of $\mathbb{Z}^2$ with vertex set $V(\mathsf{R}_{N,M}) = \{1, \dots, N\} \times \{1, \dots, M\}$ and edge-set $E(\mathsf{R}_{N,M})$ formed of all edges of $\mathbb{Z}^2$ with both endpoints in $V(\mathsf{R}_{N,M})$. Define the *edge-boundary* of $\mathsf{R}_{N,M}$ as the set $\partial_e \mathsf{R}_{N,M}$ of edges of $\mathbb{Z}^2$ with exactly one endpoint in $V(\mathsf{R}_{N,M})$.

A six-vertex configuration on $\mathsf{R}_{N,M}$ is an assignment of directions to each edge of $E(\mathsf{R}_{N,M}) \cup \partial_e \mathsf{R}_{N,M}$. For such a configuration $\vec{\omega}$, the weight is computed as on the torus:

$$w(\vec{\omega}) = a^{n_1+n_2} \cdot b^{n_3+n_4} \cdot c^{n_5+n_6},$$

where $n_1, \dots, n_6$ are the numbers of vertices of $\mathsf{R}_{N,M}$ of types $1, \dots, 6$ respectively (as on the torus, we implicitly assign weight 0 to configurations not obeying the ice rule). As in the rest of the paper, we fix $a = b = 1$ and $c > 0$.

A *boundary condition* ξ for $\mathsf{R}_{N,M}$ is an assignment of directions to each edge of $\partial_e R_{N,M}$. Let

$$Z^{\xi}_{N,M} = \sum_{\vec{\omega}} w(\vec{\omega}) \mathbf{1}_{\{\vec{\omega}(e)=\xi(e)\ \forall e \in \partial_e \mathsf{R}_{N,M}\}}.$$

Here, we are effectively summing only over configurations which agree with ξ on the edge-boundary. Observe that configurations obeying the ice-rule and consistent with ξ exist only when $\mathsf{R}_{N,M}$ has as many incoming as outgoing edges in ξ.

Some boundary conditions ξ are called *toroidal* if $\xi(e) = \xi(f)$ for any boundary edges e and f of $\mathsf{R}_{N,M}$ such that f is a translate of e by $(0, M)$ or $(N, 0)$. When N is even, some toroidal boundary conditions ξ are called *balanced* if they contain exactly $N/2$ up arrows on the lower row of $\partial_e \mathsf{R}_{N,M}$. Using this notation, the partition function of the six-vertex model on $\mathbb{T}_{N,M}$ may be expressed as

$$Z_{N,M} = \sum_{\vec{\omega} \in \mathbb{T}_{N,M}} w(\vec{\omega}) = \sum_{\xi:\,\text{toroidal}} Z^{\xi}_{N,M},$$

where the second sum is over all toroidal boundary conditions ξ on $\mathsf{R}_{N,M}$. Moreover, set

$$Z^{(\text{bal})}_{N,M} = \sum_{\xi:\,\text{balanced}} Z^{\xi}_{N,M},$$

the sum now being only over balanced toroidal boundary conditions. Our goal is to prove that the following limits exist:

$$\lim_{N,M\to\infty} \frac{1}{MN} \log Z_{N,M} = \lim_{\substack{N,M\to\infty \\ N \text{ even}}} \frac{1}{MN} \log Z^{(\text{bal})}_{N,M}. \tag{3.9}$$

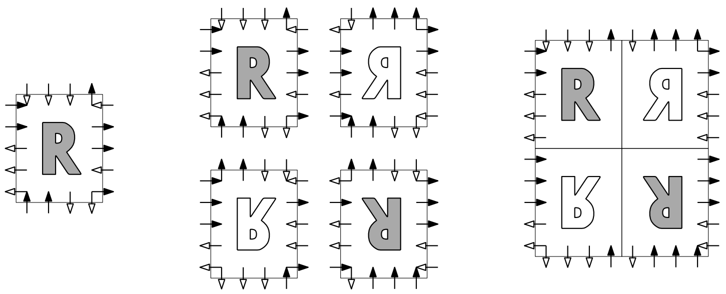

FIGURE 4. The passage from boundary conditions ξ to balanced toroidal boundary conditions ζ. The letter R inside the rectangles is only to indicate the performed transformations (rotations, reflections and reversal of all arrows).

Above, the limits can be taken in whichever order we desire. We leave it as a simple exercise to the reader to check that (3.9) can be easily deduced from the following lemma.

LEMMA 3.6. – (i) *The following inequality holds:*

$$Z_{2N,2M} \geq Z^{(\mathrm{bal})}_{2N,2M} \geq (\tfrac{1}{16})^{M+N}\big(Z_{N,M}\big)^4.$$

(ii) *There exists $C > 0$ such that for all integers $n > N$ and $m > M$ with n and N even,*

$$\frac{1}{nm}\log Z^{(\mathrm{bal})}_{n,m} \geq \frac{1}{MN}\log Z^{(\mathrm{bal})}_{N,M} - C\Big(\frac{N}{n} + \frac{M}{m}\Big).$$

REMARK 3.7. – This lemma may also be used to show that the free energy of the six-vertex model with "free boundary conditions" (i.e., with partition function $\sum_\xi Z^\xi_{N,M}$ with sum over all boundary conditions) is equal to $f(1,1,c)$.

Lemma 3.6. – (i) Before proceeding to the proof, observe that the weight of a configuration is invariant under horizontal and vertical reflections and rotations by π of the configuration, as well as under the inversion of all arrows. It follows that if ξ is some boundary condition on some rectangle $\mathsf{R}_{N,M}$ and ξ' is the boundary condition obtained from ξ via one of the operations mentioned above, then

$$Z^{\xi'}_{N,M} = Z^{\xi}_{N,M}.$$

Let N, M be integers and ξ be boundary conditions on $\mathsf{R}_{N,M}$. The construction below is described in Fig. 4. Let ξ_1 be the boundary conditions on $\mathsf{R}_{N,M}$ obtained from ξ by horizontal reflection and arrow reversal, ξ_2 be obtained from ξ by vertical reflection and arrow reversal and ξ_3 be obtained from ξ by rotation by π. Let ζ be the toroidal boundary condition on $\mathsf{R}_{2N,2M}$ composed as follows:

— the top half of the left side agrees with ξ,

— the bottom half of the left side agrees with ξ_2,

— the left half of the top side agrees with ξ and

— the right half of the top side agrees with ξ_1.

Note that ζ is balanced so that

$$Z_{2N,2M} \geq Z^{(\mathrm{bal})}_{2N,2M} \geq Z^{\zeta}_{2N,2M}.$$

Upon inspection of Fig. 4, one may easily deduce that for any boundary condition ξ on $\mathsf{R}_{N,M}$,

$$Z^{(\mathrm{bal})}_{2N,2M} \geq Z^{\zeta}_{2N,2M} \geq Z^{\xi}_{N,M} Z^{\xi_1}_{N,M} Z^{\xi_2}_{N,M} Z^{\xi_3}_{N,M} = \big(Z^{\xi}_{N,M}\big)^4.$$

By summing over the 2^{N+M} toroidal boundary condition ξ, the result follows.

(ii) Write $n = aN + r$ and $m = bM + q$ with $0 \leq r < N$ and $0 \leq q < M$. Fix a balanced toroidal boundary condition ξ on $\mathsf{R}_{N,M}$. The construction that follows is described in Fig. 3.2.

Let ξ_1 be the toroidal boundary conditions on $\mathsf{R}_{aN,bM}$ obtained by repeating a times each horizontal side and b times each vertical side of ξ on the corresponding sides of ξ_1.

Let ξ_2 be the toroidal boundary conditions on $\mathsf{R}_{r,bM}$, equal to ξ_1 on the vertical sides, with $r/2$ down arrows amassed to the left of the bottom (and top) side, completed by $r/2$ up arrows at the right of the bottom (and top) side.

Finally, define ξ_3 to be the toroidal boundary conditions on $\mathsf{R}_{n,q}$ with only left-pointing arrows on the vertical sides, equal to the top of ξ_1 for the left-most aN arrows of both the top and bottom sides and equal to top of ξ_2 for the remaining r right-most arrows of the top and bottom sides.

Set ζ to be the boundary conditions obtained from the gluing of ξ_1, ξ_2 and ξ_3, that is:

- the top and bottom sides of ζ are equal to those of ξ_3,
- the bottom bM arrows of the left and right sides of ζ are equal to those of ξ_1,
- the top q arrows of the left and right sides of ζ are pointing leftwards.

We thus easily deduce that

$$Z^{\zeta}_{n,m} \geq Z^{\xi_1}_{aN,bM} Z^{\xi_2}_{r,bM} Z^{\xi_3}_{n,q} \geq \big(Z^{\xi}_{N,M}\big)^{ab} Z^{\xi_2}_{r,bM} Z^{\xi_3}_{n,q}.$$

It remains to prove a lower bound on the last two terms. Observe that there exists at least one configuration $\vec{\omega}_3$ on $\mathsf{R}_{aN+r,q}$, agreeing with the boundary conditions ξ_3 and having non-zero weight. It is obtained by setting all horizontal edges pointing left and all rows of vertical edges being identical to the top of ξ_3. This proves that

$$Z^{\xi_3}_{aN+r,q} \geq w(\vec{\omega}_3) \geq \min\{1, c\}^{(aN+r)q}.$$

A slightly more involved construction is necessary to exhibit a configuration $\vec{\omega}_2$ on $\mathsf{R}_{r,bM}$, consistent with ξ_2 and with non-zero weight. We represent it in Fig. 3.2 and leave it to the meticulous reader to check the details of its construction. It follows that

$$Z^{\xi_2}_{r,bM} \geq w(\vec{\omega}_2) \geq \min\{1, c\}^{bMr}.$$

We conclude that

$$Z^{(\mathrm{bal})}_{n,m} \geq Z^{\zeta}_{n,m} \geq \big(Z^{\xi}_{N,M}\big)^{ab} \min\{1, c\}^{bMr+aNq+rq}.$$

By choosing ξ maximizing $Z^{\xi}_{N,M}$, we deduce that

$$Z^{(\mathrm{bal})}_{n,m} \geq \big(Z^{(\mathrm{bal})}_{N,M}\big)^{ab} \min\{1, c\}^{bMr+aNq+rq} \big(\tfrac{1}{2}\big)^{ab(M+N)}.$$

The result follows by taking the logarithm. $\square$

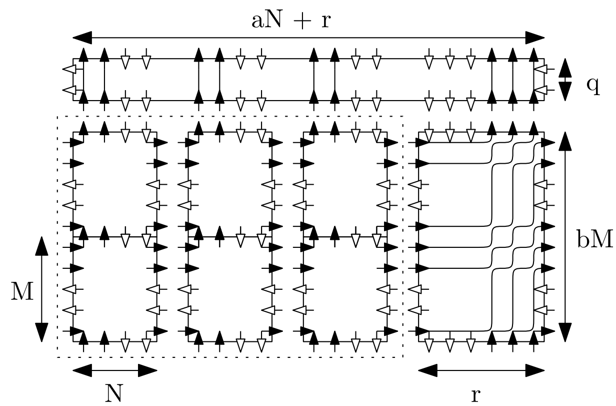

FIGURE 5. The block of size $\mathsf{R}_{aN,bM}$ with balanced toroidal boundary conditions ξ_1 is encircled; on its right is the block $\mathsf{R}_{r,bM}$ with boundary conditions ξ_2 and above is the block $\mathsf{R}_{aN+r,q}$ with boundary conditions ξ_3. In the two latter blocks, examples of configurations with positive weight are given (only the up and right-pointing edges are drawn in the interior of the blocks).

Calculation of the free energy. – Recall Proposition 2.1 of [8] and more specifically equation (3.1), that expresses $Z_{N,M}$ as the trace of V^M. A straightforward adaptation shows that, for all N multiple of 4 and even,

$$Z^{(\mathrm{bal})}_{N,M} = \mathrm{Tr}\Big[\big(V^{[N/2]}\big)^M\Big] = \lambda_0^M + \lambda_1^M + \cdots ,$$

where $\lambda_0, \lambda_1, \ldots$ are the $\binom{N}{N/2}$ eigenvalues of the diagonalizable matrix $V^{[N/2]}$, listed with multiplicity and indexed such that $|\lambda_0| \ge |\lambda_1| \ge \cdots$. Since $V^{[N/2]}$ is a Perron-Frobenius matrix, $|\lambda_0| > |\lambda_1|$ and $\lambda_0 = \Lambda_0(N)$ (the eigenvalue computed in Theorem 1.3), so that

$$\lim_{M\to\infty} \frac{1}{M} \log \mathrm{Tr}\,\big(V^{[N/2]}\big)^M = \log \Lambda_0(N).$$

In light of (3.9), the limit defining $f(1,1,c)$ may be taken with $M \to \infty$ first, then $N \to \infty$ along multiples of 4. Thus we find,

$$f(1,1,c) = \lim_{\substack{N\to\infty \\ N\in 4\mathbb{N}}} \frac{1}{N} \log \Lambda_0(N) \overset{(1.6)}{=} \frac{\lambda}{2} + \sum_{m=1}^{\infty} \frac{e^{-m\lambda}\tanh(m\lambda)}{m}. \qquad \square$$

REMARK 3.8. – We have shown that the free energy for the torus is the same as that for "free" boundary conditions. However, it is possible to construct boundary conditions on rectangles that lead to nonzero, but strictly smaller free energy. One prominent example of this is the Domain Wall boundary conditions, which have been studied extensively due to their relations to combinatorial objects, such as Young diagrams. Under these boundary conditions, the six-vertex model partition functions satisfy recursion relations that make it possible to exactly compute them for finite lattices (see [26] for more detail). This technique gives a formula for the free energy of this model (see[19]), which is different from the one we showed above.

3.3. From the six-vertex to the random-cluster model: proof of Theorem 1.2

The proof is split into two main steps. First, we present a classical correspondence between the six-vertex and random-cluster models using a series of intermediate representations (this correspondence may be found in [2]). Then, certain estimates on the random-cluster model are provided, that are used to relate its correlation length to quantities obtained via the six-vertex model.

3.3.1. *Correspondence between the random-cluster and six-vertex models*. – Fix two integers M, N, both even and $q > 4$. Notice that the torus $\mathbb{T}_{N,M}$ is then a bipartite graph. Let $V_\circ(\mathbb{T}_{N,M})$ and $V_\bullet(\mathbb{T}_{N,M})$ be a partition of the vertices of the graph $\mathbb{T}_{N,M} + (\frac{1}{2}, \frac{1}{2})$ (that is, $\mathbb{T}_{N,M}$ translated by $(\frac{1}{2}, \frac{1}{2})$), each containing no adjacent vertices. Define the graphs $\mathbb{T}^\diamond_{N,M}$ and $(\mathbb{T}^\diamond_{N,M})^*$ as having vertex sets $V_\bullet(\mathbb{T}_{N,M})$ and $V_\circ(\mathbb{T}_{N,M})$, respectively, and having an edge between vertices u and v if u is a translation of v by $(1, 1)$ or $(-1, 1)$ (see Fig. 6). By construction, $(\mathbb{T}^\diamond_{N,M})^*$ is the dual graph of $\mathbb{T}^\diamond_{N,M}$.

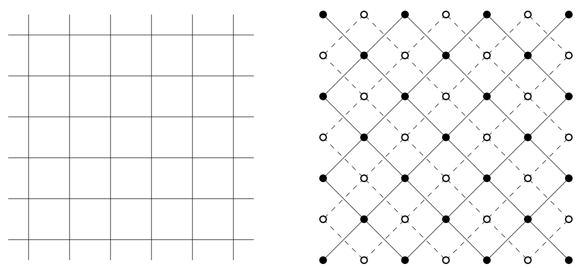

FIGURE 6. *Left:* the lattice $\mathbb{T}_{N,M}$ used for the six-vertex model. *Right:* the corresponding lattice for the random-cluster model, $\mathbb{T}^\diamond_{N,M}$ (in solid lines), and its dual (with dotted lines).

Let Ω_{RC} be the set of random-cluster configurations on $\mathbb{T}^\diamond_{N,M}$ and $\Omega_{6\mathrm{V}}$ be the set of six-vertex configurations on $\mathbb{T}_{N,M}$. We will exhibit a correspondence between Ω_{RC} and $\Omega_{6\mathrm{V}}$ that will allow us to relate the free energy and correlation length of the two models. The correspondence consists of several intermediate steps embodied by Lemmas 3.9 – 3.12; the whole process is depicted in Fig. 7. The ultimate goal of this part is Corollary 3.13, which will be the only result used in the proof of Theorem 1.2.

In linking the random-cluster and six-vertex models, we will use another type of configurations, called *loop configurations*. An oriented loop on $\mathbb{T}_{N,M}$ is a cycle on $\mathbb{T}_{N,M}$ which is edge-disjoint and non-self-intersecting. We may view oriented loops as ordered collections of edges of $E(\mathbb{T}_{N,M})$, quotiented by cyclic permutations of the indices. Un-oriented loops (or simply loops) are oriented loops considered up to reversal of the indices. A (oriented) loop configuration on $\mathbb{T}_{N,M}$ is a partition of $E(\mathbb{T}_{N,M})$ into (oriented) loops.

To each $\omega \in \Omega_{\mathrm{RC}}$ we associate a loop configuration $\omega^{(\ell)}$ as in Fig. 2. In order to do so, we first construct the dual configuration ω^* on $(\mathbb{T}^\diamond_{N,M})^*$ by setting $\omega^*(e^*) = 1 - \omega(e)$, where e^* is the edge of $(\mathbb{T}^\diamond_{N,M})^*$ intersecting the edge e of $\mathbb{T}^\diamond_{N,M}$ in its middle (in words, a dual edge is in ω^* if the corresponding edge of $\mathbb{T}^\diamond_{N,M}$ is not in ω, and vice versa). Then, consider the loop configuration $\omega^{(\ell)}$ on $\mathbb{T}_{N,M}$ created by loops that do not cross the edges of ω or ω^*. It is easy to see that $\omega \mapsto \omega^{(\ell)}$ is a bijection between Ω_{RC} and the set of all loop configurations.

Call $\ell(\omega)$ the number of different loops of $\omega^{(\ell)}$, and $\ell_0(\omega)$ the number of such loops that are not retractable (on the torus) to a point. Call $\ell_c(\omega) := \ell(\omega) - \ell_0(\omega)$, the number of retractable loops. We say that ω has a *net* if it has a cluster that winds around $\mathbb{T}^{\diamond}_{N,M}$ in both directions. Set

$$s(\omega) = \begin{cases} 0 & \text{if } \omega \text{ has no net,} \\ 1 & \text{if } \omega \text{ has a net.} \end{cases}$$

Fix $q > 4$. For $\omega \in \Omega_{\mathrm{RC}}$, define the weight of ω in the critical random-cluster model as

$$w_{\mathrm{RC}}(\omega) = p_c^{o(\omega)}(1-p_c)^{c(\omega)}q^{k(\omega)},$$

where we recall that $p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$ (see [3]).

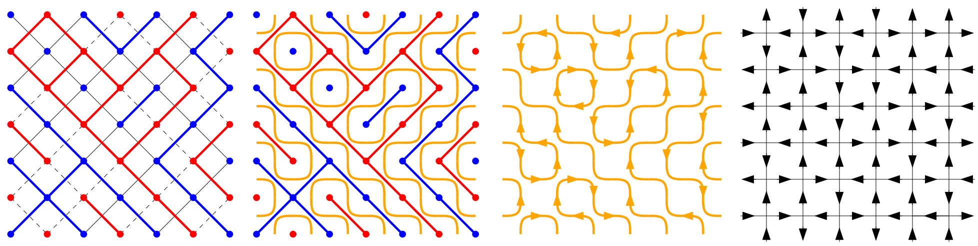

FIGURE 7. The different steps in the correspondence between the random-cluster and six-vertex models on a torus. From left to right: A random-cluster configuration and its dual, the corresponding loop configuration, an orientation of the loop configuration, the resulting six-vertex configuration. Note that in the first picture, there exist both a primal and dual cluster winding vertically around the torus; this leads to two loops that wind vertically (see second picture); if these loops are oriented in the same direction (as in the third picture), then the number of up arrows on every row of the six-vertex configuration is equal to $\frac{N}{2} \pm 1$.

LEMMA 3.9. – *For all $\omega \in \Omega_{\mathrm{RC}}$,*

$$w_{\mathrm{RC}}(\omega) = C\,\sqrt{q}^{\,\ell(\omega)+2s(\omega)},$$

where $C = q^{\frac{MN}{4}}(1+\sqrt{q})^{-MN}$ is a constant not depending on ω.

Proof. – Set $V_\bullet = V_\bullet(\mathbb{T}_{N,M})$ and $E_\bullet$ to be the set of edges of $\mathbb{T}_{N,M}$. Fix $\omega \in \Omega_{\mathrm{RC}}$. Observe that, due to the Euler formula,

$$2k(\omega) = \ell(\omega) - o(\omega) + 2s(\omega) + |V_\bullet|.$$

This relation offers us an alternative way of writing the random-cluster weight of a configuration:

$$w_{\mathrm{RC}}(\omega) = (1-p_c)^{|E_\bullet|}\left(\frac{p_c}{1-p_c}\right)^{o(\omega)}\sqrt{q}^{\,\ell(\omega)-o(\omega)+2s(\omega)+|V_\bullet|}.$$

Since $p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$, the above becomes

$$w_{\mathrm{RC}}(\omega) = \left(\frac{1}{1+\sqrt{q}}\right)^{|E_\bullet|}\sqrt{q}^{\,|V_\bullet|}\sqrt{q}^{\,\ell(\omega)+2s(\omega)} = C\,\sqrt{q}^{\,\ell(\omega)+2s(\omega)},$$

where we have used that $|E_\bullet| = MN$ and $|V_\bullet| = MN/2$. □

Write $\omega^{\circlearrowleft}$ for oriented loop configurations, $\ell_0(\omega^{\circlearrowleft})$ for the number of non-retractable loops of $\omega^{\circlearrowleft}$ and $\ell_-(\omega^{\circlearrowleft})$ and $\ell_+(\omega^{\circlearrowleft})$ for the number of retractable loops of $\omega^{\circlearrowleft}$ which are oriented clockwise and counterclockwise, respectively. We introduce $\lambda > 0$ defined by

$$e^{\lambda} + e^{-\lambda} = \sqrt{q}. \tag{3.10}$$

For an oriented loop configuration $\omega^{\circlearrowleft}$, write

$$w_{\circlearrowleft}(\omega^{\circlearrowleft}) = e^{\lambda \ell_+(\omega^{\circlearrowleft})}\, e^{-\lambda \ell_-(\omega^{\circlearrowleft})}.$$

LEMMA 3.10. – *For any $\omega \in \Omega_{\mathrm{RC}}$,*

$$w_{\mathrm{RC}}(\omega) = C\Big(\frac{\sqrt{q}}{2}\Big)^{\ell_0(\omega)} q^{s(\omega)} \sum_{\omega^{\circlearrowleft}} w_{\circlearrowleft}(\omega^{\circlearrowleft}),$$

where the sum is over the $2^{\ell(\omega)}$ oriented loop configurations $\omega^{\circlearrowleft}$ obtained by orienting each loop of $\omega^{(\ell)}$ in one of two possible ways.

Proof. – Fix $\omega \in \Omega_{\mathrm{RC}}$ and consider its associated loop configuration $\omega^{(\ell)}$. In summing the $2^{\ell(\omega)}$ oriented loop configurations $\omega^{\circlearrowleft}$ associated with $\omega^{(\ell)}$, each loop appears with both orientations. Thus,

$$\sum_{\omega^{\circlearrowleft}} w_{\circlearrowleft}(\omega^{\circlearrowleft}) = \Big(1+1\Big)^{\ell_0(\omega)}(e^{\lambda}+e^{-\lambda})^{\ell_c(\omega)} = 2^{\ell_0(\omega)}\sqrt{q}^{\ell_c(\omega)} = \frac{1}{C}\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} w_{\mathrm{RC}}(\omega).$$

$\square$

Notice now that an oriented loop configuration gives rise to 8 different configurations at each vertex. These are depicted in Fig. 8. For an oriented loop configuration $\omega^{\circlearrowleft}$, write $n_i(\omega^{\circlearrowleft})$ for the number of vertices of type i in $\omega^{\circlearrowleft}$, with $i = 1, 2, 3, 4, 5A, 5B, 6A, 6B$.

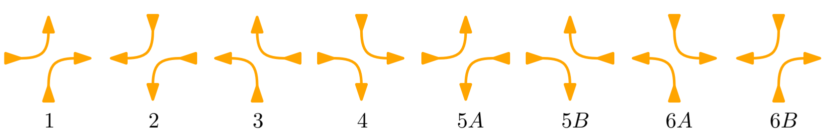

FIGURE 8. The 8 different types of vertices encountered in an oriented loop configuration.

LEMMA 3.11. – *For any oriented loop configuration $\omega^{\circlearrowleft}$,*

$$w_{\circlearrowleft}(\omega^{\circlearrowleft}) = e^{\frac{\lambda}{2}[n_{5A}(\omega^{\circlearrowleft})+n_{6A}(\omega^{\circlearrowleft})]}\, e^{-\frac{\lambda}{2}[n_{5B}(\omega^{\circlearrowleft})+n_{6B}(\omega^{\circlearrowleft})]}.$$

Proof. – Fix an oriented loop configuration $\omega^{\circlearrowleft}$. Notice that the retractable loops of $\omega^{\circlearrowleft}$ which are oriented clockwise have total winding -2π, while those oriented counterclockwise have winding 2π. Loops which are not retractable have total winding 0. Write $W(\ell)$ for the winding of a loop $\ell \in \omega^{\circlearrowleft}$. Then

$$w_{\circlearrowleft}(\omega^{\circlearrowleft}) = \exp\Big(\frac{\lambda}{2\pi}\sum_{\ell\in\omega^{\circlearrowleft}} W(\ell)\Big), \tag{3.11}$$

where the sum is over all loops ℓ of $\omega^{\circlearrowleft}$. The winding of each loop may be computed by summing the winding of every turn along the loop. The compounded winding of the two pieces of paths appearing in the different diagrams of Fig. 8 are

— vertices of type $1,\dots,4$: total winding 0;

— vertices of type $5A$ and $6A$: total winding π;

— vertices of type $5B$ and $6B$: total winding $-\pi$.

The total winding of all loops may therefore be expressed as

$$\sum_{\ell\in\omega^{\circlearrowleft}} W(\ell) = \pi\left[n_{5A}(\omega^{\circlearrowleft}) + n_{6A}(\omega^{\circlearrowleft}) - n_{5B}(\omega^{\circlearrowleft}) - n_{6B}(\omega^{\circlearrowleft})\right].$$

The lemma follows from the above and (3.11). □

For the final step of the correspondence, notice that each diagram in Fig. 8 corresponds to a six-vertex local configuration (as those depicted in Fig. 2). Indeed, configurations $5A$ and $5B$ correspond to configuration 5 in Fig. 2 and configurations $6A$ and $6B$ correspond to configuration 6 in Fig. 2. The first four configurations of Fig. 8 correspond to the first four in Fig. 2, respectively.

Thus, to each oriented loop configuration $\omega^{\circlearrowleft}$ is associated a six-vertex configuration $\vec{\omega}$. Note that the map associating $\vec{\omega}$ to $\omega^{\circlearrowleft}$ is not injective since there are $2^{n_5(\vec{\omega})+n_6(\vec{\omega})}$ oriented loop configurations corresponding to each $\vec{\omega}$.

Define the parameter c of the six-vertex model by

$$c = e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}} = \sqrt{2+\sqrt{q}}. \tag{3.12}$$

(The latter equality is obtained from (3.10) by straightforward computation.) As in the rest of the paper, $a = b = 1$ are fixed. Write $w_{6V}(\vec{\omega})$ instead of simply $w(\vec{\omega})$ for the weight of a six-vertex configuration $\vec{\omega}$ as defined in (1.4).

Lemma 3.12. – *For all six-vertex configurations $\vec{\omega}$ (that is configurations obeying the ice rule),*

$$w_{6V}(\vec{\omega}) = \sum_{\omega^{\circlearrowleft}} w_{\circlearrowleft}(\omega^{\circlearrowleft}),$$

where the sum is over all oriented loop configurations $\omega^{\circlearrowleft}$ corresponding to $\vec{\omega}$.

Proof. – Fix a six-vertex configuration $\vec{\omega}$. Let $N_{5,6}(\vec{\omega})$ be the set of vertices of type 5 and 6 in $\vec{\omega}$. Then, due to the choice of c,

$$w_{6V}(\vec{\omega}) = \prod_{u\in N_{5,6}(\vec{\omega})} \left(e^{\frac{\lambda}{2}} + e^{-\frac{\lambda}{2}}\right) = \sum_{\varepsilon\in\{\pm1\}^{N_{5,6}(\vec{\omega})}} \prod_{u\in N_{5,6}(\vec{\omega})} e^{\frac{\lambda}{2}\varepsilon(u)} = \sum_{\omega^{\circlearrowleft}} w_{\circlearrowleft}(\omega^{\circlearrowleft}).$$

For the last equality above, notice that each choice of $\varepsilon\in\{\pm1\}^{N_{5,6}(\vec{\omega})}$ corresponds to a choice of type A or B for every vertex of $N_{5,6}(\vec{\omega})$, and hence to one of the $2^{n_5(\vec{\omega})+n_6(\vec{\omega})}$ oriented loop configurations corresponding to $\vec{\omega}$. □

For a six-vertex configuration $\vec{\omega}$ on $\mathbb{T}_{N,M}$, write $|\vec{\omega}|$ for the number of up arrows on each row (recall that this number is the same on all rows). The notation obviously extends to oriented loop configurations. Moreover, for $r\ge 0$, set

$$Z_{6V}^{(r)}(N,M) = \sum_{\vec{\omega}:\,|\vec{\omega}|=\frac{N}{2}-r} w_{6V}(\vec{\omega}).$$

For $\omega \in \Omega_{\mathrm{RC}}$, let $2U(\omega)$ be the total number of times loops of $\omega^{(\ell)}$ wind vertically around $\mathbb{T}^{\diamond}_{N,M}$ (due to periodicity, this number is necessarily even).

COROLLARY 3.13. – *Let $q > 4$ and set $c = \sqrt{2+\sqrt{q}}$. Fix $r \geq 1$. For N, M even, set $C = q^{\frac{MN}{4}}(1+\sqrt{q})^{-MN}$. Then*

(i) $$\sum_{\omega\in\Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = C\ Z_{6V}(N,M);$$

(ii) $$\sum_{\omega\in\Omega_{\mathrm{RC}}:\,U(\omega)=1} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} \leq 4C\ Z^{(1)}_{6V}(N,M);$$

(iii) $$\sum_{\omega\in\Omega_{\mathrm{RC}}:\,U(\omega)\geq r} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} \geq C\ Z^{(r)}_{6V}(N,M).$$

Note that Items (ii) and (iii) imply that for $r = 1$, the left and right sides of (ii) are of the same order. Item (iii) may appear technical for $r > 1$, but will be used later to bound the correlation length of the random-cluster model from below.

Proof. – Let us start by proving (i). Due to Lemmas 3.10 and 3.12, we have

$$\sum_{\omega\in\Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = C\sum_{\omega^{\circlearrowleft}} w_{\circ}(\omega^{\circlearrowleft}) = C\sum_{\vec{\omega}} w_{6V}(\vec{\omega}) = Z_{6V}(N,M),$$

where the sums in the second and third terms run over all oriented loop configurations and six-vertex configurations, respectively.

Let us now prove (ii). We restrict ourselves to random-cluster configurations with $U(\omega) = 1$. For such configuration ω, $\omega^{(\ell)}$ has two loops winding vertically around $\mathbb{T}$. Moreover, for any oriented loop configuration $\omega^{\circlearrowleft}$ which is compatible with $\omega^{(\ell)}$, we may consider the oriented loop configuration $\tilde{\omega}^{\circlearrowleft}$, obtained from $\omega^{\circlearrowleft}$ by orienting the two vertically-winding loops downwards. Then, $w_{\circ}(\omega^{\circlearrowleft}) = w_{\circ}(\tilde{\omega}^{\circlearrowleft})$ and there are four oriented loop configurations corresponding to any $\tilde{\omega}^{\circlearrowleft}$. Thus,

$$w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} = 4C\sum_{\omega^{\circlearrowleft}} w_{\circ}(\omega^{\circlearrowleft}),$$

where the sum in the right-hand side is over oriented loop configurations corresponding to ω in which the two vertically-winding loops are oriented downwards. Since all other loops do not wind vertically around $\mathbb{T}$, the total number of up arrows on any given row of such an oriented loop configuration is $N/2-1$. Thus

$$\sum_{\omega\in\Omega_{\mathrm{RC}}:\,U(\omega)=1} w_{\mathrm{RC}}(\omega)\Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)} \leq 4C\sum_{\omega^{\circlearrowleft}:\,|\omega^{\circlearrowleft}|=N/2-1} w_{\circ}(\omega^{\circlearrowleft}) = 4C\ Z^{(1)}_{6V}(N,M).$$

Finally we show (iii). If ω° is an oriented loop configuration with $|\omega^{\circ}| = N/2 - r$, then, by the same up-arrow counting argument as above, the corresponding random-cluster configuration ω has $U(\omega) \geq r$. Thus,

$$C Z_{6V}^{(r)}(N, M) = C \sum_{\omega^{\circ}\colon |\omega^{\circ}| = N/2 - r} w_{\circ}(\omega^{\circ}) \leq \sum_{\omega \in \Omega_{\mathrm{RC}}\colon U(\omega) \geq r} w_{\mathrm{RC}}(\omega) \Big(\frac{2}{\sqrt{q}}\Big)^{\ell_0(\omega)} q^{-s(\omega)}. \qquad \square$$

3.3.2. *Random-cluster computations.* – In this section, we relate the correlation length of the random-cluster model to the rates of growth of the quantities $Z_{6V}^{(r)}(N, M)$ defined in the previous section. We will need some notation.

Let a, b be two vertices and C be a subset of vertices. Let $\{a \overset{C}{\leftrightarrow} b\}$ be the event that there exists a path of vertices in C, starting at a and finishing at b composed of edges in ω only. In this case, we say that a is *connected* to b in C. We also set $\{A \overset{C}{\leftrightarrow} B\}$ for the union on $a \in A$ and $b \in B$ of $\{a \overset{C}{\leftrightarrow} b\}$. When C is the whole graph, we omit it from the notation.

Consider the sub-lattice $\mathbb{L}$ of $\mathbb{Z}^2$ made of vertices with sum of coordinates even, and edges between two vertices if one is the translate of the other by $(1, 1)$ or $(1, -1)$ [15]. This is not the same as in the introduction, but we believe that since this change is restricted to this section, it should not lead to any confusion. We will view $\mathbb{T}^{\diamond}_{N,M}$ as having vertices (i, j) with i, j integers of even sum, taken modulo N and M respectively. Also, we write $[a, b] \times [c, d]$ for the subgraph of $\mathbb{L}$ composed of vertices (i, j) with $a \leq i \leq b$ and $c \leq j \leq d$. Let $\phi^0_{\mathbb{L}, p_c, q}$ be the infinite-volume random-cluster measure on $\mathbb{L}$ with free boundary conditions.

Write $\xi(q)$ for the correlation length of the critical random-cluster model on this rotated lattice defined by

$$\xi(q)^{-1} = \lim_{n \to \infty} -\tfrac{1}{2n} \log \phi^0_{\mathbb{L}, p_c, q}[0 \longleftrightarrow (0, 2n)]. \tag{3.13}$$

By the definition of the lattice $\mathbb{L}$ on which $\phi^0_{\mathbb{Z}^2, p_c, q}$ is defined, the right-hand side corresponds to the left-hand side of (1.2). The limit may be shown to exist by sub-additivity arguments.

The two following lemmas will be used to prove Theorem 1.2. Unlike the rest of the paper, both lemmas below are based on probabilistic estimates specific to the random-cluster model. We refer the reader to [18] for a manuscript on the subject, and [7] for an account of recent progress. We will apply repeatedly classical facts about the random-cluster model, and give each time the precise reference in [18].

LEMMA 3.14. – *For all $q \geq 1$,*

$$\lim_{N \to \infty} \lim_{M \to \infty} \frac{1}{M} \log \phi_{\mathbb{T}^{\diamond}_{N,M}, p_c, q} \Big[\big(\tfrac{2}{\sqrt{q}}\big)^{\ell_0(\omega)} q^{-s(\omega)} \Big] = 0. \tag{3.14}$$

LEMMA 3.15. – *For all $q \geq 1$ and $r \geq 1$, we have that*

$$\liminf_{N \to \infty} \liminf_{M \to \infty} \frac{1}{M} \log \phi_{\mathbb{T}^{\diamond}_{N,M}, p_c, q}(U(\omega) = 1) \geq -\xi(q)^{-1}, \tag{3.15}$$

$$\limsup_{N \to \infty} \limsup_{M \to \infty} \frac{1}{M} \log \phi_{\mathbb{T}^{\diamond}_{N,M}, p_c, q}(U(\omega) \geq r) \leq -(r-1)\xi(q)^{-1}. \tag{3.16}$$

[15] This lattice is the local limit of the graphs $\mathbb{T}^{\diamond}_{N,M}$ as M and N tend to infinity. It is a version of $\sqrt{2}\mathbb{Z}^2$ rotated by an angle of $\pi/4$.

REMARK 3.16. – Inequality (3.15) should actually be an equality. Unfortunately, we did not manage to derive the reverse inequality using the random-cluster model only. In order to circumvent this fact, in the proof of Theorem 1.2 we will rely on (3.16) (see Remark 3.18).

In both proofs below, $q \geq 1$ and $p = p_c(q)$ are fixed, and we drop them from the notation of the random-cluster measure.

Lemma 3.14. – Fix $q \geq 1$. Since $q^{-s(\omega)} \geq q^{-1}$, it is sufficient to prove that

$$\lim_{N\to\infty}\lim_{M\to\infty}\tfrac{1}{M}\log\phi_{\mathbb{T}^\diamond_{N,M}}\Big[\big(\tfrac{2}{\sqrt{q}}\big)^{\ell_0(\omega)}\Big] = 0.$$

Fix $\delta > 0$. To start, we will bound $\phi_{\mathbb{T}^\diamond_{N,M}}(\ell_0(\omega) \geq \delta M)$.

By closing all the edges intersecting $\mathbb{R} \times \{-\frac{1}{2}\}$ and $\{-\frac{1}{2}\} \times \mathbb{R}$, we transform the random-cluster model on $\mathbb{T}^\diamond_{N,M}$ into the random-cluster model with free boundary conditions on the rectangle $\mathrm{R}^\diamond_{N,M} = [0, N-1] \times [0, M-1]$. The finite-energy property [18, Eq. (3.4)] implies the existence of a constant $\mathbf{c} > 0$ independent of N, M and δ such that

$$\phi_{\mathbb{T}^\diamond_{N,M}}(\ell_0(\omega) \geq \delta M) \leq \mathbf{c}^{M+N}\phi^0_{\mathrm{R}^\diamond_{N,M}}(\exists n \text{ disjoint clusters crossing } \mathrm{R}_{N,M} \text{ horizontally}), \tag{3.17}$$

where $n = \delta M - N$. The appearance of $-N$ in the definition of n is due to the fact that at most N of the $\ell_0(\omega)$ non-retractable loops intersect the horizontal line $\mathbb{R} \times \{-\frac{1}{2}\}$.

For $x_1, \dots, x_r$ on the left side ∂_L of $\mathrm{R}^\diamond_{N,M}$, let $H(x_1, \dots, x_r)$ be the event that x_j is connected to the right side ∂_R of $\mathrm{R}^\diamond_{N,M}$ for $j = 1, \dots, r$ and that the clusters of $x_1, \dots, x_r$ are all distinct. If ω is a configuration contributing to the right-hand side of (3.17), then there exist n points $x_1, \dots, x_n$ on ∂_L such that $H(x_1, \dots, x_n)$ occurs.

Write C_{x_j} for the cluster of the point x_j. Then, for any $j \geq 1$ and any subset C of vertices of $\mathrm{R}^\diamond_{N,M}$, we have that

$$\begin{aligned}\phi^0_{\mathrm{R}^\diamond_{N,M}}\big[H(x_1, \dots, x_{j+1})\big|H(x_1, \dots, x_j), \bigcup_{i\leq j}\mathsf{C}_{x_i} = C\big] &= \phi^0_{\mathrm{R}^\diamond_{N,M}\setminus C}(x_{j+1} \longleftrightarrow \partial_R)\\ &\leq \phi^0_{\mathbb{L}}(0 \longleftrightarrow \partial\Lambda_N),\end{aligned}$$

where in the first equality, we used the domain Markov property [16] [18, Lem. 4.13] and in the second, the comparison between boundary conditions [18, Lem. 4.14] and the invariance under translations of $\phi^0_{\mathbb{L}}$ [18, Thm. 4.19]. By summing over possible values of C, we deduce that

$$\phi^0_{\mathrm{R}^\diamond_{N,M}}\big[H(x_1, \dots, x_{j+1})\big|H(x_1, \dots, x_j)\big] \leq \phi^0_{\mathbb{L}}(0 \longleftrightarrow \partial\Lambda_N).$$

Induction on $j < n$ implies that

$$\phi^0_{\mathrm{R}^\diamond_{N,M}}[H(x_1, \dots, x_n)] \leq \phi^0_{\mathbb{L}}(0 \longleftrightarrow \partial\Lambda_N)^n.$$

[16] This argument is classical and involves the fact that the cluster of a point is measurable in terms of edges with one or two endpoints in that cluster (see Fig. 9 for an illustration of this argument).

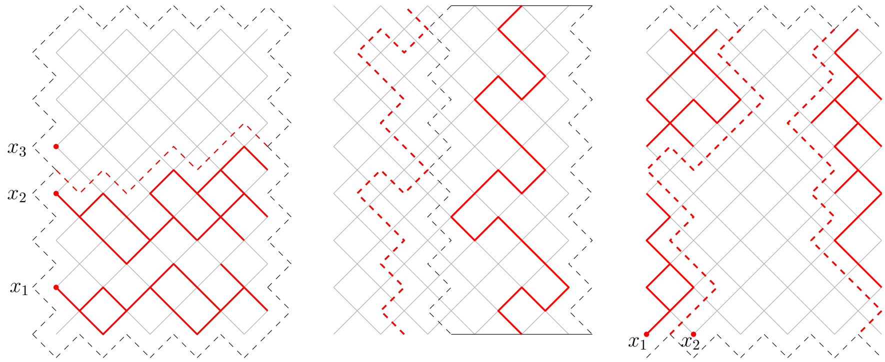

FIGURE 9. *Left:* Exploring one by one the disjoint, horizontally crossing clusters contributing to (3.17). Each new cluster (for instance the one of x_3) is surrounded by free boundary conditions. *Middle:* To create ω with $U(\omega) = 1$, it is sufficient to ensure that B occurs (dotted red line), and that, conditionally on B, C also occurs. The latter is more likely than the occurrence of a top-bottom crossing in the black rectangle with free boundary conditions on the lateral sides. *Right:* In exploring $H(x_1,\dots,x_r)$, every cluster crossing vertically the torus (except the first) is surrounded by free boundary conditions.

After taking the union over all possible $x_1,\dots,x_n$ on ∂_L, we deduce from (3.17) that

$$\begin{aligned}\phi_{\mathbb{T}^\diamond_{N,M}}(\ell_0(\omega)\geq\delta M) &\leq \mathbf{c}^{M+N}\times\binom{M}{n}\times\phi^0_{\mathbb{L}}(0\longleftrightarrow\partial\Lambda_N)^n\\ &\leq \left(2\mathbf{c}^2\left[\phi^0_{\mathbb{L}}(0\longleftrightarrow\partial\Lambda_N)\right]^{\delta/2}\right)^M,\end{aligned}$$

where we bound $\binom{M}{n}$ by 2^M, and increase M until $n > \delta M/2$. Now, it is classical [18, Thm. 6.17] that $\phi^0_{\mathbb{L}}(0\longleftrightarrow\partial\Lambda_N)$ tends to 0 as N tends to infinity so that for N large enough,

$$\phi_{\mathbb{T}^\diamond_{N,M}}(\ell_0(\omega)\geq\delta M)\leq\left(\frac{1}{2}\right)^M.$$

This implies that for any $\delta>0$, provided that N is large enough,
(3.18)

$$\limsup_{M\to\infty}\tfrac{1}{M}\left|\log\phi_{\mathbb{T}^\diamond_{N,M}}\left[\left(\tfrac{2}{\sqrt{q}}\right)^{\ell_0(\omega)}\right]\right|\leq\limsup_{M\to\infty}\tfrac{1}{M}\left|\log\left[\left(\tfrac{2}{\sqrt{q}}\right)^{\delta M}+\left(\tfrac{1}{\sqrt{q}}\right)^M\right]\right|\leq\left|\log\left(\tfrac{\sqrt{q}}{2}\right)\right|\delta,$$

which concludes the proof by letting δ tend to 0. □

Before starting the proof of Lemma 3.15, we wish to highlight the fact that the random-cluster model enjoys a self-duality relation for planar graphs when $p=p_c$ [18, Sec. 6.1]. On the torus, this self-duality can be restated as follows. Consider the measure

$$\widetilde{\phi}_{\mathbb{T}^\diamond_{N,M}}(\omega)=\frac{p_c^{o(\omega)}(1-p_c)^{c(\omega)}q^{k(\omega)}q^{-s(\omega)}}{\widetilde{Z}(N,M)},$$

where $\widetilde{Z}(N, M)$ is the appropriate partition function. If ω is sampled according to $\widetilde{\phi}_{\mathbb{T}^{\diamond}_{N,M}}(\omega)$, then ω^* is sampled according to the measure on $(\mathbb{T}^{\diamond}_{N,M})^*$ obtained by translating $\widetilde{\phi}_{\mathbb{T}^{\diamond}_{N,M}}(\omega)$ by $(1, 0)$ (this claim follows directly from Lemma 3.9).

Also note that $(\mathbb{T}^{\diamond}_{N,M})^*$ can be obtained from $\mathbb{T}^{\diamond}_{N,M}$ from reflections from either vertical or horizontal lines. We will use this observation several times in the next proof to transfer the probability of events defined in terms of ω to similar claims for ω^* (and vice versa).

(3.15) *of Lemma 3.15.* – For a rectangle R, let V_R (resp. H_R) be the event that there exists a path in ω included in R from the bottom to the top of R (resp. from the left to the right). We begin by proving [17] that there exists a constant $c > 0$ such that for any n, N, M with $3n \le \min\{N, M\}$,

$$\phi_{\mathbb{T}^{\diamond}_{N,M}}(V_{[0,3n]\times[0,n]}) \ge c. \tag{3.19}$$

Indeed, if this is not the case, then the probability that some rectangle $[0, 3n] \times [0, n]$ contains a path in ω^* from the left to the right is larger than $1 - c$. Therefore, the self-duality and the symmetry between $\mathbb{T}^{\diamond}_{N,M}$ and its dual (mentioned above) imply that

$$\phi_{\mathbb{T}^{\diamond}_{N,M}}(H_{[0,3n]\times[0,n]}) \ge \tfrac{1-c}{q^2}.$$

The FKG inequality [18, Thm. 3.8] implies that

$$\phi_{\mathbb{T}^{\diamond}_{N,M}}\big(H_{[0,3n]\times[0,n]} \cap H_{[0,3n]\times[2n,3n]}\big) \ge \big(\tfrac{1-c}{q^2}\big)^2.$$

Now consider the bottom-most (resp. top-most) path Γ (resp. Γ') in ω crossing $[0, 3n] \times [0, n]$ (resp. $[0, 3n] \times [2n, 3n]$) from left to right. Fix two possible realizations γ and γ' of Γ and Γ'. Conditioned on $\Gamma = \gamma$ and $\Gamma' = \gamma'$, the law of edges in $[0, 3n]^2$ between γ and γ' is stochastically dominating the random-cluster measure with wired boundary conditions on the bottom and top of $[0, 3n]^2$, and free on the left and right. Therefore, one may use self-duality in the square $[0, 3n]^2$ to show that the probability that there is an open path connecting γ to γ' is larger or equal to $1/(1 + q^2)$. This reasoning is classical, we refer for instance to [3]. In particular, this path crosses $[0, 3n] \times [2n, 3n]$ from bottom to top. Overall, summing over all possible γ and γ' gives

$$\begin{aligned}\phi_{\mathbb{T}^{\diamond}_{N,M}}(V_{[0,3n]\times[2n,3n]}) &\ge \tfrac{1}{1+q^2} \times \phi_{\mathbb{T}^{\diamond}_{N,M}}\big(H_{[0,3n]\times[0,n]} \cap H_{[0,3n]\times[2n,3n]}\big)\\ &\ge \tfrac{1}{1+q^2} \times \big(\tfrac{1-c}{q^2}\big)^2.\end{aligned}$$

Provided that $c = c(q) > 0$ is chosen sufficiently small, this claim contradicts the assumption that (3.19) was wrong. In conclusion, we proved (3.19) and we can proceed with the proof of (3.15).

Fix $8n \le \min\{M, N\}$. As a consequence of (3.19), there exists $x \in [0, 3n] \times \{0\}$ and $y \in [0, 3n] \times \{n\}$ such that

$$\phi_{\mathbb{T}^{\diamond}_{N,M}}\big(x \xleftrightarrow{[0,3n]\times[0,n]} y\big) \ge \frac{c}{9n^2}.$$

[17] This claim was proved in the special case $N = M$ in [3]. Here, some additional care must be taken since the torus has different vertical and horizontal sizes.

The FKG inequality [18, Thm. 3.8] and the symmetry under reflections give that

$$\phi_{\mathbb{T}^\diamond_{N,M}}\big(x \xleftrightarrow{[0,3n]\times[0,2n]} x+(0,2n)\big) \geq \phi_{\mathbb{T}^\diamond_{N,M}}\big(x \xleftrightarrow{[0,3n]\times[0,n]} y\big) \times \phi_{\mathbb{T}^\diamond_{N,M}}\big(y \xleftrightarrow{[0,3n]\times[n,2n]} x+(0,2n)\big)$$
$$\geq \Big(\frac{c}{9n^2}\Big)^2.$$

Write $M = 2nk + r$ with $k \in \mathbb{N}$ and $0 \leq r < 2n$. We can use the FKG inequality k times to deduce that

$$\phi_{\mathbb{T}^\diamond_{N,M}}\big(x \xleftrightarrow{[0,3n]\times[0,2nk]} x+(0,2nk)\big) \geq \Big(\frac{c}{9n^2}\Big)^{M/n}.$$

Let A be the event that ω contains a loop winding vertically around $\mathbb{T}^\diamond_{N,M}$ and staying in $[0,3n]\times[0,M]$ (seen as a subgraph of $\mathbb{T}^\diamond_{N,M}$), and that every edge of $\mathbb{T}^\diamond_{N,M}$ intersecting $\mathbb{R}\times\{-\frac{1}{2}\}$ but one is closed in ω.

Since this event can be obtained from $\{x \xleftrightarrow{[0,3n]\times[0,2nk]} x+(0,2nk)\}$ by opening (in ω) a self-avoiding path of length $2n-r$ zigzaging vertically between $x+(0,2nk)$ and x, and then closing all the remaining edges intersecting $\mathbb{R}\times\{-\frac{1}{2}\}$, the finite-energy property [18, Eq. (3.4)] implies that

$$\phi_{\mathbb{T}^\diamond_{N,M}}(A) \geq \mathbf{c}^{2n+N} \times \Big(\frac{c}{9n^2}\Big)^{M/n},$$

for some constant $\mathbf{c} > 0$ only depending of q. Let B be the event that ω does not contain any path from left to right in $[0,3n]\times[0,M]$, and that every edge of $\mathbb{T}^\diamond_{N,M}$ intersecting $\mathbb{R}\times\{-\frac{1}{2}\}$ but one is open in ω. Using the self-duality and the symmetry between $\mathbb{T}^\diamond_{N,M}$ and its dual, we deduce that

$$\phi_{\mathbb{T}^\diamond_{N,M}}(B) \geq \tfrac{1}{q^2} \times \mathbf{c}^{2n+N} \times \Big(\frac{c}{9n^2}\Big)^{M/n}. \tag{3.20}$$

We are near the end: the event B induces the existence of a path in ω^* winding vertically around the torus and contained in its left half. As (3.20) indicates, this comes at a (relatively) low cost. Next we also construct a vertically winding path contained in ω, which will induce a vertically winding loop.

For each $j \in \mathbb{N}$, define $y_j := (3N/4, 2nj)$ and let C be the event that y_j is connected to y_{j+1} (in ω) for every $0 \leq j \leq M/(2n)$ [18]. Notice that the event $U(\omega) = 1$ occurs if B and C occur together. Therefore,

$$\phi_{\mathbb{T}^\diamond_{N,M}}(U(\omega)=1) \geq \phi_{\mathbb{T}^\diamond_{N,M}}(B\cap C) \geq \phi_{\mathbb{T}^\diamond_{N,M}}(B) \times \phi_{\mathbb{T}^\diamond_{N,M}}(C|B).$$

We now wish to bound the term $\phi_{\mathbb{T}^\diamond_{N,M}}(C|B)$. The comparison between boundary conditions [18, Lem. 4.14] implies that the measure on $[N/2,N]\times[0,M]$ induced by $\phi_{\mathbb{T}^\diamond_{N,M}}(\cdot|B)$ dominates the random-cluster measure $\phi^{\mathrm{mix}}_{[N/2,N]\times[0,M]}$ on $[N/2,N]\times[0,M]$ with free boundary

(18) We define y_j for every $j \in \mathbb{N}$, but we see y_j as an element of $\mathbb{T}^\diamond_{N,M}$, hence we think of $2nj$ as being taken modulo M.

conditions on the left and right sides, and wired on the top and bottom sides. Using the FKG inequality and the comparison between boundary conditions one more time, we find that

$$\begin{aligned}\phi_{\mathbb{T}^{\diamond}_{N,M}}(C|B) &\geq \prod_{j=0}^{\lfloor M/(2n)\rfloor} \phi^{\mathrm{mix}}_{[N/2,N]\times[0,M]}(y_j \longleftrightarrow y_{j+1})\\ &\geq \phi^0_{\Lambda_{N/4}}(0 \longleftrightarrow (0,2n))^{1+M/(2n)}.\end{aligned}$$

Overall, we deduce that

$$\phi_{\mathbb{T}^{\diamond}_{N,M}}(U(\omega)=1) \geq \tfrac{1}{q^2} \times \mathbf{c}^{2n+N} \times \Big(\frac{c}{9n^2}\Big)^{M/n} \times \phi^0_{\Lambda_{N/4}}(0 \longleftrightarrow (0,2n))^{1+M/(2n)}.$$

This in turn implies that

$$\liminf_{M\to\infty} \frac{1}{M} \log \phi_{\mathbb{T}^{\diamond}_{N,M}}(U(\omega)=1) \geq \tfrac{1}{n}\log(\tfrac{c}{9n^2}) + \tfrac{1}{2n}\log \phi^0_{\Lambda_{N/4}}(0 \longleftrightarrow (0,2n)).$$

As N tends to infinity (while n is fixed), $\phi^0_{\Lambda_{N/4}}$ converges to $\phi^0_{\mathbb{L}}$ [18, Thm. 4.19]. Thus

$$\liminf_{N\to\infty}\liminf_{M\to\infty} \frac{1}{M} \log \phi_{\mathbb{T}^{\diamond}_{N,M}}(U(\omega)=1) \geq \tfrac{1}{n}\log(\tfrac{c}{9n^2}) + \tfrac{1}{2n}\log \phi^0_{\mathbb{L}}(0 \longleftrightarrow (0,2n)).$$

Letting n tend to infinity yields (3.15). □

(3.16) *of Lemma 3.15*. – Fix $r \geq 1$ and consider $M, N \geq 2r$ even integers. Denote by $x_i = (2i, 0)$ (for $i = 1, \dots, N/2$) the points on the lower side of the torus $\mathbb{T}^{\diamond}_{N,M}$ and set $y_j := x_j + (1, M-1)$.

Let $\phi^0_{\mathbb{H}_{N,M}}$ be the measure on $\mathbb{T}^{\diamond}_{N,M}$ conditioned on all edges intersecting $\mathbb{R}\times\{-\frac{1}{2}\}$ being closed; it may be viewed as a random-cluster measure on a cylinder $\mathbb{H}_{N,M}$ of height M with free boundary conditions on the top and bottom.

Let $V(x_1,\dots,x_r)$ be the event that $x_j \longleftrightarrow y_j$ for $j = 1,\dots,r$ and that the clusters of $x_1,\dots,x_r$ are all distinct. The finite-energy property [18, Eq. (3.4)] implies that

$$\limsup_{M\to\infty} \frac{1}{M}\log \phi_{\mathbb{T}^{\diamond}_{N,M}}(U(\omega)\geq r) = \limsup_{M\to\infty} \frac{1}{M}\log \phi^0_{\mathbb{H}_{N,M}}[V(x_1,\dots,x_r)]. \tag{3.21}$$

Write C_{x_j} for the cluster of the point x_j. Then, for any $j \geq 1$, an exploration argument similar to that of Lemma 3.14 (and therefore omitted [19]) implies that

$$\begin{aligned}\phi^0_{\mathbb{H}_{N,M}}&[V(x_1,\dots,x_{j+1})\,|\,V(x_1,\dots,x_j)]\\ &= \phi^0_{\mathbb{H}_{N,M}}[y_{j+1}\in \mathsf{C}_{x_{j+1}} \text{ and } x_1,\dots,x_j \notin \mathsf{C}_{x_{j+1}}\,|\,V(x_1,\dots,x_j)]\\ &\leq \phi^0_{\mathbb{L}}(y_{j+1} \longleftrightarrow x_{j+1})\\ &\leq \phi^0_{\mathbb{L}}(0 \longleftrightarrow y_0),\end{aligned} \tag{3.22}$$

where $y_0 = (1, M-1)$. Applying this $r-1$ times yields

$$\phi^0_{\mathbb{H}_{N,M}}[V(x_1,\dots,x_r)] \leq \phi^0_{\mathbb{L}}[0 \longleftrightarrow y_0]^{r-1} \leq \mathbf{c}^{r-1}\times \phi^0_{\mathbb{L}}[0 \longleftrightarrow (0,M)]^{r-1}.$$

(In the second inequality, we used the finite-energy one last time). The conclusion follows from (3.21), the previous inequality, and the definition of $\xi(q)$. □

[19] It involves again the domain Markov property [18, Lem. 4.13] and the comparison between boundary conditions [18, Lem. 4.14].

REMARK 3.17. – Note that in order to obtain (3.22), we need to explore the cluster C_{x_1}, i.e., we need $j \geq 1$. Indeed, we used that conditioned on $V(x_1,\dots,x_j)$, the boundary conditions in $\mathbb{T}^{\diamond}_{N,M} \setminus (\mathsf{C}_{x_1} \cup \dots \cup \mathsf{C}_{x_j})$ are dominated by free boundary conditions at infinity. The fact that we do not obtain a bound on $\phi^0_{\mathbb{H}_{N,M}}[V(x_1)]$ (in this case, the boundary conditions are cylindrical and cannot be easily compared to the free boundary conditions at infinity) is the reason why we obtain $r-1$ instead of r in (3.16).

3.3.3. *Proof of Theorem 1.2.* – Fix $q > 4$. By [12], for points 1 and 2 it is sufficient to show that $\xi(q) < \infty$. We therefore focus on point 3, that is we compute $\xi(q)^{-1}$ explicitly and show that it is equal to

$$R(q) := \lambda + 2\sum_{k=1}^{\infty} \tfrac{(-1)^k}{k} \tanh(k\lambda) > 0,$$

where $\lambda > 0$ satisfies $e^{\lambda} + e^{-\lambda} = \sqrt{q}$. We will show that this quantity is positive and analyze its asymptotics in Section 4.

We will refer to the associated six-vertex model, with $c = \sqrt{2+\sqrt{q}}$. Write $Z_{\mathrm{RC}}(N,M)$ for the partition function of the random-cluster model with parameters p_c, q on $\mathbb{T}^{\diamond}_{N,M}$, that is

$$Z_{\mathrm{RC}}(N,M) := \sum_{\omega\in\Omega_{\mathrm{RC}}} w_{\mathrm{RC}}(\omega).$$

3.3.4. *Lower bound on the inverse correlation length.* – Equation (3.15) may be rewritten as

$$\xi(q)^{-1} \geq -\liminf_{N\to\infty}\liminf_{M\to\infty} \frac{1}{M}\log\frac{\sum_{\omega:\,U(\omega)=1} w_{\mathrm{RC}}(\omega)}{Z_{\mathrm{RC}}(N,M)}.$$

Since all configurations with $U(\omega) = 1$ have exactly two non-retractable loops and no net, Corollary 3.13 (ii) implies that the numerator above is smaller than

$$\frac{\sqrt{q}}{2} q^{\frac{MN}{4}} (1+\sqrt{q})^{-MN} Z^{(1)}_{6V}(N,M).$$

Furthermore, in light of Corollary 3.13 (i), Lemma 3.14 may be rewritten as

$$\lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{M}\log\frac{q^{\frac{MN}{4}}(1+\sqrt{q})^{-MN}\, Z_{6V}(N,M)}{Z_{\mathrm{RC}}(N,M)} = 1. \tag{3.23}$$

Therefore, we may write

$$\xi(q)^{-1} \geq -\liminf_{N\to\infty}\liminf_{M\to\infty}\frac{1}{M}\log\frac{Z^{(1)}_{6V}(N,M)}{Z_{6V}(N,M)} = -\liminf_{N\to\infty}\log\frac{\Lambda_1(N)}{\Lambda_0(N)} \overset{(1.7)}{=} R(q). \tag{3.24}$$

3.3.5. *Upper bound on the inverse correlation length.* – For all $r \geq 2$, (3.16) may be written as

$$(r-1)\xi(q)^{-1} \leq -\limsup_{N\to\infty}\limsup_{M\to\infty}\frac{1}{M}\log\frac{\sum_{\omega:\,U(\omega)\geq r} w_{\mathrm{RC}}(\omega)}{Z_{\mathrm{RC}}(N,M)}.$$

Using Corollary 3.13 (iii) and (3.23) again, we find

$$(r-1)\xi(q)^{-1} \leq -\limsup_{N\to\infty}\limsup_{M\to\infty}\frac{1}{M}\log\frac{Z^{(r)}_{6V}(N,M)}{Z_{6V}(N,M)} = -\limsup_{N\to\infty}\log\frac{\Lambda_r(N)}{\Lambda_0(N)} \overset{(1.7)}{=} rR(q).$$

The bound above being valid for all $r \geq 2$, one may divide by $r-1$ and take r to infinity. The resulting upper bound on $\xi(q)^{-1}$ matches the lower bound of (3.24), and the theorem is proved.

REMARK 3.18. – As mentioned before, (3.24) should, in fact, be an equality. This would allow us to compute $\xi(q)^{-1}$ using nothing but the asymptotics of $\Lambda_0(N)$ and $\Lambda_1(N)$, and require no control of $\Lambda_r(N)$ for $r \geq 2$. However, since we did not manage to derive the reversed inequality of (3.15) (and hence of (3.24)) using only the random-cluster model, we used (3.16) and our control of $\Lambda_r(N)$, $r \geq 2$ as an indirect route to the desired bound.

In retrospect, it may be deduced from the Theorem 1.2 that (3.15) and (3.24) are actually equalities. We believe that proving the equality in (3.15) using only the random-cluster model is an interesting question.

3.4. From the random-cluster to the Potts model: proof of Theorem 1.1

Below, we consider the Potts and random-cluster models on the standard lattice $\mathbb{Z}^2$; contrarily to previous sections, no reference to the rotated lattice is used. In particular, $\phi^0_{\mathbb{Z}^2,p,q}$ and $\phi^1_{\mathbb{Z}^2,p,q}$ are infinite-volume measures on $\mathbb{Z}^2$ (like in the introduction, and unlike in the previous section).

The results for the Potts model can be obtained from those for the random-cluster model via a classical coupling, see [13, 18]. We describe the consequences of this coupling in the theorem below; for a proof, see the references. In the next statement, the operation of *attributing a spin $s \in \{1,\dots,q\}$ to a set S of vertices* means that we fix $\sigma_x = s$ for every $x \in S$.

THEOREM 3.19. – *Fix $\beta > 0$ and an integer $q \geq 2$. Set $p = 1 - e^{-\beta}$.*

- *Consider ω with law $\phi^0_{\mathbb{Z}^2,p,q}$. Then, the law of $\sigma \in \{1,\dots,q\}^{\mathbb{Z}^2}$ obtained by attributing independently and uniformly a spin in $\{1,\dots,q\}$ to each cluster of ω is μ^0_β.*
- *Fix $i \in \{1,\dots,q\}$ and consider ω with law $\phi^1_{\mathbb{Z}^2,p,q}$. Then, the law of $\sigma \in \{1,\dots,q\}^{\mathbb{Z}^2}$ obtained by attributing independently and uniformly a spin in $\{1,\dots,q\}$ to each* finite *cluster of ω, and spin i to the* infinite *clusters*[20] *of ω is μ^i_β.*

Theorem 3.19 implies immediately the following facts.

1. The critical inverse-temperature of the Potts model and the critical parameter of random-cluster model are related by the formula $p_c = 1 - e^{\beta_c}$.
2. For any $i \in \{1,\dots,q\}$,
$$\mu^i_\beta[\sigma_0 = i] = \tfrac{1}{q} + \phi^1_{\mathbb{Z}^2,p,q}[0 \text{ is in an infinite cluster}].$$
3. For any $x, y \in \mathbb{Z}^2$,
$$\mu^0_\beta[\sigma_x = \sigma_y] = \tfrac{1}{q} + \phi^0_{\mathbb{Z}^2,p,q}[x \text{ and } y \text{ are in the same cluster}].$$

(20) There is in fact a unique one almost surely, see [18, Section 4.4.].

With these properties at hand, it is elementary to deduce Theorem 1.1 from Theorem 1.2. Theorem 1.1 (2) follows directly from items 1. and 2. above combined with (2) of Theorem 1.2. Theorem 1.1 (3) follows from item 3. above and the expression for $\xi(q)$ obtained in Theorem 1.2.

For Theorem 1.1 (1), it is well-known (see for instance results in [18]) that a Gibbs measure is extremal if and only if it is ergodic. Furthermore, the measures $\phi^0_{\mathbb{Z}^2,p,q}$ and $\phi^1_{\mathbb{Z}^2,p,q}$ are ergodic for any value of $p \in [0,1]$. Since there exists no infinite cluster $\phi^0_{\mathbb{Z}^2,p_c,q}$-almost surely (by (3) of Theorem 1.2), the construction of $\mu^0_{\beta_c}$ from $\phi^0_{\mathbb{Z}^2,p_c,q}$ described in Theorem 3.19 implies that $\mu^0_{\beta_c}$ is ergodic as well. In the same way, each measure $\mu^i_{\beta_c}$, $i = 1,\dots,q$, may be shown to be ergodic (here the existence of an infinite cluster is not problematic, since it is given the fixed spin i). By Theorem 1.1 (2), the measures $\mu^i_{\beta_c}$ induce different distributions for the spin of any given vertex, hence they are all distinct.

4. Fourier computations

In this section, we gather the computations of certain Fourier-analytic identities used throughout the paper.

4.1. Evaluation of the Fourier coefficients of Ξ_λ and R

Let $m \geq 0$ and consider the contour integral

$$\frac{1}{2\pi}\int_{C_N} \frac{\sinh(\lambda)e^{-imz}}{\cosh(\lambda)-\cos(z)}dz,$$

where C_N is the boundary of $[-\pi,\pi]+i[-N,0]$, oriented clockwise. As N goes to infinity, this integral goes to $\hat{\Xi}_\lambda(m)$. Since the only residues of the integrand in the interior of C_N occur at $-i\lambda$, we conclude that

$$\hat{\Xi}_\lambda(m) = e^{-\lambda m} \qquad m \geq 0.$$

If $m<0$, we integrate around C'_N, the boundary of $[-\pi,\pi]+i[0,N]$, oriented counterclockwise. The residue will now be at $i\lambda$, and

$$\hat{\Xi}_\lambda(m) = e^{\lambda m} \qquad m < 0.$$

Via (2.2), this implies

$$\hat{R}(m) = \frac{e^{-\lambda|m|}}{1+e^{-2\lambda|m|}} = \frac{1}{2\cosh(\lambda m)}. \tag{4.1}$$

4.2. Evaluation of the Fourier coefficients of Ψ and T

To evaluate $\hat{\Psi}$, we first note that $k(\alpha)$ is an odd function, and Θ is anti-symmetric, meaning Ψ is an odd function and $\hat{\Psi}(0)=0$. As a consequence, (2.2) implies $\hat{T}(0)=0$.

For an integer $m \neq 0$, we first replace $\Theta(k(\alpha),\pi) + \Theta(k(\alpha),-\pi)$ with the equivalent expression $2[\Theta(k(\alpha),\pi) - \pi]$ (using the fact that $\Theta(x,\pi) = \Theta(x,-\pi) + 2\pi$). Then, using integration by parts, we find

$$\begin{aligned}\hat{\Psi}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[\Theta(k(\alpha),\pi) - \pi]e^{-im\alpha}d\alpha \\ &= \frac{[\Theta(\pi,\pi) - \Theta(-\pi,\pi)](-1)^m}{-2\pi i m} + \frac{1}{2\pi i m}\int_{-\pi}^{\pi}\frac{d}{d\alpha}\Theta(k(\alpha),\pi)e^{-im\alpha}d\alpha \\ &= \frac{(-1)^m}{im} - \frac{1}{2\pi i m}\int_{-\pi}^{\pi}\Xi_{2\lambda}(\alpha-\pi)e^{-im\alpha}d\alpha \\ &= \frac{(-1)^m}{im}\left(1 - \hat{\Xi}_{2\lambda}(m)\right),\end{aligned}$$

where we used $\Theta(\pi,\pi) - \Theta(-\pi,\pi) = -2\pi$, the change of variable $u = \alpha - \pi$ and the periodicity of $\Xi_{2\lambda}$ to show that the integral in the penultimate line is equal to $2\pi(-1)^m\hat{\Xi}_{2\lambda}(m)$. Thus,

$$\hat{T}(m) = \frac{(-1)^m\left(1 - e^{-2\lambda|m|}\right)}{im\left(1 + e^{-2\lambda|m|}\right)} = \frac{(-1)^m}{im}\tanh(\lambda|m|).$$

4.3. Computations of R and T

We start with T. Pairing the terms for $\pm m$, we find[21]

$$T(\alpha) = 2\sum_{m>0}\frac{(-1)^m}{m}\tanh(\lambda m)\left(\frac{e^{im\alpha} - e^{-im\alpha}}{2i}\right) = 2\sum_{m>0}\frac{(-1)^m}{m}\tanh(\lambda m)\sin(m\alpha).$$

We now turn to R. We will show that it is equal to the sum

$$\mathscr{R}(\alpha) := \frac{\pi}{2\lambda}\sum_{r\in\mathbb{Z}}\frac{1}{\cosh[\pi(2\pi r + \alpha)/(2\lambda)]}$$

by showing that the two have the same Fourier coefficients. By direct computation and the Dominated Convergence Theorem,

$$\begin{aligned}\hat{\mathscr{R}}(m) &= \frac{1}{4\lambda}\sum_{r\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{e^{-im\alpha}d\alpha}{\cosh[\pi(2\pi r + \alpha)/(2\lambda)]} \\ &= \frac{1}{4\lambda}\int_{-\infty}^{\infty}\frac{e^{-im\alpha}d\alpha}{\cosh(\pi\alpha/2\lambda)}\end{aligned}$$

using the 2π periodicity of the numerator. Observe that the hyperbolic secant function can be written as a *continuous* Fourier transform:

$$\frac{1}{\cosh(\lambda m)} = \frac{1}{2\lambda}\int_{-\infty}^{\infty}\frac{e^{-im\alpha}d\alpha}{\cosh(\pi\alpha/2\lambda)}.$$

This concludes the proof since $\hat{R}(m) = \frac{1}{2\cosh(\lambda m)}$ by (4.1).

[21] In the formula, the series is not absolutely convergent, however, $\sum_{m=1}^{N}(-1)^m\tanh(\lambda m)\sin(m\alpha)/m$ converges as $N \to \infty$, and we will consider this as the limit.

4.4. Computation of the integral on the right-hand side of (3.6)

The change of variable $x = k(\alpha)$ and some elementary algebraic manipulations give

$$\int_{-\pi}^{\pi} \log \left| M\left(e^{ix}\right) \right| \rho(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\alpha) R(\alpha) d\alpha,$$

with

$$P(\alpha) := \log |M(e^{ik(\alpha)})| = \tfrac{1}{2} \log \left(\frac{\cosh(2\lambda) - \cos(\alpha)}{1 - \cos(\alpha)} \right) = \int_0^\lambda \Xi_{2t}(\alpha) dt.$$

The final equality may be checked by noticing that the two sides have equal derivatives and are both equal to 0 when $\lambda = 0$. We note that, even though $P(\alpha)$ is not a bounded function, its singularity at $\alpha = 0$ is logarithmic, and hence it is in $L^2([-\pi, \pi])$. Thus, we can use Fubini's Theorem to deduce that

$$\hat{P}(m) = \int_0^\lambda e^{-2t|m|} dt = \begin{cases} \lambda & \text{if } m = 0, \\ \frac{1 - \exp(-2\lambda|m|)}{2|m|} & \text{if } m \neq 0. \end{cases} \tag{4.2}$$

Finally, Parseval's Theorem implies that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(\alpha) R(\alpha) d\alpha = \sum_{m \in \mathbb{Z}} \hat{P}(m) \hat{R}(-m) = \frac{\lambda}{2} + \sum_{m>0} \frac{e^{-m\lambda} \tanh(\lambda m)}{m}$$

using (4.1) in the final equality.

4.5. Computation of the integral on the right-hand side of (3.7) and (3.8)

We begin our analysis of the second integral by recalling (2.16), which implies the existence of C such that $|\tau(x)| < C|x|$ for all $x \in [-\pi, \pi]$. Thus, although $\ell'(x)$ grows as $1/|x|$ near the origin, the integrand is uniformly bounded. Using the Dominated Convergence Theorem [22] and the explicit computation of τ in Proposition 2.1, we find

$$\int_{-\pi}^{\pi} \ell'(x) \tau(x) dx = \int_{-\pi}^{\pi} P'(\alpha) \tau(k(\alpha)) d\alpha = \sum_{m>0} \frac{(-1)^m \tanh(\lambda m)}{m} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} P'(\alpha) \sin(m\alpha) d\alpha \right].$$

Calculating the integrals on the right-hand side is a simple case of integration by parts:

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} P'(\alpha) \sin(m\alpha) d\alpha &= \frac{P(\alpha) \sin(m\alpha)}{\pi} \Big|_{-\pi}^{\pi} - \frac{m}{\pi} \int_{-\pi}^{\pi} P(\alpha) \cos(m\alpha) d\alpha \\ &= -m[\hat{P}(m) + \hat{P}(-m)] \\ &= e^{-2\lambda m} - 1, \end{aligned}$$

where we use our earlier computation (4.2) for the final line. Substituting this in (3.7) yields

$$\lim_{N \to \infty} \log \frac{\Lambda_r(N)}{\Lambda_0(N)} = -r \cdot \Big[\log |\Delta| - \sum_{m>0} \frac{(-1)^m}{m} \tanh(\lambda m) (e^{-2\lambda m} - 1) \Big].$$

[22] In the formula below, the series in the right-hand side is not absolutely convergent. However, if terms are paired (each odd term with the succeeding even one) the resulting series becomes absolutely convergent. This observation is used here and below.

By expanding $\log|\Delta| = \log\cosh(\lambda)$ in powers of $e^{-\lambda}$ and manipulating the result algebraically, we find that

$$\log|\Delta| = \lambda - \sum_{m>0} \frac{(-1)^m(e^{-2\lambda m}-1)}{m}.$$

This directly implies

$$\log|\Delta| - \sum_{m>0} \frac{(-1)^m}{m}\tanh(\lambda m)(e^{-2\lambda m}-1) = \lambda + 2\sum_{m>0}\frac{(-1)^m}{m}\tanh(m\lambda). \tag{4.3}$$

4.6. Proof of (1.3)

We wish to show that

$$\lambda + 2\sum_{m\geq 1}\frac{(-1)^m}{m}\tanh(m\lambda) = \sum_{m\geq 0}\frac{4}{(2m+1)\sinh[\pi^2(2m+1)/(2\lambda)]}. \tag{4.4}$$

Let C_N be the boundary of the rectangle $[-(2N+1)/2, (2N+1)/2] + i[-\pi N/\lambda, \pi N/\lambda]$, oriented counterclockwise, and consider

$$\mathscr{I}_N := \int_{C_N} \frac{\pi\tanh(\lambda z)dz}{z\sin(\pi z)}.$$

The integrand has a simple pole at every integer m and at $i\pi(2r+1)/(2\lambda)$ for every integer r. A straightforward computation shows that the residues of the integrand at the natural numbers are:

$$\operatorname{Res}\left(\frac{\pi\tanh(\lambda z)}{z\sin(\pi z)}, m\right) = \begin{cases} \frac{\tanh(\lambda m)}{\cos(\pi m)m} & m \neq 0, \\ \lambda & m = 0. \end{cases}$$

Summing over $m \in [-N, N]\cap\mathbb{Z}$ gives the partial sums of the right-hand side of (4.4). Meanwhile,

$$\operatorname{Res}\left(\frac{\pi\tanh(\lambda z)}{z\sin(\pi z)}, i\pi(2m+1)/(2\lambda)\right) = \frac{-2}{(2m+1)\sinh[\pi^2(2m+1)/(2\lambda)]}.$$

The hyperbolic tangent is bounded away for its poles (and therefore on C_N), so we may deduce that, for some uniform constant c_0,

$$|\mathscr{I}_N| \leq \frac{c_0}{N}\left[\int_{-\pi N/\lambda}^{\pi N/\lambda}\frac{dt}{\cosh(\pi t)} + \int_{-(2N+1)/2}^{(2N+1)/2}\frac{dt}{|\sin(i\pi^2/\lambda + t)|}\right].$$

Both integrals are uniformly finite in N, hence $\mathscr{I}_N$ converges to zero. As a consequence, the sum of residues of the integrand converges to zero. Using the residues computed above, this implies [23] (4.4).

Upon inspection of the right-hand side of (4.4), we observe that the quantity in the equation is strictly positive whenever $\lambda > 0$. The asymptotic behavior of (4.4) as Δ tends to -1 (corresponding to $2\lambda \sim \sqrt{q-4}$ tending to 0) is governed by the first term or the right-hand side, namely $\frac{4}{\sinh(\pi^2/(2\lambda))} \sim 8e^{-\pi^2/(2\lambda)}$.

[23] We obtain explicitly $\lambda + 2\sum_{m=1}^{N}\frac{(-1)^m}{m}\tanh(m\lambda) - \sum_{m=0}^{N}\frac{4}{(2m+1)\sinh[\pi^2(2m+1)/(2\lambda)]} \to 0$ as $N \to \infty$.

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TENSOR PRODUCT MULTIPLICITIES VIA UPPER CLUSTER ALGEBRAS

BY Jiarui FEI

Abstract. – For each valued quiver Q of Dynkin type, we construct a valued ice quiver Δ_Q^2. Let G be a simply connected Lie group with Dynkin diagram the underlying valued graph of Q. The upper cluster algebra of Δ_Q^2 is graded by the triple dominant weights (μ,ν,λ) of G. We prove that when G is simply-laced, the dimension of each graded component counts the tensor multiplicity $c_{\mu,\nu}^{\lambda}$. We conjecture that this is also true if G is not simply-laced, and sketch a possible approach. Using this construction, we improve Berenstein-Zelevinsky's model, or in some sense generalize Knutson-Tao's hive model in type A.

Résumé. – Nous construisons un carquois valué glacé Δ_Q^2 pour chaque carquois valué de type Dynkin. Soit G un groupe de Lie simplement connexe dont le diagramme de Dynkin est le graphe valué sous-jacent de Q. L'algèbre amassée supérieure de Δ_Q^2 est graduée par le triplet de poids dominants (μ,ν,λ) de G. Lorsque G est simplement lacé, nous montrons que la dimension de chaque composante graduée compte $c_{\mu,\nu}^{\lambda}$ la multiplicité tensorielle. Nous conjecturons que c'est aussi le cas lorsque G n'est pas simplement lacé, et nous esquissons une approche possible. En utilisant cette construction, nous améliorons le modèle de Berenstein-Zelevinsky, ou en un certain sens, nous généralisons le modèle de ruche de Knutson-Tao en type A.

Introduction

Finding the polyhedral model for the tensor multiplicities in Lie theory is a long-standing problem. By *tensor multiplicities* we mean the multiplicities of irreducible summands in the tensor product of any two finite-dimensional irreducible representations of a simply connected Lie group G. The problem asks to express the multiplicity as the number of lattice points in some convex polytope.

Accumulating from the works of Gelfand, Berenstein and Zelevinsky since 1970's, a first quite satisfying model for G of type A was invented in [4]. Finally around 1999, building

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upon their work, Knutson and Tao invented their *hive model*, which led to the solution of the *saturation conjecture* [35]. In fact, the reduction of Horn's problem to the Saturation conjecture is an important driving force for the evolution of the models.

Outside type A, up to now Berenstein and Zelevinsky's models [5] are still the only known polyhedral models. Those models lose a few nice features of Knutson-Tao's hive model. We will have a short discussion on this in Section 0.1. Despite a lot of effort to improve the Berenstein-Zelevinsky model, to the author's best knowledge there is no very satisfying further result in this direction.

Recently an interesting link between the hive model and the *cluster algebra* theory was established in [13] through the Derksen-Weyman-Zelevinsky's *quiver with potential* model [8, 9] for cluster algebras. A similar but different link between the polyhedral models and tropical geometry was established by Goncharov and Shen in [29]. In fact, from the work of Berenstein, Fomin and Zelevinsky [5, 3], those links may not be a big surprise.

There are two goals in the current paper. First we want to generalize the work [13] to other types. More specifically, we hope to prove that the algebras of regular functions on certain configuration spaces are all upper cluster algebras. Second we want to improve the Berenstein-Zelevinsky's model in the spirit of Knutson-Tao. In fact, as we shall see, we accomplish these two goals almost simultaneously. Namely, we use our conjectural models to establish the cluster algebra structures. Once the cluster structures are established, the conjectural models are proved as well.

The key to making new models is the construction of the *iARt quivers*. Let Q be a valued quiver of Dynkin type. Let C^2Q be the category of projective presentations of Q. We can associate to this category an *Auslander-Reiten quiver* $\Delta(C^2Q)$ with translation (ARt quiver in short). The ice ARt quiver (iARt quiver in short) Δ_Q^2 is obtained from $\Delta(C^2Q)$ by *freezing* three sets of vertices, which correspond to the *negative*, *positive*, and *neutral* presentations in C^2Q. We can put a (quite canonical) *potential* W_Q^2 on the iARt quiver Δ_Q^2.

A quiver with potential (or QP in short) (Δ, W) is related to Berenstein-Fomin-Zelevinsky's *upper cluster algebras* [3] through *cluster characters* evaluating on *μ-supported* g-*vectors* introduced in [13] (see Definition 4.5 and 4.8). The cluster character C_W considered in this paper is the *generic* one [42, 12], but it can be replaced by fancier ones. As we have seen in many different situations [13, 14, 15] the set $G(\Delta, W)$ of μ-supported g-vectors is given by lattice points in some rational polyhedral cone. This is also the case for the iARt QPs (Δ_Q^2, W_Q^2).

The whole Part I is devoted to the construction of the iARt QP (Δ_Q^2, W_Q^2) and the polyhedral cone $\mathsf{G}_{\Delta_Q^2}$. It turns out that the cone $\mathsf{G}_{\Delta_Q^2}$ has a very neat hyperplane presentation $\{x \in \mathbb{R}^{(\Delta_Q^2)_0} \mid xH \geq 0\}$, where the columns of the matrix H are given by the dimension vectors of subrepresentations of $3|Q_0|$ representations of Δ_Q^2. These $3|Q_0|$ representations are in bijection with the frozen vertices of Δ_Q^2. They also have a very simple and nice description (see Theorem 5.3). The main result of Part I is the following.

THEOREM 5.9. – *The set* $\mathsf{G}_{\Delta_Q^2} \cap \mathbb{Z}^{(\Delta_Q^2)_0}$ *is exactly* $G(\Delta_Q^2, W_Q^2)$.

The upper cluster algebra $\overline{\mathcal{C}}(\Delta_Q^2)$ has a natural grading by the *weight vectors* of presentations. This grading can be extended to a triple-weight grading $\boldsymbol{\sigma}_Q^2 : \mathbb{Z}^{(\Delta_Q^2)_0} \to \mathbb{Z}_{\geqslant 0}^{3|Q_0|}$. This grading slices the cone $\mathsf{G}_{\Delta_Q^2}$ into polytopes

$$\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda) := \left\{\mathsf{g} \in \mathsf{G}_{\Delta_Q^2} \mid \boldsymbol{\sigma}_Q^2(\mathsf{g}) = (\mu,\nu,\lambda)\right\}.$$

Let $G := G_Q$ be the simply connected simple Lie group with Dynkin diagram the underlying valued graph of Q. Our conjectural model is that the lattice points in $\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda)$ count the tensor multiplicity $c_{\mu\,\nu}^{\lambda}$ for G. Here, $c_{\mu\,\nu}^{\lambda}$ is the multiplicity of the irreducible representation $L(\lambda)$ of highest weight λ in the tensor product $L(\mu) \otimes L(\nu)$. More often than not we identify a dominant weight by a non-negative integral vector. To prove this model, we follow a similar line as [13]. However, we do not have a quiver setting to work with in general. We replace the semi-invariant rings of triple-flag quiver representations by the ring of regular functions on a certain configuration space introduced in [19].

Fix an opposite pair of maximal unipotent subgroups (U^-, U) of G. The quotient space $\mathscr{A} := U^-\backslash G$ is called *base affine space*, and the quotient space $\mathscr{A}^\vee := G/U$ is called its *dual*. The configuration space $\mathrm{Conf}_{2,1}$ is by definition $(\mathscr{A} \times \mathscr{A} \times \mathscr{A}^\vee)/G$, where G acts multi-diagonally. The ring of regular functions $k[\mathrm{Conf}_{2,1}]$ is just the invariant ring $(k[G]^{U^-} \otimes k[G]^{U^-} \otimes k[G]^{U})^G$. The ring $k[\mathrm{Conf}_{2,1}]$ is multigraded by a triple of weights (μ,ν,λ). Each graded component $C_{\mu,\nu}^{\lambda} := k[\mathrm{Conf}_{2,1}]_{\mu,\nu,\lambda}$ is given by the G-invariant space $(L(\mu) \otimes L(\nu) \otimes L(\lambda)^\vee)^G$. So the dimension of $C_{\mu,\nu}^{\lambda}$ counts the tensor multiplicity $c_{\mu\,\nu}^{\lambda}$. Here is the main result of Part II.

THEOREM 9.1. – *Suppose that Q is trivially valued. Then the ring of regular functions on* $\mathrm{Conf}_{2,1}$ *is the graded upper cluster algebra* $\overline{\mathcal{C}}(\Delta_Q^2, \mathcal{S}_Q^2; \boldsymbol{\sigma}_Q^2)$. *Moreover, the generic character maps the lattice points in* $\mathsf{G}_{\Delta_Q^2}$ *onto a basis of this algebra. In particular, $c_{\mu\,\nu}^{\lambda}$ is counted by lattice points in* $\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda)$.

We will show by an example that the upper cluster algebra strictly contains the corresponding cluster algebra in general. We conjecture that the trivially valued assumption can be dropped in the above theorem and the theorem below. It is pointed in the end that the only missing ingredient for proving the conjecture is the analogue of [9, Lemma 5.2] for *species with potentials* [37].

Fock and Goncharov studied in [19] the similar spaces Conf_3 [(1)] as cluster varieties. However, to the author's best knowledge it is not clear from their discussion what an *initial seed* is if G is not of type A. Moreover the equality established in the theorem does not seem to follow from any result there. In fact, Fock and Goncharov later conjectured in [20] that the tropical points in their cluster $\mathscr{X}$-varieties parametrize bases in the corresponding (upper) cluster algebras. Our result can be viewed as an algebraic analog of their conjecture for the space $\mathrm{Conf}_{2,1}$. Instead of working with the tropical points, we work with the g-vectors.

To sketch our ideas, we first observe that if we forget the frozen vertices corresponding to the positive and neutral presentations, then we get a valued ice quiver denoted by Δ_Q

(1) They considered the generic part of the quotient stack $[(\mathscr{A}^\vee)^3/G]$. We will work with the categorical quotient as its partial compactification.

whose cluster algebra is isomorphic to the coordinate ring $k[U]$. Roughly speaking, this procedure corresponds to an open embedding $i : H \times H \times U \hookrightarrow \mathrm{Conf}_{2,1}$, or more precisely Corollary 8.9. We will define the cluster $\mathcal{S}^2_Q$ in Theorem through the pullback map i^*. It is then not hard to show that $k[\mathrm{Conf}_{2,1}]$ contains the upper cluster algebra $\overline{\mathcal{C}}(\Delta^2_Q, \mathcal{S}^2_Q; \boldsymbol{\sigma}^2_Q)$ as a graded subalgebra. The detail will be given in Section 8.1.

So far we have the graded inclusions

$$\mathrm{Span}\left(C_W(\mathsf{G}_{\Delta^2_Q})\right) \subseteq \overline{\mathcal{C}}(\Delta^2_Q, \mathcal{S}^2_Q; \boldsymbol{\sigma}^2_Q) \subseteq k[\mathrm{Conf}_{2,1}].$$

To finish the proof, it suffices to show the containment $k[\mathrm{Conf}_{2,1}] \subseteq \mathrm{Span}\left(C_W(\mathsf{G}_{\Delta^2_Q})\right)$. For this, we come back to the cluster structure of $k[U]$. It turns out that the analog of Theorem for U is rather easy to prove. The set $G(\Delta_Q, W_Q)$ contains exactly lattice points in the polytope G_{Δ_Q}, which is defined by one of the three sets of relations of $\mathsf{G}_{\Delta^2_Q}$. On the other hand, we have two other embeddings $i_l,\ i_r : U \hookrightarrow \mathrm{Conf}_{2,1}$. They are the map $i_u := i|_U$ followed by the *twisted cyclic shift* of $\mathrm{Conf}_{2,1}$. Another crucial ingredient in this paper is an interpretation of the twisted cyclic shift in terms of a sequence of mutations $\boldsymbol{\mu}_l$. Applying $\boldsymbol{\mu}_l$ and $\boldsymbol{\mu}_l^{-1}$ to the QP (Δ_Q, W_Q), we get two other QPs (Δ^l_Q, W^l_Q) and (Δ^r_Q, W^r_Q). The analogous polytopes $\mathsf{G}_{\Delta^l_Q}$ and $\mathsf{G}_{\Delta^r_Q}$ for them are defined by the other two sets of relations of $\mathsf{G}_{\Delta^2_Q}$. Finally, after showing the good behavior of g-vectors under the pullback of the three embeddings, the required inclusion will follow from the fact that

$$k[\mathrm{Conf}_{2,1}] \subseteq \left\{s \in \mathcal{L}(\mathcal{S}^2_Q) \mid i^*_\#(s) \in k[U] \text{ for } \# = u, l, r\right\}, \tag{0.1}$$

where $\mathcal{L}(\mathcal{S}^2_Q)$ is the Laurent polynomial ring in the cluster $\mathcal{S}^2_Q$. The detail will be given in Section 8.2.

Except for these two main results, we have a side result for the base affine spaces. The author would like to thank B. Leclerc and M. Yakimov for confirming that the following theorem was an open problem. It turns out that the cluster structure of $\mathcal{A}$ lies between that of U and $\mathrm{Conf}_{2,1}$. Let $\Delta^\sharp_Q$ be the valued ice quiver obtained from Δ^2_Q by deleting frozen vertices corresponding to neutral presentations.

THEOREM 10.2. – *Suppose that Q is trivially valued. Then the ring of regular functions on $\mathcal{A}$ is the graded upper cluster algebra $\overline{\mathcal{C}}(\Delta^\sharp_Q, \mathcal{M}^\sharp_Q; \varpi(\boldsymbol{\sigma}^\sharp_Q))$. Moreover, the generic character maps the lattice points in $\mathsf{G}_{\Delta^\sharp_Q}$ onto a basis of this algebra. In particular, the weight multiplicity* $\dim L(\mu)_\lambda$ *is counted by lattice points in $\mathsf{G}_{\Delta^\sharp_Q}(\mu, \lambda)$.*

0.1. The Models

In [35] Knutson and Tao invented a remarkable polyhedral model called hives or honeycomb. The author personally thinks that it has at least three advantages over Berenstein-Zelevinsky's model [5]. First, the hive polytopes have a nice presentation $\left\{x \in \mathbb{R}^{3n} \mid xH \geq 0,\ x\boldsymbol{\sigma} = (\mu, \nu, \lambda)\right\}$. Second, the cyclic symmetry of the type-A tensor multiplicity is lucid from the hive model. Actually other symmetries can also follow from the hive model. Last and most importantly, there is an operation called *overlaying* for honeycombs [35].

In appropriate sense, our models share these nice properties. The first one is clear from our result. Our H-matrices even have all non-negative entries. However, if readers prefer the rhombus-type inequalities of the hives, one can transform our model through a totally unimodular map as in [13]. However, the rhombus-type inequalities are not always as neat as the ones in type A. We will discuss the transformation and the analogous overlaying elsewhere. Although the cyclic symmetry is not immediately clear from H itself, we understand from our construction and Appendix 11.2 that it is just hidden there. We believe that this is probably the best we can do outside type A.

In a more general context of Kac-Moody algebras, the tensor multiplicity problem can be solved by P. Littelmann's path model [40]. As pointed out in [5], his model can be transformed into polyhedral ones (with some non-trivial work). However, in general it involves a union of several convex polytopes.

0.2. Relation to the work of Berenstein-Zelevinsky and Goncharov-Shen

In a groundbreaking work [5] Berenstein and Zelevinsky invented their polyhedral model for all Dynkin types. Their main tools are Lusztig's canonical basis and tropical relations in double Bruhat cells. The polytopes are defined explicitly in terms of their **i**-*trails*. But the author feels that **i**-trails are hard to compute especially in type E. By contrast, the subrepresentations defining our H are rather easy to list in most cases. In few difficult cases, such as type E_7 and E_8, we provide an algorithm suitable for computers.

Recently Goncharov and Shen made some further progress in [29]. Using tropical geometry and geometric Satake, they proved a more symmetric polyhedral model (see [29, Theorem 2.6 and (214)]). However, there is no further *explicit* description on the polytopes. The equality of (0.1) as an intermediate byproduct of our proof is similar to this result.

Loosely speaking, our work is independent of their results, though the author did benefit a lot from reading their papers. The construction of iARt quivers Δ_Q^2 is new. We believe that the construction and results, especially the ideas behind, are beyond just solving the tensor multiplicity problem for simple Lie groups. The proofs in Part I are similar to those in [13]. In Part II what we heavily rely on is the cluster structure of $k[U]$ and a mutation interpretation of the twisted cyclic shift. Throughout the quiver with potential model for cluster algebras is most important.

Outline of the Paper

In Section 1.1 we recall the basics on valued quivers and their representations. We define the graded upper cluster algebra attached to a valued quiver in Section 1.2 and 1.3. In Section 2.1 we recall the Auslander-Reiten theory from a functorial point of view. We specialize the theory to the category of presentations mostly for hereditary algebras in Section 2.2. In Section 3.1 we define the iARt quivers in general. We then consider the hereditary cases in more detail in Section 3.2. Proposition 3.6 compares the ARt quivers of presentations with the more familiar ARt quivers of representations. In Section 4 we review the generic cluster character in the setting of quivers with potentials. In Section 5 we study the iARt QPs and their μ-supported g-vectors. We prove the two main results of Part I—Theorem 5.3 and 5.9. In Appendix 6, we provide more examples of iARt quivers.

In Section 7 we review the rings of regular functions on base affine spaces and maximal unipotent groups, especially the cluster structure of the latter (Theorem 7.5 and Proposition 7.6). In Section 8 we study maps relating the configuration spaces to the corresponding unipotent groups. These are almost all the technical work required for proving the main result. In Section 9 we prove our main result—Theorem 9.1. In Section 10 we prove the side result—Theorem 10.2. In the end we make some remark on the possible generalization to the non-simply laced cases. In Appendix 11 we prove the mutation interpretation of the twisted cyclic shift in Theorem 11.14. As a consequence, we produce an algorithm for computing the (μ-supported) g-vector cones.

Notations and Conventions

Our vectors are exclusively row vectors. All modules are right modules. Arrows are composed from left to right, i.e., ab is the path $\cdot \xrightarrow{a} \cdot \xrightarrow{b} \cdot$. Unless otherwise stated, unadorned Hom and $\otimes$ are all over the base field k, and the superscript $*$ is the trivial dual for vector spaces. We write hom and ext for $\dim \mathrm{Hom}$ and $\dim \mathrm{Ext}$. For direct sum of n copies of M, we write nM instead of the traditional $M^{\oplus n}$.

PART I
CONSTRUCTION OF IART QPS

1. Graded Upper Cluster Algebras

1.1. Valued Quivers and their Representations

If you are familiar with the usual quiver representations and only care about our results on the simply laced cases, you can skip this subsection.

Definition 1.1. – A valued quiver is a triple $Q = (Q_0, Q_1, C)$ where

1. Q_0 is a set of vertices, usually labeled by natural numbers $1, 2, \dots, n$;
2. Q_1 is a set of arrows, which is a subset of $Q_0 \times Q_0$;
3. $C = \{(c_{i,j}, c_{j,i}) \in \mathbb{N} \times \mathbb{N} \mid (i, j) \in Q_1\}$ is called the *valuation* of Q.

It is called *symmetrizable* if there is $d = \{d_i \in \mathbb{N} \mid i \in Q_0\}$ such that $d_i c_{i,j} = c_{j,i} d_j$ for every $(i, j) \in Q_1$.

For such a valued quiver, the pair (Q_0, Q_1) is called its *ordinary* quiver. Throughout this paper, all valued quivers are assumed to have no loops or oriented 2-cycles in their ordinary quivers. If $c_{i,j} = c_{j,i}$ for every $(c_{i,j}, c_{j,i}) \in C$, then Q is called *equally valued*. To draw a valued quiver (Q_0, Q_1, C), we first draw its ordinary quiver, then put valuations above its arrows, eg. $i \xrightarrow{(c_{i,j}, c_{j,i})} j$. We will omit the valuation if (i, j) is trivially valued, i.e., $c_{i,j} = c_{j,i} = 1$. All valued quivers in this paper will be symmetrizable. We always fix a choice of d, so readers may view d as a part of the defining data for Q. We let $d_{i,j} = \gcd(d_i, d_j)$.

Let $\mathbb{F}$ be a finite field. We write $\overline{\mathbb{F}}$ for an algebraic closure of $\mathbb{F}$. For each positive integer k denote by $\mathbb{F}_k$ the degree k extension of $\mathbb{F}$ in $\overline{\mathbb{F}}$. Note that the largest subfield of $\overline{\mathbb{F}}$ contained in both $\mathbb{F}_k$ and $\mathbb{F}_l$ is $\mathbb{F}_{\gcd(k,l)} = \mathbb{F}_k \cap \mathbb{F}_l$. If $k \mid l$ we can fix a basis of $\mathbb{F}_l$ over $\mathbb{F}_k$ and thus freely identify $\mathbb{F}_l$ as a vector space over $\mathbb{F}_k$.

A representation M of Q is an assignment for each $i \in Q_0$ a $\mathbb{F}_{d_i}$-vector space $M(i)$, and for each arrow $(i, j) \in Q_1$ an $\mathbb{F}_{d_{i,j}}$-linear map $M(i, j)$. This definition is different from the original one in [10], but it is more adapted to the cluster algebra theory (see [45]). The equivalence of two definitions was established in [45, (2.2)]. The dimension vector $\underline{\dim} M$ is the integer vector $(\dim_{\mathbb{F}_{d_i}} M(i))_{i\in Q_0}$. Similar to the usual quiver representations, we can define a morphism $\phi : M \to N$ as the set

$$\left\{\phi_i \in \operatorname{Hom}_{\mathbb{F}_{d_i}}(M(i), N(i))\right\}_{i\in Q_0} \text{ such that } \phi_j M(i, j) = N(i, j)\phi_i \text{ for all } (i, j) \in Q_1.$$

The category $\operatorname{Rep}(Q)$ of all (finite-dimensional) representations of Q is an abelian category, in which the kernels and cokernels are taken vertex-wise. The category $\operatorname{Rep}(Q)$ is also *Krull-Schmidt*, that is, each object is a finite direct sum of indecomposable objects with local endomorphism rings.

Just as with usual quivers it is useful to consider an equivalent category of modules over the path algebra. Such an analog for valued quivers is the notion of $\mathbb{F}$-species. Define $\Gamma_0 = \prod_{i\in Q_0} \mathbb{F}_{d_i}$ and $\Gamma_1 = \bigoplus_{(i,j)\in Q_1} \mathbb{F}_{d_i c_{i,j}}$. Notice that $\mathbb{F}_{d_i c_{i,j}}$ contains both $\mathbb{F}_{d_i}$ and $\mathbb{F}_{d_j}$ and thus we have a Γ_0-Γ_0-bimodule structure on Γ_1. Now we define the $\mathbb{F}$-species Γ_Q to be the tensor algebra $T_{\Gamma_0}(\Gamma_1)$ of Γ_1 over Γ_0. If Γ_Q is finite-dimensional, then it is clear that the indecomposable projective (resp. injective) modules are precisely $P_i = e_i \Gamma_Q$ (resp. $I_i = (\Gamma_Q e_i)^*$) for $i \in Q_0$, where e_i is the identity element in $\mathbb{F}_{d_i}$. The category $\operatorname{Rep}(Q)$ has enough projective and injective objects. The top of P_i is the simple representation S_i supported on the vertex i, which is also the socle of I_i. The minimal projective and injective resolutions of simple S_i are given by

$$(1.1)\quad 0 \to \bigoplus_{(i,j)\in Q_1} c_{j,i} P_j \to P_i \to S_i \to 0 \quad \text{and} \quad 0 \to S_i \to I_i \to \bigoplus_{(i,j)\in Q_1} c_{i,j} I_j \to 0.$$

The algebra Γ_Q is *hereditary*, that is, it has global dimension 1. So for $M, N \in \operatorname{Rep}(Q)$,

$$\langle M, N\rangle = \dim_{\mathbb{F}} \operatorname{Hom}_Q(M, N) - \dim_{\mathbb{F}} \operatorname{Ext}^1_Q(M, N)$$

is a bilinear form only depending on the dimension vectors of M and N. This is called "Ringel-Euler" form, and we denote the matrix of this form by $E(Q)$. We also define the matrix $E_l(Q) := (e^l_{j,i})$ and $E_r(Q) := (e^r_{i,j})$ by

$$e^l_{i,j} = \begin{cases} 1 & i = j; \\ -c_{j,i} & (i, j) \in Q_1; \\ 0 & \text{otherwise}, \end{cases} \qquad e^r_{i,j} = \begin{cases} 1 & i = j; \\ -c_{i,j} & (i, j) \in Q_1; \\ 0 & \text{otherwise}. \end{cases}$$

These matrices are related by $E(Q) = E_l(Q)D = DE_r(Q)$, where D is the diagonal matrix with diagonal entries $d_{i,i} = d_i$.

EXAMPLE 1.2 (G_2). – Consider the valued quiver $1 \xrightarrow{3,1} 2$ of type G_2 with $d = (1, 3)$. Its module category has six indecomposable objects

- The simple injective $S_1 : \mathbb{F} \to 0$, and its projective cover $P_1 : \mathbb{F} \hookrightarrow \mathbb{F}_3$;
- The simple projective $S_2 : 0 \to \mathbb{F}_3$, and its injective hull $I_2 : \mathbb{F}^3 \hookrightarrow \mathbb{F}_3$;
- The module $M_1 : \mathbb{F}^2 \hookrightarrow \mathbb{F}_3$, which is presented by $P_2 \hookrightarrow 2P_1$;
- The module $M_2 : \mathbb{F}^3 \hookrightarrow \mathbb{F}_3^2$, which is presented by $P_2 \hookrightarrow 3P_1$.

In this paper, we will encounter two kinds of valued quivers. One is valued quivers Q of Dynkin type, and the other is bigger valued quivers Δ_Q and Δ_Q^2 constructed from Q (see Section 3). We will define upper cluster algebras attached to the latter.

1.2. Upper Cluster Algebras

We mostly follow [3, 24, 21]. To define the upper cluster algebra, we need to introduce the notion of the quiver mutation. The *mutation* of valued quivers is defined through Fomin-Zelevinsky's mutation of the associated skew-symmetrizable matrix.

Every symmetrizable valued quiver Δ corresponds to a skew symmetrizable integer matrix $B(\Delta) := -E_l(\Delta) + E_r(\Delta)^T$. So the entries $(b_{u,v})_{u,v\in\Delta_0}$ are given by

$$b_{u,v} = \begin{cases} c_{u,v}, & \text{if } (u,v) \in \Delta_1, \\ -c_{u,v}, & \text{if } (v,u) \in \Delta_1, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $B(\Delta)$ is *skew symmetrizable* because DB is skew-symmetric for the diagonal matrix D. Conversely, given a skew symmetrizable matrix B, a unique valued quiver Δ can be easily defined such that $B(\Delta) = B$.

Definition 1.3. – The *mutation* of a skew symmetrizable matrix B on the direction $u \in \Delta_0$ is given by $\mu_u(B) = (b'_{v,w})$, where

$$b'_{v,w} = \begin{cases} -b_{v,w}, & \text{if } u \in \{v,w\}, \\ b_{v,w} + \operatorname{sign}(b_{v,u}) \max(0, b_{v,u}b_{u,w}), & \text{otherwise.} \end{cases}$$

We denote the induced operation on its valued quiver also by μ_u.

The cluster algebras that we will consider in this paper are skew-symmetrizable cluster algebras of geometric type. The combinatorial data defining such a cluster algebra is encoded in a symmetrizable valued quiver Δ with *frozen vertices*. Frozen vertices are forbidden to be mutated, and the remaining vertices are *mutable*. Such a valued quiver is called valued *ice* quiver (or VIQ in short). The *mutable part* Δ^μ is the full subquiver of Δ consisting of mutable vertices. In general, to define a(n) (upper) cluster algebra only Δ^μ is required to be symmetrizable. However, in this paper all VIQs happen to be "globally" symmetrizable. We usually label the mutable vertices as the first p out of q vertices of Δ. The *restricted* B-matrix B_Δ of Δ is the first p rows of $B(\Delta)$.

Let k be a field, not necessarily related in any sense to the finite field $\mathbb{F}$ or the base field in the rest of Part I.

DEFINITION 1.4. – Let $\mathcal{F}$ be a field containing k. A *seed* in $\mathcal{F}$ is a pair $(\Delta, \mathbf{x})$ consisting of a VIQ Δ as above together with a collection $\mathbf{x} = \{x_1, x_2, \ldots, x_q\}$, called an *extended cluster*, consisting of algebraically independent (over k) elements of $\mathcal{F}$, one for each vertex of Δ. The elements of $\mathbf{x}$ associated with the mutable vertices are called *cluster variables*; they form a *cluster*. The elements associated with the frozen vertices are called *frozen variables*, or *coefficient variables*.

A *seed mutation* μ_u at a (mutable) vertex u transforms $(\Delta, \mathbf{x})$ into the seed $(\Delta', \mathbf{x}') = \mu_u(\Delta, \mathbf{x})$ defined as follows. The new VIQ is $\Delta' = \mu_u(\Delta)$. The new extended cluster is $\mathbf{x}' = \mathbf{x} \cup \{x'_u\} \setminus \{x_u\}$ where the new cluster variable x'_u replacing x_u is determined by the *exchange relation*

$$x_u x'_u = \prod_{(v,u)\in\Delta_1} x_v^{c_{v,u}} + \prod_{(u,w)\in\Delta_1} x_w^{c_{u,w}}. \tag{1.2}$$

We note that the mutated seed $(\Delta', \mathbf{x}')$ contains the same coefficient variables as the original seed $(\Delta, \mathbf{x})$. It is easy to check that one can recover $(\Delta, \mathbf{x})$ from $(\Delta', \mathbf{x}')$ by performing a seed mutation again at u. Two seeds $(\Delta, \mathbf{x})$ and $(\Delta', \mathbf{x}')$ that can be obtained from each other by a sequence of mutations are called *mutation-equivalent*, denoted by $(\Delta, \mathbf{x}) \sim (\Delta', \mathbf{x}')$.

DEFINITION 1.5. – The *cluster algebra* $\mathcal{C}(\Delta, \mathbf{x})$ associated to a seed $(\Delta, \mathbf{x})$ is defined as the subring of $\mathcal{F}$ generated by all elements of all extended clusters of the seeds mutation-equivalent to $(\Delta, \mathbf{x})$.

Note that the above construction of $\mathcal{C}(\Delta, \mathbf{x})$ depends only, up to a natural isomorphism, on the mutation equivalence class of the initial VIQ Δ. In fact, it only depends on the mutation equivalence class of the restricted B-matrix of Δ. So we may drop $\mathbf{x}$ and simply write $\mathcal{C}(\Delta)$ or $\mathcal{C}(B_\Delta)$.

An amazing property of cluster algebras is the *Laurent Phenomenon*.

THEOREM 1.6 ([23, 3]). – *Any element of a cluster algebra $\mathcal{C}(\Delta, \mathbf{x})$ can be expressed in terms of the extended cluster $\mathbf{x}$ as a Laurent polynomial, which is polynomial in coefficient variables.*

Since $\mathcal{C}(\Delta, \mathbf{x})$ is generated by cluster variables from the seeds mutation equivalent to $(\Delta, \mathbf{x})$, Theorem 1.6 can be rephrased as

$$\mathcal{C}(\Delta, \mathbf{x}) \subseteq \bigcap_{(\Delta',\mathbf{x}')\sim(\Delta,\mathbf{x})} \mathcal{L}_{\mathbf{x}'},$$

where $\mathcal{L}_{\mathbf{x}} := k[x_1^{\pm 1}, \ldots, x_p^{\pm 1}, x_{p+1}, \ldots x_q]$. Note that our definition of $\mathcal{L}_{\mathbf{x}}$ is slightly different from the original one in [3], where $\mathcal{L}_{\mathbf{x}}$ is replaced by the Laurent polynomial $\mathcal{L}(\mathbf{x}) := k[x_1^{\pm 1}, \ldots, x_p^{\pm 1}, x_{p+1}^{\pm 1}, \ldots, x_q^{\pm 1}]$.

DEFINITION 1.7. – The *upper cluster algebra* with seed $(\Delta, \mathbf{x})$ is

$$\overline{\mathcal{C}}(\Delta, \mathbf{x}) := \bigcap_{(\Delta',\mathbf{x}')\sim(\Delta,\mathbf{x})} \mathcal{L}_{\mathbf{x}'}.$$

Any (upper) cluster algebra, being a subring of a field, is an integral domain (and under our conventions, a k-algebra). Conversely, given such a domain R, one may be interested in identifying R as a(n) (upper) cluster algebra. The following useful lemma is a specialization of [21, Proposition 3.6] to the case where R is a unique factorization domain.

LEMMA 1.8. – *Let R be a finitely generated UFD over k. Suppose that $(\Delta, \mathbf{x})$ is a seed contained in R, and each adjacent cluster variable x_u' is also in R. Moreover, each pair in $\mathbf{x}$ and each pair (x_u, x_u') are relatively prime. Then $R \supseteq \overline{\mathcal{C}}(\Delta, \mathbf{x})$.*

1.3. g-vectors and Gradings

Let $\mathbf{x} = \{x_1, x_2, \ldots, x_q\}$ be a(n) (extended) cluster. For a vector $\mathsf{g} \in \mathbb{Z}^q$, we write $\mathbf{x}^{\mathsf{g}}$ for the monomial $x_1^{\mathsf{g}(1)} x_2^{\mathsf{g}(2)} \cdots x_q^{\mathsf{g}(q)}$. For $u = 1, 2, \ldots, p$, we set $y_u = \mathbf{x}^{-b_u}$ where b_u is the u-th row of the matrix B_Δ, and let $\mathbf{y} = \{y_1, y_2, \ldots, y_p\}$.

Suppose that an element $z \in \mathcal{L}(\mathbf{x})$ can be written as

$$z = \mathbf{x}^{\mathsf{g}(z)} F(y_1, y_2, \ldots, y_p), \tag{1.3}$$

where F is a rational polynomial not divisible by any y_i, and $\mathsf{g}(z) \in \mathbb{Z}^q$. If we assume that the matrix B_Δ has full rank, then the elements $y_1, y_2, \ldots, y_p$ are algebraically independent so that the vector $\mathsf{g}(z)$ is uniquely determined [24]. We call the vector $\mathsf{g}(z)$ the (extended) g-*vector* of z with respect to the pair $(\Delta, \mathbf{x})$. Definition implies at once that for two such elements z_1, z_2 we have that $\mathsf{g}(z_1 z_2) = \mathsf{g}(z_1) + \mathsf{g}(z_2)$. So the set of all g-vectors in any subalgebra of $\mathcal{L}(\mathbf{x})$ forms a sub-semigroup of $\mathbb{Z}^q$.

LEMMA 1.9 ([13, Lemma 5.5], *cf.* [42]). – *If the matrix B_Δ has full rank, then any subset of $\mathcal{L}(\mathbf{x})$ with distinct well-defined* g-*vectors is linearly independent over k.*

DEFINITION 1.10. – A *weight configuration* $\boldsymbol{\sigma}$ of a lattice $\mathbb{L} \subseteq \mathbb{R}^m$ on a VIQ Δ is an assignment for each vertex v of Δ a weight vector $\boldsymbol{\sigma}(v) \in \mathbb{L}$ such that for each mutable vertex u, we have that

$$\sum_{(v,u)\in\Delta_1} c_{v,u}\boldsymbol{\sigma}(v) = \sum_{(u,w)\in\Delta_1} c_{u,w}\boldsymbol{\sigma}(w). \tag{1.4}$$

The *mutation* μ_u also transforms $\boldsymbol{\sigma}$ into a weight configuration $\boldsymbol{\sigma}'$ on the mutated quiver $\mu_u(\Delta)$ defined as

$$\boldsymbol{\sigma}'(v) = \begin{cases} \displaystyle\sum_{(u,w)\in\Delta_1} c_{u,w}\boldsymbol{\sigma}(w) - \boldsymbol{\sigma}(u) & \text{if } v = u, \\ \boldsymbol{\sigma}(v) & \text{otherwise.} \end{cases}$$

By slight abuse of notation, we can view $\boldsymbol{\sigma}$ as a matrix whose v-th row is the weight vector $\boldsymbol{\sigma}(v)$. In this matrix notation, the condition (1.4) is equivalent to that $B_\Delta \boldsymbol{\sigma}$ is a zero matrix. So we call the cokernel of B_Δ as the *grading space* of $\overline{\mathcal{C}}(\Delta)$. A weight configuration $\boldsymbol{\sigma}$ is called *full* if the corank of B_Δ is equal to the rank of $\boldsymbol{\sigma}$. It is easy to see that for any weight configuration of Δ, the mutation can be iterated.

Given a weight configuration $(\Delta; \boldsymbol{\sigma})$, we can assign a multidegree (or weight) to the upper cluster algebra $\overline{\mathcal{C}}(\Delta, \mathbf{x})$ by setting $\deg(x_v) = \boldsymbol{\sigma}(v)$ for $v = 1, 2, \ldots, q$. Then mutation preserves multihomogeneity. We say that this upper cluster algebra is $\boldsymbol{\sigma}$-graded, and denoted by $\overline{\mathcal{C}}(\Delta, \mathbf{x}; \boldsymbol{\sigma})$. We refer to $(\Delta, \mathbf{x}; \boldsymbol{\sigma})$ as a graded seed. Note that the variables in $\mathbf{y}$ have zero degrees. So if z has a well-defined g-vector as in (1.3), then z is homogeneous of degree $\mathsf{g}\boldsymbol{\sigma}$.

2. AR-theory of Presentations

2.1. Review of Auslander-Reiten theory

We briefly review Auslander-Reiten theory for Krull-Schmidt exact categories following [11]. The theory was developed originally for module categories of Artin algebras, but without much difficulty most of the theory can be generalized to Krull-Schmidt exact categories. Readers should consult [11, Section 2.2] or the standard textbook [2] for the basic notions in Auslander-Reiten theory, such as the left and right (minimal) almost split morphisms.

Let k be a field, and $\mathscr{A}$ be a k-linear, Hom-finite, and Krull-Schmidt category with an exact structure $\mathcal{E}$. So $\mathcal{E}$ is a class of exact pairs which is closed under isomorphisms satisfying Gabriel-Roiter's axiom (see [11, 1.1]). Recall that a pair (i, d) of composable morphisms $L \xrightarrow{i} M \xrightarrow{d} N$ in $\mathscr{A}$ is called *exact* if i is a kernel of d and d is a cokernel of i. If the underlying exact structure $\mathcal{E}$ is clear, we speak of projective and injective objects rather than $\mathcal{E}$-projective and $\mathcal{E}$-injective objects. The proof of the following proposition coincides with the usual one for module categories.

Proposition 2.1 ([11, Proposition 2.3]). – *Suppose that $L \xrightarrow{i} M \xrightarrow{d} N$ is an exact pair in $\mathcal{E}$. Then the following assertions are equivalent.*

1. *i is left minimal almost split.*
2. *d is right minimal almost split.*
3. *i is left almost split and d is right almost split.*

Definition 2.2. – An exact pair $L \xrightarrow{i} M \xrightarrow{d} N$ in $\mathcal{E}$ as in the above proposition is called an *almost split pair*. In this case, L is called the *translation* of N denoted by τN, and N is called the inverse translation of L denoted by $\tau^{-1} L$.

Such an almost split pair can only exist provided L is indecomposable non-injective and N is indecomposable non-projective. The exact category $(\mathscr{A}, \mathcal{E})$ is said to have almost split pairs if $\mathscr{A}$ has almost split morphisms and moreover for all indecomposable non-projective objects N there exists an almost split pair $L \xrightarrow{i} M \xrightarrow{d} N$ and dually for all indecomposable non-injective objects L there exists an almost split pair $L \xrightarrow{i} M \xrightarrow{d} N$. The uniqueness of minimal almost split maps shows that almost split pairs $L \xrightarrow{i} M \xrightarrow{d} N$ are uniquely determined by L or N.

Example 2.3. – Let A be a finite dimensional k-algebra, and $\operatorname{mod} A$ be the category of finite dimensional (right) A-modules. [2, Theorem V.1.15] says that $\operatorname{mod} A$ has almost split pairs, so the translation τ is defined for every indecomposable non-projective A-module. It is given by the trivial dual of *Auslander's transpose functor* (see [2, IV.1]).

Recall that a morphism $f \in \mathrm{Hom}_{\mathscr{A}}(M,N)$ is called *radical* if $\mathrm{Id}_M + gf$ is invertible for each $g \in \mathrm{Hom}_{\mathscr{A}}(N,M)$. If M and N are indecomposable, then this is equivalent to say that f is a non-isomorphism. We denote by $\mathrm{rad}_{\mathscr{A}}(M,N)$ the space of all radical morphisms in $\mathrm{Hom}_{\mathscr{A}}(M,N)$. We define $\mathrm{rad}^2_{\mathscr{A}}(M,N)$ to consist of all morphisms of form gf, where $f \in \mathrm{rad}_{\mathscr{A}}(M,L)$ and $g \in \mathrm{rad}_{\mathscr{A}}(L,N)$ for some $L \in \mathscr{A}$. We denote by $\mathrm{ind}(\mathscr{A})$ the full subcategory of all indecomposable objects in $\mathscr{A}$.

Definition 2.4. – For $M,N \in \mathrm{ind}(\mathscr{A})$, an *irreducible morphism* $f : M \to N$ is an element in $\mathrm{rad}_{\mathscr{A}}(M,N) \setminus \mathrm{rad}^2_{\mathscr{A}}(M,N)$. We denote

$$\mathrm{Irr}_{\mathscr{A}}(M,N) := \mathrm{rad}_{\mathscr{A}}(M,N)/\,\mathrm{rad}^2_{\mathscr{A}}(M,N).$$

For $M \in \mathrm{ind}(\mathscr{A})$, $\mathrm{End}_{\mathscr{A}}(M)$ is local, then

$$D_M := \mathrm{End}_{\mathscr{A}}(M)/\,\mathrm{rad}\,\mathrm{End}_{\mathscr{A}}(M)$$

is a division k-algebra.

Let $M = \bigoplus_{i=1}^t m_i M_i$ be an object in $\mathscr{A}$ with M_i indecomposable and pairwise non-isomorphic. For $f \in \mathrm{Hom}_{\mathscr{A}}(M,N)$ with N indecomposable, we can write f as $f = (f_1,\dots,f_t)$ where $f_i = (f_{i,1},\dots,f_{i,m_i}) : m_i M_i \to N$. The following proposition was originally proved for module categories of Artin algebras (see [2, Proposition VII.1.3]) but the proof there also works in our setting.

Proposition 2.5. – *The morphism f is* right minimal almost split *iff the residual classes of $f_{i,j}$'s in $\mathrm{Irr}_{\mathscr{A}}(M_i,N)$ form a D_N^{op}-basis for all i. There is a similar statement for left minimal almost split morphisms.*

We also recall a basic fact [2] that if $L \xrightarrow{i} M \xrightarrow{d} N$ almost split, then

$$\dim_{D_{M_i}} \mathrm{Irr}_{\mathscr{A}}(M_i,N) = \dim_{D_{M_i}^{\mathrm{op}}} \mathrm{Irr}_{\mathscr{A}}(L,M_i). \tag{2.1}$$

2.2. Presentations

In this subsection we briefly review some results from [44] in our setting. Let A be some finite dimensional k-algebra with valued quiver Q (see [2, III.1]). If you do not know what a valued quiver associated to A is, then you can just take A to be the $\mathbb{F}$-species defined in Section 1.1. Let $C^2A := \mathrm{Ch}_2(\mathrm{proj}\text{-}A)$ be the category of projective presentations. To be more precise, the objects in C^2A are 2-term complexes $P_+ \xrightarrow{f} P_-$ in $\mathrm{proj}\text{-}A$ (with P_+ and P_- in some fixed degrees). The morphisms are commutative diagrams. Let $\mathcal{E}$ be the class of pairs of morphisms in C^2A, which is split exact in both degrees. It is well known (eg. [44]) that the category C^2A is Krull-Schmidt and $\mathcal{E}$ is an exact structure on C^2A. By abuse of notation we will denote an exact pair in C^2A by an exact sequence $0 \to f \to g \to h \to 0$.

Let P_i be the indecomposable projective module corresponding to $i \in Q_0$.

Definition 2.6. – For any $\beta \in \mathbb{Z}_{\geqslant 0}^{Q_0}$ we denote $\bigoplus_{i\in Q_0} \beta(i)P_i$ by $P(\beta)$. If $P_\pm = P(\beta_\pm)$, then the *weight vector* $(\mathsf{f}_-,\mathsf{f}_+)$ of f is (β_-,β_+). The *reduced weight vector* f is the difference $\mathsf{f}_+ - \mathsf{f}_-$.

Definition 2.7. – Presentations of forms $0 \to P$, $P \to 0$, and $P \xrightarrow{\mathrm{Id}} P$ are called *negative, positive*, and *neutral*. They are also denoted by O_P^-, O_P^+ and Id_P respectively. If $P = P_i$, then they are called i-th negative, positive, and neutral presentation, and denoted by O_i^-, O_i^+ and Id_i respectively.

Lemma 2.8 ([7]). – *Any presentation f decomposes as $f = f_+ \oplus f_{\mathrm{Id}} \oplus f'$, where f_+ is positive, f_{Id} is neutral, and f' is the minimal presentation of* $\mathrm{Coker}(f)$.

Corollary 2.9. – *An indecomposable presentation is one of the following four kinds. They are i-th negative, positive, neutral presentations, and minimal presentations of indecomposable non-projective representations of A.*

The following lemma is easy to verify.

Lemma 2.10 ([44, Proposition 3.1]). – *For any $P_+ \xrightarrow{f} P_- \in C^2A$ and $P \in \mathrm{mod}\, A$, we have*

1. $\mathrm{Hom}_{C^2A}(\mathrm{O}_P^-, f) \cong \mathrm{Hom}_A(P, P_-)$;
2. $\mathrm{Hom}_{C^2A}(f, \mathrm{O}_P^+) \cong \mathrm{Hom}_A(P_+, P)$;
3. $\mathrm{Hom}_{C^2A}(\mathrm{Id}_P, f) \cong \mathrm{Hom}_A(P, P_+)$;
4. $\mathrm{Hom}_{C^2A}(f, \mathrm{Id}_P) \cong \mathrm{Hom}_A(P_-, P)$.

Corollary 2.11 ([44, Corollary 3.1, 3.2]). – *The indecomposable $\mathcal{E}$-projective objects in C^2A are precisely O_i^- and Id_i. The indecomposable $\mathcal{E}$-injective objects in C^2A are precisely O_i^+ and Id_i.*

Let f and g be two presentations of representations M and N, namely, $M = \mathrm{Coker}\, f$ and $N = \mathrm{Coker}\, g$. For any morphism in $\varphi \in \mathrm{Hom}_{C^2A}(f, g)$, we get an induced morphism $\phi \in \mathrm{Hom}_A(M, N)$:

$$\begin{tikzcd} P_+ \arrow[r, "f"] \arrow[d, "\varphi_+"] & P_- \arrow[r] \arrow[d, "\varphi_-"] & M \arrow[r] \arrow[d, "\phi"] & 0 \\ R_+ \arrow[r, "g"] & R_- \arrow[r] & N \arrow[r] & 0. \end{tikzcd}$$

Conversely, any $\phi \in \mathrm{Hom}_A(M, N)$ lifts to a morphism in $\mathrm{Hom}_{C^2A}(f, g)$. So we obtain a surjection

$$\pi : \mathrm{Hom}_{C^2A}(f, g) \twoheadrightarrow \mathrm{Hom}_A(\mathrm{Coker}\, f, \mathrm{Coker}\, g).$$

π maps to a zero morphism if and only if the image of φ_- is contained in the image of g. In this case, φ_- lifts to a map in $\mathrm{Hom}_A(P_-, R_+)$ because P_- is projective. Hence the kernel of π is the image of the map ι

$$\iota : \mathrm{Hom}_A(P_-, R_+) \to \mathrm{Hom}_{C^2A}(f, g),\ h \mapsto gh - hf.$$

Recall that $\mathrm{Hom}_A(P_-, R_+) \cong \mathrm{Hom}_{C^2A}(f, \mathrm{Id}_{R_+})$. So we can summarize the above discussion as follows. The functor $\mathrm{Coker} : C^2A \to \mathrm{mod}\, A$ is full and dense with the kernel consisting of those morphisms which are factored through positive and neutral presentations. Let $\overline{C^2A}$ be the category C^2A modulo the morphisms which are factorized through $\mathcal{E}$-injectives.

PROPOSITION 2.12 ([44, Proposition 3.3]). – *The functor* Coker *induces an isomorphism* $\overline{C^2A} \cong \operatorname{mod} A$.

Here is a main result in [44].

THEOREM 2.13 ([44, Theorem 5.1]). – *The exact category C^2A has almost split pairs.*

The next two propositions enable us to construct almost split pairs in C^2A. For an A-module M, we write $f(M) : P_+(M) \to P_-(M)$ for its minimal presentation.

PROPOSITION 2.14 ([44, Proposition 5.6]). – *If $x : 0 \to f \to e \to g \to 0$ is exact in C^2A with* $\operatorname{Coker}(f) \neq 0$ *and* $\operatorname{Coker}(g) \neq 0$*, then x is almost split iff. the induced sequence* $\operatorname{Coker}(x) : 0 \to \operatorname{Coker}(f) \to \operatorname{Coker}(e) \to \operatorname{Coker}(g) \to 0$ *is almost split.*

PROPOSITION 2.15 ([44, Proposition 5.9 and Corollary 5.3]). – *The almost split pair starting at $f(I_i)$ has the form:*

$$0 \to f(I_i) \to f(I_i/\operatorname{soc}(I_i)) \oplus \mathrm{O}_R^+ \to \mathrm{O}_i^+ \to 0, \qquad \textit{if } I_i \textit{ is not simple;}$$
$$0 \to f(I_i) \to \mathrm{Id}_i \oplus \mathrm{O}_{P_+(I_i)}^+ \to \mathrm{O}_i^+ \to 0, \qquad \textit{if } I_i \textit{ is simple,}$$

where $\operatorname{Hom}_A(R, A)^*$ *is the maximal injective summand of E, with $E \to I_i$ a right minimal almost split morphism in* $\operatorname{mod} A$.

COROLLARY 2.16. – *We have that $\tau f(M) = f(\tau M)$ for M non-projective and $\tau(\mathrm{O}_i^+) = f(I_i)$ in C^2A.*

From now on let us assume A is the $\mathbb{F}$-species Γ_Q. We denote $C^2\Gamma_Q$ by C^2Q.

LEMMA 2.17. – *Suppose that $0 \to L \to M \to N \to 0$ is an almost split sequence in* $\operatorname{Rep}(Q)$. *Then we have the following almost split pairs in C^2Q.*

$$0 \to f(L) \to f(M) \to f(N) \to 0 \qquad \textit{if } L \neq S_i; \tag{2.2}$$
$$0 \to f(L) \to f(M) \oplus \mathrm{Id}_i \to f(N) \to 0 \qquad \textit{if } L = S_i. \tag{2.2'}$$

Proof. – Suppose that L is not simple. We can splice the minimal presentations of L and N together to form a presentation of M

$$P_+(L) \oplus P_+(N) \xrightarrow{f} P_-(L) \oplus P_-(N) \to M \to 0.$$

By construction, we have the exact sequence $0 \to f(L) \to f \to f(N) \to 0$. We claim that f is minimal. This is equivalent to that $\hom_Q(M, S_i) = \hom_Q(L, S_i) + \hom_Q(N, S_i)$ and $\operatorname{ext}^1_Q(M, S_i) = \operatorname{ext}^1_Q(L, S_i) + \operatorname{ext}^1_Q(N, S_i)$ for each S_i. Since $0 \to L \to M \to N \to 0$ is almost split and L is non-simple, it follows that

$$0 \to \operatorname{Hom}_Q(N, S_i) \to \operatorname{Hom}_Q(M, S_i) \to \operatorname{Hom}_Q(L, S_i) \to 0,$$
$$0 \to \operatorname{Ext}^1_Q(N, S_i) \to \operatorname{Ext}^1_Q(M, S_i) \to \operatorname{Ext}^1_Q(L, S_i) \to 0$$

are both exact.

In the case where $L = S_i$, by Auslander-Reiten formula [2, Corollary IV.4.7]

$$\operatorname{Hom}_Q(\tau^{-1}L, S_i) = \operatorname{Ext}^1_Q(S_i, S_i)^* = 0, \quad \operatorname{Ext}^1_Q(\tau^{-1}L, S_i) = \operatorname{Hom}_Q(S_i, S_i)^* = k.$$

So we have the exact sequence

$$0 \to \mathrm{Hom}_Q(M, S_i) \to \mathrm{Hom}_Q(L, S_i) \cong \mathrm{Ext}^1_Q(\tau^{-1}L, S_i) \to \mathrm{Ext}^1_Q(M, S_i) \to 0.$$

Hence $\mathrm{Hom}_Q(M, S_i) = \mathrm{Ext}^1_Q(M, S_i) = 0$. This implies the exactness of (2.2′). Finally the claim follows from Proposition 2.14. □

Similarly the next lemma follows directly from Proposition 2.15.

LEMMA 2.18. – *We have the following almost split pairs in C^2Q*

$$0 \to f(I_i) \to \bigoplus_{(k,i)\in Q_1} c_{i,k} f(I_k) \oplus \bigoplus_{(i,j)\in Q_1} c_{j,i}\, \mathrm{O}_j^+ \to \mathrm{O}_i^+ \to 0 \quad \textit{if } i \textit{ is not a source}; \tag{2.3}$$

$$0 \to f(I_i) \to \mathrm{Id}_i \oplus \bigoplus_{(i,j)\in Q_1} c_{j,i}\, \mathrm{O}_j^+ \to \mathrm{O}_i^+ \to 0 \qquad \textit{if } i \textit{ is a source}. \tag{2.3′}$$

3. iARt Quivers

3.1. iARt Quivers

We slightly upgrade the classical Auslander-Reiten quiver by adding the translation arrows. The following definition is basically taken from [2, VII.1]. Let $\mathscr{A}$ be a category as in Section 2.1. Recall that for each $M \in \mathrm{ind}(\mathscr{A})$, $D_M := \mathrm{End}_{\mathscr{A}}(M)/\operatorname{rad}\mathrm{End}_{\mathscr{A}}(M)$ is a division k-algebra.

DEFINITION 3.1 (ARt quiver). – The ARt valued quiver $\Delta(\mathscr{A})$ of $\mathscr{A}$ is defined as follows:

1. The vertex of $\Delta(\mathscr{A})$ are the isomorphism classes of objects in ind $\mathscr{A}$.
2. There is a *morphism arrow* $M \to N$ if $\mathrm{Irr}_{\mathscr{A}}(M, N)$ is non-empty. We assign the valuation (a, b) to this arrow, where $a = \dim_{D_M} \mathrm{Irr}_{\mathscr{A}}(M, N)$ and $b = \dim_{D_N^{\mathrm{op}}} \mathrm{Irr}_{\mathscr{A}}(M, N)$.
3. There is a *translation arrow* from N to τN with trivial valuation if τN is defined.

A vertex u in an ARt quiver is called *transitive* if the translation and its inverse are both defined at u.

Note that the number a in the valuation (a, b) can be alternatively interpreted as the (direct sum) multiplicity of M in E for $E \to N$ right minimal almost split. Similarly b is the multiplicity of N in E' for $M \to E'$ left minimal almost split. Moreover, if $\mathscr{A}$ is k-*elementary*, i.e., $D_M = k$ for any $M \in \mathrm{ind}(\mathscr{A})$, then all morphism arrows are equally valued.

DEFINITION 3.2 (iARt quivers). – We have that:

- the iARt quiver $\Delta_{\mathscr{A}}$ is obtained from the ARt quiver $\Delta(\mathscr{A})$ by freezing all vertices whose translations are not defined;
- the iARt quiver $\Delta^2_{\mathscr{A}}$ is obtained from the ARt quiver $\Delta(C^2\mathscr{A})$ by freezing all non-transitive vertices.

REMARK 3.3. – When $\mathscr{A}$ is the module category of a finite-dimensional algebra, the frozen vertices of $\Delta_{\mathscr{A}}$ are precisely indecomposable projective modules. By Theorem 2.13 and Corollary 2.11, the frozen vertices in $\Delta^2_{\mathscr{A}}$ are precisely the negative, positive and neutral presentations.

We use the notation

$L \to M$	if there is a morphism arrow from L to M;
$L \dashrightarrow M$	if there is a translation arrow from L to M;
$L \rightarrowtail M$	if there is an arrow from L to M in the ARt quiver.

LEMMA 3.4. – *Let θ be an additive function from $\mathscr{A}$ to some abelian group, that is, $\theta(L) + \theta(N) = \theta(M)$ for each exact sequence $0 \to L \to M \to N \to 0$ in $\mathscr{A}$. Then at each transitive vertex L, we have that $\sum_{M \rightarrowtail L} c_{M,L}\theta(M) = \sum_{L \rightarrowtail N} c_{L,N}\theta(N)$.*

Proof. – We have two almost split sequences

$$0 \to \tau L \to \bigoplus_{M \to L} c_{M,L} M \to L \to 0, \qquad 0 \to L \to \bigoplus_{L \to N} c_{L,N} N \to \tau^{-1} L \to 0.$$

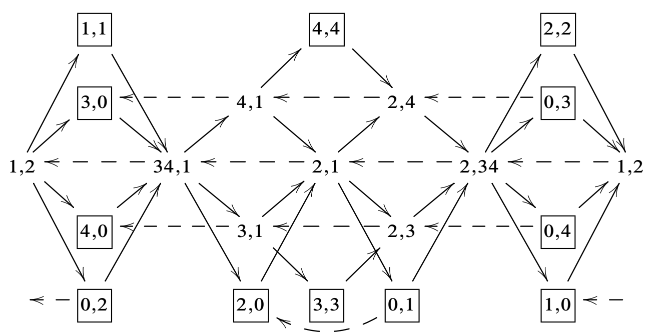
By the additivity of θ, we have that

$$\begin{aligned} \theta(L) = \sum_{M \to L} c_{M,L}\theta(M) - \theta(\tau L) &= \sum_{L \to N} c_{L,N}\theta(N) - \theta(\tau^{-1}L), \\ \Rightarrow \quad \sum_{M \rightarrowtail L} c_{M,L}\theta(M) &= \sum_{L \rightarrowtail N} c_{L,N}\theta(N). \end{aligned}$$

□

A typical additive function in C^2A is the weight vector. In some special cases including examples below, indecomposable presentations are uniquely determined by their weight vectors. So we can label them on an iARt quiver by their weight vectors. We will use the "exponential form" as a shorthand. For example, a vector $(3, 1, 0, 0, -2, 0, -1)$ is written as $(5^2 7, 1^3 2)$.

EXAMPLE 3.5. – Let A be the Jacobian algebra of the quiver with potential (Q, W) (see Section 4.1), where $Q = \begin{tikzcd}[row sep=small, column sep=small] & 3 \arrow[dr] & \\ 2 \arrow[ur] \arrow[dr] & & 1 \arrow[ll] \\ & 4 \arrow[ur] & \end{tikzcd}$ and W is the difference of two oriented triangles. The iARt quiver $\Delta^2_{\mathrm{mod}\,A}$ is drawn below. We always put frozen vertices in boxes.



Two vertices with the same weight label $(1,2)$ are identified. The translation arrow going out from $(0,2)$ ends in $(1,0)$.

3.2. Hereditary Cases

In particular, if we take $\mathscr{A} := \operatorname{Rep}(Q)$ for some valued quiver Q, we get two ARt quivers $\Delta(Q) := \Delta(\operatorname{Rep}(Q))$ and $\Delta(C^2Q)$. We denote the corresponding iARt quivers by Δ_Q and Δ_Q^2.

PROPOSITION 3.6. – *The ARt quiver $\Delta(C^2Q)$ can be obtained from $\Delta(Q)$ as follows.*

1. *We add $|Q_0|$ vertices corresponding to O_i^+ and also $|Q_0|$ vertices corresponding to Id_i.*
2. *For each $i \xrightarrow{(a,b)} j$ in Q, we draw morphism arrows $\mathrm{O}_j^+ \xrightarrow{(a,b)} \mathrm{O}_i^+$, and $f(I_i) \xrightarrow{(b,a)} \mathrm{O}_j^+$. We add translation arrows from O_i^+ to $f(I_i)$.*
3. *We draw morphism arrows $f(S_i) \to \mathrm{Id}_i$, and $\mathrm{Id}_i \to \tau^{-1} f(S_i)$.*

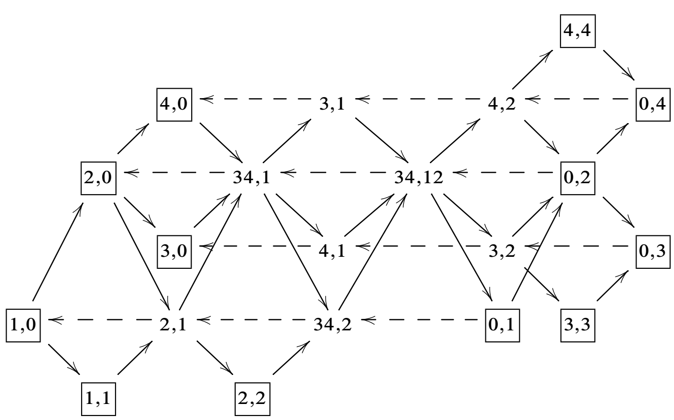
Proof. – The vertices of $\Delta(Q)$ are identified with vertices of $\Delta(C^2Q)$ via minimal presentations. By Corollary 2.9, we only need to add the vertices as in (1). Step (2) is due to (2.3) and (2.3′). Note that $\tau^{-1}f(S_i)$ is equal to $f(\tau^{-1}S_i)$ if i is not a source, otherwise it is equal to O_i^+. So Step (3) is due to (2.2′) and (2.3′). We do not need anything else because of (2.2) and the easy fact that $\bigoplus_{(i,j)\in Q_1} c_{j,i}\,\mathrm{O}_j^- \to \mathrm{O}_i^-$ is right minimal almost split. □

Due to this proposition, we will freely identify Δ_Q as a subquiver of Δ_Q^2.

From now on, we let Q be a valued quiver of Dynkin type. In this case, any indecomposable presentation f is uniquely determined by its weight vector. The quiver Δ_Q was already considered in [3, 26]. In [3] the authors associated an ice quiver to any reduced expression of the longest element w_0 in the Weyl group of Q. The iARt quiver Δ_Q only corresponds to those reduced expressions *adapted* to Q.

EXAMPLE 3.7. – The iARt quiver Δ_Q^2 for Q of type A_n is the ice hive quiver Δ_n constructed in [13] up to some arrows between frozen vertices.

EXAMPLE 3.8. – The iARt quiver Δ_Q^2 for Q a D_4-quiver:



Readers can find a few other iARt quivers in Appendix 6.

Remark 3.9. – One natural question is whether the iARt quivers Δ_Q^2 and $\Delta_{Q'}^2$ (or Δ_Q and $\Delta_{Q'}$) are mutation-equivalent if Q and Q' are reflection-equivalent. The answer is positive at least in trivially valued cases. We conjecture that this is also true in general. As pointed in [3, Remark 2.14], for Δ_Q with Q trivially valued, by the Tits lemma every two reduced words can be obtained form each other by a sequence of elementary 2- and 3-moves (see [46, Section 2.1]); by [46, Theorem 3.5] every such move either leaves the seed unchanged, or replaces it by an adjacent seed. Finally, similar to the proof of Corollary 11.9, the result can be extended from Δ_Q to Δ_Q^2.

However, reflection-equivalent quivers cannot be replaced with mutation-equivalent Jacobian algebras (see Section 4.1). The Jacobian algebra in Example 3.5 is obtained from the above path algebra of D_4 by mutating at the vertex 2. According to [33], the iARt quiver in Example 3.8 is mutation-equivalent to a finite mutation type quiver $E_6^{(1,1)}$, while the one in Example 3.5 is mutation equivalent to a wild acyclic quiver, which is of infinite mutation type.

It follows from (2.1) and the fact that each Q is symmetrizable that

Lemma 3.10. – *The B-matrix of Δ_Q^2 is skew-symmetrizable.*

It was constructed in [5, Theorem 8.3] a family of compatible pairs $\{(B(\mathrm{i}), \Lambda(\mathrm{i}))\}_{\mathrm{i}}$, i.e, $B(\mathrm{i})$ and $\Lambda(\mathrm{i})$ satisfy that $B(\mathrm{i})\Lambda(\mathrm{i}) = (I, 0)$. The family $\{B(\mathrm{i})\}_{\mathrm{i}}$ contains the restricted B-matrix of Δ_Q. It follows that

Lemma 3.11. – *The restricted B-matrices of Δ_Q, and thus of Δ_Q^2, have full ranks.*

By Lemma 3.4, the assignment $f \mapsto (\mathsf{f}_-, \mathsf{f}_+)$ is a weight configuration of Δ_Q^2. However, it is not full (see Section 1.3). We want to extend it to a full one which is useful for the second half of the paper. Since Q is of finite representation type, each non-neutral $f \in \operatorname{ind}(C^2Q)$ is translated from a unique indecomposable positive presentation, that is, $f = \tau^t(\mathrm{O}_i^+)$ for some $i \in Q_0$ and $t \in \mathbb{Z}_{\geqslant 0}$. Now for each $f \in \operatorname{ind}(C^2Q)$, we assign a triple-weight vector as follows.

Definition 3.12. – If f is translated from O_i^+, then the *triple weight* $\widetilde{\mathsf{f}} \in \mathbb{Z}^{3|Q_0|}$ of f is given by $(\mathsf{e}(f), \mathsf{f}_-, \mathsf{f}_+)$ where $\mathsf{e}(f) := \mathsf{e}_i$ the unit vector supported on i. If $f = \mathrm{Id}_i$, then we set $\widetilde{\mathsf{f}} := (0, \mathsf{e}_i, \mathsf{e}_i)$. We also define another weight vector $\bar{\mathsf{f}} \in \mathbb{Z}^{|Q_0|}$ attached to f by $\bar{\mathsf{f}} := \mathsf{e}_i + \mathsf{f}_- - \mathsf{f}_+$.

Corollary 3.13. – *The assignment $\boldsymbol{\sigma}_Q^2 : f \mapsto \widetilde{\mathsf{f}}$ (resp. $\boldsymbol{\sigma}_Q : f \mapsto \bar{\mathsf{f}}$) defines a full weight configuration for the iARt quiver Δ_Q^2 (resp. Δ_Q).*

Proof. – Due to Lemma 3.4, it suffices to show that for each mutable f,

$$\sum_{g \rightarrowtail f} c_{g,f}\,\mathsf{e}(g) = \sum_{f \rightarrowtail h} c_{f,h}\,\mathsf{e}(h).$$

We call a vertex u *regular* if it is transitive and $\tau^{-1}v$ is defined for each $v \to u$, and τw is defined for each $u \to w$. It is clear from (2.1) that the equation holds at each regular vertex. From the description of Proposition 3.6, we see that all transitive vertices are regular except for $f(S_i)$ and $\tau^{-1}f(S_i)$. The problem is that these vertices may have (morphism) arrows to neutral frozen vertices, whose translation is not defined. But the first component of the triple weights of Id_i is a zero vector so the equality still holds at these vertices.

For the case of Δ_Q, it is enough to observe that the weight vector $\bar{\mathsf{f}}$ is zero on the positive and neutral frozen vertices of Δ_Q^2, and Δ_Q is obtained from Δ_Q^2 by deleting these vertices. □

We shall consider the graded upper cluster algebra $\overline{\mathcal{C}}(\Delta_Q^2;\boldsymbol{\sigma}_Q^2)$ and the graded cluster algebra $\mathcal{C}(\Delta_Q;\boldsymbol{\sigma}_Q)$ later.

4. Cluster Character from Quivers with Potentials

4.1. Quivers with Potentials

The mutation of quivers with potentials is invented in [8] and [9] to model the cluster algebras. In this and next section and Appendix 11, we switch back to the usual quiver notation. A quiver Δ is a quadruple $(\Delta_0, \Delta_1, h, t)$, where the maps h and t map an arrow $a \in \Delta_1$ to its head and tail $h(a), t(a) \in \Delta_0$. Following [8], we define a potential W on an ice quiver Δ as a (possibly infinite) linear combination of oriented cycles in Δ. More precisely, a *potential* is an element of the *trace space* $\mathrm{Tr}(\widehat{k\Delta}) := \widehat{k\Delta}/[\widehat{k\Delta},\widehat{k\Delta}]$, where $\widehat{k\Delta}$ is the completion of the path algebra $k\Delta$ and $[\widehat{k\Delta},\widehat{k\Delta}]$ is the closure of the commutator subspace of $\widehat{k\Delta}$. The pair (Δ, W) is an *ice quiver with potential*, or IQP for short. For each arrow $a \in \Delta_1$, the *cyclic derivative* ∂_a on $\widehat{k\Delta}$ is defined to be the linear extension of

$$\partial_a(a_1\cdots a_d) = \sum_{k=1}^{d} a^*(a_k)a_{k+1}\cdots a_d a_1\cdots a_{k-1},$$

where $a^*(b) = 1$ if $a = b$ and zero otherwise. For each potential W, its *Jacobian ideal* ∂W is the (closed two-sided) ideal in $\widehat{k\Delta}$ generated by all $\partial_a W$. The *Jacobian algebra* $J(\Delta, W)$ is the quotient algebra $\widehat{k\Delta}/\partial W$.(2) If W is polynomial and the quotient of $k\Delta$ by the unclosed ideal generated by all $\partial_a W$ is finite-dimensional, then the completion is unnecessary to define $J(\Delta, W)$. This is the case throughout this paper.

The key notion introduced in [8, 9] is the *mutation* of quivers with potentials and their decorated representations. For an ice quiver with *nondegenerate potential* (see [8]), the mutation in certain sense "lifts" the mutation in Definition 1.3. We have a short review in Appendix 11.1.

Definition 4.1. – A *decorated representation* of a Jacobian algebra $J := J(\Delta, W)$ is a pair $\mathcal{M} = (M, M^+)$, where $M \in \mathrm{Rep}(J)$, and M^+ is a finite-dimensional k^{Δ_0}-module.

Let $\mathcal{R}ep(J)$ be the set of decorated representations of $J(\Delta, W)$ up to isomorphism. Let K^2J be the homotopy category of C^2J. There is a bijection between two additive categories $\mathcal{R}ep(J)$ and K^2J mapping any representation M to its minimal presentation in $\mathrm{Rep}(J)$, and the simple representation S_u^+ of k^{Δ_0} to $P_u \to 0$. Suppose that $\mathcal{M}$ corresponds to a projective presentation $P(\beta_+) \to P(\beta_-)$.

(2) Unlike the definition in [6], we need to include $\partial_a W$ in the ideal ∂W even if a is an arrow between frozen vertices.

DEFINITION 4.2. – The g-*vector* $g(\mathcal{M})$ of a decorated representation $\mathcal{M}$ is the reduced weight vector $\beta_+ - \beta_-$.

DEFINITION 4.3. – A potential W is called *rigid* on a quiver Δ if every potential on Δ is cyclically equivalent to an element in the Jacobian ideal ∂W. Such a QP (Δ, W) is also called *rigid*. A potential W is called μ-*rigid* on an ice quiver Δ if its restriction to the mutable part Δ^μ is rigid.

It is known [8, Proposition 8.1, Corollary 6.11] that every rigid QP is 2-acyclic, and the rigidity is preserved under mutations. In particular, any rigid QP is nondegenerate.

DEFINITION 4.4. – Two QPs (Δ, W) and (Δ', W') on the same vertex set Δ_0 are called *right-equivalent* if there is an isomorphism $\varphi : k\Delta \to k\Delta'$ such that $\varphi|_{k\Delta_0} = \mathrm{Id}$ and $\varphi(W)$ is cyclically equivalent to W'. Two IQPs (Δ, W) and (Δ', W') are called μ-*right-equivalent* if they are right-equivalent when restricted to their mutable parts.

DEFINITION 4.5. – A representation is called μ-*supported* if its supporting vertices are all mutable. We denote by $\mathcal{Rep}^\mu(J)$ the full subcategory of all μ-supported decorated representations of J.

REMARK 4.6. – If (Δ, W) and (Δ', W') are μ-right-equivalent, then $\mathcal{Rep}^\mu(J)$ and $\mathcal{Rep}^\mu(J')$ are equivalent. Indeed, we write $W = W_\mu + W_\diamond$ where W_μ is the restriction of W to the mutable part of Δ. We find that any cyclic derivative $\partial_a W_\diamond$ is a sum of paths passing some frozen vertices. Such a sum gives rise to a trivial relation on the μ-supported representations.

4.2. The Generic Cluster Character

DEFINITION 4.7. – To any $g \in \mathbb{Z}^{\Delta_0}$ we associate the *reduced* presentation space

$$\mathrm{PHom}_J(g) := \mathrm{Hom}_J(P([g]_+), P([-g]_+)),$$

where $[g]_+$ is the vector satisfying $[g]_+(u) = \max(g(u), 0)$. We denote by $\mathrm{Coker}(g)$ the cokernel of a general presentation in $\mathrm{PHom}_J(g)$.

The reader should be aware that $\mathrm{Coker}(g)$ is just a notation rather than a specific representation. If we write $M = \mathrm{Coker}(g)$, this simply means that we take a presentation general enough (according to context) in $\mathrm{PHom}_J(g)$, then let M to be its cokernel.

DEFINITION 4.8. – A g-vector g is called μ-*supported* if $\mathrm{Coker}(g)$ is μ-supported. Let $G(\Delta, W)$ be the set of all μ-supported g-vectors in $\mathbb{Z}^{\Delta_0}$.

It turns out that for a large class of IQPs the set $G(\Delta, W)$ is given by lattice points in some rational polyhedral cone. Such a class includes the IQPs introduced in [13, 14, 15], and the ones to be introduced in Section 5.1.

DEFINITION 4.9 ([42, 12]). – We define the *generic character* $C_W : G(\Delta, W) \to \mathbb{Z}(\mathbf{x})$ by

$$C_W(g) = \mathbf{x}^{g} \sum_{e} \chi\big(\mathrm{Gr}^{e}(\mathrm{Coker}(g))\big)\mathbf{y}^{e}, \tag{4.1}$$

where $\mathrm{Gr}^e(M)$ is the variety parametrizing e-dimensional quotient representations of M, and $\chi(-)$ denotes the topological Euler-characteristic.

THEOREM 4.10 ([13, Corollary 5.14], *cf.* [42, Theorem 1.1]). – *Suppose that IQP* (Δ, W) *is non-degenerate and* B_Δ *has full rank. The generic character* C_W *maps* $G(\Delta, W)$ *(bijectively) to a set of linearly independent elements in* $\overline{\mathcal{C}}(\Delta)$ *containing all cluster monomials.*

DEFINITION 4.11. – We say that an IQP (Δ, W) *models* an algebra $\mathcal{C}$ if the generic cluster character maps $G(\Delta, W)$ (bijectively) onto a basis of $\mathcal{C}$. If $\mathcal{C}$ is the upper cluster algebra $\overline{\mathcal{C}}(\Delta)$, then we simply say that (Δ, W) is a *cluster model.*

REMARK 4.12. – Suppose that (Δ, W) and (Δ', W') are μ-right-equivalent. By Remark 4.6, $\mathcal{R}ep^{\mu}(J)$ and $\mathcal{R}ep^{\mu}(J')$ are equivalent via some isomorphism $\varphi : k\Delta^{\mu} \to k(\Delta')^{\mu}$. By abuse of notation we denote the equivalence also by φ. Since $\varphi\ |_{\Delta_0} = \mathrm{Id}$, $\varphi(M)$ and M are isomorphic and have the same g-vector. We see from (4.1) that if (Δ, W) is a cluster model, then so is (Δ', W').

5. iARt QPs

5.1. The iARt QP (Δ_Q^2, W_Q^2)

For the time being, let us assume Q is a trivially valued Dynkin quiver. A *translation triangle* in an iARt quiver is an oriented cycle of the form

$$\begin{tikzcd} & M \arrow[dr] & \\ \tau L \arrow[ur] & & L. \arrow[ll, dashed] \end{tikzcd}$$

For each iARt quiver Δ_Q^2, we define the potential W_Q^2 as an alternating sum of all translation triangles. We make this more precise as follows. We can also label each non-neutral $f = \tau^{-t}\,\mathrm{O}_i^-$ by the pair $(i,t)^* = (i^*(f), t^*(f))$. The arrows of Δ_Q^2 are thus classified into three classes

Type A arrows	$(i,t)^* \to (j,t+1)^*,\ f(S_i) \to \mathrm{Id}_i;$
Type B arrows	$(i,t)^* \to (j,t)^*,\ \mathrm{Id}_i \to \tau^{-1} f(S_i);$
Type C arrows	$(i,t)^* \dashrightarrow (i,t-1)^*.$

Let $\dot a$ (resp. $\dot b$ and $\dot c$) denote the sum of all type A (B and C) arrows. The potential W_Q^2 is defined as $\dot a\dot c\dot b - \dot a\dot b\dot c$. Thus the Jacobian ideal is generated by the elements

$$e_u(\dot a\dot c - \dot c\dot a)e_v,\ e_u(\dot c\dot b - \dot b\dot c)e_v, \tag{5.1}$$

$$e_u(\dot b\dot a - \dot a\dot b)e_v. \tag{5.2}$$

Let $J := J(\Delta_Q^2, W_Q^2)$ be the Jacobian algebra of (Δ_Q^2, W_Q^2). For the rest of this section, we denote a single arrow by the lowercase letter of its type with some superscript (eg. a and a'). We observe that for each non-neutral vertex in Δ_Q^2 there are exactly one incoming arrow and one outgoing arrow of type C. Moreover, if non-neutral u and v are connected by an arrow of type B or A, then the relations (11.11) say that

$$e_u a c e_v = e_u c' a' e_v \ \text{ or } \ e_u c b e_v = e_u b' c' e_v \ \text{ for some } a, a', b, b', c, c'. \tag{5.1.1}$$

In general, the relations (5.2) do not have a similar implication because there is a trivalent vertex for Q of type D or E. We have that

$$e_u b a e_v = \sum_{a',b'} (e_u a' b' e_v \text{ or } e_u b' a' e_v). \tag{5.2.1}$$

If u and v are not trivalent, then the right sum has only one summand. If both u and v are negative (resp. positive), then some translation arrows are undefined so the relations (5.1.1) reduce to the following

$$e_u a c e_v = 0 \text{ (resp. } e_u c a e_v = 0 \text{ or } e_u c b e_v = 0). \tag{5.1.2}$$

Similarly if u (resp. v) is neutral, then

$$e_u b c e_v = 0 \text{ (resp. } e_u c a e_v = 0). \tag{5.1.3}$$

LEMMA 5.1. – *The IQP (Δ_Q^2, W_Q^2) is rigid and $J(\Delta_Q^2, W_Q^2)$ is finite-dimensional.*

Proof. – To show that $J(\Delta_Q^2, W_Q^2)$ is finite-dimensional, it suffices to observe that any nonzero path from f to g in J can be uniquely identified as an element in $\bigoplus_{t=0}^{t^*(f)} \mathrm{Hom}_{C^2Q}(\tau^t f, g)$. Indeed, suppose that p is a path from f to g. By (5.2.1) and (5.1.3), we can make p avoid any neutral vertex. By (5.1.1) we can move all arrows of type C to the left. If we remove all arrows of type C, then the truncated path can be interpreted as a morphism from $\tau^t f$ to g.

Due to relations (5.1.1) and (5.2.1), any cycle in the Jacobian algebra is equivalent to a sum of composition of cycles $e_u(acb)e_{u'}$ with u and u' mutable. It suffices to show that each $e_u acb e_{u'}$ is in fact zero in the Jacobian algebra. Applying the relation (5.1.1) twice (if u is not negative), we see that $e_u acb e_{u'}$ is equivalent to $e_w a'c'b' e_u$ where u and w are connected by an arrow of type C. If u is mutable, then there is some negative vertex v connected to u by arrows of type C. So $e_u acb e_{u'}$ is equivalent to $e_v a''c''b''$, which is zero by (5.1.2). □

We delete all translation arrows of Δ_Q^2, and obtain a subquiver denoted by $\wedge_Q^2$. Let R be the direct sum of all presentations in $\mathrm{ind}(C^2Q)$. It is well-known that the *Auslander algebra* $A_Q^2 := \mathrm{End}_{C^2Q}(R)$ is equal to $k\wedge_Q^2$ modulo the *mesh relations* [2, VII.1]. The Auslander algebra A_Q^2 is the quotient of Jacobian algebra J by the ideal generated by translation arrows.

Let $f : P_+ \to P_-$ be a presentation in $\mathrm{ind}(C^2Q)$. We denote by $\boldsymbol{P}_f$ (resp. $\boldsymbol{I}_f$) the indecomposable projective (resp. injective) representation of J corresponding to the vertex f.

LEMMA 5.2. – *We have the following for the module $\boldsymbol{P}_f$ in $J(\Delta_Q^2, W_Q^2)$*

1. $\boldsymbol{P}_f(\mathrm{O}_i^-) \cong \mathrm{Hom}_Q(P_{i^*(f)}, P_i)$*;*
2. $\boldsymbol{P}_f(\mathrm{O}_i^+) \cong \mathrm{Hom}_Q(P_+, P_i)$*;*
3. $\boldsymbol{P}_f(\mathrm{Id}_i) \cong \mathrm{Hom}_Q(P_-, P_i)$*.*

Proof. – (1). Recall from the proof of Lemma 5.1 that

$$\boldsymbol{P}_f(\mathrm{O}_i^-) \subset \bigoplus_{t=0}^{t^*(f)} \mathrm{Hom}_{C^2Q}(\tau^t f, \mathrm{O}_i^-).$$

We know that $\mathrm{Hom}_{C^2(Q)}(f, \mathrm{O}_i^-) = 0$ unless f is negative. This implies that $\boldsymbol{P}_f(\mathrm{O}_i^-) \subset \mathrm{Hom}_Q(P_{i^*(f)}, P_i)$. Conversely, we identify an element in $\mathrm{Hom}_Q(P_{i^*(f)}, P_i)$ by a path consisting solely of arrows of type B. By adjoining $t^*(f)$ arrows of type C we get a path from f to O_i^-, which is easily seen to be nonzero.

(2). We observe that any path from f to O_i^+ containing a translation arrow must be equivalent to zero due to the relations (11.11) and (5.1.2). So $\boldsymbol{P}_f(\mathrm{O}_i^+)$ is the same as $\boldsymbol{P}_f(\mathrm{O}_i^+)$ restricted to the Auslander algebra. The result follows from Lemma 2.10.(4).

(3). Similar to (2), any path from f to Id_i containing a translation arrow must be equivalent to zero. The result follows from Lemma 2.10.(2). □

5.2. The Cone $\mathsf{G}_{\Delta_Q^2}$

We consider the following set of representations T_v

$$0 \to T_v \to \boldsymbol{I}_v \to \bigoplus_{i\to j} \boldsymbol{I}_{\mathrm{O}_j^-} \qquad \text{for } v = \mathrm{O}_i^-, \tag{5.3}$$

$$0 \to T_v \to \boldsymbol{I}_v \to \bigoplus_{i\to j} \boldsymbol{I}_{\mathrm{O}_j^+} \qquad \text{for } v = \mathrm{O}_i^+, \tag{5.4}$$

$$0 \to T_v \to \boldsymbol{I}_v \to \bigoplus_{i\to j} \boldsymbol{I}_{\mathrm{Id}_j} \qquad \text{for } v = \mathrm{Id}_i. \tag{5.5}$$

The maps in (5.3) and (5.4) are canonical, that is, the map from $\boldsymbol{I}_{\mathrm{O}_i^\pm}$ to $\boldsymbol{I}_{\mathrm{O}_j^\pm}$ is given by the morphism arrow $\mathrm{O}_j^\pm \to \mathrm{O}_i^\pm$. For the map from $\boldsymbol{I}_{\mathrm{Id}_i}$ to $\boldsymbol{I}_{\mathrm{Id}_j}$, let us recall from Lemma 2.17.(3) that $\mathrm{Hom}(\boldsymbol{I}_{\mathrm{Id}_i}, \boldsymbol{I}_{\mathrm{Id}_j}) \cong \boldsymbol{P}_{\mathrm{Id}_j}(\mathrm{Id}_i) \cong \mathrm{Hom}_Q(P_j, P_i)$. We take the map to be the irreducible map in $\mathrm{Hom}_Q(P_j, P_i)$. It will follow from the proof of Theorem 5.3 that the rightmost maps in (5.3)–(5.5) are all surjective.

For $j \in Q_0$, let $j^* = i^*(\mathrm{O}_j^+) \in Q_0$. It is well-known that $j \mapsto j^*$ is a (possibly trivial) involution. The involution does not depend on the orientation of Q. Its formula is listed in [26, Section 2.3]. For any map between projective modules $f : P(\beta_1) \to P(\beta_2)$, the *$i$-th top restriction* of f is the induced map $\mathrm{top}\, P(\beta_1(i)) \to \mathrm{top}\, P(\beta_2(i))$.

THEOREM 5.3. – *We have the following description for the modules T_v.*

1. *The module $T_{\mathrm{O}_i^-}$ is the indecomposable module supported on all vertices translated from $\mathrm{O}_{i^*}^+$ with dimension vector $(1, 1, \dots, 1)$;*
2. *The defining linear map $f \to g$ in $T_{\mathrm{O}_i^+}$ is given by the i-th top restriction of φ_+;*
3. *The defining linear map $f \to g$ in T_{Id_i} is given by the i-th top restriction of φ_-,*

where $\varphi = (\varphi_+, \varphi_-)$ is the irreducible morphism from f to g.

In particular, the dimension vector θ_v of T_v is given by

$$\theta_v(f) = \mathsf{e}(f)(i^*) \qquad \textit{for } v = \mathrm{O}_i^-,$$
$$\theta_v(f) = \mathsf{f}_+(i) \qquad \textit{for } v = \mathrm{O}_i^+,$$
$$\theta_v(f) = \mathsf{f}_-(i) \qquad \textit{for } v = \mathrm{Id}_i\,.$$

Proof. – By Lemma 5.2.(1), we can identify $\boldsymbol{I}_{\mathrm{O}_i^-}(f)$ with $\mathrm{Hom}_Q(P_{i^*(f)}, P_i)^*$. From the definition of $T_{\mathrm{O}_i^-}$ and the exact sequence

$$0 \to \bigoplus_{i\to j} \mathrm{Hom}_Q(P_{i^*(f)}, P_j) \to \mathrm{Hom}_Q(P_{i^*(f)}, P_i) \to \mathrm{Hom}_Q(P_{i^*(f)}, S_i) \to 0,$$

we conclude that $T_{\mathrm{O}_i^-}(f) = \mathrm{Hom}_Q(P_{i^*(f)}, S_i)^*$.

By Lemma 5.2.(2), we can identify $\boldsymbol{I}_{\mathrm{O}_i^+}(f)$ with $\mathrm{Hom}_Q(P_+, P_i)^*$. From the definition of $T_{\mathrm{O}_i^+}$ and the exact sequence

$$0 \to \bigoplus_{i\to j} \mathrm{Hom}_Q(P_+, P_j) \to \mathrm{Hom}_Q(P_+, P_i) \to \mathrm{Hom}_Q(P_+, S_i) \to 0,$$

we conclude that $T_{\mathrm{O}_i^+}(f) = \mathrm{Hom}_Q(P_+, S_i)^*$. The description of maps follows from the naturality.

By Lemma 5.2.(3), we can identify $\boldsymbol{I}_{\mathrm{Id}_i}(f)$ with $\mathrm{Hom}_Q(P_-, P_i)^*$. From the definition of T_{Id_i} and the exact sequence

$$0 \to \bigoplus_{i\to j} \mathrm{Hom}_Q(P_-, P_j) \to \mathrm{Hom}_Q(P_-, P_i) \to \mathrm{Hom}_Q(P_-, S_i) \to 0,$$

we conclude that $T_{\mathrm{Id}_i}(f) = \mathrm{Hom}_Q(P_-, S_i)^*$ and the description of maps follows from the naturality. □

LEMMA 5.4. – *Let* $M = \mathrm{Coker}(\mathsf{g})$*, then* $\mathrm{Hom}_J(M, T_v) = 0$ *for each frozen* v *if and only if* $\mathrm{Hom}_J(M, I_v) = 0$ *for each frozen* v.

Proof. – Since each subrepresentation of T_v is also a subrepresentation of I_v, one direction is clear. Conversely, let us assume that $\mathrm{Hom}_J(M, T_v) = 0$ for each frozen vertex v. We prove that $\mathrm{Hom}_J(M, I_v) = 0$ by induction.

We first notice that for $v = \mathrm{O}_i^-, \mathrm{O}_i^+, \mathrm{Id}_i$, $T_v = I_v$ if i is a sink. In general, for an injective presentation of $T\colon 0 \to T \to I_1 \to I_0$ with $\mathrm{Hom}_J(M, I_0) = 0$, we have that $\mathrm{Hom}_J(M, I_1) = 0$ is equivalent to $\mathrm{Hom}_J(M, T) = 0$. Now we perform the induction from a sink i in an appropriate order using (5.3)–(5.5). We can conclude that $\mathrm{Hom}_J(M, I_v) = 0$ for all frozen v. □

For the rest of this article, when we write $\mathsf{g}(-)$, we view g as a linear functional via the usual dot product.

DEFINITION 5.5 ([18]). – The *tropical* F*-polynomial* f_T of a representation T is the function

$$\mathsf{g} \mapsto \max_{S \hookrightarrow T} -\mathsf{g}(\underline{\dim} S).$$

We shall show that all these T_v are *reachable* (see 11.1 for the definition) in Appendix 11.3.

THEOREM 5.6 ([18, Theorem 6.5]). – *If* T *is reachable, then for any* $\mathsf{g} \in \mathbb{Z}^{\Delta_0}$ *we have that*

$$f_T(\mathsf{g}) = \hom(\mathrm{Coker}(\mathsf{g}), T).$$

DEFINITION 5.7. – A vertex v is called *maximal* in a representation M if $\dim M(v) = 1$ and all strict subrepresentations of M are not supported on v.

We note that every T_v contains a maximal vertex w, which is another frozen vertex. For example, $w = \mathrm{O}_{i^*}^+$ (resp. Id_i and O_i^-) for $v = \mathrm{O}_i^-$ (resp. O_i^+ and Id_i).

Definition 5.8. – We define a cone $\mathsf{G}_{\Delta_Q^2} \subset \mathbb{R}^{(\Delta_Q^2)_0}$ by $\mathsf{g}(\underline{\dim} S) \geq 0$ for all strict subrepresentations $S \subseteq T_v$ and all frozen v.

Theorem 5.9. – *The set of lattice points* $\mathsf{G}_{\Delta_Q^2} \cap \mathbb{Z}^{(\Delta_Q^2)_0}$ *is exactly* $G(\Delta_Q^2, W_Q^2)$.

Proof. – Due to Lemma 5.6 and 5.4, it suffices to show that $\mathsf{G}_{\Delta_Q^2}$ is defined by $\mathsf{g}(\underline{\dim} S) \geq 0$ for all subrepresentations S of T_v and all v frozen. We notice that these conditions are the union of the defining conditions of $\mathsf{G}_{\Delta_Q^2}$ and $\mathsf{g}(\underline{\dim} T_v) \geq 0$. But the latter conditions are redundant because $\underline{\dim} T_v = \mathsf{e}_w + (\underline{\dim} T_v - \mathsf{e}_w)$ and $\underline{\dim} T_v - \mathsf{e}_w$ is the dimension vector of a strict subrepresentation of T_v, where w is the maximal (frozen) vertex of T_v. □

Example 5.10 (Example 3.8 continued). – In this case, it is almost trivial to list all strict subrepresentations for all T_v. Readers can easily find that there are 3 subrepresentations for all O_i^-, and $7, 6, 1, 1$ subrepresentations for $\mathrm{O}_1^+, \mathrm{O}_2^+, \mathrm{O}_3^+, \mathrm{O}_4^+$ respectively, and $1, 2, 7, 7$ subrepresentations for $\mathrm{Id}_1, \mathrm{Id}_2, \mathrm{Id}_3, \mathrm{Id}_4$ respectively. All these subrepresentations are needed to define $\mathsf{G}_{\Delta_Q^2}$, so there are 44 inequalities. Readers can find an extended version of this example in [16]. This example can be easily generalized to Q of type D_n with a similar orientation.

Remark 5.11 (The Valued Cases). – To deal with the general valued quiver Q, we could have worked with species analogue of QP, but this require some lengthy preparation. To avoid this we can define the analogous Jacobian algebra by the Ext-completion algebra $\prod_{i\geq 0} \mathrm{Ext}_A^2(A^*, A)^{\otimes_A i}$ for the Auslander algebra $A := A_Q^2$.

We can even define $\mathsf{G}_{\Delta_Q^2}$ without introducing the Jacobian algebra. Without the Jacobian algebra we are unable to define the module T_v by injective presentations, but it still makes perfect sense to define T_v via Theorem 5.3. Once T_v's are defined, we define the cones $\mathsf{G}_{\Delta_Q^2}$ by Definition 5.8. The same remark also applies for the cone G_{Δ_Q} to be defined below.

In general, some of the defining conditions of $\mathsf{G}_{\Delta_Q^2}$ may be redundant as shown in the following example.

Example 5.12. – The most complicated T_v for Q of type G_2 (see Example 1.2 and Appendix 6) is T_{Id_1}. Its dimension vector is

$$\begin{tikzcd}
 & & 1 \arrow[dr] & \\
1 \arrow[dr] & 2 \arrow[l, dashed] \arrow[dr] & 1 \arrow[l, dashed] \arrow[u] \arrow[dr] & 0 \arrow[l, dashed] \\
0 \arrow[u] \arrow[dr] & 3 \arrow[l, dashed] \arrow[u] & 3 \arrow[l, dashed] \arrow[u] & 0 \arrow[l, dashed] \arrow[u] \\
 & 0 \arrow[u] & &
\end{tikzcd}$$

, and its strict subrepresentations have 13 distinct dimension vectors. However, we only need 5 of them to define $\mathsf{G}_{\Delta_Q^2}$. Readers can find a full list of inequalities in [16]. In fact, we can download the full H-matrices of all exceptional types from the author's web page [17].

Now we come back to the iARt quiver Δ_Q. We define a potential W_Q on Δ_Q by the same formula defining W_Q^2. The iARt QP (Δ_Q, W_Q) is nothing but the restriction of (Δ_Q^2, W_Q^2) to the subquiver Δ_Q. For each O_i^-, we define the representation $T_{\mathrm{O}_i^-}$ of $J(\Delta_Q, W_Q)$ by the same injective presentation (5.3). Similar to Theorem 5.3.(1), the module $T_{\mathrm{O}_i^-}$ is the indecomposable module supported on all vertices translated from $\mathrm{O}_{i^*}^+$ with dimension vector $(1, 1, \ldots, 1)$.

We define a cone $\mathsf{G}_{\Delta_Q} \subset \mathbb{R}^{(\Delta_Q)_0}$ by $\mathsf{g}(\underline{\dim} S) \geq 0$ for all subrepresentations $S \subseteq T_{\mathrm{O}_i^-}$ and all i. Note that we ask all subrepresentations not just strict ones. We observe that the defining conditions of G_{Δ_Q} and those of $\mathsf{G}_{\Delta_Q^2}$ are related as follows. We group the defining conditions of $\mathsf{G}_{\Delta_Q^2}$ into three sets

$$\mathsf{g}H_u \geq 0, \ \mathsf{g}H_l \geq 0, \ \mathsf{g}H_r \geq 0.$$

They arise from the subrepresentations of T_v for v negative, neutral and positive respectively. Then the defining conditions of G_{Δ_Q} are exactly $\mathsf{g}H_u \geq 0$.

Similar to Theorem 5.9, we have the following proposition.

Proposition 5.13. – *The set of lattice points* $\mathsf{G}_{\Delta_Q} \cap \mathbb{Z}^{(\Delta_Q)_0}$ *is exactly* $G(\Delta_Q, W_Q)$.

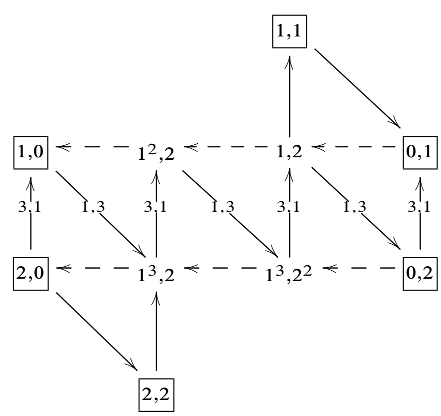
Definition 5.14. – Given a weight configuration $\boldsymbol{\sigma}$ of a quiver Δ and a convex polyhedral cone $\mathsf{G} \subset \mathbb{R}^{\Delta_0}$, we define the (not necessarily bounded) convex polytope $\mathsf{G}(\sigma)$ as G cut out by the hyperplane sections $\mathsf{g}\boldsymbol{\sigma} = \sigma$.

Our conjectural model for the tensor multiplicity is that the multiplicity $c_{\mu,\nu}^{\lambda}$ is counted by $\mathsf{G}_{\Delta_Q^2}(\mu, \nu, \lambda)$ for the weight configuration $\boldsymbol{\sigma}_Q^2$. We will prove this model for G of type ADE in Part II.

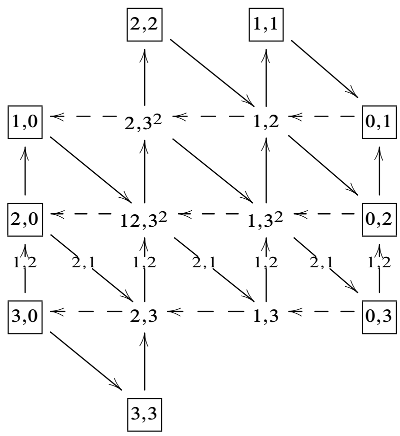
6. A List of some iARt Quivers

For Q of type B, C and D, we only draw the iARt quiver Δ_Q^2 for B_3, C_3 and D_5 (D_4 is our running example). The reader should have no difficulty to draw the general ones. The cases of E_7 and E_8 can be found in [16]. We label the vertices of Q in the same way as the software LiE [39] so that you can check things .

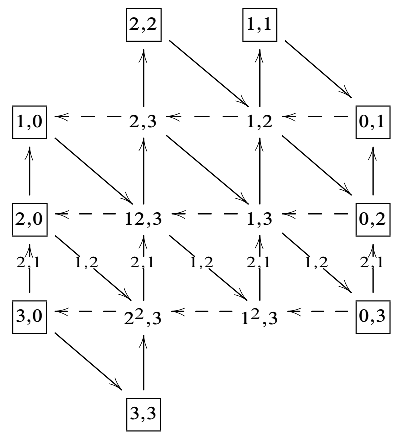
G_2



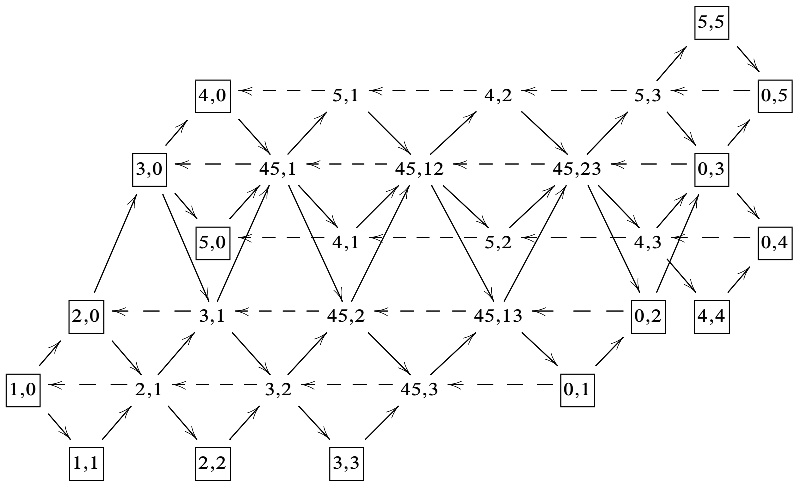
B_3


2,2
1,1
1,0
2,3²
1,2
0,1
2,0
12,3²
1,3²
0,2
1,2
2,1
3,0
2,3
1,3
0,3
3,3

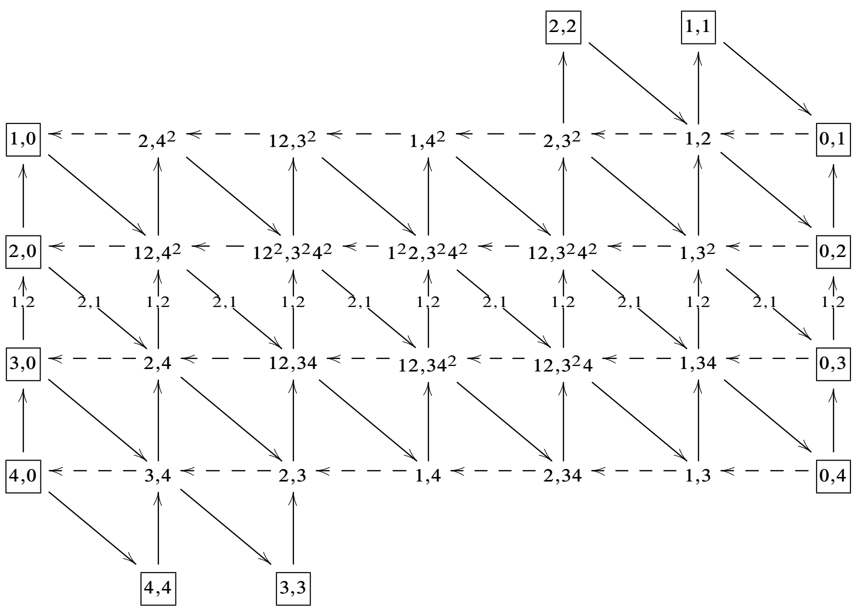
C_3

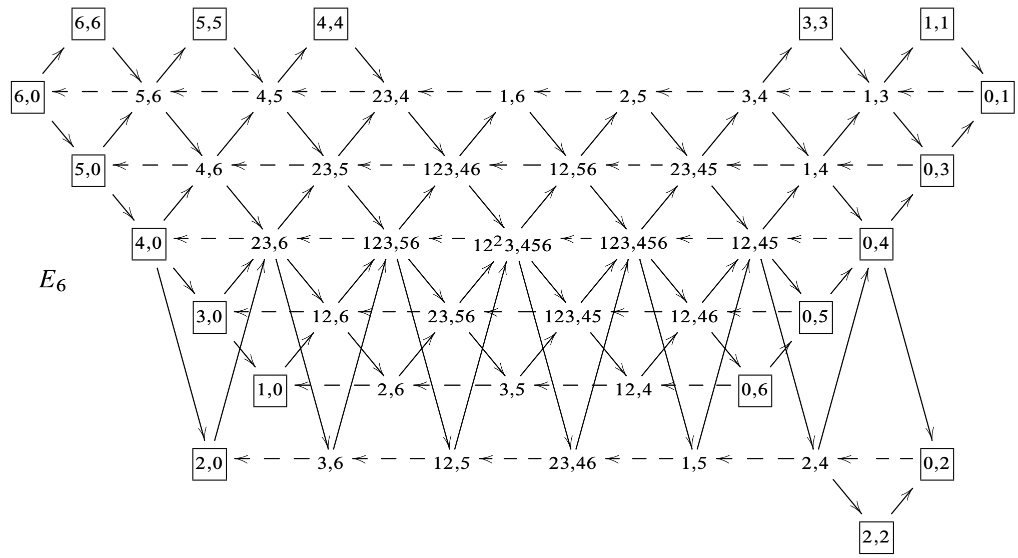

2,2
1,1
1,0
2,3
1,2
0,1
2,0
12,3
1,3
0,2
2,1
1,2
3,0
2²,3
1²,3
0,3
3,3

D_5


5,5
4,0
5,1
4,2
5,3
0,5
3,0
45,1
45,12
45,23
0,3
5,0
4,1
5,2
4,3
0,4
2,0
3,1
45,2
45,13
0,2
4,4
1,0
2,1
3,2
45,3
0,1
1,1
2,2
3,3

F_4


2,2
1,1
1,0
2,4²
12,3²
1,4²
2,3²
1,2
0,1
2,0
12,4²
12²,3²4²
1²2,3²4²
12,3²4²
1,3²
0,2
1,2
2,1
3,0
2,4
12,34
12,34²
12,3²4
1,34
0,3
4,0
3,4
2,3
1,4
2,34
1,3
0,4
4,4
3,3



PART II
ISOMORPHISM TO $k[\mathrm{Conf}_{2,1}]$

7. Cluster Structure of Maximal Unipotent Groups

7.1. Basic Notation for a Simple Lie Group

Let k be an algebraically closed field of characteristic zero. From now on, G will always be a simply connected linear algebraic group over k with Lie algebra $\mathfrak{g}$. We assume that the Dynkin diagram of $\mathfrak{g}$ is the underlying valued graph of Q. The Lie algebra $\mathfrak{g}$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Let $e_i, \alpha_i^\vee$, and f_i ($i \in Q_0$) be the Chevalley generators of $\mathfrak{g}$. The *simple coroots* $\alpha_i^\vee$ of $\mathfrak{g}$ form a basis of a Cartan subalgebra $\mathfrak{h}$. The *simple roots* α_i ($i \in Q_0$) form a basis in the dual space $\mathfrak{h}^*$ such that $[h, e_i] = \alpha_i(h)e_i$, and $[h, f_i] = -\alpha_i(h) f_i$ for any $h \in \mathfrak{h}$ and $i \in Q_0$. The structure of $\mathfrak{g}$ is uniquely determined by the *Cartan matrix* $C(G) = (c'_{i,j})$ given by $c'_{i,j} = \alpha_j(\alpha_i^\vee)$. We have that $c'_{i,i} = 2$ and $c'_{i,j} = -c_{j,i}$ so we have that $C(G) = E_l(Q) + E_r(Q)^T$.

Let U^-, H and $U := U^+$ be closed subgroups of G with Lie algebras $\mathfrak{n}^-$, $\mathfrak{h}$ and $\mathfrak{n}$. Thus H is a *maximal torus*, and U^- and U are two opposite *maximal unipotent subgroups* of G. Let $U_i^\pm$ ($i \in Q_0$) be the simple root subgroup of $U^\pm$. By abuse of notation, we let $\alpha_i^\vee : \mathbb{G}_m \to H$ be the simple coroot corresponding to the root $\alpha_i : H \to \mathbb{G}_m$. For all $i \in Q_0$, there are isomorphisms $x_i : \mathbb{G}_a \to U_i^+$ and $y_i : \mathbb{G}_a \to U_i^-$ such that the maps

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_i(a), \quad \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mapsto y_i(b), \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto \alpha_i^\vee(t)$$

provide homomorphisms $\phi_i : \mathrm{SL}_2 \to G$. We denote by $g \mapsto g^T$ the transpose anti-automorphism of G defined by

$$x_i(a)^T = y_i(a),\ y_i(b)^T = x_i(b),\ h^T = h,\ (i \in Q_0, h \in H).$$

Let s_i $(i \in Q_0)$ be the simple reflections generating the Weyl group of G. Set

$$\overline{s_i} := x_i(-1)y_i(1)x_i(-1).$$

The elements $\overline{s_i}$ satisfy the braid relations. So we can associate to each $w \in W$ its representative $\overline{w}$ in such a way that for any reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$ one has $\overline{w} = \overline{s_{i_1}} \cdots \overline{s_{i_\ell}}$. Denote by w_0 the longest element of the Weyl group. In general $s_G := \overline{w_0}^2$ is not the identity but an order two central element in G. It is well-known that $w_0(\alpha_i) = -\alpha_{i^*}$, where $i \mapsto i^*$ is the same involution in Section 5.1.

The *weight lattice* $P(G)$ of G consists of all $\gamma \in \mathfrak{h}^*$ such that $\gamma(\alpha_i^\vee) \in \mathbb{Z}$ for all i. Thus $P(G)$ has a $\mathbb{Z}$-basis $\{\varpi_i\}_{i\in Q_0}$ of *fundamental weights* given by $\varpi_j(\alpha_i^\vee) = \delta_{i,j}$. We can thus identify a weight by an integral vector $\lambda \in \mathbb{Z}^{Q_0}$. We write $\varpi(\lambda)$ for $\sum_i \lambda(i)\varpi_i$. In this notation, $\varpi_i = \varpi(\mathsf{e}_i)$. We stress that throughout the paper, this identification is widely used. A weight $\lambda \in \mathbb{Z}^{Q_0}$ is *dominant* if it is non-negative.

7.2. The Base Affine Space

The natural $G \times G$-action on G:

$$(g_1, g_2) \cdot g = g_1 g g_2^{-1},$$

induces the left and right translation of G on $k[G]$:

$$(g_1, g_2)\varphi(g) = \varphi(g_1^{-1} g g_2) \ \text{ for } \varphi \in k[G].$$

The algebraic Peter-Weyl theorem [31, Theorem 4.2.7] says that as a $G \times G$-module $k[G]$ decomposes as

$$k[G] = \bigoplus_{\lambda \in \mathbb{Z}_{\geqslant 0}^{Q_0}} L(\lambda)^\vee \otimes L(\lambda),$$

where $L(\lambda)$ is the irreducible G-module of highest-weight $\varpi(\lambda)$, and $L(\lambda)^\vee$ is its dual. Quotienting out the left translation of U^-, we get a G-module decomposition of the ring of regular functions on the *base affine space* $\mathcal{A} := U^- \backslash G$:

$$k[\mathcal{A}] = \{\varphi \in k[G] \mid \varphi(u^- g) = \varphi(g) \text{ for } u^- \in U^-,\ g \in G\} = \bigoplus_{\lambda \in \mathbb{Z}_{\geqslant 0}^{Q_0}} L(\lambda).$$

Each G-module $L(\lambda)$ can be realized as the subspace of $k[\mathcal{A}]$:

$$L(\lambda) \cong \left\{\varphi \in k[\mathcal{A}] \mid \varphi(hg) = h^{\varpi(\lambda)}\varphi(g) \text{ for } h \in H,\ g \in G\right\}. \tag{7.1}$$

Similarly for the *dual* base affine space $\mathcal{A}^\vee := G/U$, we have that

$$k[\mathcal{A}^\vee] = \{\varphi \in k[G] \mid \varphi(gu) = \varphi(g) \text{ for } u \in U,\ g \in G\} = \bigoplus_{\lambda \in \mathbb{Z}_{\geqslant 0}^{Q_0}} L(\lambda)^\vee,$$

$$L(\lambda)^\vee \cong \left\{\varphi \in k[\mathcal{A}^\vee] \mid \varphi(gh) = h^{\varpi(\lambda)}\varphi(g) \text{ for } h \in H,\ g \in G\right\}. \tag{7.2}$$

Keep in mind that the G-actions on $k[U^-\backslash G]$ and $k[G/U]$ are via the right and left translations respectively.

We fix an additive character $U^- \to \mathbb{G}_a$, then a coset U^-g (resp. gU) determines a point in $\mathscr{A}$ (resp. $\mathscr{A}^\vee$). We refer readers to [29, 1.1.1] for the detail.

For each fundamental representation $L(\mathsf{e}_i)$ and its dual $L(\mathsf{e}_i)^\vee$, we choose a U^--fixed vector u_i and a U-fixed vector $u_i^\vee$, normalized by $\langle u_i^\vee \mid u_i\rangle = 1$. Following [22] we define for each fundamental weight ϖ_i and each pair $w, w' \in W$ the *generalized minor* $m_{w,w'}^{\varpi_i} \in k[G]$

$$g \mapsto \langle \overline{w}u_i^\vee \mid g\overline{w'}u_i\rangle. \tag{7.3}$$

More concretely, the Bruhat decomposition $G = \coprod_{w\in W} U^-HwU$ implies that any $g \in G$ can be written as $g = u^-h\overline{w}u$. Suppose that g lies in the open set $G_0 := U^-HU$, then the regular function $m^{\varpi_i} := m_{e,e}^{\varpi_i}$ restricted to G_0 is given by $m^{\varpi_i}(g) = h^{\varpi_i}$ (For the type A_r, this is a principal minor of the matrix g). Then the definition of m^{ϖ_i} can be extended to whole G as in [22, Proposition 2.4].

Proposition 7.1. – *For $g = u^-h\overline{w}u \in G$, we have that*

$$m^{\varpi_i}(g) = h^{\varpi_i} \text{ if } w(\varpi_i) = \varpi_i \text{ and } m^{\varpi_i}(g) = 0 \text{ otherwise.}$$

The function $m_{w,w'}^{\varpi_i}$ is then given by

$$m_{w,w'}^{\varpi_i}(g) = m^{\varpi_i}(\overline{w}^{-1}g\overline{w'}). \tag{7.4}$$

It follows from (7.4) or (7.3) that

Lemma 7.2. – *The weight for the $H \times H$-action on $m_{w,w'}^{\varpi^i}$ is $(w(\varpi_i), -w'(\varpi_i))$, i.e.,*

$$m_{w,w'}^{\varpi_i}(h_1gh_2^{-1}) = h_1^{w(\varpi_i)}h_2^{-w'(\varpi_i)}m_{w,w'}^{\varpi_i}(g).$$

7.3. Cluster Structure on $k[U]$

Now we are ready to recall the cluster algebra structure of $k[U]$. For this, we associate to each non-neutral indecomposable presentation f, a generalized minor m_f as follows. Suppose that $\mathsf{e}(f) = \mathsf{e}_i$, i.e., f is translated from O_i^+, and moreover $w(\varpi_i) = \varpi(\mathsf{f})$, then we put $m_f := m_{e,w}^{\varpi_i}$. Note that if f is positive, then $f = m_{e,e}^{\varpi_i}$ is a principal minor. We take the iARt quiver Δ_Q, and let $\mathscr{M}_Q := \{m_f\}_{f\in(\Delta_Q)_0}$. It is known [3, 26] that $k[U]$ is the upper cluster algebra with this *standard* seed $(\Delta_Q, \mathscr{M}_Q)$. Here we again identify Δ_Q as the subquiver of Δ_Q^2. Moreover, the cluster algebra is equal to its upper cluster algebra when Q is simply-laced [27].

Recall the weight configuration $\boldsymbol{\sigma}_Q$ in Definition 3.12. For each vertex $f \in (\Delta_Q)_0$, we set $\boldsymbol{\sigma}_Q(f) = \mathsf{e}(f) - \mathsf{f}$. By Lemma 7.2 the degree of m_f for the conjugation action of H is exactly $\varpi(\boldsymbol{\sigma}_Q(f))$. Let $\boldsymbol{\rho}_Q$ be the matrix with rows also indexed by $(\Delta_Q)_0$ such that $\boldsymbol{\rho}_Q(f)$ is the positive root of G corresponding to f, i.e., $\boldsymbol{\rho}_Q(f) = \sum_i \dim(M(i))\alpha(i)$ where $M = \mathrm{Coker}(f)$. Note that by the *generalized Gabriel's theorem* [10], $\boldsymbol{\rho}_Q$ contains exactly all positive roots of G. The *Kostant's partition function* $p_Q(\gamma)$ by definition counts the lattice points in the polytope

$$\mathsf{K}(\gamma) := \{\mathsf{h} \geq 0 \mid h\boldsymbol{\rho}_Q = \gamma\}.$$

Now we consider a labeling dual to the one in Section 5.1. Let $(i,t) = (i(f), t(f))$ be the number such that $f = \tau^t(\mathrm{O}_i^+)$. Note that $i(f)^* = i^*(f)$ and $t(f) + t^*(f) = t\big(\mathrm{O}_{i^*(f)}^-\big) = t^*\big(\mathrm{O}_{i(f)}^+\big)$.

Lemma 7.3. – *We have* $\varpi\,(\boldsymbol{\sigma}_Q(f)) = \sum_{t=0}^{t(f)-1} \boldsymbol{\rho}_Q(\tau^{-t} f)$.

Proof. – Let $M = \operatorname{Coker}(f)$. The equality is equivalent to

$$\mathsf{e}(f) - \mathsf{f} = \left(\sum_{t=0}^{t(f)-1} \underline{\dim}(\tau^{-t} M)\right) C(G).$$

Recall that the Cartan matrix $C(G)$ is equal to $E_l(Q) + E_r(Q)^T$. It follows from (1.1) that matrix $E_l(Q)$ transforms $-\underline{\dim} M$ to the reduced weight vector of $f(M)$, and $E_r(Q)^T$ transforms $\underline{\dim} M$ to the reduced weight vector $\mathsf{f}^\vee(M)$ of the minimal injective presentation of M, or equivalently, $\mathsf{f}^\vee(M) = -\mathsf{f}(\tau^{-1} M)$. So the righthand side is equal to

$$\sum_{t=0}^{t(f)-1} -\mathsf{f}(\tau^{-t} M) - \mathsf{f}^\vee(\tau^{-t} M).$$

After some cancelations, only two terms survive. They are $\mathsf{e}(f)$ and $-\mathsf{f} = -\mathsf{f}(M)$. □

For the weight configuration $\varpi(\boldsymbol{\sigma}_Q) := \{\varpi(\boldsymbol{\sigma}_Q(f))\}_{f \in (\Delta_Q)_0}$ and the polyhedral cone G_{Δ_Q}, we consider the polytope $\mathsf{G}_{\Delta_Q}(\gamma)$ as in Definition 5.14.

Lemma 7.4. – *The function* $|\mathsf{G}_{\Delta_Q}(-) \cap \mathbb{Z}^{(\Delta_Q)_0}|$ *is the partition function* $p_Q(-)$.

Proof. – Recall that $\mathsf{G}_{\Delta_Q}(\gamma)$ is defined by $\sum_{t=0}^{t^*(f)} \mathsf{g}(\tau^t f) \geq 0$ for $f \in (\Delta_Q)_0$ and $\mathsf{g}\varpi(\boldsymbol{\sigma}_Q) = \gamma$. We introduce new variables $\mathsf{h}(f)$ for each f satisfying that $\mathsf{h}(f) = \sum_{t=0}^{t^*(f)} \mathsf{g}(\tau^t f)$. Then

$$\mathsf{h}\boldsymbol{\rho}_Q = \sum_f \sum_{t=0}^{t^*(f)} \mathsf{g}(\tau^t f)\boldsymbol{\rho}_Q(f) = \sum_f \sum_{t=0}^{t(f)-1} \mathsf{g}(f)\boldsymbol{\rho}_Q(\tau^{-t} f) = \mathsf{g}\varpi(\boldsymbol{\sigma}_Q).$$

The second equality is established through an easy bijection and the last one is due to Lemma 7.3. So the defining condition for $\mathsf{G}_{\Delta_Q}(\gamma)$ is equivalent to that for the polytope $\mathsf{K}(\gamma)$. Finally, we observe that the transformation from g to h is totally unimodular. In particular, the transformation and its inverse preserve lattice points. □

Theorem 7.5. – *The coordinate ring of U is equal to the graded cluster algebra* $\mathcal{C}\big(\Delta_Q, \mathcal{M}_Q; \varpi(\boldsymbol{\sigma}_Q)\big)$. *Moreover, if Q is trivially valued, then (Δ_Q, W_Q) is a cluster model (Definition 4.11).*

Proof. – We only need to prove the second statement. Recall that the universal enveloping algebra $U(\mathfrak{n})$ has a standard grading by $\deg(e_i) = \alpha_i$. It is a classical fact [36] that the partition function $p_Q(\gamma)$ counts the dimension of the homogeneous component $U(\mathfrak{n})_\gamma$. The algebra $k[U]$ is graded dual to $U(\mathfrak{n})$ with (see [47])

$$k[U]_\gamma = \left\{\varphi \in k[U] \mid \varphi(huh^{-1}) = h^\gamma \varphi(u) \text{ for } h \in H,\ u \in U\right\}.$$

We have seen that the degree of $\mathcal{M}_Q(f) = m_f$ is $\varpi(\boldsymbol{\sigma}_Q(f))$. So the second statement follows from Theorem 4.10, Lemma 7.4, and the fact that $\mathcal{C}(\Delta_Q) = \overline{\mathcal{C}}(\Delta_Q)$. □

There are another two seeds of this cluster structure of $k[U]$. One is called *left standard*, and the other is called *right standard*. Both are obtained from the standard seed by a sequence of mutations. The sequences of mutations $\boldsymbol{\mu}_l$ and $\boldsymbol{\mu}_r$ will be defined and studied in Appendix 11. Let

$$\big(\widetilde{\Delta}_Q^{\#}, \widetilde{W}_Q^{\#}, \mathcal{M}_Q^{\#}; \boldsymbol{\sigma}_Q^{\#}\big) = \boldsymbol{\mu}_{\#}(\Delta_Q, W_Q, \mathcal{M}_Q; \boldsymbol{\sigma}_Q) \text{ for } \# = l, r.$$

For what follows in this section, # always represents l and r. Let Δ_Q^l (resp. Δ_Q^r) be the ice quiver obtained from Δ_Q^2 by deleting the negative and positive (resp. negative and neutral) frozen vertices. By Corollary 11.11, $\widetilde{\Delta}_Q^{\#}$ and $\Delta_Q^{\#}$ are equal up to arrows between frozen vertices. Since the QP (Δ_Q, W_Q) is rigid, so is $(\widetilde{\Delta}_Q^{\#}, \widetilde{W}_Q^{\#})$. Let $W_Q^{\#}$ be the potential W_Q^2 restricted to $\Delta_Q^{\#}$. By [8, Proposition 8.9] $(\Delta_Q^{\#}, W_Q^{\#})$ is also rigid. In particular, both $(\widetilde{\Delta}_Q^{\#}, \widetilde{W}_Q^{\#})$ and $(\Delta_Q^{\#}, W_Q^{\#})$ are μ-rigid. By the equivalent definition of rigidity ([8, Definition 6.10, Remark 6.8]), $(\widetilde{\Delta}_Q^{\#}, \widetilde{W}_Q^{\#})$ is μ-right-equivalent to $(\Delta_Q^{\#}, W_Q^{\#})$.

[13, Proposition 5.15] implies that being a cluster model is mutation-invariant. It follows that $(\widetilde{\Delta}_Q^{\#}, \widetilde{W}_Q^{\#})$ is also a cluster model. In view of Remark 4.12, we can replace $(\widetilde{\Delta}_Q^{\#}, \widetilde{W}_Q^{\#})$ by $(\Delta_Q^{\#}, W_Q^{\#})$.

Proposition 7.6. – *$(\Delta_Q^l, \mathcal{M}_Q^l; \boldsymbol{\sigma}_Q^l)$ and $(\Delta_Q^r, \mathcal{M}_Q^r; \boldsymbol{\sigma}_Q^r)$ are another two graded seeds for the cluster algebra $k[U]$. Moreover, if Q is trivially valued, both (Δ_Q^l, W_Q^l) and (Δ_Q^r, W_Q^r) are cluster models.*

Remark 7.7. – Later we will also need a concrete description of $G(\Delta_Q^l, W_Q^l)$ and $G(\Delta_Q^r, W_Q^r)$. Both are given by lattice points in rational polyhedral cones. Recall the three sets of defining conditions of the polyhedral cone $\mathsf{G}_{\Delta_Q^2}$

$$\mathsf{g}H_u \geq 0, \ \mathsf{g}H_l \geq 0, \ \mathsf{g}H_r \geq 0.$$

We define the polyhedral cones $\mathsf{G}_{\Delta_Q^l} \subset \mathbb{R}^{(\Delta_Q^l)_0}$ and $\mathsf{G}_{\Delta_Q^r} \subset \mathbb{R}^{(\Delta_Q^l)_0}$ by the relations $\mathsf{g}H_l \geq 0$ and $\mathsf{g}H_r \geq 0$ respectively. Almost the same proof as Theorem 5.9 can show that $G(\Delta_Q^l, W_Q^l) = \mathsf{G}_{\Delta_Q^l} \cap \mathbb{Z}^{(\Delta_Q^l)_0}$ and $G(\Delta_Q^r, W_Q^r) = \mathsf{G}_{\Delta_Q^r} \cap \mathbb{Z}^{(\Delta_Q^r)_0}$.

8. Maps Relating Unipotent Groups

8.1. Standard Maps

Recall the base affine space $\mathcal{A}$ and its dual $\mathcal{A}^\vee$ defined in Section 7.2. Let $\mathrm{Conf}_{n,1} := (\mathcal{A}^n \times \mathcal{A}^\vee)/G$ be the categorical quotient in the category of varieties [(3)].

Lemma 8.1. – *The ring of regular functions on* $\mathrm{Conf}_{n,1}$ *is a unique factorization domain.*

Proof. – It is well-known [34] that $k[G]$ is a UFD. Since the groups U and G have no multiplicative characters, by [43, Theorem 3.17] $k[\mathrm{Conf}_{n,1}]$ is also a UFD. □

(3) For our purpose, it is enough to work with the categorical quotient, which has the same ring of regular functions on the corresponding quotient stack.

By the Bruhat decomposition, any pair $(A_1, A_0^\vee) \in \mathrm{Conf}_{1,1}$ has a representative $(U^- h\overline{w}, U)$ for some $h \in H, w \in W$. A pair $(A_1, A_0^\vee) \in \mathrm{Conf}_{1,1}$ is called *generic* if the $w \in W$ can be chosen as the identity. Let $\mathrm{Conf}_{n,1}^\circ$ be the open subset of $\mathrm{Conf}_{n,1}$ where each pair $(A_i, A_0^\vee)$ is generic (but we do not impose any condition among A_i's). By definition, we have an isomorphism $H \cong \mathrm{Conf}_{1,1}^\circ$. Let $\iota : H \hookrightarrow \mathrm{Conf}_{1,1}$ be the open embedding $h \mapsto (U^- h, U)$. For each $i \in Q_0$ we define the regular function $\widetilde{\varpi}_i$ on $\mathrm{Conf}_{1,1}$ by $\widetilde{\varpi}_i(U^- g_1, g_0 U) = m^{\varpi_i}(g_1 g_0)$. It is clear that the definition does not depend on the representatives. It follows from Proposition 7.1 that

LEMMA 8.2. – *$k[H]$ is exactly the localization of $k[\mathrm{Conf}_{1,1}]$ at all $\widetilde{\varpi}_i$.*

From now on, we will focus on the space $\mathrm{Conf}_{2,1}$. Its ring of regular functions has a triple-weight decomposition

$$k[\mathrm{Conf}_{2,1}] = \bigoplus_{\mu,\nu,\lambda \in Z_{\geqslant 0}^{Q_0}} \left(L(\mu) \otimes L(\nu) \otimes L(\lambda)^\vee\right)^G.$$

We write very often $C_{\mu,\nu}^\lambda$ for the graded component $k[\mathrm{Conf}_{2,1}]_{\mu,\nu,\lambda}$. It is clear that $c_{\mu\,\nu}^\lambda = \dim C_{\mu,\nu}^\lambda$.

We recall several rational maps defined in [19, 29]. Let $i : H \times H \times U \hookrightarrow \mathrm{Conf}_{2,1}$ be the open embedding

$$(h_1, h_2, u) \mapsto (U^- h_1, U^- h_2 u, U).$$

It is an embedding because the stabilizer of the generic pair $(U^- h, U)$ is 1_G. It is clear that the image of i is exactly $\mathrm{Conf}_{2,1}^\circ$. By restriction we get an embedding $i^* : k[\mathrm{Conf}_{2,1}] \hookrightarrow k[H \times H \times U]$. We will view two H's and U as subgroups of $H \times H \times U$ in the natural way, and write i_1, i_2 and i_u for the restriction of i on first H, second H and U respectively. A function s in $C_{\mu,\nu}^\lambda$ is uniquely determined by its restriction $i_u^*(s)$ on U because

$$s(h_1, h_2, u) = h_1^{\varpi(\mu)} h_2^{\varpi(\nu)} s(u) \quad \text{for } (h_1, h_2, u) \in H \times H \times U. \tag{8.1}$$

So i^* embeds $C_{\mu,\nu}^\lambda$ into $k[U]$ (in fact into $k[U]_{\varpi(\mu+\nu-\lambda)}$ by easy calculation). We note that each $C_{\mu,\nu}^\lambda$ is *not* disjoint under this embedding.

We recall a classical interpretation of the multiplicity $c_{\mu\,\nu}^\lambda$. Let $L(\mu)_\gamma$ be the weight-γ subspace of the irreducible G-module $L(\mu)$. We denote

$$L(\mu)_\gamma^\nu := \left\{\varphi \in L(\mu)_\gamma \mid e_i^{\nu(i)+1}(\varphi) = 0 \text{ for } i \in Q_0\right\}.$$

LEMMA 8.3 ([41]). – *We have that $c_{\mu\,\nu}^\lambda = \dim L(\mu)_{\varpi(\lambda-\nu)}^\nu$.*

LEMMA 8.4. – *For any $f \in \mathrm{ind}(C^2 Q)$, the generalized minor m_f spans the space $L\left(\mathsf{e}(f)\right)_{\varpi(\mathsf{f})}^{\mathsf{f}_-} \subset k[\mathscr{A}]$. In particular, $c_{\mathsf{e}(f),\mathsf{f}_-}^{\mathsf{f}_+} = 1$.*

Proof. – Recall that the Chevalley generator e_i acts on $k[G]$ by

$$e_i \varphi(g) = \frac{d}{dt}\Big|_{t=0} \varphi(g x_i(t)).$$

For any $\varphi \in k[G]$, the coefficient of each t^n in $\varphi(x_i(t))$ is equal to $e_i^n\varphi(1_G)/n!$. If φ is bihomogeneous of degree (γ,γ'), then $\varphi(1_G)$ can be nonzero only if $\gamma = \gamma'$. It follows that $\varphi(x_i(t))$ contains only t^n with $n\alpha_i = \gamma-\gamma'$. So for $(\gamma,\gamma') = (\varpi(\mathsf{e}(f)), \varpi(\mathsf{f}))$

$$e_i^{\mathsf{f}_-(i)+1}\varphi = 0 \text{ if and only if } (\mathsf{f}_-(i)+1)c_i \neq \mathsf{e}(f)-\mathsf{f},$$

where c_i is the i-th column of the Cartan matrix $C(G)$. Since $\mathsf{e}(f)(i) \le 1$ and $c_i(i) = 2$, we can never have $(\mathsf{e}(f)+\mathsf{f}_- - \mathsf{f}_+)(i) = (\mathsf{f}_-(i)+1)c_i(i)$. Hence we proved that $e_i^{\mathsf{f}_-(i)+1}m_f = 0$ for $i \in Q_0$ so that $m_f \in L\,(\mathsf{e}(f))^{\mathsf{f}_-}_{\varpi(\mathsf{f})}$.

Suppose that $m_f = m_{e,w}^{\varpi_j}$. The generalized minor m_f spans the space because

$$\dim L(\mathsf{e}_j)_{w(\varpi_j)} = \dim L(\mathsf{e}_j)_{\varpi_j} = 1. \qquad \square$$

We recall that each $m_{\mathsf{O}_i^+}$ is a principal minor, which evaluates to 1 on U; and now we set $m_{\mathrm{Id}_i} = 1$.

Lemma-Definition 8.5. – *For any $f \in \operatorname{ind}(C^2Q)$, there is a unique function $s_f \in k[\mathrm{Conf}_{2,1}]_{\mathsf{e}(f),\mathsf{f}_-,\mathsf{f}_+}$ such that*

$$i_u^*(s_f) = m_f.$$

Moreover, $s \in C_{\mu,\nu}^{\lambda}$ satisfies $i_u^*(s) = 1$ if and only if $s = s_+^{\mu}s_0^{\nu}$, where $s_+^{\mu} := \prod_i s_{\mathsf{O}_i^+}^{\mu(i)}$ and $s_0^{\nu} := \prod_i s_{\mathrm{Id}_i}^{\nu(i)}$.

Proof. – The first statement follows from Lemma 8.4. For the last statement, it suffices to prove the "only if" part. If $i_u^*(s) = 1$, then $\mu+\nu = \lambda$ because $i_u^*(s)$ has degree $\varpi(\mu+\nu-\lambda)$. It is clear that $c_{\mu,\nu}^{\mu+\nu} = 1$ (consider tensoring the highest weight vectors of $L(\mu)$ and $L(\nu)$), but $s_+^{\mu}s_0^{\nu}$ also has degree $(\mu,\nu,\mu+\nu)$, so $s = s_+^{\mu}s_0^{\nu}$. $\square$

Remark 8.6. – For Q of type A, the analogous map i^* was also considered in a quiver-invariant theory setting (see [13, Example 3.10, 3.12]). In that setting, the map comes from a semi-orthogonal decomposition of the module category of a triple flag quiver.

Corollary 8.7. – *For each $f \in \operatorname{ind}(C^2Q)$, s_f is irreducible in $k[\mathrm{Conf}_{2,1}]$.*

Proof. – It is clear that the zero-degree component of $k[\mathrm{Conf}_{2,1}]$ is k. If f is positive or neutral, then the degree of f is obviously indecomposable. The same argument as in [13, Lemma 1.8] shows that s_f is irreducible.

We remain to consider the case where f is the minimal presentation of a general representation. It is known that all generalized minors m_f are irreducible in $k[U]$. Suppose that s_f factors as $s_f = s_1s_2$, then $i_u^*(s_f) = i_u^*(s_1)i_u^*(s_2) = m_f$. So one of them, say $i_u^*(s_1)$, has to be a unit. According to Lemma 8.5, s_1 is a polynomial in $s_{\mathsf{O}_i^+}$'s and s_{Id_i}'s. So there is a homogenous component s_2' of s_2 and some μ and ν such that $s_2's_+^{\mu}s_0^{\nu}$ has the same degree as s_f, i.e., $s_2's_+^{\mu}s_0^{\nu} \in C_{\mathsf{e}(f),\mathsf{f}_-}^{\mathsf{f}_+}$. Since f is a minimal presentation of a general representation, we must have $\nu = 0$. If μ is nonzero, then we must have $\mu = \mathsf{e}(f)$. Then the weight of s_2' is $(0,\mathsf{f}_-,\mathsf{f}_+-\mathsf{e}(f))$, and f_- has to be equal to $\mathsf{f}_+-\mathsf{e}(f)$. But i_u^* embeds $C_{\mathsf{e}(f),\mathsf{f}_-}^{\mathsf{f}_+}$ into $k[U]_{\mathsf{e}(f)-\mathsf{f}}$. Since f is not positive or neutral, $\mathsf{e}(f) \neq \mathsf{f}$, which is a contradiction. $\square$

Let p be a rational inverse of i, which is regular on $\mathrm{Conf}_{2,1}^\circ = i(H \times H \times U)$. Let p_1, p_2 and p_u be the composition of p with the natural projection to first H, second H, and U respectively.

COROLLARY 8.8. – *On the open subset* $\mathrm{Conf}_{2,1}^\circ \subset \mathrm{Conf}_{2,1}$

$$p_u^*(m_f) = s_f \left(s_+^{\mathsf{e}(f)} s_0^{\mathsf{f}_-} \right)^{-1},$$
$$p_1^*(\varpi_i) = s_{\mathrm{O}_i^+},\ p_2^*(\varpi_i) = s_{\mathrm{Id}_i},$$

for any indecomposable presentation f.

Proof. – They follow from straightforward calculation

$$(p_u^* m_f)\,(i(h_1, h_2, u)) = m_f\,(p_u i(h_1, h_2, u)) = m_f(u);$$
$$s_f(i(h_1, h_2, u)) = h_1^{\varpi(\mathsf{e}(f))} h_2^{\varpi(\mathsf{f}_-)} m_f(u),\ (s_+^{\mathsf{e}(f)} s_0^{\mathsf{f}_-})(i(h_1, h_2, u)) = h_1^{\varpi(\mathsf{e}(f))} h_2^{\varpi(\mathsf{f}_-)}.$$

The statement for p_1^*, p_2^* is rather obvious. For example,

$$s_{\mathrm{O}_i^+}(i(h_1, h_2, u)) = h_1^{\varpi_i} h_2^0 i_u^*(s_{\mathrm{O}_i^+})(u) = h_1^{\varpi_i} = p_1^*(\varpi_i)\,(i(h_1, h_2, u))\,. \qquad \square$$

COROLLARY 8.9. – *The localization of* $k[\mathrm{Conf}_{2,1}]$ *at all* s_f*'s for* f *positive and neutral is exactly* $k[H \times H \times U]$.

Proof. – Let $\widetilde{p}_1$ and $\widetilde{p}_2$ be the natural projections $\mathrm{Conf}_{2,1} \to \mathrm{Conf}_{1,1}$ given by $(A_1, A_2, A_0^\vee) \mapsto (A_1, A_0^\vee)$ and $(A_1, A_2, A_0^\vee) \mapsto (A_2, A_0^\vee)$. Then $\mathrm{Conf}_{2,1}^\circ$ is the intersection $\widetilde{p}_1^{-1}(\mathrm{Conf}_{1,1}^\circ) \cap \widetilde{p}_2^{-1}(\mathrm{Conf}_{1,1}^\circ)$. It suffices to show that $\widetilde{p}_1^*(\widetilde{\varpi}_i) = s_{\mathrm{O}_i^+}$ and $\widetilde{p}_2^*(\widetilde{\varpi}_i) = s_{\mathrm{Id}_i}$. The map ιp_i agrees with $\widetilde{p}_i$ on a dense subset of $\mathrm{Conf}_{2,1}$ for $i = 1, 2$. So $\widetilde{p}_1^*(\widetilde{\varpi}_i) = p_1^* \iota^*(\widetilde{\varpi}_i) = p_1^*(\varpi_i) = s_{\mathrm{O}_i^+}$ and $\widetilde{p}_2^*(\widetilde{\varpi}_i) = p_2^* \iota^*(\widetilde{\varpi}_i) = p_2^*(\varpi_i) = s_{\mathrm{Id}_i}$. $\square$

COROLLARY 8.10. – *The set of functions* $\mathcal{S}_Q^2 := \{s_f\}_{f \in \mathrm{ind}(C^2 Q)}$ *is algebraically independent over the base field* k.

Proof. – Since the map $p : \mathrm{Conf}_{2,1} \to H \times H \times U$ is birational, the pullbacks of (two copies of) $\varpi_i \in H$ and $m_f \in U$ are algebraically independent. We have seen that up to a factor of some monomial in $s_{\mathrm{O}_i^+}, s_{\mathrm{Id}_i} \in \mathcal{S}_Q^2$, they are exactly the functions in $\mathcal{S}_Q^2$. Our claim follows. $\square$

Suppose that f as above is mutable. Let m_f' be a function in $k[U]$ in the exchange relation

$$m_f m_f' = \prod_{g \rightarrowtail f} m_g^{c_{g,f}} + \prod_{f \rightarrowtail h} m_h^{c_{f,h}}. \tag{8.2}$$

Let $(\mu', \nu', \lambda') = \sum_{f \rightarrowtail h} c_{f,h}(\mathsf{e}(h), \mathsf{h}_-, \mathsf{h}_+) - (\mathsf{e}(f), \mathsf{f}_-, \mathsf{f}_+)$.

LEMMA 8.11. – *The function* $s_f' := p_u^*(m_f') s_+^{\mu'} s_0^{\nu'}$ *is regular on* $\mathrm{Conf}_{2,1}$ *satisfying* $i_u^*(s_f') = m_f'$ *and the exchange relation*

$$s_f s_f' = \prod_{g \rightarrowtail f} s_g^{c_{g,f}} + \prod_{f \rightarrowtail h} s_h^{c_{f,h}}.$$

Moreover, s_f' *and* s_f *are relatively prime in* $k[\mathrm{Conf}_{2,1}]$.

Proof. – Since i is a regular section of p, we have that $i_u^*(s'_f) = m'_f$ and s'_f is regular on $\mathrm{Conf}^\circ_{2,1} = i(H \times H \times U)$. We pull back (8.2) through p_u, and multiply $s_+^{\mu'} s_0^{\nu'}$ on both sides. Then we get from Corollary 8.8 that

$$s_f\left(p_u^*(m'_f)s_+^{\mu'}s_0^{\nu'}\right) = \prod_{g \rightarrowtail f} s_g^{c_{g,f}} + \prod_{f \rightarrowtail h} s_h^{c_{f,h}}. \tag{8.3}$$

We remain to show that s'_f is in fact regular on whole $\mathrm{Conf}_{2,1}$. By (8.3), the locus of indeterminacy of s'_f is contained in the zero locus $Z(s_f)$ of s_f. Since $s_f|_U = m_f$, the intersection of $Z(s_f)$ and $\mathrm{Conf}^\circ_{2,1}$ is nonempty. In other words, $Z(s_f)$ is not contained in the complement of $\mathrm{Conf}^\circ_{2,1}$. Since $Z(s_f)$ is irreducible (Corollary 8.7), we conclude that s'_f is regular outside a codimension 2 subvariety of $\mathrm{Conf}_{2,1}$. Since $\mathrm{Conf}_{2,1}$ is factorial (Lemma 8.1) and thus normal, by the algebraic Hartogs s'_f is a regular function on $\mathrm{Conf}_{2,1}$.

For the last statement, we suppose the contrary. Since s_f is irreducible, s_f is then a factor of s'_f. But this is clearly impossible by comparing the first component of their triple-weight. □

8.2. Twisted Maps

Let $\vee$ (written exponentially) be the morphism

$$U^-\backslash G \to G/U \text{ given by } (U^-g)^\vee \mapsto \overline{\omega_0}^{-1}g^{-1}\overline{\omega_0}U.$$

Let $\tilde{l}$ be the twisted cyclic shift on $\mathscr{A} \times \mathscr{A} \times \mathscr{A}^\vee$:

$$(A_1, A_2, A_3^\vee) \mapsto (A_2, A_3\overline{w_0}, (A_1\overline{w_0})^\vee).$$

Suppose that $(A_1, A_2, A_3^\vee) = (U^-g_1, U^-g_2, g_3U)$, then

$$\begin{aligned} \tilde{l}(g(A_1, A_2, A_3^\vee)) &= \tilde{l}(U^-g_1g^{-1}, U^-g_2g^{-1}, gg_3U) \\ &= (U^-g_2g^{-1}, U^-\overline{w_0}(gg_3)^{-1}, s_G(g_1g^{-1})^{-1}\overline{w_0}U) \\ &= g(U^-g_2, U^-\overline{w_0}g_3^{-1}, g_1^{-1}\overline{w_0}^{-1}U). \end{aligned} \tag{8.4}$$

By the universal property of categorical quotients, $\tilde{l}$ descends to an endomorphism l on $\mathrm{Conf}_{2,1}$, which is an automorphism on the generic part of $\mathrm{Conf}_{2,1}$.

We have that

$$\tilde{l}^2(A_1, A_2, A_3^\vee) = (A_3\overline{w_0}, s_GA_1, (A_2\overline{w_0})^\vee),$$

and $\tilde{l}^3 = (s_GA_1, s_GA_2, s_GA_3^\vee) = (A_1, A_2, A_3^\vee)$ is the identity. We define $i_l := li_u : U \hookrightarrow \mathrm{Conf}_{2,1}$, that is

$$u \mapsto (U^-u, U^-\overline{w_0}, \overline{w_0}^{-1}U).$$

We write r for l^2. Similarly we denote $i_r := r i_u : U \hookrightarrow \mathrm{Conf}_{2,1}$.

$$\begin{tikzcd}
 & \mathrm{Conf}_{2,1} \arrow[ddl, "l"'] & \\
 & U \arrow[u, hook, "i_u"] \arrow[dl, hook, "i_l"'] \arrow[dr, hook, "i_r"] & \\
\mathrm{Conf}_{2,1} \arrow[rr, "l"'] & & \mathrm{Conf}_{2,1} \arrow[uul, "l"']
\end{tikzcd}$$

It turns out that the twisted cyclic shift is also related to the sequence $\boldsymbol{\mu}_l$ of mutations considered in Section 7.3. We will see in Theorem 11.14 that $l^*(s_f) = \boldsymbol{\mu}_l(s_f)$ for f mutable. Note that l^* permutes frozen variables

$$s_{\mathrm{O}_i^-} \mapsto s_{\mathrm{O}_i^+},\; s_{\mathrm{O}_i^+} \mapsto s_{\mathrm{Id}_{i^*}},\; s_{\mathrm{Id}_i} \mapsto s_{\mathrm{O}_{i^*}^-}.$$

Lemma 8.12. – *We have the following:*

$$\begin{aligned}
&i_l^*(s_f) = \boldsymbol{\mu}_l(m_f), && i_r^*(s_f) = \boldsymbol{\mu}_r(m_f) \text{ for } f \text{ mutable};\\
&i_l^*(s_{\mathrm{Id}_i}) = m_{\mathrm{O}_{i^*}^-}, && i_l^*(s_{\mathrm{O}_i^-}) = i_l^*(s_{\mathrm{O}_i^+}) = 1;\\
&i_r^*(s_{\mathrm{O}_i^+}) = m_{\mathrm{O}_i^-}, && i_r^*(s_{\mathrm{O}_i^-}) = i_r^*(s_{\mathrm{Id}_i}) = 1.
\end{aligned}$$

Proof. – First consider the case where f is mutable, then l^* is given by a sequence of mutations $\boldsymbol{\mu}_l$. The pullback i_u^* clearly commutes with any sequence of mutations, so by Lemma 8.5

$$i_l^*(s_f) = i_u^* l^*(s_f) = i_u^*(\boldsymbol{\mu}_l(s_f)) = \boldsymbol{\mu}_l(i_u^*(s_f)) = \boldsymbol{\mu}_l(m_f).$$

For f frozen, by Lemma 8.5 we have that

$$i_l^*(s_{\mathrm{Id}_i}) = i_u^*(s_{\mathrm{O}_{i^*}^-}) = m_{\mathrm{O}_{i^*}^-},\; i_l^*(s_{\mathrm{O}_i^-}) = i_u^*(s_{\mathrm{O}_i^+}) = 1,\; i_l^*(s_{\mathrm{O}_i^+}) = i_u^*(s_{\mathrm{Id}_{i^*}}) = 1.$$

The argument for i_r^* is similar. □

We set $\Delta_Q^u := \Delta_Q$ and $\mathcal{M}_Q^u := \mathcal{M}_Q$. We define three linear natural projections $\mathrm{i}_\#^* : \mathbb{R}^{(\Delta_Q^2)_0} \to \mathbb{R}^{(\Delta_Q^\#)_0}$ for $\# \in \{u, l, r\}$, where i_u^* (resp. i_l^*; i_r^*) forgets the coordinates corresponding to the neutral and positive (resp. positive and negative; negative and neutral) frozen vertices.

Recall the definition of the g-vector with respect to a pair $(\Delta, \mathbf{x})$ as in Section 1.3. Let $\mathcal{L}(\mathcal{S}_Q^2)$ be the Laurent polynomial ring in $\mathcal{S}_Q^2$. Note that $\mathcal{L}(\mathcal{S}_Q^2)$ has a basis parameterized by all possible g-vectors (with respect to $(\Delta_Q^2, \mathcal{S}_Q^2)$), which can be identified with $\mathbb{Z}^{(\Delta_Q^2)_0}$.

Lemma 8.13. – *If $s \in \mathcal{L}(\mathcal{S}_Q^2) \subset k(\mathrm{Conf}_{2,1})$ has a well-defined* g*-vector* g *with respect to $(\Delta_Q^2, \mathcal{S}_Q^2)$, then so is the* g*-vector of $\mathrm{i}_\#^*(s)$ with respect to $(\Delta_Q^\#, \mathcal{M}_Q^\#)$, which is equal to $\mathrm{i}_\#^*(\mathrm{g})$ for $\# \in \{u, l, r\}$.*

Proof. – We will only prove the statement for i_l because the argument for the other two is similar. Suppose that $s = \mathbf{x}^{\mathrm{g}} F(\mathbf{y})$ where $\mathbf{x}(f) = s_f$ and $\mathbf{y}$ is as in Section 1.3. We have seen in Lemma 8.12 that $i_l^*(s_f) = \boldsymbol{\mu}_l(m_f) = \mathcal{M}_Q^l(f)$ if f is not negative or positive. If f is negative or positive, then $i_l^*(s_f) = 1$ so $i_l^*(\mathbf{x}^{\mathrm{g}}) = (\mathbf{x}_l)^{\mathrm{i}_l^*(\mathrm{g})}$ where $\mathbf{x}_l(f) = \boldsymbol{\mu}_l(m_f)$. Recall that the quiver Δ_Q^l is obtained from Δ_Q^2 by deleting the negative and positive frozen vertices. According to this description, we have that $i_l^*(\mathbf{y}(f)) = i_l^*(\mathbf{x}^{-b_f}) = \mathbf{x}_l^{-b_f^l} = \mathbf{y}_l(f)$, where b_f (resp. b_f^l) is the row of $B_{\Delta_Q^2}$ (resp. $B_{\Delta_Q^l}$) corresponding to f. Hence, we get $i_l^*(s) = (\mathbf{x}_l)^{\mathrm{i}_l^*(\mathrm{g})} F(\mathbf{y}_l)$. □

LEMMA 8.14. – *The polytope $\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda)$ has lattice points no less than $c_{\mu\nu}^{\lambda}$.*

Proof. – By Corollary 8.9, we have that $k[\mathrm{Conf}_{2,1}] \subset \mathcal{L}(\mathcal{S}_Q^2)$. Since $i_u,\ i_l,\ i_r$ are all regular, we trivially have that

$$k[\mathrm{Conf}_{2,1}] \subseteq \left\{ s \in \mathcal{L}(\mathcal{S}_Q^2) \mid i_\#^*(s) \in k[U] \text{ for } \# = u,l,r \right\}. \tag{8.5}$$

If $s \in \mathcal{L}(\mathcal{S}_Q^2)$ has a well-defined g-vector, then by the previous lemma the g-vector of $i_\#^*(s)$ is equal to $\mathrm{i}_\#^*(\mathrm{g})$. So if $i_\#^*(s) \in k[U]$, then $\mathrm{i}_\#^*(\mathrm{g}) \in \mathsf{G}_{\Delta_Q^\#}$ by Lemma 1.9, Proposition 5.13, 7.6 and Remark 7.7.

Due to (8.5), it suffices to show that the lattice points in $\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda)$ can be identified with points $\mathrm{g} \in \mathbb{Z}^{(\Delta_Q^2)_0}$ of weight (μ,ν,λ) such that $\mathrm{i}_\#^*(\mathrm{g}) \in \mathsf{G}_{\Delta_Q^\#}$ for $\# \in \{u,l,r\}$. But this follows from the description of the cones $\mathsf{G}_{\Delta_Q^2}$, G_{Δ_Q}, $\mathsf{G}_{\Delta_Q^l}$ and $\mathsf{G}_{\Delta_Q^r}$ (Theorem 5.9, Proposition 5.13, and Remark 7.7). □

9. Cluster Structure on $k[\mathrm{Conf}_{2,1}]$

THEOREM 9.1. – *Suppose that Q is trivially valued. Then the ring of regular functions on* $\mathrm{Conf}_{2,1}$ *is the graded upper cluster algebra $\overline{\mathcal{C}}(\Delta_Q^2, \mathcal{S}_Q^2; \boldsymbol{\sigma}_Q^2)$. Moreover, (Δ_Q^2, W_Q^2) is a cluster model. In particular, $c_{\mu\nu}^{\lambda}$ is counted by lattice points in $\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda)$.*

Proof. – By Lemma 8.10 and Corollary 3.13, $(\Delta_Q^2, \mathcal{S}_Q^2; \boldsymbol{\sigma}_Q^2)$ form a graded seed. Due to Lemma 8.11 and Corollary 8.7, we can apply Lemma 1.8 to conclude that $\overline{\mathcal{C}}(\Delta_Q^2, \mathcal{S}_Q^2; \boldsymbol{\sigma}_Q^2)$ is a graded subalgebra of $k[\mathrm{Conf}_{2,1}]$.

By Theorem 4.10, $C_W\left(\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda) \cap \mathbb{Z}^{(\Delta_Q^2)_0}\right)$ is a linearly independent set in $C_{\mu,\nu}^{\lambda}$ for any triple weights. But Lemma 8.14 says that the cardinality of $\mathsf{G}_{\Delta_Q^2}(\mu,\nu,\lambda) \cap \mathbb{Z}^{(\Delta_Q^2)_0}$ is at least $c_{\mu\nu}^{\lambda}$. So they actually span the vector space $C_{\mu,\nu}^{\lambda}$. We conclude that $\overline{\mathcal{C}}(\Delta_Q^2, \mathcal{S}_Q^2; \boldsymbol{\sigma}_Q^2)$ is equal to $k[\mathrm{Conf}_{2,1}]$, and (Δ_Q^2, W_Q^2) is a cluster model. □

It follows that (8.5) is in fact an equality. The proof shows that $\overline{\mathcal{C}}(\Delta_Q^2, \mathcal{S}_Q^2; \boldsymbol{\sigma}_Q^2)$ is a graded subalgebra of $k[\mathrm{Conf}_{2,1}]$ no matter if Q is trivially valued.

CONJECTURE 9.2. – The first and last statement of Theorem 9.1 is true even if Q is not trivially valued.

We illustrate by example that in general the upper cluster algebra $\overline{\mathcal{C}}(\Delta_Q^2)$ strictly contains the cluster algebra $\mathcal{C}(\Delta_Q^2)$.

EXAMPLE 9.3. – Let Q be of type D_4 as in Example 3.8. It can be checked by the 44 inequalities in Example 5.10 that $\mathsf{g} = \mathsf{e}_{34,12} - \mathsf{e}_{34,1} + \mathsf{e}_{2,0}$ (in weight labeling) is μ-supported. Its triple weight is $(\mathsf{e}_2, \mathsf{e}_2, \mathsf{e}_2)$, and moreover $c_{\mathsf{e}_2,\mathsf{e}_2}^{\mathsf{e}_2} = 1$. This is clearly an extremal weight. So it suffices to show that no cluster variable has this g-vector. For this, we need a result in [7], which says that the g-vector of a cluster variable is a g-vector of a *rigid* presentation in the Jacobian algebra. The rigidity is characterized by the vanishing of the E-invariant introduced in [9, 7]. It is not hard to show that $\mathrm{E}(f, f) := \mathrm{Hom}_{K^b(J)}(f, f[1]) \neq 0$ where f is a general presentation in $\mathrm{Hom}_J(P([\mathsf{g}]_+), P([\mathsf{g}]_-))$.

10. Epilog on G/U

10.1. Cluster Structure of Base Affine Spaces

We denote the ice quiver obtained from Δ_Q^2 by deleting neutral frozen vertices by $\Delta_Q^\sharp$. Let $\mathcal{M}_Q^\sharp := \{m_f\}_{f \in (\Delta_Q^\sharp)_0}$. We define the weight configuration $\boldsymbol{\sigma}_Q^\sharp$ on $\Delta_Q^\sharp$ by $\boldsymbol{\sigma}_Q^\sharp(f) = (\mathsf{e}(f), \mathsf{f})$. Recall from Lemma 7.2 that the degree of m_f is $\varpi(\boldsymbol{\sigma}_Q^\sharp(f))$. As a side result, we will show that the ring of regular functions on the base affine space $\mathcal{A} := U^- \backslash G$ is the graded upper cluster algebra $\overline{\mathcal{C}}(\Delta_Q^\sharp, \mathcal{M}_Q^\sharp; \varpi(\boldsymbol{\sigma}_Q^\sharp))$.

The proof is similar to but easier than the one for $\mathrm{Conf}_{2,1}$ so our treatment may be a little sketchy. Recall the open set $G_0 = U^- H U$ of G. We have an open embedding

$$\iota_0 : H \times U = U^- \backslash G_0 \hookrightarrow U^- \backslash G.$$

The localization of $k[\mathcal{A}]$ at all m_f's for f positive is exactly $k[H \times U]$. In particular, $k[\mathcal{A}]$ is contained in the Laurent polynomial ring in $\mathcal{M}_Q^\sharp$. Consider the open embedding

$$\iota = (\iota_1, \iota_2) : H \times \mathcal{A} \hookrightarrow \mathrm{Conf}_{2,1}, \ (h, U^- g) \mapsto (U^- h, U^- g, U).$$

Note that the map $\iota(\mathrm{Id}_H, \iota_0) : H \times H \times U \to \mathrm{Conf}_{2,1}$ is the map i defined in Section 8.1. Let p be a rational inverse of ι, and p_2 be the map p followed by the second component projection. We define the birational map $r' : U^- \backslash G \to U^- \backslash G$ to be the composition $r' := p_2 r \iota_2$. The map r' can be viewed as a variation of Fomin-Zelevinsky's *twist automorphism* [22] on the big cell $U^- \backslash G_0$. Readers can check that they differ by a (fibrewise) rescaling along the toric fiber of $H \times U$, but we do not need this fact.

Let $i_{u'} := \iota_0|_U$, and $i_{r'} := r' i_{u'}$. Then we have the following commutative diagram. The map $i_{r'}$ is in fact regular because r' is regular on the image of $i_{u'}$.

$$\begin{tikzcd}
\mathrm{Conf}_{2,1} \arrow[rr, "r"] \arrow[dd, "p_2"'] & & \mathrm{Conf}_{2,1} \arrow[dd, "p_2"] \\
& U \arrow[ul, hook, "i_u"'] \arrow[ur, hook, "i_r"] \arrow[dl, hook, "i_{u'}"] \arrow[dr, hook, "i_{r'}"'] & \\
U^- \backslash G \arrow[rr, "r'"] & & U^- \backslash G.
\end{tikzcd}$$

To finish the proof, we only need three maps $i_{u'}$, $i_{r'}$ and r'. Analogous to Theorem 11.14, we have that the pullback $(r')^*$ is related to the sequence of mutations $\boldsymbol{\mu}_r$. More precisely, we have that

$$\boldsymbol{\mu}_r(m_f) = (r')^*(m_f) \text{ for any } f \text{ mutable.}$$

So analogous to Lemma 8.12, we have that $i_{r'}^*(m_f) = (r')^*(m_f) = \boldsymbol{\mu}_r(m_f)$ for f mutable. Obviously we also have that $i_{r'}^*(m_{\mathrm{O}_i^+}) = m_{\mathrm{O}_i^-}$ and $i_{r'}^*(m_{\mathrm{O}_i^-}) = 1$.

Analogous to i_u^* and i_r^*, we have the maps $\mathsf{i}_{u'}^* : \mathbb{R}^{(\Delta_Q^\sharp)_0} \to \mathbb{R}^{(\Delta_Q)_0}$ and $\mathsf{i}_{r'}^* : \mathbb{R}^{(\Delta_Q^\sharp)_0} \to \mathbb{R}^{(\Delta_Q^r)_0}$. The map $\mathsf{i}_{u'}^*$ (resp. $\mathsf{i}_{r'}^*$) forgets the coordinates corresponding to the positive (resp. negative) frozen vertices. We have the following analog of Lemma 8.13. If $m \in \mathscr{A}$ has a well-defined g-vector g with respect to $(\mathscr{M}_Q^\sharp, \Delta_Q^\sharp)$, then so are the g-vectors of $i_{u'}^*(m)$ and $i_{r'}^*(m)$ with respect to $(\mathscr{M}_Q, \Delta_Q)$ and $(\mathscr{M}_Q^r, \Delta_Q^r)$. They are equal to $\mathsf{i}_{u'}^*(\mathsf{g})$ and $\mathsf{i}_{r'}^*(\mathsf{g})$ respectively.

We restrict the iARt QP (Δ_Q^2, W_Q^2) to the subquiver $\Delta_Q^\sharp$. We denote the restricted potential by $W_Q^\sharp$. Almost the same proof as in Theorem 5.9 shows that the set $G(\Delta_Q^\sharp, W_Q^\sharp)$ of μ-supported g-vectors is given by the lattice points in the polyhedral cone $\mathsf{G}_{\Delta_Q^\sharp}$. The cone is defined by the two sets of defining conditions of $\mathsf{G}_{\Delta_Q^2}$, namely, $\mathsf{g}H_u \geq 0$ and $\mathsf{g}H_r \geq 0$. By almost the same argument, we get the following analog of Lemma 8.14. For the weight configuration $\varpi(\boldsymbol{\sigma}_Q^\sharp)$ we define the polytope $\mathsf{G}_{\Delta_Q^\sharp}(\mu,\lambda)$ as in Definition 5.14.

LEMMA 10.1. – *The polytope* $\mathsf{G}_{\Delta_Q^\sharp}(\mu,\lambda)$ *has lattice points no less than* $\dim k[\mathscr{A}]_{\mu,\lambda}$.

THEOREM 10.2. – *Suppose that Q is trivially valued. Then the ring of regular functions on $\mathscr{A}$ is the graded upper cluster algebra $\overline{\mathcal{C}}(\Delta_Q^\sharp, \mathscr{M}_Q^\sharp; \varpi(\boldsymbol{\sigma}_Q^\sharp))$. Moreover, $(\Delta_Q^\sharp, W_Q^\sharp)$ is a cluster model. In particular, the weight multiplicity* $\dim L(\mu)_{\varpi(\lambda)}$ *is counted by lattice points in* $\mathsf{G}_{\Delta_Q^\sharp}(\mu,\lambda)$.

Proof. – We know from [3] that $(\Delta_Q^\sharp, \mathscr{M}_Q^\sharp)$ is a seed satisfying the condition of Lemma 1.8. By Lemma 1.8, $\overline{\mathcal{C}}(\Delta_Q^\sharp, \mathscr{M}_Q^\sharp; \varpi(\boldsymbol{\sigma}_Q^\sharp))$ is a graded subalgebra of $k[\mathscr{A}]$.

By Theorem 4.10, $C_W\left(\mathsf{G}_{\Delta_Q^\sharp}(\mu,\lambda) \cap \mathbb{Z}^{(\Delta_Q^\sharp)_0}\right)$ is a linearly independent set in $k[\mathscr{A}]_{\mu,\lambda}$ for any μ,λ. But Lemma 10.1 implies that they actually span the vector space $k[\mathscr{A}]_{\mu,\lambda}$. We conclude that $\overline{\mathcal{C}}(\Delta_Q^\sharp, \mathscr{M}_Q^\sharp; \varpi(\boldsymbol{\sigma}_Q^\sharp))$ is equal to $k[\mathscr{A}]$, and $(\Delta_Q^\sharp, W_Q^\sharp)$ is a cluster model. □

REMARK 10.3. – For Q of type A, the theorem was proved in [14, Example 3.10] using the technique of projection. The map p_2 is induced from a semi-orthogonal decomposition of the module category of a triple flag quiver.

In general, the cluster algebra $\mathcal{C}(\Delta_Q^\sharp)$ is also strictly contained in its upper cluster algebra. The example is still given by the Q of type D_4. We take the same g-vector as in Example 9.3.

REMARK 10.4. – Let $K^2Q := K^2(\text{proj-}Q)$ be the homotopy category of C^2Q as in [7]. Then the ice quiver $\Delta_Q^\sharp$ can be obtained from the ARt quiver $\Delta(K^2Q)$ by freezing the negative and positive vertices.

There are two other seeds mutation equivalent to this seed via $\boldsymbol{\mu}_l$ and $\boldsymbol{\mu}_r$. Their quivers are obtained from Δ_Q^2 by deleting the positive and negative frozen vertices respectively.

10.2. Remark on the Non-simply Laced Cases

Suppose that G is not simply laced, or equivalently Q is not trivially valued. To prove Conjecture 9.2, let us examine the arguments in proving Theorem 9.1. We find that there are two missing ingredients. One is the straightforward generalization of Theorem 11.14, and the other is an analogous cluster character for species with potentials. Although the proof of Theorem 11.14 depends on a result involving preprojective algebras, the author has a much longer proof for all cases using only a conjectural generalization of Lemma 11.3 to the species with potentials. For the missing cluster character, as proved in [13] for the usual QPs, the existence of such a map to the upper cluster algebra is also equivalent to Lemma 11.3. The argument there can be generalized. So the upshot is that the only missing part for Conjecture 9.2 is a certain generalization of Lemma 11.3. But to the author's best knowledge, the existing theory of species with potentials, such as [37], is far from reaching this goal.

11. The Twisted Cyclic Shift via Mutations

11.1. Mutation of Representations, g-vectors, and F-polynomials

We review some material from [9]. Let (Δ, W) be a QP as in Section 4.1. The *mutation* μ_u of (Δ, W) at a vertex u is defined as follows. The first step is to define the following new QP $\widetilde{\mu}_u(\Delta, W) = (\widetilde{\Delta}, \widetilde{W})$. We put $\widetilde{\Delta}_0 = \Delta_0$ and $\widetilde{\Delta}_1$ is the union of three different kinds

- all arrows of Δ not incident to u,
- a composite arrow $[ab]$ from $t(a)$ to $h(b)$ for each a, b with $h(a) = t(b) = u$,
- an opposite arrow a^* (resp. b^*) for each incoming arrow a (resp. outgoing arrow b) at u.

Note that this $\widetilde{\Delta}$ is the result of the first two steps in Definition 1.3. The new potential on $\widetilde{\Delta}$ is given by

$$\widetilde{W} := [W] + \sum_{h(a)=t(b)=u} b^* a^* [ab],$$

where $[W]$ is obtained by substituting $[ab]$ for each word ab occurring in W. Finally we define $(\Delta', W') = \mu_u(\Delta, W)$ as the *reduced part* ([8, Definition 4.13]) of $(\widetilde{\Delta}, \widetilde{W})$. For this last step, we refer readers to [8, Section 4,5] for details.

Now we start to define the mutation of decorated representations of (Δ, W). Consider the resolution of the simple module S_u

$$\cdots \to \bigoplus_{h(a)=u} P_{t(a)} \xrightarrow{a(\partial_{[ab]})_b} \bigoplus_{t(b)=u} P_{h(b)} \xrightarrow{b(b)} P_u \to S_u \to 0, \tag{11.1}$$

$$0 \to S_u \to I_u \xrightarrow{(a)_a} \bigoplus_{h(a)=u} I_{t(a)} \xrightarrow{a(\partial_{[ab]})_b} \bigoplus_{t(b)=u} I_{h(b)} \to \cdots, \tag{11.2}$$

where I_u is the indecomposable injective representation of $J(\Delta, W)$ corresponding to a vertex u. We thus have the triangle of linear maps with $\beta_u\gamma_u = 0$ and $\gamma_u\alpha_u = 0$.

$$\begin{tikzcd} & M(u) \arrow[dr, "\beta_u"] & \\ \bigoplus_{h(a)=u} M(t(a)) \arrow[ur, "\alpha_u"] & & \bigoplus_{t(b)=u} M(h(b)). \arrow[ll, "\gamma_u"] \end{tikzcd}$$

We first define a decorated representation $\widetilde{\mathcal{M}} = (\widetilde{M}, \widetilde{M}^+)$ of $\widetilde{\mu}_u(\Delta, W)$. We set

$$\widetilde{M}(v) = M(v), \qquad \widetilde{M}^+(v) = M^+(v) \quad (v \neq u);$$
$$\widetilde{M}(u) = \frac{\operatorname{Ker}\gamma_u}{\operatorname{Im}\beta_u} \oplus \operatorname{Im}\gamma_u \oplus \frac{\operatorname{Ker}\alpha_u}{\operatorname{Im}\gamma_u} \oplus M^+(u), \qquad \widetilde{M}^+(u) = \frac{\operatorname{Ker}\beta_u}{\operatorname{Ker}\beta_u \cap \operatorname{Im}\alpha_u}.$$

We then set $\widetilde{M}(a) = M(a)$ for all arrows not incident to u, and $\widetilde{M}([ab]) = M(ab)$. It is defined in [8] a choice of linear maps $\widetilde{M}(a^*), \widetilde{M}(b^*)$ making $\widetilde{M}$ a representation of $(\widetilde{\Delta}, \widetilde{W})$. We refer readers to [9, Section 10] for details. Finally, we define $\mathcal{M}' = \mu_u(\mathcal{M})$ to be the *reduced part* ([8, Definition 10.4]) of $\widetilde{\mathcal{M}}$. We say a representation M of (Δ, W) *reachable* if there is a sequence of mutations $\boldsymbol{\mu}$ such that $\boldsymbol{\mu}(M)$ has only the decorated part.

Recall the g-vector form Definition 4.2. We can also define the dual g-vectors using the injective presentations. Let M be a representation of $J(\Delta, W)$ with minimal injective presentation $0 \to M \to I(\beta_-^\vee) \to I(\beta_+^\vee)$, then the *dual* g*-vector* $\mathrm{g}^\vee(M)$ of M is the reduced weight vector $\mathrm{g}^\vee = \beta_+^\vee - \beta_-^\vee$. The definition can also be extended to all decorated representations similar to g-vectors.

Recall the y-variables $y_u = \mathbf{x}^{-b_u}$ as in Section 1.3. The seed mutation of Definition 1.4 induces the **y**-*seed mutation*. We recall the mutation rule [24, (3.8)]. Let $(\mathbf{y}', \Delta') := \mu_u(\mathbf{y}, \Delta)$ and $B(\Delta) = (b_{u,v})$, then $\Delta' = \mu_u(\Delta)$ and

$$y'_v = \begin{cases} y_u^{-1} & \text{if } v = u; \\ y_v y_u^{[-b_{u,v}]_+}(y_u+1)^{b_{u,v}} & \text{if } v \neq u. \end{cases} \tag{11.3}$$

Definition 11.1 ([9]). – We define the dual F-polynomial of a representation M by

$$F_M^\vee(\mathbf{y}) = \sum_{\mathsf{e}} \chi(\mathrm{Gr}_{\mathsf{e}}(M))\mathbf{y}^{\mathsf{e}}, \tag{11.4}$$

where $\mathrm{Gr}_{\mathsf{e}}(M)$ is the variety parameterizing e-dimensional subrepresentations of M.

Remark 11.2. – The F-polynomial of M is $F_M(\mathbf{y}) = \sum_{\mathsf{e}} \chi(\mathrm{Gr}^{\mathsf{e}}(M))\mathbf{y}^{\mathsf{e}}$. We only need the dual version in 11.3. Our dual g-vector and dual F-polynomial is the g-vector and F-polynomial in [24, 9].

Here is the key lemma in [9].

Lemma 11.3. – *Let $\mathcal{M}$ be an arbitrary representation of a nondegenerate QP (Δ, W), and let $\mathcal{M}' = \mu_u(\mathcal{M})$, then*

1. *The F-polynomials of M and M' are related by*

$$(y_u+1)^{-\beta_-^\vee(u)} F_M^\vee(\mathbf{y}) = (y'_u+1)^{-(\beta_-^\vee)'(u)} F_{M'}^\vee(\mathbf{y}'). \tag{11.5}$$

2. *The* g*-vector and* $\mathrm{g}^\vee$*-vector of* $\mathcal{M}$ *and* $\mathcal{M}'$ *are related by*

$$\mathrm{g}'(v) = \begin{cases} -\mathrm{g}(u) & \text{if } u = v, \\ \mathrm{g}(v) + [b_{v,u}]_+\mathrm{g}(u) + b_{v,u}\beta_-(u) & \text{if } u \neq v, \end{cases} \tag{11.6}$$

$$(\mathrm{g}^\vee)'(v) = \begin{cases} -\mathrm{g}^\vee(u) & \text{if } u = v, \\ \mathrm{g}^\vee(v) + [-b_{v,u}]_+\mathrm{g}^\vee(u) - b_{v,u}\beta_-^\vee(u) & \text{if } u \neq v. \end{cases} \tag{11.7}$$

REMARK 11.4. – If $\mathcal{M}$ is g*-coherent* (that is $\min(\beta_+(u), \beta_-(u)) = 0$ for all vertices u, or equivalently, $\beta_+ = [\mathrm{g}]_+$ and $\beta_- = [-\mathrm{g}]_+$), then (11.6) reads

$$\mathrm{g}'(v) := \begin{cases} -\mathrm{g}(u) & \text{if } u = v; \\ \mathrm{g}(v) + b_{v,u}[\mathrm{g}(u)]_+ & \text{if } b_{v,u} \geq 0; \\ \mathrm{g}(v) + b_{v,u}[-\mathrm{g}(u)]_+ & \text{if } b_{v,u} \leq 0. \end{cases} \tag{11.8}$$

It is known [13] that if $\mathcal{M}$ corresponds to a general presentation, then $\mathrm{g}(\mathcal{M})$ is g-coherent. If $\mathcal{M}$ is obtained from positive simple $(0, S_u)$ via a sequence of mutations, then $\mathcal{M}$ corresponds to an indecomposable rigid presentation. In particular, it is general [7].

11.2. The Twisted Cyclic Shift via Mutations

We recall from Section 8.2 the twisted cyclic shift l on $\mathrm{Conf}_{2,1}$ induced by

$$(A_1, A_2, A_3^\vee) \mapsto (A_2, A_3\overline{w_0}, (A_1\overline{w_0})^\vee),$$

where $\vee : \mathcal{A} \to \mathcal{A}^\vee$, $(U^-g)^\vee \mapsto \overline{\omega_0}^{-1}g^{-1}\overline{\omega_0}U$. To understand this map, we introduce a "half" of this map. Consider the involution $* : G \to G$ as in [30] defined by

$$x_i(a)^* = x_{i^*}(a),\ y_i(b)^* = y_{i^*}(b),\ h^* = \overline{w_0}^{-1}h^{-1}\overline{w_0},\ (i \in Q_0, h \in H).$$

The involution $*$ preserves U, U^- and H so it induces an involution of $\mathcal{A}$ and $\mathcal{A}^\vee$. Thus it acts on $\mathrm{Conf}_{2,1}$. We consider the automorphism of $\mathrm{Conf}_{2,1}$ induced by

$$(A_1, A_2, A_3^\vee) \mapsto (A_3^*, (A_1\overline{w_0})^*, (A_2^*)^\vee).$$

We denote this map by $\sqrt{l}$. It is clear that $l = (\sqrt{l})^2$. It will turn out (see Remark 11.15) that this map is the *Donaldson-Thomas transformation* of $\mathrm{Conf}_{2,1}$ in the sense of [30].

Let us recall a sequence of mutations constructed in [27, Section 13]. The sequence is originally defined for the ice quiver Δ_Q in terms of reduced expression of w_0. We now translate it into our setting. Recall that we can label the non-neutral vertices of Δ_Q^2 by a pair (i, t) (before Lemma 7.3). Let

$$t_i := t^*(\mathrm{O}_i^+) = \max\{t \mid (i, t) \in \Delta_Q^2\}.$$

We first assume that the vertices of Q are ordered such that $i < j$ if $(i, j) \in Q_1$. Then we totally order the mutable vertices of Δ_Q^2 by the relation that $(i, t) < (i', t')$ if $t < t'$, or $t = t'$ and $i < i'$.

Starting from the minimal vertex $(1, 1)$ in the ascending order just defined, we perform a sequence of mutations $\boldsymbol{\mu}_{i,t}$ for each (mutable) vertex of Δ_Q^2. For the vertex (i, t), the sequence of mutations is defined to be $\boldsymbol{\mu}_{i,t} := \mu_{i,t_i-t} \cdots \mu_{i,2}\mu_{i,1}$. So the whole sequence of mutations $\boldsymbol{\mu}_{\sqrt{l}} := \cdots \boldsymbol{\mu}_{2,1}\boldsymbol{\mu}_{1,1}$.

Let π be the permutation on $(\Delta_Q^2)_0$ defined by

$$(i,t)\mapsto(i,t_i-t)\ \text{ if } (i,t) \text{ is mutable,}$$
$$\mathrm{O}_i^-\mapsto\mathrm{O}_{i^*}^+,\ \mathrm{O}_i^+\mapsto\mathrm{Id}_i,\ \mathrm{Id}_i\mapsto\mathrm{O}_i^-.$$

It is clear that π is an involution on the set of mutable vertices. When applying it to the quiver Δ_Q^2 or its B-matrix, we view π as a relabeling of the vertices.

For a (trivially valued) Dynkin quiver Q, let Π_Q be its associated *preprojective algebra*. Recall that the module category $\operatorname{mod}\Pi_Q$ is a *Frobenius category*. Let $\underline{\operatorname{mod}}\Pi_Q$ be its *stable category*. Recall that such a stable category is naturally triangulated with the shift functor given by the (relative) inverse Syzygy functor Ω^{-1}. Readers can find these standard terminology in, for example, [27]. We will write $\operatorname{Hom}_{\underline{\Pi}_Q}$ for $\operatorname{Hom}_{\underline{\operatorname{mod}}\Pi_Q}$. In [26] the authors defined a tilting module $V:=\bigoplus_{u\in(\Delta_Q)_0}V_u$ in $\operatorname{mod}\Pi_Q$ (it is denoted by I_Q in [26]). They proved that the quiver of the endomorphism algebra $\operatorname{End}_{\Pi_Q}(V)^{\mathrm{op}}$ is exactly Δ_Q with its frozen vertices of corresponding to the projective-injective objects in $\operatorname{mod}\Pi_Q$. Moreover, it is known [6] that the *stable* endomorphism algebra $\operatorname{End}_{\underline{\Pi}_Q}(V)^{\mathrm{op}}$ is the Jacobian algebra J^{μ} of the QP (Δ_Q^2,W_Q^2) restricted to its mutable part Δ_Q^{μ}.

For each rigid Π_Q-module M, we can associate a function $\varphi_M\in k[U]$ as in [25]. This function turns out to be a cluster variable for the standard seed $(\Delta_Q,\mathcal{M}_Q)$. In [25] the *mutation of maximal rigid modules* of Π_Q is defined so that it is compatible with the seed mutation. In particular, it is compatible with the quiver mutation, so the quiver of $\operatorname{End}_{\Pi_Q}(\mu_u(V))$ is exactly $\mu_u(\Delta_Q)$.

Proposition 11.5 ([27, Proposition 13.4]). – *The sequence of mutations $\boldsymbol{\mu}_{\sqrt{l}}$ takes the modules $V_{\pi(u)}$ to $\Omega^{-1}(V_u)$ for u mutable in Δ_Q.*

Corollary 11.6. – *Identifying Δ_Q as a subquiver of Δ_Q^2, we have that $\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q)$ and $\pi(\Delta_Q)$ have the same restricted B-matrix.*

Proof. – Since Ω is an autoequivalence, $\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q)$ and $\pi(\Delta_Q)$ have the same mutable part. Let $\overline{\boldsymbol{\mu}}_{i,t}:=\boldsymbol{\mu}_{i,t}\cdots\boldsymbol{\mu}_{1,1}$. In [27, Section 13.1], the authors described each mutated quiver after applying each $\overline{\boldsymbol{\mu}}_{i,t}$. It follows easily from their description that after forgetting arrows between frozen vertices, $\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q)$ is a subquiver of Δ_Q^2, whose frozen vertex $\mathrm{O}_{i^*}^+$ is identified with the frozen vertex O_i^- of $\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q)$. □

Let $(\Delta_Q^{\mu},W_Q^{\mu})$ be the restriction of (Δ_Q,W_Q) to its mutable part. Next, we compute the g-vectors of cluster variables after applying $\boldsymbol{\mu}_{\sqrt{l}}$ to the seed $(\Delta_Q^{\mu},\mathbf{x})$. The g-vector of the initial cluster variable $\mathbf{x}_{i,t}$ at the vertex (i,t) is the unit vector $\mathsf{e}_{i,t}\in\mathbb{Z}^{(\Delta_Q^{\mu})_0}$. By [9, Theorem 5.2], the g-vector of $\boldsymbol{\mu}_{\sqrt{l}}(\mathbf{x}_{i,t})$ is nothing but the g-vector of $\boldsymbol{\mu}_{\sqrt{l}}^{-1}(0,S_{i,t})$, where $(0,S_{i,t})$ is the positive simple representation of $\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q^{\mu},W_Q^{\mu})$.

Lemma 11.7. – *We have that $\boldsymbol{\mu}_{\sqrt{l}}^{-1}(\mathsf{e}_{i,t})=-\mathsf{e}_{i,t_i-t}$ for each $\mathsf{e}_{i,t}$. Here, we view $\mathsf{e}_{i,t}$ as the* g-*vector of the simple representation $(0,S_{i,t})$ of $\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q^{\mu},W_Q^{\mu})$.*

Proof. – It is equivalent to show that $\boldsymbol{\mu}_{\sqrt{l}}(-\mathsf{e}_{i,t_i-t}) = \mathsf{e}_{i,t}$. Let $\overline{\boldsymbol{\mu}}_{i,t} := \boldsymbol{\mu}_{i,t} \cdots \boldsymbol{\mu}_{1,1}$. We study the mutated g-vector $\overline{\boldsymbol{\mu}}_{i,t}(-\mathsf{e}_{i,t_i-t})$ for each (i,t). From the description of mutated quiver $\overline{\boldsymbol{\mu}}_{i,t}(\Delta_Q)$ [27, Section 13.1] and the Formula (11.8), we find that for $j \neq i$, each $\boldsymbol{\mu}_{j,s}$ does not play any role in $\boldsymbol{\mu}_{\sqrt{l}}(-\mathsf{e}_{i,t_i-t})$. In other words, it suffices to compute $\boldsymbol{\mu}_{\sqrt{l}}(-\mathsf{e}_{i,t_i-t})$ on the full linear subquiver with vertices (i,t) for $t = 1, \cdots, t_i - 1$. Using the Formula (11.8), we can easily prove by induction that

$$\overline{\boldsymbol{\mu}}_{i,s}(-\mathsf{e}_{i,t_i-t}) = \begin{cases} \mathsf{e}_{i,t_i-s} - \mathsf{e}_{i,t_i-s-t} & \text{if } s+t < t_i, \\ \mathsf{e}_{i,t} & \text{otherwise.} \end{cases}$$

We conclude from this that $\boldsymbol{\mu}_{\sqrt{l}}(-\mathsf{e}_{i,t_i-t}) = \mathsf{e}_{i,t}$. □

Let $\mathbf{g}^{\pi}_{\sqrt{l}}$ be the set of g-vectors given by

$$\mathbf{g}^{\pi}_{\sqrt{l}}(f) = -\mathsf{e}_f + \sum_{i \in Q_0} \left(\mathsf{e}(f)(i)\mathsf{e}_{\mathrm{O}_i^-} + \mathsf{f}_+(i)\mathsf{e}_{\mathrm{O}_i^+} + \mathsf{f}_-(i)\mathsf{e}_{\mathrm{Id}_i} \right). \tag{11.9}$$

Let $\boldsymbol{\sigma}^{\pi}_{\sqrt{l}}$ be the weight configuration defined by

$$\boldsymbol{\sigma}^{\pi}_{\sqrt{l}}(f) = (\mathsf{f}_+, \mathsf{e}(f)^*, \mathsf{f}_-),$$

where $\mathsf{f} \mapsto \mathsf{f}^*$ is the involution on $\mathbb{Z}^{Q_0}$ induced by $\mathsf{e}_i \mapsto \mathsf{e}_{i^*}$, so $-w_0(\varpi(\mathsf{f})) = \varpi(\mathsf{f}^*)$.

Corollary 11.8. – *For any f mutable, we have that $\boldsymbol{\mu}^{-1}_{\sqrt{l}}(\mathsf{e}_f) = \mathbf{g}^{\pi}_{\sqrt{l}}(\pi(f))$. Here, we view e_f as the* g*-vector of the simple representation $(0, S_f)$ of $\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q^2, W_Q^2)$. So $\boldsymbol{\mu}_{\sqrt{l}}(\boldsymbol{\sigma}_Q^2)(f) = \boldsymbol{\sigma}^{\pi}_{\sqrt{l}}(\pi(f))$.*

Proof. – In Lemma 11.7 we have obtained the principal part of $\boldsymbol{\mu}^{-1}_{\sqrt{l}}(\mathsf{e}_f)$, which is $-\mathsf{e}_{\pi(f)}$. Now we are going to recover its extended part from its principal part.

We recall from [13, Proposition 5.15] that being μ-supported for a g-vector is mutation invariant. The general presentation with g-vector equal to $-\mathsf{e}_f$ corresponds to the indecomposable projective representation $\boldsymbol{P}_f$. By Lemma 5.2 $\boldsymbol{P}_f(\mathrm{O}_i^+) \cong \mathrm{Hom}_Q(P_+, P_i)$. So to make the g-vector $-\mathsf{e}_f$ not supported on O_i^+, we must add at least $\mathsf{f}_+(i)\mathsf{e}_{\mathrm{O}_i^+}$ to it. Similarly we must add at least $\mathsf{f}_-(i)\mathsf{e}_{\mathrm{Id}_i}$ and $\mathsf{e}(f)(i)\mathsf{e}_{\mathrm{O}_i^-}$ as well. On the other hand, it is easy to check that (11.9) is μ-supported. So adding any additional positive component on the frozen vertices will make the general presentation of this g-vector decomposable, which is not the case. Hence, $\boldsymbol{\mu}_{\sqrt{l}}(\mathsf{e}_f) = \mathbf{g}^{\pi}_{\sqrt{l}}(\pi(f))$.

For the last statement, recall that $\boldsymbol{\sigma}_Q^2(f) = (\mathsf{e}(f), \mathsf{f}_-, \mathsf{f}_+)$. Then

$$\begin{aligned} \mathbf{g}^{\pi}_{\sqrt{l}}(f)\boldsymbol{\sigma}_Q^2 &= -(\mathsf{e}(f), \mathsf{f}_-, \mathsf{f}_+) + (\mathsf{e}(f), \mathsf{e}(f)^*, 0) + (\mathsf{f}_+, 0, \mathsf{f}_+) + (0, \mathsf{f}_-, \mathsf{f}_-) \\ &= (\mathsf{f}_+, \mathsf{e}(f)^*, \mathsf{f}_-). \end{aligned}$$

Hence, $\boldsymbol{\mu}_{\sqrt{l}}(\boldsymbol{\sigma}_Q^2)(f) = \boldsymbol{\mu}_{\sqrt{l}}(\mathsf{e}_f)\boldsymbol{\sigma}_Q^2 = \mathbf{g}^{\pi}_{\sqrt{l}}(\pi(f))\boldsymbol{\sigma}_Q^2 = \boldsymbol{\sigma}^{\pi}_{\sqrt{l}}(\pi(f))$. □

Corollary 11.9. – *$\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q^2)$ and $\pi(\Delta_Q^2)$ have the same restricted B-matrix.*

Proof. – By Corollary 11.6, $\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q^2)$ and $\pi(\Delta_Q^2)$ have the same mutable part as well. It remains to show that the blocks of their restricted B-matrices corresponding to positive and neutral frozen vertices are equal. But this follows from an easy linear algebra consideration. Indeed, let $B^2_{\sqrt{l}}$ and $B_{\sqrt{l}}$ be the restricted B-matrices of $\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q^2)$ and $\boldsymbol{\mu}_{\sqrt{l}}(\Delta_Q)$ and denote $\boldsymbol{\sigma}^2_{\sqrt{l}} := \boldsymbol{\mu}_{\sqrt{l}}(\boldsymbol{\sigma}_Q^2)$. We have that $B^2_{\sqrt{l}}\boldsymbol{\sigma}^2_{\sqrt{l}} = 0$. We write $B^2_{\sqrt{l}}$ and $\boldsymbol{\sigma}^2_{\sqrt{l}}$ in blocks: $(B_{\sqrt{l}}, C_{\sqrt{l}})$ and $\begin{pmatrix} \boldsymbol{\sigma}_{\sqrt{l}} \\ \boldsymbol{\sigma}_{+,\mathrm{Id}} \end{pmatrix}$, where $\boldsymbol{\sigma}_{+,\mathrm{Id}} = \begin{pmatrix} I & 0 & I \\ 0 & I & I \end{pmatrix}$ contains weights corresponding to the positive and neutral frozen vertices. We have that

$$(B_{\sqrt{l}}, C_{\sqrt{l}})\begin{pmatrix} \boldsymbol{\sigma}_{\sqrt{l}} \\ \boldsymbol{\sigma}_{+,\mathrm{Id}} \end{pmatrix} = B_{\sqrt{l}}\boldsymbol{\sigma}_{\sqrt{l}} + C_{\sqrt{l}}\boldsymbol{\sigma}_{+,\mathrm{Id}} = 0. \tag{11.10}$$

Since $\boldsymbol{\sigma}_{+,\mathrm{Id}}$ is of full rank, this linear equation is underdetermined (for solving $C_{\sqrt{l}}$). We have seen that $B_{\sqrt{l}} = B_{\pi(\Delta_Q)}$ and $\boldsymbol{\sigma}_{\sqrt{l}}(f) = \boldsymbol{\sigma}^{\pi}_{\sqrt{l}}(\pi(f))$, then it is clear that $C_{\sqrt{l}} = \pi(C)$ satisfies (11.10), where C is the block in $B_{\Delta_Q^2} = (B_{\Delta_Q}, C)$. □

DEFINITION 11.10. – Let $\boldsymbol{\mu}^{\pi}_{\sqrt{l}} := \pi\boldsymbol{\mu}_{\sqrt{l}}\pi^{-1}$. We define $\boldsymbol{\mu}_l := \boldsymbol{\mu}^{\pi}_{\sqrt{l}}\boldsymbol{\mu}_{\sqrt{l}}$ and $\boldsymbol{\mu}_r = \boldsymbol{\mu}_l^2$.

COROLLARY 11.11. – *$\boldsymbol{\mu}_l(\Delta_Q^2)$ and $\pi^2(\Delta_Q^2)$ have the same restricted B-matrix. Moreover, $\boldsymbol{\mu}_l(\boldsymbol{\sigma}_Q^2)(f) = (\mathsf{f}_-, \mathsf{f}_+^*, \mathsf{e}(f)^*)$ for f mutable.*

Note that π^2 fixes the mutable vertices and shuffles the frozen vertices of Δ_Q^2

$$\mathrm{O}_i^+ \mapsto \mathrm{O}_i^-,\ \mathrm{O}_i^- \mapsto \mathrm{Id}_{i^*},\ \mathrm{Id}_{i^*} \mapsto \mathrm{O}_i^+ .$$

COROLLARY 11.12. – *We have that $\boldsymbol{\mu}_l^3(B_{\Delta_Q^2}, \mathbf{x}) = (B_{\Delta_Q^2}, \mathbf{x})$.*

Proof. – Since $\pi^6 = \mathrm{Id}$, the fact that $\boldsymbol{\mu}_l^3(B_{\Delta_Q^2}) = B_{\Delta_Q^2}$ follows from Corollary 11.11. As pointed in [26], it is well-known [1] that the functor Ω is also 6-periodic. [28, Proposition 6.3] says that the mutation of maximal rigid modules of Π_Q is compatible with the mutation of the corresponding QP-representations (see remarks before Proposition 11.5). So by Proposition 11.5 the positive simples $(0, S_f)$ of (Δ_Q, W_Q) are invariant under $\boldsymbol{\mu}_l^3$. The invariance obviously extends to (Δ_Q^2, W_Q^2). The desired result follows. □

By experiment with [33] we believe the following conjecture.

CONJECTURE 11.13. – Corollary 11.9 (and thus Corollaries 11.11, 11.12 and Theorem 11.14 below) hold for non-trivially valued Q as well.

We define a rational self-map μ_l of $\overline{\mathcal{C}}(\Delta_Q^2, \mathcal{S}_Q^2)$ induced by

$$\begin{cases} s_f \mapsto \boldsymbol{\mu}_l(s_f) & \text{if } f \text{ is mutable,} \\ s_{\mathrm{O}_i^+} \mapsto s_{\mathrm{O}_i^-},\ s_{\mathrm{O}_i^-} \mapsto s_{\mathrm{Id}_{i^*}},\ s_{\mathrm{Id}_{i^*}} \mapsto s_{\mathrm{O}_i^+}. \end{cases}$$

Since our main theorem (Theorem 9.1) uses the following theorem, we cannot assume that $\overline{\mathcal{C}}(\Delta_Q^2, \mathcal{S}_Q^2) = k[\mathrm{Conf}_{2,1}]$. But from the proof of Theorem 9.1, we have that $\overline{\mathcal{C}}(\Delta_Q^2, \mathcal{S}_Q^2) \subseteq k[\mathrm{Conf}_{2,1}]$. So the pullback l^* is defined on $\overline{\mathcal{C}}(\Delta_Q^2, \mathcal{S}_Q^2)$.

THEOREM 11.14. – *The map μ_l is equal to the pullback l^* of the twisted cyclic shift.*

Proof. – We need to show that $\mu_l(\varphi) = \varphi l$. It suffices to prove the statement for φ's in some extended cluster because of the definition of the upper cluster algebra. The cluster we shall take is $\boldsymbol{\mu}_l^2(\mathcal{S}_Q^2)$, which is the same as $\boldsymbol{\mu}_l^{-1}(\mathcal{S}_Q^2)$ by Corollary 11.12. The statement can be easily checked for the frozen variables. For example, $s_{\mathrm{O}_i^+} = s_{\mathrm{Id}_{i*}} l$, that is

$$s_{\mathrm{O}_i^+}(A_1, A_2, A_3^\vee) = s_{\mathrm{Id}_{i*}}(A_2, A_3\overline{w_0}, (A_1\overline{w_0})^\vee). \tag{11.11}$$

Indeed, by Corollary 8.8 we have that $s_{\mathrm{O}_i^+} = p_1^*(\varpi_i)$ and $s_{\mathrm{Id}_i} = p_2^*(\varpi_i)$. So by (8.4)

$$s_{\mathrm{O}_i^+}(U^-h_1, U^-h_2u, U) = h_1^{\varpi_i},$$
$$s_{\mathrm{Id}_{i*}}(U^-h_2u, U^-\overline{w_0}, h_1^{-1}\overline{w_0}^{-1}U) = (\overline{w_0}h_1^{-1}\overline{w_0}^{-1})^{\varpi_{i^*}} = h_1^{\varpi_i}.$$

Next let us assume that $\varphi = \boldsymbol{\mu}_l^2(s_f)$ for some mutable $f \in \operatorname{ind}(C^2Q)$. We need to show that $\boldsymbol{\mu}_l(\varphi)(A_1, A_2, A_3^\vee) = \varphi(A_2, A_3\overline{w_0}, \overline{w_0}A_1^\vee)$. We argue by multidegrees. By Corollary 11.11, the degree of φ is $(\mathsf{f}_+^*, \mathsf{e}(f), \mathsf{f}_-^*)$. Then according to (7.1) and (7.2), the degree of $\varphi(A_2, A_3\overline{w_0}, \overline{w_0}A_1^\vee)$ is $(\mathsf{e}(f), \mathsf{f}_-, \mathsf{f}_+)$, which is also the degree of $\boldsymbol{\mu}_l(\varphi) = s_f$. So by Lemma 8.4, we must have that $\boldsymbol{\mu}_l(\varphi) = c\varphi l$ for some $c \in k$. Again by the relation (11.11) we must have $c = 1$. □

REMARK 11.15. – The similar argument can show that $\pi\boldsymbol{\mu}_{\sqrt{l}}$ is equal to the pull-back $\sqrt{l}^*$. In view of Lemma 11.7, the automorphism $\sqrt{l}$ is the Donaldson-Thomas transformation of $\mathrm{Conf}_{2,1}$ in the sense of [30].

11.3. An Algorithm

In this section, we present an algorithm to find all subrepresentations of T_v defined in Section 5.1. As said in the introduction, only few T_v's for type E_7 and E_8 need this algorithm. With a little more effort, one can show that Corollary 11.11 can be strengthened to $\boldsymbol{\mu}_l(\Delta_Q^2) = \pi^2(\Delta_Q^2)$ for any orientation of type D_n and E_7, E_8, and for some orientations of type A_n and E_6. For any particular case whether $\boldsymbol{\mu}_l(\Delta_Q^2) = \pi^2(\Delta_Q^2)$ holds can be checked by computer.

We first observe that by the description of T_v's in Theorem 5.3, all subrepresentations of $T_{\mathrm{O}_i^-}$ are known. In particular, we have the dual F-polynomial of $T_{\mathrm{O}_i^-}$. Even better, $T_{\mathrm{O}_i^-}$ can be mutated from the positive simple $(0, S_{\mathrm{O}_i^-})$ of the (unfrozen) QP $\boldsymbol{\mu}_i\left(\Delta(C^2Q), W_Q^2\right)$ via $\boldsymbol{\mu}_i^{-1}$, where $\boldsymbol{\mu}_i := \boldsymbol{\mu}_{i,0}\mu_{i,0}$ and the bold $\boldsymbol{\mu}_{i,0}$ is defined in the beginning of Appendix 11.2. Indeed, it can be easily checked by (11.7) that the $\mathsf{g}^\vee$-vector of such a mutated representation is exactly $\mathsf{e}_{\mathrm{O}_i^-} - \sum_{i\to j}\mathsf{e}_{\mathrm{O}_j^-}$. Since $T_{\mathrm{O}_i^-}$ is the cokernel of a general presentation of such a weight, our claim follows from Remark 11.4. The idea of the algorithm is that we can generate the dual F-polynomial of $T_{\mathrm{Id}_{i*}}$ (resp. $T_{\mathrm{O}_i^+}$) from that of $T_{\mathrm{O}_i^-}$ through the sequence of mutations $\boldsymbol{\mu}_l$ (resp. $\boldsymbol{\mu}_r$) just defined.

PROPOSITION 11.16. – *If $\boldsymbol{\mu}_l(\Delta_Q^2) = \pi^2(\Delta_Q^2)$, then $\boldsymbol{\mu}_l(T_{\mathrm{O}_i^-}) = T_{\mathrm{Id}_{i*}}$ and $\boldsymbol{\mu}_r(T_{\mathrm{O}_i^-}) = T_{\mathrm{O}_i^+}$. Here, we view $T_{\mathrm{Id}_{i*}}$ and $T_{\mathrm{O}_i^+}$ as representations of the original QP via the automorphisms π^{-2} and π^2.*

Proof. – It is clear from (11.7) that the $\mathrm{g}^\vee$-vector $\mathrm{e}_{\mathrm{O}_i^-} - \sum_{i\to j} \mathrm{e}_{\mathrm{O}_j^-}$ is unchanged under $\boldsymbol{\mu}_l$. If $\boldsymbol{\mu}_l(\Delta_Q^2) = \pi^2(\Delta_Q^2)$, by Corollary 11.11 the mutated $\mathrm{g}^\vee$-vector can be viewed as $\mathrm{e}_{\mathrm{Id}_{i*}} - \sum_{i\to j} \mathrm{e}_{\mathrm{Id}_{j*}}$ for the original QP (Δ_Q^2, W_Q^2) via the automorphism π^{-2}. Note that $i \to j$ if and only if $i^* \to j^*$. Clearly $T_{\mathrm{Id}_{i*}}$ is the cokernel of the general presentation of weight $\mathrm{e}_{\mathrm{Id}_{i*}} - \sum_{i^*\to j^*} \mathrm{e}_{\mathrm{Id}_j^*}$. Hence, $\boldsymbol{\mu}_l(T_{\mathrm{O}_i^-}) = T_{\mathrm{Id}_{i*}}$. The argument for $\boldsymbol{\mu}_r(T_{\mathrm{O}_i^-}) = T_{\mathrm{O}_i^+}$ is similar. □

By the positivity of the cluster variables [32, 38], the coefficients of F-polynomials of these cluster variables are all positive. So we can compute $F^\vee_{T_{\mathrm{Id}_{i*}}}$ (resp. $F^\vee_{T_{\mathrm{O}_i^+}}$) by applying $\boldsymbol{\mu}_l$ (resp. $\boldsymbol{\mu}_r$) to $F^\vee_{T_{\mathrm{O}_i^-}}$ using the Formula (11.5). In this way, we find all subrepresentations of T_v's.

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LOCAL NORMAL FORMS FOR C-PROJECTIVELY EQUIVALENT METRICS AND PROOF OF THE YANO-OBATA CONJECTURE IN ARBITRARY SIGNATURE. PROOF OF THE PROJECTIVE LICHNEROWICZ CONJECTURE FOR LORENTZIAN METRICS

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Abstract. – Two Kähler metrics on a complex manifold are called c-projectively equivalent if their J-planar curves coincide. These curves are defined by the property that the acceleration is complex proportional to the velocity. We give an explicit local description of all pairs of c-projectively equivalent Kähler metrics of arbitrary signature and use this description to prove the classical Yano-Obata conjecture: we show that on a closed connected Kähler manifold of arbitrary signature, any c-projective vector field is an affine vector field unless the manifold is $\mathbb{C}P^n$ with (a multiple of) the Fubini-Study metric. As a by-product, we prove the projective Lichnerowicz conjecture for metrics of Lorentzian signature: we show that on a closed connected Lorentzian manifold, any projective vector field is an affine vector field.

Résumé. – Deux métriques kählériennes sur une variété complexe sont appelées c-projectivement équivalentes si leurs courbes J-planaires coïncident. Ces courbes sont définies par la propriété que l'accélération est proportionnelle (au sens complexe) à la vitesse. Nous donnons une description locale de tous les paires de métriques kählériennes c-projectivement équivalentes de signature arbitraire et utilisons cette description pour prouver la conjecture classique de Yano-Obata: nous montrons que sur une variété kählérienne de signature arbitraire, connexe et fermée, tout champ de vecteurs c-projectif est un champ de vecteur affine sauf si la variété est $\mathbb{C}P^n$, munie de la métrique de Fubini-Study. En tant que sous-produit, nous prouvons la conjecture de Lichnerowicz pour les métriques de signature lorentzienne. Plus précisément, sur une variété lorentzienne connexe fermée tout champ de vecteurs projectif est un champ de vecteurs affine.

1. Introduction

1.1. Definitions and description of results

Let (M, g, J) be a Kähler manifold of arbitrary signature of real dimension $2n \geq 4$. We denote by ∇ the Levi-Civita connection of g and let $\omega = g(J\cdot, \cdot)$ denote the Kähler form. All objects under consideration are assumed to be sufficiently smooth.

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A regular curve $\gamma : \mathbb{R} \supseteq I \to M$ is called *J-planar* if there exist functions $\alpha, \beta : I \to \mathbb{R}$ such that

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = \alpha\dot\gamma(t) + \beta J(\dot\gamma(t)) \text{ for all } t \in I, \tag{1.1}$$

where $\dot\gamma = \frac{\mathrm{d}}{\mathrm{d}t}\gamma$.

From the definition we see immediately that the property of J-planarity is independent of the parameterisation of the curve, and that geodesics are J-planar curves. We also see that J-planar curves form a much bigger family than the family of geodesics; at every point and in every direction there exist infinitely many geometrically different J-planar curves.

Two metrics g and $\hat g$ of arbitrary signature that are Kähler w.r.t the same complex structure J are *c-projectively equivalent* if any J-planar curve of g is a J-planar curve of $\hat g$. Actually, the condition that the metrics are Kähler with respect to the same complex structure is not essential; it is an easy exercise to show that if any J-planar curve of a Kähler structure (g, J) is a $\hat J$-planar curve of another Kähler structure $(\hat g, \hat J)$, then $\hat J = \pm J$.

C-projective equivalence was introduced (under the name "h-projective equivalence") by Otsuki and Tashiro in [37, 43]. Their motivation was to generalize the notion of projective equivalence to the Kähler situation. Since the notion of projective equivalence plays an essential role in our paper let us recall it. Two metrics g and $\hat g$ of arbitrary signature are *projectively equivalent*, if each g-geodesic is, up to an appropriate reparameterisation, a $\hat g$-geodesic.

Otsuki and Tashiro have shown that projective equivalence is not interesting in the Kähler situation, since only simple examples are possible, and suggested c-projective equivalence as an interesting object of study instead. This suggestion appeared to be very fruitful and between the 1960s and the 1970s, the theory of c-projectively equivalent metrics and c-projective transformations was one of the main research topics in Japanese and Soviet (mostly Odessa and Kazan) differential geometry schools. For a collection of results of these times, see for example the survey [34] with more than 150 references. Moreover, two classical books [42, 46] contain chapters on c-projectively equivalent metrics and connections.

Relatively recently c-projective equivalence was re-introduced, under different names and because of different motivation. In fact, c-projectively equivalent metrics are essentially the same as Hamiltonian 2-forms, defined and investigated in Apostolov *et al.* [1, 2, 3, 4] for positive definite metrics, see also [17]. Though the definition of Hamiltonian 2-forms is visually different from that of c-projectively equivalent metrics, the defining equation [1, equation (12)] of a Hamiltonian 2-form is algebraically equivalent to a reformulation (see (1.4) below) of the condition "$\hat g$ is c-projectively equivalent to g" into the language of PDE. The motivation of Apostolov *et al.* to study Hamiltonian 2-forms is different from that of Otsuki and Tashiro. Roughly speaking, in [1, 2] Apostolov *et al.* observe that many interesting problems in Kähler geometry lead to Hamiltonian 2-forms and suggest studying them. The motivation is justified in [3, 4], where the authors indeed construct interesting and useful examples of Kähler manifolds. In dimension ≥ 6, c-projectively equivalent metrics are also essentially the same as Hermitian conformal Killing (or twistor) $(1, 1)$-forms studied in [35, 40, 41], see [1, Appendix A] or [33, §1.3] for details. Finally, such metrics are closely related to the so-called Kähler-Liouville integrable systems of type *A* introduced by Kiyohara in [24], see also [25].

We would also like to mention a recent review on c-projective geometry [16] that contains many new and old results in this area. Let us however make clear that our paper is totally independent of [16]. The work on these two papers was carried out more or less simultaneously and the second author of the present paper, being also one of the authors of [16], was quite careful about possible intersections between them. The only exception is Lemma 2.2 for which we suggest an alternative proof, for more details see discussion just after Lemma 2.2. It is worth noticing that the present paper and paper [16] represent two rather different approaches in c-projective geometry. Our approach is based on the reduction to the *real* projective setting, we explain it in §1.2, whereas [16] studies c-projectively equivalent metrics using ideas and methods of *parabolic* geometry.

Our paper contains three main results. The first result is a local description (near a generic point) of c-projectively equivalent Kähler metrics of arbitrary signature, see Example 5 and Theorem 1.6. If g is positive definite, such a description follows from the local description of Hamiltonian 2-forms due to Apostolov *et al.* [1]. Although the precise statements are slightly lengthy, we indeed provide an explicit description of the components of the metrics and of the Kähler form $\omega = g(J\cdot,\cdot)$. The parameters in this description are almost arbitrary numbers and functions of one variable and, in certain cases, almost arbitrary affinely equivalent Kähler metrics of smaller dimension (note that the description of affinely equivalent Kähler metrics was recently obtained by Boubel in [14]).

It is hard to overestimate the future role of a local description in the local and global theory of c-projectively equivalent metrics. Almost all known local results can easily be proved using it. Roughly speaking, using the local description, one can reduce any problem that can be stated using geometric PDEs (for example, any problem involving the curvature) to the analysis of a system of ODEs. As we mentioned above, in the positive definite case, the description of c-projectively equivalent metrics in the language of Hamiltonian 2-forms is due to Apostolov *et al.* [1], and with the help of such a description they did a lot. In particular they described possible topologies of closed manifolds admitting c-projectively equivalent Kähler metrics, described Bochner-flat Kähler metrics and constructed new examples of Einstein and extremal Kähler metrics on closed manifolds, see [1, 2, 3, 4]. We expect similar applications of our description and some have been already obtained, e.g., in [13] the local description of c-projectively equivalent metrics has been used to describe all Bochner-flat (pseudo-)Kähler metrics, generalizing results of [1] and [15] to the case of arbitrary signature. We plan to look for other applications and in particular to study the topology of c-projectively equivalent closed Kähler manifolds of arbitrary signature in further papers.

A demonstration of the importance of the local description is our second main result, which is a proof of the natural generalization of the Yano-Obata conjecture for Kähler manifolds of arbitrary signature. A vector field on a Kähler manifold is called *c-projective* if its local flow sends J-planar curves to J-planar curves, and *affine* if its local flow preserves the Levi-Civita connection.

THEOREM 1.1 (Yano-Obata conjecture). – *Let* (M, g, J) *be a closed connected Kähler manifold of arbitrary signature and of real dimension* $2n \geq 4$ *such that it admits a c-projective* *vector field that is not an affine vector field. Then the manifold is isometric to* $(\mathbb{C}P^n, c \cdot g_{\mathrm{FS}}, J_{\mathrm{standard}})$ *for some non-zero constant* c*, where* g_{FS} *is the Fubini-Study metric.*

For positive definite metrics, Theorem 1.1 was proved in [32], where also a history including a list of previously proven special cases can be found. Generalizations of [32] to the case of complete positive definite metrics is in [16, Theorem 7.6] and [31, Theorem 1.2]. The 4-dimensional version of Theorem 1.1 was proved in [11].

We see that a closed Kähler manifold with a non-affine c-projective vector field has definite signature. This phenomenon is, of course, essentially global since locally we can construct counterexamples in any signature. In dimension 4, such examples are described in [11], and in Proposition 5.7 we explicitly construct Kähler metrics of any dimension and any signature admitting non-affine c-projective vector fields. Let us also mention (see, e.g., [32, Example 2]) that $(\mathbb{C}P^n, c \cdot g_{\mathrm{FS}}, J_{\mathrm{standard}})$ admits many non-affine c-projective vector fields.

As a by-product of our proof of the Yano-Obata conjecture (we explain in the next section why it is a by-product), we establish the possibly more popular *projective Lichnerowicz* conjecture for metrics of Lorentzian signature. Recall that a vector field is *projective* with respect to a (arbitrary, not necessarily Kähler) metric g, if its local flow sends geodesics viewed as unparameterised curves to geodesics.

THEOREM 1.2 (Projective Lichnerowicz conjecture for metrics of Lorentzian signature). *Let* (M, g) *be a closed connected Lorentzian manifold of dimension* $n \geq 2$*. Then any projective vector field on* M *is an affine vector field.*

For Riemannian metrics, the analogue of Theorem 1.2 was proved in [27] (dimension 2) and [28] (dimension greater that 2—this paper also contains a historical overview and a list of previously known special cases), see also [48]. In Japanese mathematics, this statement, at least in the Riemannian setting, is also known as *projective Obata conjecture* and was published many times as an important conjecture, see introduction to [28] for details and precise references. For 2-dimensional Lorentzian manifolds, Theorem 1.2 was proved in [29].

We would like to emphasize here that our proofs of the Yano-Obata and Lichnerowicz conjectures are not generalizations of the proofs from [16, 27, 28, 32, 48], and are based on a different circle of ideas. In general, it is difficult to extend global statements about Riemannian metrics to the pseudo-Riemannian setting, since many "global" methods require definiteness of the metrics. This is also the case in our situation; the main ingredients of the proofs of [27, 28, 32, 48, 16, 31] are the global ordering of the eigenvalues of the endomorphisms A and L (given by (1.2) and (1.6)—these endomorphisms play an important role in our paper), and an investigation of the behavior of curvature invariants (scalar curvature in [28], holomorphic sectional curvature in [32], norm of the projective and c-projective Weyl tensors in [16, 31, 48]) along the orbits of the group of projective and c-projective transformations. None of these ingredients exists in the case of indefinite signature. Examples show that in the pseudo-Kähler case the eigenvalues of A (resp. L) are not globally ordered anymore, holomorphic sectional curvature is usually unbounded even on closed manifolds, and vanishing of the norm of a tensor does not imply that the tensor is zero. Moreover, as follows from our calculations in §5.6, in the indefinite case, all curvature invariants along integral curves of projective and c-projective vector fields can be bounded. In Remark 5.1 we give more details on what ideas of [28, 32, 48, 16, 31] were used in our paper, and also on some new methods developed here. Other proofs of special cases of the Yano-Obata and Lichnerowicz conjecture (see e.g., [45, 47]) are based on the Bochner technique, which also requires that the metric is definite.

1.2. Main idea

The local description of c-projectively equivalent metrics will be given in Theorem 1.6 in §1.3 (which does not require this paragraph so a hurried reader can directly go there). The goal of this section is to explain the main idea of our solution. We hope that this allows the reader to see the geometry behind the formulas and also may be used in many other problems related to c-projectively equivalent metrics.

Experts always expected that projectively equivalent metrics must have a close relation with c-projectively equivalent metrics. The expectation is based on the following informal observation: most mathematicians that studied c-projectively equivalent metrics and c-projective vector fields studied projectively equivalent metrics and projective vector fields before. It appears that many ideas and many results in the theory of projectively equivalent metrics have their counterparts in the c-projective setting, though most of the proofs in the c-projective setting are longer and are more involved than their projective analogues. (1)

We suggest an explanation about why the theories are closely related, which is simultaneously the main idea of our description. The following observation, which we formalize (and give a self-contained proof) in §2, is crucial: c-projectively, but not affinely equivalent metrics g and $\hat{g}$ allow us to construct vector fields $K_1, \dots, K_\ell$ which preserve the complex structure and which are Killing with respect to both metrics. For Hamiltonian 2-forms (at least for a positive definite metric), the existence of these Killing vector fields was shown by Apostolov *et al.* [1], and in the framework of Kähler-Liouville manifolds (under certain nondegeneracy assumptions) their existence was observed by Kiyohara and Topalov [25].

We consider the local action of these vector fields and the local quotient Q of M with respect to this action (it will be shown that such a quotient is well-defined near a generic point). Let us denote the quotient metrics by g_Q and $\hat{g}_Q$. Notice that Q is not a Kähler quotient and the metrics g_Q and $\hat{g}_Q$ are in general not Kähler.

Main Observation. The following statements hold:

1. g_Q and $\hat{g}_Q$ are projectively equivalent;
2. the metrics g and $\hat{g}$ can be reconstructed from g_Q and $\hat{g}_Q$ in a relatively straightforward way.

Recently, projectively equivalent metrics have been explicitly locally described in [10]. We obtain our description of c-projectively equivalent metrics by taking the formulas from [10] for the quotient metrics g_Q and $\hat{g}_Q$ and then "reconstructing" g and $\hat{g}$.

(1) This analogy between c-projective and projective geometry fails at the level of affine connections (note that the definition of c-projective equivalence makes also sense for affine connections which are not necessarily Levi-Civita connections): though both affine projective and affine c-projective geometries are parabolic geometries, there are essential differences between these theories if only connections are involved, see e.g., [16].

However, not every pair of projectively equivalent metrics g_Q, $\hat{g}_Q$ as considered in [10] can be obtained from a pair g, $\hat{g}$ of c-projectively equivalent metrics: we will describe the conditions that g_Q and $\hat{g}_Q$ have to satisfy in order to arise as quotients from c-projectively equivalent metrics. These additional conditions actually simplify the formulas for the metrics g_Q, $\hat{g}_Q$ as compared to the formulas from [10] for the general case. Moreover, we show, assuming these conditions are satisfied, how to effectively reconstruct the initial metrics g and $\hat{g}$. This yields our description of c-projectively equivalent Kähler metrics.

The relation between projectively and c-projectively equivalent metrics plays also an important role in the proof of the Yano-Obata conjecture. We will see that under the additional assumption that the degree of mobility is 2 (which means that the "space of c-projectively equivalent metrics" is two-dimensional—the formal definition is in §5.1 where it is also explained why it is the most non-trivial case in the proof of the Yano-Obata conjecture), a c-projective vector field on the initial manifold reduces to a projective vector field on the quotient.

We expect further applications of this observation which suggests, in the metric setting, an almost algorithmic way to produce results in c-projective geometry from results in projective geometry and the latter is much better developed.

Unfortunately, this almost algorithmic way does not automatically work in the other (c-projective $\to$ projective) direction. The reason is that the quotient metrics g_Q and $\hat{g}_Q$, as already noticed, satisfy certain additional conditions. The most important of them is as follows: for the metrics $h = g_Q$ and $\hat{h} = \hat{g}_Q$ the endomorphism L given by (1.6) below has no Jordan blocks with non-constant eigenvalues. For general projectively equivalent metrics, L may have non-trivial Jordan blocks with non-constant eigevalues. This is the only reason why we can not modify the proof of the Yano-Obata conjecture to obtain the proof of the projective Lichnerowicz conjectures *for metrics of all signatures*. For the metrics *of Lorentzian signature*, at most one non-trivial Jordan block may occur and after some additional work in §6 we exclude this case in the proof of the projective Lichnerowicz conjecture. The rest of the proof of the projective Lichnerowicz conjecture is a straightforward modification (actually, a simplification) of the proof of the Yano-Obata conjecture and when proving the projective Lichnerowicz conjecture in the "no-Jordan-blocks" case (Theorem 5.1), we confine ourselves with a series of remarks explaining necessary amendments.

1.3. Local description of c-projectively equivalent metrics

Let (M, g, J) be a Kähler manifold of real dimension $2n \geq 4$ and let ∇ and $\omega = g(J\cdot,\cdot)$ denote the Levi-Civita connection and Kähler form respectively. We do not require that g or any other Kähler metric that appears has positive signature.

Instead of the pair $(g, \hat{g})$ of c-projectively equivalent metrics it is appropriate to consider the pair (g, A), where $A : TM \to TM$ is a Hermitian (i.e., g-selfadjoint and J-commuting) endomorphism constructed from g and $\hat{g}$ by

$$A = A(g, \hat{g}) = \left(\frac{\det \hat{g}}{\det g}\right)^{\frac{1}{2(n+1)}} \hat{g}^{-1} g. \tag{1.2}$$

In this formula, we view $g, \hat{g} : TM \to T^*M$ as bundle isomorphisms. In tensor notation (with summation convention in force),

$$A^i_j = \left(\frac{\det \hat{g}}{\det g}\right)^{\frac{1}{2(n+1)}} \hat{g}^{ik} g_{kj},$$

where $\hat{g}^{ij}$ denotes the inverse to $\hat{g}_{ij}$, i.e., $\hat{g}^{ik}\hat{g}_{kj} = \delta^i_j$.

Clearly, one can reconstruct $\hat{g}$ from the pair (g, A) and obtains

$$\hat{g} = (\det A)^{-\frac{1}{2}} g(A^{-1}\cdot, \cdot). \tag{1.3}$$

The endomorphism A, introduced in [19], plays an important role in the theory of c-projectively equivalent metrics. One of the reasons for this is that the condition that g and $\hat{g}$ are c-projectively equivalent amounts to the fact that the tensor A satisfies the *linear* partial differential equation

$$\nabla_X A = X^\flat \otimes \Lambda + \Lambda^\flat \otimes X + (JX)^\flat \otimes J\Lambda + (J\Lambda)^\flat \otimes JX, \tag{1.4}$$

for all $X \in TM$, where $\Lambda = \frac{1}{4}\mathrm{grad}(\mathrm{tr}\, A)$ and $X^\flat = g(X, \cdot)$. We say that g and A are *compatible in the c-projective sense* or just *c-compatible* if A is a Hermitian endomorphism solving (1.4). In particular, any Hermitian endomorphism A with nowhere vanishing determinant and c-compatible with g gives us a c-projectively equivalent metric $\hat{g}$ by (1.3), this metric is automatically Kähler with respect to J.

Another reason for working with A instead of $\hat{g}$ is that in our local description, the formulas for (g, A) are much simpler than those for $(g, \hat{g})$.

We describe locally all Kähler structures (g, J, ω) admitting solutions A to (1.4) in Theorem 1.6 below. Since the description is relatively complicated, we first consider two special cases corresponding to the "weakest" (Theorem 1.3) and "strongest" (Theorem 1.4) case of c-projective equivalence.

Note that any parallel Hermitian endomorphism A (i.e., satisfying $\nabla A = 0$), in particular the identity $\mathrm{Id} : TM \to TM$, is a solution to (1.4). Such solutions correspond to Kähler metrics $\hat{g}$ which are affinely equivalent to g, i.e., which have the same Levi-Civita connection as g.

THEOREM 1.3 (Well-known special case of Theorem 1.6). – *Let (M, g, J) be a Kähler manifold of arbitrary signature and $A : TM \to TM$ a parallel Hermitian endomorphism. Then locally (M, g, J) is a direct product of Kähler manifolds $(M_\gamma, g_\gamma, J_\gamma)$, $\gamma = 1, \ldots, N$, and A decomposes as $A = A_1 + \cdots + A_N$, where $A_\gamma : TM_\gamma \to TM_\gamma$ is a parallel Hermitian endomorphism on $(M_\gamma, g_\gamma, J_\gamma)$ having either a single real eigenvalue c_γ or a pair of complex-conjugate eigenvalues $c_\gamma, \bar{c}_\gamma$.*

The above theorem is just the de Rham–Wu decomposition [39, 44] of the Kähler manifold into components corresponding to the parallel distributions given by the generalized eigenspaces of A. This is not a complete description of pairs $((g, J), A)$, where (g, J) is Kähler and A is a parallel Hermitian endomorphism: what is left is an explicit description of the blocks (g_γ, J_γ) and A_γ. In the positive definite case, the description of these blocks is trivial since in this case A_γ is a constant multiple of $\mathrm{Id} : TM_\gamma \to TM_\gamma$. If the signature of g_γ is arbitrary, the local description of (g_γ, J_γ) and A_γ has recently been obtained by C.

Boubel in [14]. Boubel's description of $(g_\gamma, J_\gamma, A_\gamma)$ is quite complicated, we will not repeat it here and refer to [14] for more details.

Remark 1.1. – Let us reformulate the statement from Theorem 1.3 in matrix notation: we can find local coordinates such that the matrices of g, J and A in these coordinates are block-diagonal with the same structure of blocks:

$$g = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_N \end{pmatrix}, \quad J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_N \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix}. \tag{1.5}$$

In all matrices, the components of each block only depend on the corresponding coordinates and for each $\gamma = 1, \dots, N$ the endomorphism A_γ is Hermitian and parallel w.r.t. the Kähler structure (g_γ, J_γ).

The "main idea" and "main observation" described in §1.2 become vacuous in the setting of Theorem 1.3: the number of "canonical" Killing vector fields $K_1, \dots, K_\ell$ is zero, hence, the quotient of the manifold is the manifold itself. The "main observation" remains, of course, formally true but in this case projective equivalence is affine equivalence.

A special feature of the situation described in Theorem 1.3 is that the eigenvalues of A are constant, and may have high multiplicities. Let us now consider the "strongest" special case of c-projective equivalence: all eigenvalues of A are non-constant (when considered as functions on M), and their multiplicity is minimal possible.

Consider two projectively equivalent pseudo-Riemannian metrics h and $\hat h$ (i.e., metrics having the same unparametrised geodesics) and define the endomorphism L by

$$L = L(h, \hat h) = \left| \frac{\det \hat h}{\det h} \right|^{\frac{1}{n+1}} \hat h^{-1} h. \tag{1.6}$$

It is well known that L satisfies the equation

$$\nabla_X L = X^\flat \otimes \Lambda + \Lambda^\flat \otimes X, \quad \text{for all } X \in TM, \tag{1.7}$$

where $\Lambda = \frac{1}{2}\mathrm{grad}(\mathrm{tr}\, L)$, $X^\flat = h(X, \cdot)$ and ∇ denotes the Levi-Civita connection of h. Moreover, if L is h-selfadjoint and non-degenerate, then (1.7) is equivalent to the projective equivalence of h and $\hat h$ given by

$$\hat h = |\det L|^{-1} h(L^{-1}\cdot, \cdot) \tag{1.8}$$

see [42] and e.g., [9]. To emphasize both the difference and similarity with c-compatibility introduced above, we will say that h and an h-selfadjoint endomorphism L satisfying (1.7) are *compatible in the projective sense* or just *compatible*.

Example 1. – Assume that on a certain domain $U \subset \mathbb{R}^\ell$ we have a compatible pair h and L for which the following conditions hold:

A1. The eigenvalues $\rho_1, \dots, \rho_\ell$ of L are all distinct at each point of U (complex conjugate pairs ρ, $\bar\rho$ with $\mathrm{Im}\,\rho \neq 0$ are allowed too), which allows us to view them as smooth functions on U;

A2. $\mathrm{d}\rho_i \neq 0$ at each point of U, $i = 1, \dots, \ell$.

We now explain how, starting from such a compatible pair

$$h = \sum_{i,j=1}^{\ell} B_{ij}(x)\mathrm{d}x_i\mathrm{d}x_j \quad \text{and} \quad L = \sum_{i,j=1}^{\ell} L^i_j \mathrm{d}x_j \otimes \frac{\partial}{\partial x_i}, \tag{1.9}$$

one can naturally construct a c-compatible pair, i.e., a Kähler structure (g, J, ω) and a Hermitian endomorphism A satisfying (1.4). By $\mu_1, \dots, \mu_\ell$ we denote the elementary symmetric polynomials in $\rho_1, \dots, \rho_\ell$ (i.e., $(\tau + \rho_1)\cdots(\tau + \rho_\ell) = \tau^\ell + \mu_1\tau^{\ell-1} + \cdots + \mu^\ell$). Notice that under the assumption that ρ_i are all distinct and $\mathrm{d}\rho_i \neq 0$, the differentials of ρ_i are linearly independent, i.e., $\mathrm{d}\rho_1 \wedge \mathrm{d}\rho_2 \wedge \cdots \wedge \mathrm{d}\rho_\ell \neq 0$. This follows from [9, Theorems 2, 3] and was essentially known to Levi-Civita [26]. Thus, both systems of functions ρ_i's and μ_i's can be considered as local coordinates on U.

Consider a domain $V \subset \mathbb{R}^\ell$ with local coordinates $t_1, \dots, t_\ell$ and define g, ω on $V \times U$ in the following way

$$g = \sum_{\alpha,\beta=1}^{\ell} H_{\alpha\beta}(x)\mathrm{d}t_\alpha\mathrm{d}t_\beta + \sum_{i,j=1}^{\ell} B_{ij}(x)\mathrm{d}x_i\mathrm{d}x_j, \qquad \omega = \sum_{\alpha=1}^{\ell} \mathrm{d}\mu_\alpha \wedge \mathrm{d}t_\alpha, \tag{1.10}$$

where $H_{\alpha\beta} = \sum_{ij} B^{ij}(x)\frac{\partial\mu_\alpha}{\partial x_i}\frac{\partial\mu_\beta}{\partial x_j}$ and B^{ij} are the components of the matrix inverse to B_{ij}, i.e., $\sum_k B_{ik}B^{kj} = \delta^j_i$. We also set

$$A = \sum_{\alpha,\beta=1}^{\ell} M^\beta_\alpha(x)\mathrm{d}t_\beta \otimes \frac{\partial}{\partial t_\alpha} + \sum_{i,j=1}^{\ell} L^i_j(x)\mathrm{d}x_j \otimes \frac{\partial}{\partial x_i}, \tag{1.11}$$

where $M^\beta_\alpha = \sum_{i,j} L^i_j \frac{\partial\mu_\beta}{\partial x_i}\frac{\partial x_j}{\partial\mu_\alpha}$.

Equivalently, in matrix form w.r.t. the coordinates $t_1, \dots, t_\ell, x_1, \dots, x_\ell$, the above expressions take the form

$$g = \begin{pmatrix} Ph^{-1}P^\top & 0 \\ 0 & h \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & -P \\ P^\top & 0 \end{pmatrix}, \quad A = \begin{pmatrix} (PLP^{-1})^\top & 0 \\ 0 & L \end{pmatrix} \tag{1.12}$$

where $P = \left(\frac{\partial\mu_\alpha}{\partial x_i}\right)$ is the Jacobi matrix of the system of functions $\mu_1, \dots, \mu_\ell$ (w.r.t. the local coordinates $x_1, \dots, x_\ell$).

The following theorem, which describes c-projectively equivalent metrics under the assumption that $A(g, \hat{g})$ has the maximal number of non-constant eigenvalues, shows that in this case the relation between projective equivalence and c-projective equivalence is rather straightforward.

THEOREM 1.4. – *Let (h, L) be a compatible pair on U satisfying* A1 *and* A2*. Then the above Formulas* (1.10) *and* (1.11) *(or equivalently* (1.12) *in matrix form) define a Kähler structure (g, ω) and a Hermitian endomorphism A which are c-compatible, i.e., satisfy* (1.4)*. Conversely, if a Kähler structure (g, ω) and a Hermitian endomorphism A are c-compatible and the eigenvalues of A (as a complex endomorphism) satisfy* A1 *and* A2 *in the neighborhood of some point, then locally, in the neighborhood of this point, g, ω and A can be written in the form* (1.10) *and* (1.11)*, where $h = \sum_{i,j} B_{ij}(x)\mathrm{d}x_i\mathrm{d}x_j$ and $L = \sum_{i,j} L^i_j\mathrm{d}x_j \otimes \partial_{x_i}$ are compatible.*

Example 2. – The simplest example of the situation described in Theorem 1.4 is obtained by starting with a 2-dimensional compatible pair (h, L) such that L has two real non-constant eigenvalues ρ, σ satisfying A1 and A2. The description of such a pair is due to Dini [18], see also [10]: locally, we find coordinates x, y such that $\rho = \rho(x)$ and $\sigma = \sigma(y)$ and

$$h = (\rho - \sigma)(\mathrm{d}x^2 \pm \mathrm{d}y^2), \quad L = \rho \mathrm{d}x \otimes \frac{\partial}{\partial x} + \sigma \mathrm{d}y \otimes \frac{\partial}{\partial y}.$$

Applying Theorem 1.4 to these formulas, we obtain the formulas for the Kähler structure (g, ω) and the c-compatible endomorphism A. These formulas can be found in [11, (3.1) and (3.2)].

We see that in the situation of Theorem 1.4, the entries of g, ω and A do not depend on the coordinates $t_1, \dots, t_\ell$. This implies that $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_\ell}$ are J-preserving Killing vector fields, and they are precisely the Killing vector fields $K_1, \dots, K_\ell$ which we mentioned in §1.2. The quotient with respect to the local action of these vector fields is n-dimensional with local coordinates $x_1, \dots, x_\ell$, and the metric g descends to the metric $g_Q = h$ on the quotient. As claimed in the “main observation” of §1.2, g_Q admits a projectively equivalent metric $\hat{g}_Q$ defined by the endomorphism L which also can be treated as the quotient $L = A_Q$ of the Hermitian endomorphism A.

In the next example and theorem, we present the most general local expression which a Kähler structure (g, ω) together with a solution A to Equation (1.4) can take. The construction below combines the previous two cases from Theorem 1.3 and Example 1.

Example 3. – We start with two ingredients:

— a compatible pair h and L defined on a domain $U \subset \mathbb{R}^\ell$ and satisfying the conditions A1 and A2 as in Example 1, see (1.9);

— a Kähler structure $(g_{\mathrm{c}}, \omega_{\mathrm{c}})$ defined on some domain S with a parallel Hermitian endomorphism A_{c} (notice that the eigenvalues of A_{c} are constant).

In addition, we assume that the eigenvalues of L at each point $p \in U$ are all different from those of A_{c}.

Consider the direct product $V \times U \times S$, where $V \subset \mathbb{R}^\ell$ is a certain domain of the same dimension ℓ as U, and denote local coordinates on V, U and S by $(t_1, \dots, t_\ell)$, $(x_1, \dots, x_\ell)$ and $(y_1, \dots, y_{2k})$ respectively. On this product $V \times U \times S$, we now define a pseudo-Riemannian metric g and a 2-form ω:

$$\begin{aligned} g &= \sum_{\alpha,\beta=1}^{\ell} H_{\alpha\beta}(x)\theta_\alpha\theta_\beta + \sum_{i,j=1}^{\ell} B_{ij}(x)\mathrm{d}x_i\mathrm{d}x_j + g_{\mathrm{c}}\big(\chi_L(A_{\mathrm{c}})\cdot,\cdot\big), \\ \omega &= \sum_{\alpha=1}^{\ell} \mathrm{d}\mu_\alpha \wedge \theta_\alpha + \omega_{\mathrm{c}}\big(\chi_L(A_{\mathrm{c}})\cdot,\cdot\big), \end{aligned} \tag{1.13}$$

where $\chi_L(t) = \det(t{\cdot}\mathrm{Id} - L)$ is the characteristic polynomial of L, $\theta_i = \mathrm{d}t_i + \alpha_i$ and the 1-forms α_i on S are chosen in such a way that $\mathrm{d}\alpha_i = (-1)^i \omega_{\mathrm{c}}(A_{\mathrm{c}}^{\ell-i}\cdot,\cdot)$ (which is possible since $\omega_{\mathrm{c}}(A_{\mathrm{c}}^{\ell-i}\cdot,\cdot)$ is a parallel 2-form on S). The other ingredients, $H_{\alpha\beta}$ and μ_i, are defined as above in Example 1 and in addition we set $\mu_0 = 1$.

Further we define the endomorphism
(1.14)
$$A=\sum_{\alpha,\beta=1}^{\ell} M_\alpha^\beta(x)\,\theta_\beta\otimes\frac{\partial}{\partial t_\alpha}+\sum_{i,j=1}^{\ell} L_j^i(x)\,\mathrm{d}x_j\otimes\frac{\partial}{\partial x_i}+\sum_{p,q=1}^{2k}(A_{\mathrm{c}})_p^q\,\mathrm{d}y_p\otimes\left(\frac{\partial}{\partial y_q}-\sum_{i=1}^{\ell}\alpha_{iq}\frac{\partial}{\partial t_i}\right),$$

where $M_\alpha^\beta=\sum_{i,j}L_j^i\frac{\partial\mu_\beta}{\partial x_i}\frac{\partial x_j}{\partial\mu_\alpha}$ and α_{iq} resp. $(A_{\mathrm{c}})_p^q$ denote the components of α_i resp. A_{c} w.r.t. the coordinates $y_1,\dots,y_{2k}$, i.e., $\alpha_i=\sum_q\alpha_{iq}\mathrm{d}y_q$ and $A_{\mathrm{c}}=\sum_{p,q}(A_{\mathrm{c}})_p^q\,\mathrm{d}y_p\otimes\partial_{y_q}$.

Equivalently, in matrix form (w.r.t. the basis $\theta_1,\dots,\theta_\ell,\mathrm{d}x_1,\dots,\mathrm{d}x_\ell,\mathrm{d}y_1,\dots,\mathrm{d}y_{2k}$), the above formulas take the form:

$$g=\begin{pmatrix}Ph^{-1}P^\top & 0 & 0\\ 0 & h & 0\\ 0 & 0 & g_{\mathrm{c}}\cdot\chi_L(A_{\mathrm{c}})\end{pmatrix},\quad \omega=\begin{pmatrix}0 & -P & 0\\ P^\top & 0 & 0\\ 0 & 0 & \omega_{\mathrm{c}}\cdot\chi_L(A_{\mathrm{c}})\end{pmatrix},\quad A=\begin{pmatrix}(PLP^{-1})^\top & 0 & 0\\ 0 & L & 0\\ 0 & 0 & A_{\mathrm{c}}\end{pmatrix},$$

where $P=\left(\frac{\partial\mu_\alpha}{\partial x_i}\right)$ is the Jacobi matrix of the system of functions $\mu_1,\dots,\mu_\ell$ (w.r.t. the local coordinates $x_1,\dots,x_\ell$).

Remark 1.2. – Each of the 1-forms α_i on S is determined by $(\omega_{\mathrm{c}},A_{\mathrm{c}})$ up to adding the differential of a function. However, replacing θ_i by the 1-forms $\tilde\theta_i=\theta_i+\mathrm{d}f_i$ in the formulas of Example 3 for functions f_i on S, it is easy to construct a local transformation $f:M\to M$ identifying the formulas in Example 3 written down w.r.t. θ_i and $\tilde\theta_i$ respectively, see also the discussion after Proposition 4.3 below.

We will call a point $p\in M$ *regular* with respect to a solution A of (1.4), if in the neighborhood of this point the number of different eigenvalues of A is constant (which implies that the eigenvalues are smooth functions in some neighborhood of p), and for each eigenvalue ρ either $\mathrm{d}\rho\neq 0$, or ρ is constant in the neighborhood of p. Clearly, the set M^0 of regular points is open and dense in M. Further (see Lemma 2.2 (4) below) we will see that the number of non-constant eigenvalues of A is the same near every regular point. The following theorem generalizes Theorems 1.3 and 1.4:

Theorem 1.5. – *The metric g and 2-form ω defined by* (1.13) *are a Kähler structure and A defined by* (1.14) *is a Hermitian solution of* (1.4).

Conversely, let (M,g,ω) be a Kähler manifold of arbitrary signature and A be a Hermitian solution of (1.4). *Then in the neighborhood of a regular point, the Kähler structure (g,ω) and the endomorphism A can be written in the form* (1.13) *and* (1.14) *from Example* 3.

Example 4. – The simplest example of the situation described in Theorem 1.5 is obtained by starting with a 1-dimensional compatible pair $h=\mathrm{d}x^2$, $L=\rho\mathrm{d}x\otimes\partial_x$ for a function $\rho=\rho(x)$ satisfying $\mathrm{d}\rho\neq 0$ and a 2-dimensional Kähler structure $(g_{\mathrm{c}},\omega_{\mathrm{c}})$ with parallel Hermitian endomorphism $A_{\mathrm{c}}=c\cdot\mathrm{Id}$ for a constant c. Applying Theorem 1.4 to these formulas, we obtain the formulas for the Kähler structure (g,ω) and the c-compatible endomorphism A given by [11, formulas (3.5) and (3.6)] (up to a slight change of notation).

Theorem 1.5 gives us a description of a c-compatible pair (g,ω) and A (at a generic point) provided we know a description of compatible pairs (h,L) and also of Kähler structures $(g_{\mathrm{c}},\omega_{\mathrm{c}})$ admitting a parallel Hermitian endomorphism A_{c}. As we already mentioned above, the latter have been described in [14]. The local normal forms for compatible pairs (h,L) have been obtained in [10] and this combined with Theorem 1.5 implies the local normal forms for a c-compatible pair (g,ω) and A, see Example 5 and Theorem 1.6 below. We also refer to [11, Theorem 3.1] for the formulas in the 4-dimensional case.

Example 5 (Main example). – Let $2n \geq 4$ and consider an open subset W of $\mathbb{R}^{2n}$ of the form $W = V \times U \times S_1 \times \cdots \times S_N$ for open subsets $V,U \subseteq \mathbb{R}^{\ell}$ and $S_\gamma \subseteq \mathbb{R}^{2m_\gamma}$. Let $t_1,\dots,t_\ell$ denote the coordinates on V and let the coordinates on U be separated into r complex coordinates $z_1,\dots,z_r$ and $q=\ell-2r$ real coordinates $x_{r+1},\dots,x_{r+q}$.

Suppose the following data is given on these open subsets:

— Kähler structures $(g_\gamma,J_\gamma,\omega_\gamma)$ on S_γ for $\gamma=1,\dots,N$.

— For each $\gamma=1,\dots,N$, a parallel Hermitian endomorphism $A_\gamma : TS_\gamma \to TS_\gamma$ for (g_γ,J_γ) having a pair of complex conjugate eigenvalues $c_\gamma,\bar{c}_\gamma \in \mathbb{C}\setminus\mathbb{R}$ for $\gamma=1,\dots,R$ and a single real eigenvalue $c_\gamma\in\mathbb{R}$ for $\gamma=R+1,\dots,N$ such that the algebraic multiplicity of c_γ equals $m_\gamma/2$ for $\gamma=1,\dots,R$ and m_γ for $\gamma=R+1,\dots,N$.

— Holomorphic functions $\rho_j(z_j)$ of z_j for $1\leq j\leq r$ and smooth functions $\rho_j(x_j)$ for $r+1\leq j\leq r+q$.

Moreover, we choose 1-forms $\alpha_1,\dots,\alpha_\ell$ on $S=S_1\times\cdots\times S_N$ which satisfy

$$\mathrm{d}\alpha_i = (-1)^i \sum_{\gamma=1}^{N} \omega_\gamma(A_\gamma^{\ell-i}\cdot,\cdot). \tag{1.15}$$

We introduce some notation to be used throughout the paper. The function Δ_i for $1\leq i\leq r+q$ is given by $\Delta_i=\prod_{\rho\in E_{\mathrm{nc}}\setminus\{\rho_i\}}(\rho_i-\rho)$, where $E_{\mathrm{nc}}=\{\rho_1,\bar{\rho}_1,\dots,\rho_r,\bar{\rho}_r,\rho_{r+1},\dots,\rho_{r+q}\}$. The 1-forms $\theta_1,\dots,\theta_\ell$ on W are defined by $\theta_i=\mathrm{d}t_i+\alpha_i$. The function μ_i denotes the ith elementary symmetric polynomial in the ℓ variables E_{nc}, $\mu_i(\hat{\rho}_s)$ denotes the ith elementary symmetric polynomial in the $\ell-1$ variables $E_{\mathrm{nc}}\setminus\{\rho_s\}$ and the notation "$c.c.$" refers to the conjugate complex of the preceding term.

Suppose that at every point of W the values of the functions $\rho_1,\bar{\rho}_1,\dots,\rho_{r+q}$ are mutually different and different from the constants $c_1,\bar{c}_1,\dots,c_N$ and their differentials are non-zero (which, as explained above, implies that they are linearly independent). Then (g,ω,J) given by the formulas

$$\begin{aligned} g = & -\frac{1}{4}\sum_{i=1}^{r}\big(\Delta_i \mathrm{d}z_i^2 + c.c.\big) + \sum_{i=r+1}^{r+q}\varepsilon_i\Delta_i\mathrm{d}x_i^2 + \sum_{i=0}^{\ell}(-1)^i\mu_i\sum_{\gamma=1}^{N} g_\gamma(A_\gamma^{\ell-i}\cdot,\cdot) \\ & + \sum_{i,j=1}^{\ell}\Big[-4\sum_{s=1}^{r}\left(\frac{\mu_{i-1}(\hat{\rho}_s)\mu_{j-1}(\hat{\rho}_s)}{\Delta_s}\left(\frac{\partial\rho_s}{\partial z_s}\right)^2 + c.c.\right) \\ & + \sum_{s=r+1}^{r+q}\varepsilon_s\frac{\mu_{i-1}(\hat{\rho}_s)\mu_{j-1}(\hat{\rho}_s)}{\Delta_s}\left(\frac{\partial\rho_s}{\partial x_s}\right)^2\Big]\theta_i\theta_j, \end{aligned} \tag{1.16}$$

$$\omega = \sum_{i=1}^{\ell} \mathrm{d}\mu_i \wedge \theta_i + \sum_{i=0}^{\ell}(-1)^i \mu_i \sum_{\gamma=1}^{N} \omega_\gamma(A_\gamma^{\ell-i}\cdot,\cdot),$$

$$\mathrm{d}z_i \circ J = 4\frac{1}{\Delta_i}\frac{\partial\rho_i}{\partial z_i}\sum_{j=1}^{\ell}\mu_{j-1}(\hat\rho_i)\theta_j, \quad \mathrm{d}x_i \circ J = -\frac{\varepsilon_i}{\Delta_i}\frac{\partial\rho_i}{\partial x_i}\sum_{j=1}^{\ell}\mu_{j-1}(\hat\rho_i)\theta_j,$$

$$\theta_i \circ J = \frac{(-1)^i}{4}\sum_{j=1}^{r}\rho_j^{\ell-i}\left(\frac{\partial\rho_j}{\partial z_j}\right)^{-1}\mathrm{d}z_j + c.c. + (-1)^{i-1}\sum_{j=r+1}^{r+q}\varepsilon_j\rho_j^{\ell-i}\left(\frac{\partial\rho_j}{\partial x_j}\right)^{-1}\mathrm{d}x_j \tag{1.17}$$

is Kähler, where $\varepsilon_i = \pm 1$ depending on the signature of g. Moreover, writing $\alpha_i = \sum_q \alpha_{iq}\mathrm{d}y_q$ and $A_\gamma = \sum_{p,q}(A_\gamma)_p^q\,\mathrm{d}y_p \otimes \partial_{y_q}$ w.r.t. local coordinates $y_1,\dots,y_{2k}$ on $S = \prod_\gamma S_\gamma$, we have that the endomorphism A given by

(1.18)

$$\begin{aligned} A = {} & \sum_{i,j=1}^{\ell}\left(\mu_i\delta_{1j} - \delta_{i(j-1)}\right)\theta_i \otimes \frac{\partial}{\partial t_j} + \sum_{s=1}^{r}(\rho_s\mathrm{d}z_s \otimes \frac{\partial}{\partial z_s} + c.c.) + \sum_{s=r+1}^{r+q}\rho_s\mathrm{d}x_s \otimes \frac{\partial}{\partial x_s} \\ & + \sum_{\gamma=1}^{N}\sum_{p,q=1}^{2k}(A_\gamma)_p^q\,\mathrm{d}y_p \otimes \left(\frac{\partial}{\partial y_q} - \sum_{i=1}^{\ell}\alpha_{iq}\frac{\partial}{\partial t_i}\right) \end{aligned}$$

is a Hermitian solution to (1.4).

Example 5 is an explicit construction of a Kähler metric (1.16) and a solution (1.18) of (1.4). This fact can be verified by a straightforward, though non-trivial computation. Another proof will be given in Sections 4.2 and 4.3. The next theorem shows that in the neighborhood of a generic point, a Kähler metric g (of any signature) and a solution A of (1.4) are, in a certain coordinate system, as in Example 5.

THEOREM 1.6 (Local description of c-projectively equivalent metrics).

Let (M,g,J) be a Kähler manifold of arbitrary signature and A be a Hermitian solution of (1.4). *If in a small neighborhood $W \subseteq M^0$ of a regular point, A has*

- *$\ell = 2r + q$ non-constant eigenvalues on W which separate into r pairs of complex-conjugate eigenvalues $\rho_1,\bar\rho_1,\dots,\rho_r,\bar\rho_r : W \to \mathbb{C}$ and q real eigenvalues $\rho_{r+1},\dots,\rho_{r+q} : W \to \mathbb{R}$,*
- *$N + R$ constant eigenvalues which separate into R pairs of complex conjugate eigenvalues $c_1,\bar c_1,\dots,c_R,\bar c_R$ and $N - R$ real eigenvalues $c_{R+1},\dots,c_N$,*

then the Kähler structure (g,J,ω) and A are given on W by the Formulas (1.16) *and* (1.18) *from Example 5.*

REMARK 1.3. – As stated above, the corresponding local description of a positive definite Kähler structure (g,J,ω) admitting a Hermitian solution A of (1.4) has been obtained in [1] in the language of Hamiltonian 2-forms.

REMARK 1.4. – As mentioned above, Theorem 1.6 yields an "almost" explicit description (in the neighborhood of a regular point) of a Kähler metric g admitting a c-projectively equivalent metric. What is not described explicitly are the Kähler structures $(g_\gamma, \omega_\gamma)$ that admit parallel Hermitian endomorphisms A_γ. The formulas for such a triple $(g_\gamma, \omega_\gamma, A_\gamma)$ in local coordinates can be found in [14].

REMARK 1.5. – As explained above, for positive definite metrics the local classification was obtained in [1]. The main reason why the proofs from [1] cannot be generalized to metrics of arbitrary signature is rather simple. Many calculations and arguments in [1] use the frame in which both metrics are simultaneously diagonal. This is impossible in the pseudo-Kähler case, since self-adjoint operators in pseudo-Hermitian vector spaces are not necessarily semisimple. The above examples demonstrate that this phenomenon effectively shows up.

Recall that even the simplest cases of such a situation are nontrivial. Indeed, in the Riemannian signature affinely equivalent metrics locally split into a direct product of proportional metrics (with constant coefficient on each factor) and were completely understood by Cartan and Eisenhart 100 years ago. For arbitrary signature, affinely equivalent metrics have been described only very recently, in [14]. Similarly, in the Riemannian signature, projectively equivalent metrics were described already by Levi-Civita in 1896. The case of arbitrary signature is much more complicated and has been solved only recently in [10, 9].

Note that in view of the discussion in §1.2, a local description of c-projectively equivalent metrics "includes" (i.e., essentially implies) the results of [14] and [9]. A straightforward attempt to generalize the proofs from [1] to an arbitrary signature would make it necessary to re-obtain, in a different language, the main results of [14] and [9]. Note also that in [15] (which in fact studies c-projectively equivalent metrics with special properties) it was explicitly pointed out that the case of arbitrary signature is essentially more complicated due to certain algebraic difficulties.

1.4. Structure of the paper

In §2, we recall that the existence of a c-projectively equivalent metric $\hat{g}$ for a Kähler metric g implies the existence of a family of independent commuting Hamiltonian (w.r.t. the Kähler form ω) Killing vector fields $K_1, \dots, K_\ell$. These vector fields are also Killing w.r.t. $\hat{g}$.

We can form the quotient of M w.r.t. the local $\mathbb{R}^\ell$-action induced by these vector fields and obtain a bundle structure $M \to Q$ with fibers being the leaves of the foliation generated by the vector fields K_i. Since $g, \hat{g}$ are invariant w.r.t. the action of the vector fields K_i and the orthogonal complements to the fibers w.r.t. g and $\hat{g}$ coincide, they descend to metrics $g_Q, \hat{g}_Q$ on the quotient. This reduction will be explained in detail in §3. As we already mentioned, in §1.2, the crucial observation is that the metrics on the quotient are *projectively equivalent*. We prove this property in §3.2. More precisely, as explained in Example 1, instead of $\hat{g}_Q$ we consider the endomorphism A_Q obtained from g_Q and $\hat{g}_Q$ by (1.6) and check the compatibility condition for the pair g_Q, A_Q.

The local classification of pseudo-Riemannian projectively equivalent metrics, or equivalently, compatible pairs g_Q and A_Q has been derived in [10]. We apply these results in §4.1 to obtain the normal forms for g_Q, A_Q on the quotient. These normal forms are, in fact, simpler than the generic ones for projectively equivalent metrics: the tensor $A = A(g, \hat{g})$

from Theorem 1.6 has non-constant eigenvalues of (complex) algebraic multiplicity equal to one such that the corresponding tensor A_Q on the quotient has no non-trivial Jordan blocks corresponding to the non-constant eigenvalues. This makes the formulas from [10] much easier.

The requirement that g is Kähler and A is Hermitian implies that they are completely determined by the reduced objects g_Q and A_Q on the quotient. In §4.1 we derive the formulas for (g, ω) and A which are in essence equivalent to (1.13) and (1.14) (Proposition 4.3). The next step is to show that there are no further restrictions on g, ω and A so that the formulas from Proposition 4.3 give us a desired local description, see §4.2.

Finally in §4.3, we complete the proof of Theorem 1.6 by deriving the explicit Formulas (1.16) and (1.18) from Example 5.

The second part of the article contains the proof of Theorems 1.1 and 1.2. As we already pointed out, there is a close relationship between c-projective and projective equivalence. This makes the proofs of these theorems rather similar. In Section 5 we focus on the proof of the Yano-Obata conjecture (Theorem 1.1) and explain in a series of remarks how this proof can be adapted for the Lichnerowicz conjecture (Theorem 1.2). This is done under one additional algebraic condition: the endomorphism A compatible with the metric g and induced by the projective vector field v has no Jordan blocks with non-constant eigenvalues [(2)]. The latter case when A admits a "non-constant" Jordan block is treated in Section 6.

2. Canonical Killing vector fields for c-projectively equivalent metrics

Let (M, g, J) be a connected Kähler manifold of real dimension $2n \geq 4$. Since by definition any A which is c-compatible with (g, J) commutes with J, we can consider A as an endomorphism of the n-dimensional complex vector space T_pM (with complex multiplication given by $(a + ib)X = aX + bJX$). The determinant of A considered as complex endomorphism will be denoted by $\det_{\mathbb{C}} A$. It is a smooth function on M and since A is Hermitian, it is real valued. Up to a sign $\det_{\mathbb{C}} A$ equals $\sqrt{\det A}$, though the latter is always positive and smoothness may fail at the points where it vanishes.

Recall that a vector field is called a *Killing vector field* w.r.t. the metric g, if its local flow preserves g. Similarly, a vector field is called *holomorphic* if its local flow preserves the complex structure J.

Since the local flow of a holomorphic Killing vector field K preserves the symplectic form $\omega = g(J\cdot, \cdot)$, the vector field K is Hamiltonian in the neighborhood of any point or, more generally, on every simply connected open subset. Recall that a vector field K is called Hamiltonian if there exists a function f such that

$$i_K\omega = -\mathrm{d}f \text{ or, equivalently, } K = J\,\mathrm{grad}\, f.$$

Such a function f is called a Hamiltonian for K and it is only unique up to adding a constant. Conversely, since every Hamiltonian vector field preserves ω, it is Killing if and only if it is holomorphic. Recall also that holomorphic vector fields are characterized by the property that their covariant differential is complex-linear (when considered as endomorphism

(2) According to the local description given by Theorem 1.6, in the c-projective setting such blocks do not occur.

of T_pM) and therefore a Hamiltonian vector field $K = J \operatorname{grad} f$ is holomorphic if and only if the hessian $\nabla^2 f$ is Hermitian.

LEMMA 2.1. – *For any A which is c-compatible with (g, J) the function* $\det_{\mathbb{C}} A$ *is a Hamiltonian for a Killing vector field.*

We do not pretend that Lemma 2.1 is new: for positive definite metrics it is equivalent to [1, Proposition 3] and this proof can be generalized to all signatures. We give a different and shorter proof, which is based on the same observation as the proof given in [16, Proposition 4.10] but does not require introducing c-projectively invariant objects.

Proof. – Since the statement is local, w.l.o.g. we may assume that $\det A \neq 0$, otherwise we can locally replace A by $A + \mathrm{const} \cdot \mathrm{Id}$. Then, as explained in §1.3, the metric $\hat{g}$ given by (1.3) is c-projectively equivalent to g. We denote by ∇ and $\hat{\nabla}$ the Levi-Civita connections of g and $\hat{g}$. It is well known (see for example the survey [34]), and follows directly from the definition of c-projective equivalence, that the connections ∇ and $\hat{\nabla}$ are related by the equation

$$\hat{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(JX)JY - \Phi(JY)JX, \tag{2.1}$$

where Φ is an exact 1-form equal to the differential of the function

$$\phi = \tfrac{1}{4(n+1)} \ln\left(\frac{\det \hat{g}}{\det g}\right). \tag{2.2}$$

Combining (1.3) and (2.2), we see that

$$\exp(-2\phi) = |\det_{\mathbb{C}} A|\,.$$

Now, it follows from straightforward calculations using (2.1) (see e.g., [34]), that the Ricci tensors Ric and $\widehat{\mathrm{Ric}}$ of the metrics g and $\hat{g}$ are related by

$$\widehat{\mathrm{Ric}} - \mathrm{Ric} = -2(n+1)(\nabla\Phi - \Phi^2 + (\Phi \circ J)^2).$$

Note that $\nabla\Phi$ is a symmetric $(0,2)$-tensor. For a Kähler metric, the Ricci tensor is Hermitian w.r.t. the complex structure. Then the above equation implies that $\nabla\Phi - \Phi^2 + (\Phi \circ J)^2$ is Hermitian. Hence,

$$\begin{aligned}\nabla^2 |\det_{\mathbb{C}} A| &= \nabla^2 \exp(-2\phi) = 2\exp(-2\phi)(-\nabla\Phi + 2\Phi^2)\\ &= 2\exp(-2\phi)(-(\nabla\Phi - \Phi^2 + (\Phi \circ J)^2) + \Phi^2 + (\Phi \circ J)^2)\end{aligned}$$

is Hermitian as well. This implies that $J \operatorname{grad} |\det_{\mathbb{C}} A|$ is Killing. □

For each A which is c-compatible with (g, J), Lemma 2.1 gives us a Hamiltonian Killing vector field with the Hamiltonian function $\det_{\mathbb{C}} A$. If A is not parallel, this function is non-constant and therefore the Killing vector field is non-trivial.

Since Equation (1.4) is linear in A and admits $\mathrm{Id} : TM \to TM$ as a solution, we actually have a whole family $A(t) = t \cdot \mathrm{Id} - A$ of endomorphisms c-compatible with (g, J). For any fixed t, the function $\det_{\mathbb{C}} A(t)$ is a Hamiltonian for a Killing vector field which we denote by $K(t)$. We will call these vector fields, and also all their linear combinations with constant coefficients, *canonical Killing vector fields corresponding to the solution A of* (1.4) (or to the c-projectively equivalent metric $\hat{g}$), or simply *canonical Killing vector fields*.

LEMMA 2.2. – *The following statements hold for any endomorphism A which is c-compatible with (g, J):*

1. *Suppose for a smooth function ρ on an open subset $U \subseteq M$ and for any point $p \in U$ the number $\rho(p)$ is an eigenvalue of A at p of algebraic multiplicity ≥ 4. Then this function ρ is a constant on U. Moreover, for any point of the manifold the constant ρ is an eigenvalue of A.*
2. *The vectors* grad ρ *and* J grad ρ *are eigenvectors of A with eigenvalue ρ at the points where the eigenvalue ρ is a smooth function.*
3. *At a generic point, the number of linearly independent canonical Killing vector fields coincides with the number of non-constant eigenvalues of A.*
4. *At each regular point the number of eigenvalues ρ with* $\mathrm{d}\rho \neq 0$ *is the same.*
5. *At regular points, the restriction of g to the distribution spanned by the canonical Killing vector fields is non-degenerate.*
6. *The canonical Killing vector fields $K(t)$, and also the vector fields $JK(t)$ commute: for any $t_1, t_2 \in \mathbb{R}$ we have*

$$[K(t_1), K(t_2)] = [K(t_1), JK(t_2)] = [JK(t_1), JK(t_2)] = 0 \quad \textit{and} \quad \omega(K(t_1), K(t_2)) = 0.$$

7. *The local flow of every canonical Killing vector field preserves A.*
8. *For any two canonical Killing vector fields $K(t_1), K(t_2)$ the vector $J\nabla_{K(t_1)}K(t_2)$ at any point is contained in the span of the vector fields $K(t)$, $t \in \mathbb{R}$.*

Most statements of the lemma can be found in [16]. More precisely, the first statement is [16, Lemma 5.16], the second statement is [16, Corollary 5.17], the third statement follows from [16, Theorem 5.18(1)], the fourth statement is [16, Proposition 5.12], the sixth statement is explained in [16, §5.6] and the seventh statement follows from [16, Theorem 5.18(1)]. The proofs in the present paper are different from those in [16], shorter and do not require introducing c-projectively invariant objects. For g positive definite, most statements of the lemma have been obtained in the language of Hamiltonian 2-forms in [1]. It is not possible (that is, we did not find an easy way to do it, see also discussion in Remark 1.5) to directly generalize the proofs from [1] to metrics of all signatures. Note also that Lemma 2.2(1), i.e., the non-existence of "non-constant" Jordan-blocks, was shown before in [11, Lemma 2.5] for $\dim_{\mathbb{C}}(M) = 2$.

Proof. – Let $\lambda_1(x), \ldots, \lambda_k(x)$ be the eigenvalues of A at a point $x \in M$. In the proof of the 1st statement of Lemma 2.2 we will work in the neighborhood of a generic point, which implies that we may assume w.l.o.g. that the algebraic multiplicities of the eigenvalues are $2m_1, \ldots, 2m_k$, they do not change in this neighborhood and all λ_i are smooth, possibly complex-valued functions. Now, evidently $f(t) = \det_{\mathbb{C}}(t \cdot \mathrm{Id} - A) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_k)^{m_k}$. Note that the formula for $f(t)$ makes sense also if $t \in \mathbb{C} \setminus \mathbb{R}$. Indeed, because of linearity of the Killing equation, for a Hamiltonian function $f(t)$ with $t \in \mathbb{C} \setminus \mathbb{R}$ the Hamiltonian vector field, which is now complex-valued, is still a holomorphic Killing vector field in the sense that its real and imaginary parts are holomorphic Killing vector fields. To see this, note that $f(t)$ is a polynomial in t so all of its coefficients are Hamiltonians for holomorphic

Killing vector fields. Thus, for complex-valued t the real and imaginary parts of $f(t)$ are still linear combinations of the coefficients.

Consider now the family $\mathrm{d}f(t)$ of differentials of Hamiltonians of the canonical Killing vector fields. It is given by

$$\begin{aligned} &m_1(t-\lambda_1)^{m_1-1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k}\mathrm{d}\lambda_1 \\ &\qquad + m_2(t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2-1}\cdots(t-\lambda_k)^{m_k}\mathrm{d}\lambda_2 \\ &\qquad + \cdots + m_k(t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k-1}\mathrm{d}\lambda_k. \end{aligned} \tag{2.3}$$

Suppose now that a (possibly complex-valued) eigenvalue λ_i has algebraic multiplicity $2m_i \geq 4$. W.l.o.g. we may think that $i = 1$. We take an arbitrary point p, set $\tilde{\lambda} = \lambda_1(p)$ and consider the Hamiltonian $f(t)$ with $t = \tilde{\lambda}$. Since $m_1 \geq 2$, we see that $\mathrm{d}f(\tilde{\lambda}) = 0$ at p. Then the components of the matrix of the hessian $\nabla^2 f(\tilde{\lambda})$ at p in any coordinate system x_i are simply given by the components $\partial_i\partial_j f(\tilde{\lambda})$ of the usual hessian at p and, hence,

$$\nabla^2 f(\tilde{\lambda})(p) = m_1(m_1-1)(\tilde{\lambda}-\lambda_1)^{m_1-2}(\tilde{\lambda}-\lambda_2)^{m_2}\cdots(\tilde{\lambda}-\lambda_k)^{m_k}\mathrm{d}\lambda_1^2\,.$$

We see that if λ_1 is actually real-valued, the hessian $\nabla^2 f(\tilde{\lambda})$ at the point p vanishes or has rank 1. But it cannot have rank 1 because it is Hermitian. Thus, $\nabla^2 f(\tilde{\lambda})$ has to vanish.

Suppose now $\lambda_1 = \alpha + i\beta$, where α and β are real-valued functions. Then,

$$\mathrm{d}\lambda_1^2 = \mathrm{d}\alpha^2 - \mathrm{d}\beta^2 + 2i\,\mathrm{d}\alpha\mathrm{d}\beta.$$

If $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ are linearly dependent, $\mathrm{d}\alpha^2 - \mathrm{d}\beta^2$ and $\mathrm{d}\alpha\mathrm{d}\beta$ have rank 1 or 0. Since rank 1 is impossible (this would imply that $\nabla^2 f(\tilde{\lambda})(p)$ has rank 1 leading us to a contradiction) they vanish. The case where $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ are linearly independent cannot occur because in this case the bilinear forms $\mathrm{d}\alpha^2 - \mathrm{d}\beta^2$ and $\mathrm{d}\alpha\mathrm{d}\beta$ have signature $(1, 1, 2n-2)$, which contradicts the fact that they are Hermitian. Finally, $\nabla^2 f(\tilde{\lambda}) = 0$ at p.

It is well known that the first jet (i.e, the vector field and its first covariant derivative) of a Killing vector field at a point determines the Killing vector field on the whole manifold. As we just proved, the first jet of the Killing vector field corresponding to the Hamiltonian $f(\tilde{\lambda})$ vanishes at p. Then it vanishes on the whole manifold which implies that the function $f(\tilde{\lambda})$ is a constant. It is clearly zero at the point p so it is identically zero and $\tilde{\lambda}$ is an eigenvalue at every point of the manifold. The 1st statement of Lemma 2.2 is proved.

Let us now prove the 2nd statement. Denote by X a vector field of eigenvectors corresponding to a non-constant eigenvalue ρ (viewed as a function on the manifold). First observe that for any vector Y we have

$$\begin{aligned} (A-\rho\cdot\mathrm{Id})\nabla_Y X &= \mathrm{d}\rho(Y)X - g(X,Y)\Lambda - g(X,\Lambda)Y \\ &\quad - g(X,JY)J\Lambda - g(X,J\Lambda)JY. \end{aligned} \tag{2.4}$$

To obtain (2.4), take the covariant derivative in the direction of Y of the equation $(A-\rho\cdot\mathrm{Id})X = 0$, substitute (1.4) and rearrange the terms.

Taking Y orthogonal to X and to JX, we see that the right hand side of (2.4) is a linear combination of the vectors X, Y and JY. Note that since the algebraic multiplicity of ρ is two, we have $g(X,X) = g(JX,JX) \neq 0$ implying that X, Y and JY are linearly independent. Since the left hand side $(A-\rho\cdot\mathrm{Id})\nabla_Y X$ is orthogonal to the kernel of $(A-\rho\cdot\mathrm{Id})$, the coefficient of X, which is $\mathrm{d}\rho(Y)$, is zero. Thus, the function ρ is constant in any direction orthogonal to X and to JX. By the 1st statement of Lemma 2.2, the algebraic multiplicity of ρ is 2 and it follows that $\operatorname{grad}\rho$ and $J\operatorname{grad}\rho$ are eigenvectors of A corresponding to the eigenvalue ρ.

To prove the 3rd statement, consider the non-constant eigenvalues of A and denote them by $\rho_1, \dots, \rho_\ell$. We will work near a generic point so we may assume that $\rho_1, \dots, \rho_\ell$ are smooth functions with non-zero differentials. Observe that for any t the function $(t - \lambda_i)^{m_i}$ is a constant if the eigenvalue λ_i is a constant and, in view of the proved first statement, if $m_i \geq 2$. Then each $f(t)$ is proportional with a constant coefficient to $(t-\rho_1)\cdots(t-\rho_\ell)$. The function $\tilde{f}(t) = (t-\rho_1)\cdots(t-\rho_\ell)$ is a polynomial of degree ℓ with leading coefficient equal to 1 and has at most ℓ non-constant coefficients. Thus, the number of linearly independent canonical Killing vector fields is at most ℓ.

Since the gradients $\operatorname{grad}\rho_i$ belong to different eigenspaces, they are linearly independent and in view of (2.3), the differentials of $\tilde{f}(t_1)$ and $\tilde{f}(t_2)$ are linearly independent for $t_1 \neq t_2$ so the number of linearly independent canonical Killing vector fields is precisely ℓ.

To prove the 4th statement, recall that a Killing vector field which vanishes on an open set vanishes everywhere. Then, by the 3rd statement of the lemma, the number of non-constant eigenvalues is the same on every open subset of regular points and the claim follows.

In order to prove the 5th statement, observe that the distribution spanned by the canonical Killing vector fields, at regular points, coincides with the distribution spanned by the Hamiltonian vector fields generated by the non-constant eigenvalues. By the 2nd statement, such Hamiltonian vector fields have non-zero length at regular points and are mutually orthogonal, and the claim follows.

Let us prove the 6th statement. By the 2nd statement, we have

$$\omega(K(t_1), K(t_2)) = g(JK(t_1), K(t_2)) = 0$$

for any real numbers t_1, t_2. By definition of a Poisson bracket, this equation is equivalent to say that the Hamiltonian functions $f(t_1), f(t_2)$ corresponding to $K(t_1), K(t_2)$ Poisson commute, $\{f(t_1), f(t_2)\} = 0$. On the other hand, recall that $[K(t_1), K(t_2)]$ is the Hamiltonian vector field corresponding to the Hamiltonian $\{f(t_1), f(t_2)\}$. We obtain $[K(t_1), K(t_2)] = 0$. The remaining equations follow from the fact that the vector field $K(t)$ is holomorphic and therefore $JK(t)$ is also holomorphic.

To prove the 7th statement, assume w. l. o. g. that A is non-degenerate (the statement of part (7) is local and we may change $A \to A + \mathrm{const}\cdot \mathrm{Id}$). Then we can consider the metric $\hat{g}$ from (1.3) c-projectively equivalent to g. It is sufficient to show that the canonical Killing vector fields for g are also canonical Killing vector fields for $\hat{g}$. W.l.o.g. we may work in the neighborhood of a regular point. Let $\rho_1, \dots, \rho_\ell$ denote the non-constant eigenvalues of A. If we swap the metrics g and $\hat{g}$ in the Definition (1.2), the tensor constructed by the pair of metrics $\hat{g}$, g is clearly the inverse of the initial A, therefore its non-constant eigenvalues are $\frac{1}{\rho_1}, \dots, \frac{1}{\rho_\ell}$.

We will show that the canonical Killing vector field $K(t)$ for g, whose Hamiltonian is $\det_{\mathbb{C}}(t\cdot \mathrm{Id} - A)$, is proportional with a non-zero constant coefficient to the canonical Killing vector field $\hat{K}\left(\frac{1}{t}\right)$ for $\hat{g}$, whose Hamiltonian is $\det_{\mathbb{C}}\left(\frac{1}{t}\cdot \mathrm{Id} - A^{-1}\right)$.

Since the multiplicity of the non-constant eigenvalues of A is two, up to multiplication by a constant, for any t, the differential of $\det_{\mathbb{C}}(t\cdot \mathrm{Id} - A)$ coincides with the differential of $(t-\rho_1)\cdots(t-\rho_\ell)$ which is

$$(t-\rho_2)\cdots(t-\rho_\ell)\mathrm{d}\rho_1+(t-\rho_1)(t-\rho_3)\cdots(t-\rho_\ell)\mathrm{d}\rho_2+\cdots+(t-\rho_1)\cdots(t-\rho_{\ell-1})\mathrm{d}\rho_\ell.$$

Similarly, for any $t\neq 0$, the differential of $\det_{\mathbb{C}}\left(\frac{1}{t}\cdot\mathrm{Id}-A^{-1}\right)$ is proportional with a constant coefficient to the differential of $(\frac{1}{t}-\frac{1}{\rho_1})\cdots(\frac{1}{t}-\frac{1}{\rho_\ell})$ which is, up to multiplication by a non-zero constant, given by

$$\frac{1}{\det_{\mathbb{C}}(A)}\big((\rho_2-t)\cdots(\rho_\ell-t)\tfrac{1}{\rho_1}\mathrm{d}\rho_1 \\ +(\rho_1-t)(\rho_3-t)\cdots(\rho_\ell-t)\tfrac{1}{\rho_2}\mathrm{d}\rho_2+\cdots+(\rho_1-t)\cdots(\rho_{\ell-1}-t)\tfrac{1}{\rho_\ell}\mathrm{d}\rho_\ell\big).$$

Now, the canonical vector fields $K(t)$ and $\hat K\left(\frac{1}{t}\right)$ are related to the differentials of $\det_{\mathbb{C}}(t\cdot\mathrm{Id}-A)$ and $\det_{\mathbb{C}}\left(\frac{1}{t}\cdot\mathrm{Id}-A^{-1}\right)$ by

$$K(t)=J\operatorname{grad}_g\det_{\mathbb{C}}(t\cdot\mathrm{Id}-A)\text{ and }\hat K\left(\tfrac{1}{t}\right)=J\operatorname{grad}_{\hat g}\det_{\mathbb{C}}\left(\tfrac{1}{t}\cdot\mathrm{Id}-A^{-1}\right).$$

Combining this with (1.3) and the 2nd statement, we conclude that $K(t)$ is proportional to $\hat K\left(\frac{1}{t}\right)$ with a constant factor.

Let us now prove the 8th statement. It is sufficient to prove it on the dense and open subset M^0 of regular points. As usual, by $\rho_1,\dots,\rho_\ell$ we denote the non-constant eigenvalues of A. From the definition, it follows that the integrable distribution $\mathcal{V}$ spanned by the canonical Killing vector fields $K(t)$, $t\in\mathbb{R}$, coincides with the distribution spanned by the vector fields $J\operatorname{grad}\rho_i$, $i=1,\dots,\ell$. Consider the distribution $\mathcal{F}=\mathcal{V}\oplus J\mathcal{V}$. It is spanned by the family of vector fields $K(t), JK(t)$, $t\in\mathbb{R}$, is integrable by the 6th statement and coincides with the span of $\operatorname{grad}\rho_i$, $J\operatorname{grad}\rho_i$, $i=1,\dots,\ell$. From Formula (2.4) combined with the 2nd statement, it follows immediately that the distribution $\mathcal{F}$ is totally geodesic. By the 5th statement, the restriction $g_{|_{\mathcal{L}}}$ of g to an integral leaf $\mathcal{L}\subseteq M^0$ of $\mathcal{F}$ is nondegenerate. Then it follows that the integral leafs $\tilde{\mathcal{L}}\subseteq\mathcal{L}$ of the integrable subdistribution $J\mathcal{V}\subset\mathcal{F}$ are totally geodesic since they are orthogonal in $(\mathcal{L},g_{|_{\mathcal{L}}})$ to a distribution spanned by Killing vector fields. This implies that $\nabla_{JK(t_1)}JK(t_2)$ is tangent to $J\mathcal{V}$, or equivalently (since J is parallel and $K(t)$ is holomorphic), that $J\nabla_{K(t_1)}K(t_1)$ is tangent to $\mathcal{V}$ as we claimed. This completes the proof of Lemma 2.2. □

Let μ_i denote the ith elementary symmetric polynomial in $\rho_1,\dots,\rho_\ell$, i.e., in the non-constant eigenvalues of A c-compatible with (g,J). Note that although the ρ_i may fail to be smooth at certain points, the μ_i are globally defined smooth functions on M: clearly we have $\det_{\mathbb{C}}(t\cdot\mathrm{Id}-A)=P(t)\sum_{i=0}^{\ell}(-1)^i\mu_i t^{\ell-i}$, where $P(t)$ is a polynomial of degree $n-\ell$ with constant coefficients and we put $\mu_0=1$. In what follows we will mainly work with a special set of canonical Killing vector fields $K_1,\dots,K_\ell$ corresponding to A, where K_i is defined to be the Hamiltonian vector field with μ_i as a Hamiltonian function, i.e.,

$$K_i=J\operatorname{grad}\mu_i. \tag{2.5}$$

These Killing vector fields have been considered in [1]. By Lemma 2.2, the span of these vector fields at each point coincides with the span of the vector fields $K(t)$, $t\in\mathbb{R}$, and therefore, they share all the properties that have been proven for the vector fields $K(t)$ in Lemma 2.2. For instance, $\omega(K_i,K_j)=0$, hence, $[K_i,K_j]=[JK_i,K_j]=[JK_i,JK_j]=0$ and $K_1\wedge\cdots\wedge K_\ell\neq 0$ at each point of M^0.

3. Reduction to the real projective setting

We recall the description of a Kähler manifold with a local isometric Hamiltonian $\mathbb{R}^\ell$-action in §3.1. In the setting of c-projectively equivalent Kähler metrics, this action is given by the commuting Killing vector fields $K_1, \dots, K_\ell$ from (2.5) induced by a Hermitian solution A of (1.4). As stated in §1.2, the quotient of the Kähler manifold (M, g, J) w.r.t. to this action yields a manifold (Q, g_Q) and g_Q admits a projectively equivalent metric. This will be described in detail in §3.2.

3.1. The Kähler quotient w.r.t. a local isometric Hamiltonian $\mathbb{R}^\ell$-action

Recall from [1, §3.1] that a *local isometric Hamiltonian* $\mathbb{R}^\ell$*-action* on a Kähler manifold (M, g, J) is given by holomorphic Killing vector fields $K_1, \dots, K_\ell$ satisfying

$$\omega(K_i, K_j) = 0$$

and $K_1 \wedge \dots \wedge K_\ell \neq 0$ on a dense and open subset $M^0 \subset M$ called the set of regular points.

Note that in [1], the name "ℓ-torus action" was used instead of "$\mathbb{R}^\ell$-action". The point is that the metrics in [1, 2] are positive definite so that under the additional assumption of compactness, the isometric $\mathbb{R}^\ell$-action described above generates a commutative subgroup of the compact group of isometries, its closure being a torus.

Since the vector fields $K_1, \dots, K_\ell$ are symplectic, they are also locally Hamiltonian, i.e., we have $K_i = J \operatorname{grad} \mu_i$ for certain local functions μ_i, $i = 1, \dots, \ell$. The condition $\omega(K_i, K_j) = 0$ moreover implies that the vector fields mutually commute.

By Lemma 2.2, the canonical Killing vector fields (2.5) coming from a solution A of (1.4) generate a local isometric Hamiltonian $\mathbb{R}^\ell$-action, where ℓ is the number of non-constant eigenvalues of A at a regular point. The notion of regular points as introduced above coincides with the notion of regular points introduced in §1.3. However, for the time being, we will forget about c-projective geometry and will first restrict to the general setting of a local isometric Hamiltonian $\mathbb{R}^\ell$-action.

Since we are dealing with metrics of arbitrary signature, we have to take care of the non-degeneracy of orbits of an $\mathbb{R}^\ell$-action. A local isometric Hamiltonian $\mathbb{R}^\ell$-action given by Killing vector fields $K_1, \dots, K_\ell$ is called *non-degenerate* if the restriction of the metric g to the (regular) distribution

$$\mathcal{V} = \operatorname{span}\{K_1, \dots, K_\ell\}$$

on the regular set M^0 is non-degenerate.

As shown in Lemma 2.2 (5), the $\mathbb{R}^\ell$-action coming from the canonical Killing vector fields corresponding to a solution A of (1.4) is non-degenerate in the above sense.

Given a local isometric non-degenerate Hamiltonian $\mathbb{R}^\ell$-action, we will now reduce the setting by considering the quotient of M w.r.t. the action of the Killing vector fields. The procedure of this reduction and the local description of Kähler metrics admitting a local isometric Hamiltonian $\mathbb{R}^\ell$-action can be found in [1, §3.1] and [38]. For the sake of completeness, we will recall these results. The only difference in our case is that the metric g is allowed to have arbitrary signature but assuming non-degeneracy, there is actually no difference to the procedure described in [1].

Consider a non-degenerate local isometric Hamiltonian $\mathbb{R}^\ell$-action on (M, g, J) by holomorphic Killing vector fields $K_1, \dots, K_\ell$. Let us restrict our attention to the regular set M^0 and let G denote the commutative (pseudo-)group generated by the local flows of $K_1, \dots, K_\ell$. Consider the (local) quotient $Q = M/G$ of M w.r.t. the G-action and the (local) fiber bundle

$$\pi : M \to Q = M/G.$$

The vertical distribution of this bundle coincides with the distribution $\mathcal{V}$ and we define a (G-invariant) horizontal distribution $\mathcal{Q} = \mathcal{V}^\perp$. Let $\theta = (\theta_1, \dots, \theta_\ell) : TM \to \mathbb{R}^\ell$ be the corresponding connection 1-form on M, where the components θ_i have been chosen to be dual to the Killing vector fields K_i, that is, the 1-forms θ_i are defined by

$$\theta_i(K_j) = \delta_{ij} \text{ and } \theta_i(\mathcal{Q}) = 0.$$

As above, the local generators for the vector fields K_i will be denoted by μ_i (so that $K_i = J \operatorname{grad} \mu_i$) and we can gather these functions into a (locally defined) moment map $\mu = (\mu_1, \dots, \mu_\ell) : M \to (\mathbb{R}^\ell)^*$ for the Hamiltonian action of G. Lemma 2.2 (6) implies that the moment map μ is G-invariant, thus it descends to a mapping $\mu : Q \to (\mathbb{R}^\ell)^*$ on the quotient. The level sets S_μ in Q of this mapping are the Kähler quotients of (M, g, J) w.r.t. the isometric Hamiltonian action of G. We refer the reader to [22, §3] for some background on symplectic reduction and Kähler quotients.

On the other hand, we can also take the (local) quotient of M w.r.t. the action of the commutative (pseudo-)group $G^\mathbb{C}$ generated by the local flows of the commuting vector fields $K_1, \dots, K_\ell, JK_1, \dots, JK_\ell$. The result is a manifold $S = M/G^\mathbb{C}$ and since the tangent spaces of the fibers of the bundle $M \to S$ are J-invariant and the action of $G^\mathbb{C}$ is by holomorphic transformations, S inherits a canonical complex structure J_S. As a complex manifold, S can canonically be identified with the Kähler quotients S_μ. In view of this, S carries a family of Kähler structures (g_μ, ω_μ) which are compatible with the complex structure J_S. The quotient Q may locally be written in the form $Q = S \times U$, where the open subset $U \subseteq (\mathbb{R}^\ell)^*$ parametrises the level sets of μ. In this picture, the subset U can be viewed as the parameter space for the family of compatible Kähler structures (g_μ, ω_μ) on S.

Since the forms $\theta_i \circ J$ and $\mathrm{d}\mu_i$ span the same subspace of T^*M, we can define a point-wise non-degenerate matrix of functions G_{ij} and its inverse with components H_{ij} by

$$\theta_i \circ J = \sum_{j=1}^{\ell} G_{ij} \mathrm{d}\mu_j \text{ and } \mathrm{d}\mu_i \circ J = -\sum_{j=1}^{\ell} H_{ij} \theta_j. \tag{3.1}$$

Note that it follows from (3.1) that

$$H_{ij} = g(K_i, K_j),$$

in particular, H_{ij} and G_{ij} are symmetric in i, j.

The Kähler structure can now be written in the form

$$\begin{aligned} g &= g_\mu + \sum_{i,j=1}^{\ell} H_{ij} \theta_i \theta_j + \sum_{i,j=1}^{\ell} G_{ij} \mathrm{d}\mu_i \mathrm{d}\mu_j, \\ \omega &= \omega_\mu + \sum_{i=1}^{\ell} \mathrm{d}\mu_i \wedge \theta_i. \end{aligned} \tag{3.2}$$

In our case, the $\mathbb{R}^\ell$-action induced by a solution A of (1.4) satisfies one additional property called *rigidity* that essentially simplifies the above local formulas for g and ω.

Recall from [1, §3.2] that a local Hamiltonian $\mathbb{R}^\ell$-action on a Kähler manifold (M, g, J) given by holomorphic Killing vector fields $K_1, \dots, K_\ell$ is called *rigid* if the leaves of the distribution

$$\mathcal{F} = \mathcal{V} \oplus J\,\mathcal{V}$$

are totally geodesic (where $\mathcal{V}$ is the vertical distribution of $M \to Q$).

There are a number of equivalent conditions to this rigidity property, see [1, Proposition 8]. We recall this result for convenience of the reader. Note that, although it has been proven in [1] for positive signature, the proof still works in arbitrary signature assuming non-degeneracy of the $\mathbb{R}^\ell$-action.

Proposition 3.1 ([1]). – *Consider a local isometric Hamiltonian $\mathbb{R}^\ell$-action given by holomorphic Killing vector fields $K_1, \dots, K_\ell$. The following assumptions are equivalent:*

1. *The action is rigid.*
2. *The functions $H_{ij} = g(K_i, K_j)$ are constant on the level surfaces of the moment map $\mu : M \to (\mathbb{R}^\ell)^*$.*
3. *$\nabla_{K_i} K_j \in \mathcal{F}$ for all $i, j = 1, \dots, \ell$.*
4. *The Kähler quotient forms ω_μ depend affinely on the components μ_i of the moment map $\mu : Q \to (\mathbb{R}^\ell)^*$ and their linear part pulls back to the curvature of $(\theta_1, \dots, \theta_\ell)$.*

Remark 3.1. – Condition (3) of the proposition can be replaced by "$\nabla_{K_i} K_j \in J\,\mathcal{V}$ for all $i, j = 1, \dots, \ell$". Indeed, if this holds, $\mathcal{F}$ is obviously totally geodesic, hence, the action is rigid. The converse direction follows from the same line of arguments that has been used in the proof of Lemma 2.2 (8). We see that "$J\,\mathcal{V}$ is totally geodesic" is another condition equivalent to rigidity of the action.

Proposition 3.1 gives rise to some simplifications in (3.2) and we come to the following local description:

Proposition 3.2. – *Let (M, g, J, ω) be a Kähler $2n$-manifold together with a rigid non-degenerate (local) isometric Hamiltonian $\mathbb{R}^\ell$-action generated by Hamiltonian Killing vector fields $K_i = J \operatorname{grad} \mu_i$, $i = 1, \dots, \ell$. Then locally M can be presented as direct product*

$$V(t_1, \dots, t_\ell) \times U(\mu_1, \dots, \mu_\ell) \times S(y_1, \dots, y_{2k}),$$

and g, ω and J take the following form:

$$g = \sum_{i,j=1}^{\ell} H_{ij}(\mu)\theta_i\theta_j + \sum_{i,j=1}^{\ell} G_{ij}(\mu)\mathrm{d}\mu_i\mathrm{d}\mu_j + \sum_{i=1}^{\ell} \mu_i g_i + g_0, \tag{3.3}$$

$$\sum_{j=1}^{\ell} H_{ij} G_{jk} = \delta_{ik}$$

$$\omega = \sum_{i=1}^{\ell} d\mu_i \wedge \theta_i + \sum_{i=1}^{\ell} \mu_i \omega_i + \omega_0 \tag{3.4}$$

and

$$\theta_i \circ J = \sum_{j=1}^{\ell} G_{ij} \mathrm{d}\mu_j, \ \ \mathrm{d}\mu_i \circ J = -\sum_{j=1}^{\ell} H_{ij}\theta_j, \ \ \mathrm{d}y_i \circ J = \mathrm{d}y_i \circ J_S, \tag{3.5}$$

where the ingredients in these formulas are as follows:

1. $\theta_i = \mathrm{d}t_i + \alpha_i$ *with* $\mathrm{d}\alpha_i = \omega_i$*;*
2. $\big(g_\mu = \sum_i \mu_i g_i + g_0,\ \omega_\mu = \sum_i \mu_i \omega_i + \omega_0,\ J_S\big)$*, is a Kähler structure on* S *for any* $\mu \in U$ *(compatible with the same complex structure* J_S *independent of* μ*);*
3. $\partial_{\mu_i} G_{jk} = \partial_{\mu_j} G_{ik}$.

Conversely, if on $M = V \times U \times S$ *we consider* g, ω, J *as above, then* (g,ω) *is a Kähler structure on* M *and the generators* $\mu_1, \dots, \mu_\ell$ *define a rigid non-degenerate (local) isometric Hamiltonian* $\mathbb{R}^\ell$*-action. In particular, the vector fields* $K_i = J \operatorname{grad} \mu_i$ *are holomorphic Killing vector fields.*

3.2. Reduction from the c-projective to the projective setting

We continue to use the notation introduced in the preceding section but assume that the nondegenerate local isometric Hamiltonian $\mathbb{R}^\ell$-action is given by the Killing vector fields $K_1, \dots, K_\ell$ from (2.5) that come from a certain solution A of (1.4). Let g_Q denote the metric on the local quotient $Q = M/G$ obtained from g. In the notation of Proposition 3.2, Q can locally be identified with $U \times S$ and g_Q can be obtained from (3.3) by removing the first term, i.e.,

$$g_Q = \sum_{i,j=1}^{\ell} G_{ij}(\mu)\mathrm{d}\mu_i \mathrm{d}\mu_j + \sum_{i=1}^{\ell} \mu_i g_i + g_0. \tag{3.6}$$

Recall that the vertical distribution $\mathcal{V} = \operatorname{span}\{K_1, \dots, K_\ell\}$ coincides with the span of the vector fields $J \operatorname{grad} \rho_1, \dots, J \operatorname{grad} \rho_\ell$, where $\rho_1, \dots, \rho_\ell$ are the non-constant eigenvalues of A. Since by Lemma 2.2, the vector fields $J \operatorname{grad} \rho_i$ take values in the eigenspaces of A, the distribution $\mathcal{V}$ is A-invariant and consequently, A preserves also $\mathcal{Q} = \mathcal{V}^\perp$. On the other hand, according to Lemma 2.2 (7), A is preserved by the Killing vector fields K_i and it follows that A descends to a g_Q-selfadjoint endomorphism $A_Q : TQ \to TQ$.

Recall the O'Neill formula [36] for a Riemannian submersion relating the Levi-Civita connection ∇^Q of the quotient metric g_Q to the Levi-Civita connection ∇ of g by

$$\nabla^Q_X Y = \mathrm{pr}_{\mathcal{Q}}(\nabla_X Y), \tag{3.7}$$

where $\mathrm{pr}_{\mathcal{Q}} : TM \to \mathcal{Q}$ is the projection onto the horizontal distribution $\mathcal{Q}$ and we adopted the convention to denote vector fields on Q and their horizontal lifts to $\mathcal{Q}$ by the same symbol. Note that the vector field Λ in (1.4) is tangent to the horizontal distribution $\mathcal{Q}$ and it is invariant w.r.t. the action of the K_i's. Thus, Λ is (the horizontal lift of) a vector field on Q.

LEMMA 3.3. – *The endomorphism* $A_Q : TQ \to TQ$ *obtained from* A *by reduction satisfies the equation*

$$\nabla^Q_X A_Q = X^\flat \otimes \Lambda + \Lambda^\flat \otimes X \tag{3.8}$$

for all $X \in TQ$, where $X^\flat = g_Q(X, \cdot)$. In other words, g_Q and A_Q are compatible in the projective sense.

Proof. – From the O'Neill Formula (3.7), the definition of A_Q and commutativity of A with $\mathrm{pr}_{\mathcal{Q}}$, it follows that

$$(\nabla^Q_X A_Q)Y = \nabla^Q_X(A_Q Y) - A_Q(\nabla^Q_X Y) = \mathrm{pr}_{\mathcal{Q}}((\nabla_X A)Y).$$

Inserting (1.4) into this equation and using the fact that $J\Lambda$ is tangent to $\mathcal{V} = \mathcal{Q}^\perp$, we obtain

$$(\nabla^Q_X A_Q)Y = g(X,Y)\Lambda + g(\Lambda, Y)X = g_Q(X,Y)\Lambda + g_Q(\Lambda, Y)X$$

as we claimed. □

The local description of compatible pairs (g_Q, A_Q) has been recently obtained in [10] (see also [9]). These results, after some adaptation, will lead us to the local description of pairs (g, A).

4. Local description of c-projectively equivalent metrics

In this section we prove Theorems 1.5 and 1.6.

4.1. Local description of the quotients of c-projectively equivalent metrics and lifting

We have shown above that by taking the quotient of M w.r.t. the action of the Killing vector fields $K_1, \dots, K_\ell$, the local description of a Kähler manifold (M, g, J) of arbitrary signature admitting a Hermitian solution A of (1.4) is reduced to the classification of pseudo-Riemannian manifolds (Q, g_Q) admitting a g_Q-selfadjoint solution A_Q to (3.8). In other words, a description of c-compatible pairs g, A is reduced to a similar problem for compatible pairs g_Q, A_Q on the quotient $Q = M/G$ which has been solved in [10] and we apply this result to our situation.

Before deriving the local description for the pair (g_Q, A_Q) in our specific situation, we briefly recall the "splitting and gluing constructions" from [9] appropriately reformulated for our purposes, we refer to [10, §1.2] for a more detailed summary.

Let (Q, g_Q) be a pseudo-Riemannian manifold and $A_Q : TQ \to TQ$ be a g_Q-selfadjoint endomorphism compatible with g_Q in the projective sense, i.e., satisfying (3.8).

In the neighborhood of a generic point, the eigenvalues of A_Q are smooth (possibly complex valued functions). Some of them, say $c_1, \dots, c_n$, are constant. Then the characteristic polynomial $\chi(t) = \det(t \cdot \mathrm{Id} - A_Q)$ of A_Q can be written as

$$\chi(t) = \chi_{\mathrm{nc}}(t) \cdot \chi_{\mathrm{c}}(t)$$

where the roots of χ_{c} are the constant eigenvalues of A_Q (with multiplicities), whereas the roots of χ_{nc} are the non-constant eigenvalues. Assume that these polynomials $\chi_{\mathrm{nc}}(t)$ and $\chi_{\mathrm{c}}(t)$ are relatively prime, i.e., the non-constant eigenvalues cannot take the values $c_1, \dots, c_n$. In other words, we divide the spectrum of $A_Q(p)$, $p \in Q$ into the "constant" and "non-constant" parts, and assume that these parts are disjoint for any $p \in Q$.

PROPOSITION 4.1. – [9] *Locally Q can be presented as $U(x_1,\dots,x_\ell)\times S(y_1,\dots,y_s)$ so that the endomorphism A_Q and the metric g_Q take the following block-diagonal form*

$$A_Q(x,y)=\begin{pmatrix} L(x) & 0\\ 0 & A_{\rm c}(y)\end{pmatrix} \quad \text{and} \quad g_Q(x,y)=\begin{pmatrix} h(x) & 0\\ 0 & g_{\rm c}(y)\cdot\chi_{\rm nc}\big(A_{\rm c}(y)\big)\end{pmatrix}, \tag{4.1}$$

where L and h are compatible (that is, satisfy (1.7)*) on U, and $A_{\rm c}$ is parallel on S w.r.t. $g_{\rm c}$.*

Conversely, $A_Q(x,y)$ and $g_Q(x,y)$ defined by (4.1) *are compatible in the projective sense, i.e., satisfy* (3.8) *on $U\times S$, whenever h and L are compatible, $A_{\rm c}$ is parallel w.r.t. $g_{\rm c}$ and the spectra of L and $A_{\rm c}$ are disjoint.*

Notice that the formula for the metric g_Q can be equivalently rewritten as follows

$$g_Q=\sum_{i,j=1}^{\ell}B_{ij}(x)\mathrm{d}x_i\mathrm{d}x_j+\sum_{i=1}^{\ell}\mu_i(x)g_i+g_0,\quad g_i=(-1)^i g_{\rm c}(A_{\rm c}^{\ell-i}\cdot,\cdot), \tag{4.2}$$

which completely agrees with the Formula (3.6) for the reduced metric g_Q. Here the first term corresponds to the metric h and the remaining terms represent the other block, i.e., the metric $g_{\rm c}(y)\cdot\chi_{\rm nc}\big(A_{\rm c}(y)\big)$ which can be understood as a family $g_\mu=\sum\mu_i g_i+g_0$ of metrics on S parametrised by the coefficients $\mu_1,\dots,\mu_\ell$ of the characteristic polynomial $\chi_{\rm nc}=\chi_L$ of the "non-constant" block L. Notice that the splitting of Q into the direct product $U\times S$ in both cases is determined by the decomposition of T_pQ into two A_Q-invariant subspaces corresponding to the partition of the spectrum of A_Q into two parts, "constant" and "non-constant". Also notice that in both cases μ_i are the same: these are the elementary symmetric polynomials of non-constant eigenvalues of L (or, which is the same, of A).

Formula (4.2) describes, however, a more general situation than (3.3). In particular, in Proposition 4.1, the non-constant eigenvalues may have arbitrary multiplicities and the "constant" block $(S,g_{\rm c},A_{\rm c})$ carries no Kähler structure. Thus, some additional specific properties of g_Q and A_Q should be taken into account. In particular, we need local formulas for the metric which simultaneously satisfies (4.2) and (3.3).

As we know from Lemma 2.2 (1), the multiplicities of the non-constant eigenvalues $\rho_1,\dots,\rho_\ell$ of A_Q equal one and moreover $\mathrm{d}\rho_1\wedge\dots\wedge\mathrm{d}\rho_\ell\neq 0$ on Q by Lemma 2.2 (3). This condition guarantees that both the eigenvalues $\rho_1,\dots,\rho_\ell$ and the symmetric polynomials $\mu_1,\dots,\mu_\ell$ can be taken as local coordinates on U. Also we know from (3.3) that S is endowed with a natural complex structure J_S and for each $\mu\in U$, the metric

$$g_\mu=\sum_{i=1}^{\ell}\mu_i g_i+g_0$$

on S is Kähler and $A_{\rm c}$ on S is Hermitian w.r.t. (g_μ,J_S). In addition $A_{\rm c}$ is parallel w.r.t. $g_{\rm c}=g_\mu(\chi_{\rm nc}(A_{\rm c})^{-1}\cdot,\cdot)$ by Proposition 4.1. This obviously implies that the metrics $g_{\rm c}$ and g_μ are affinely equivalent for each μ, i.e., their Levi-Civita connections coincide. Hence, if we introduce $\omega_{\rm c}=g_{\rm c}(J_S\cdot,\cdot)=\omega_\mu((\chi_{\rm nc}(A_{\rm c}))^{-1}\cdot,\cdot)$, then $\omega_{\rm c}$ is parallel and therefore $(g_{\rm c},\omega_{\rm c},J_S)$ is a Kähler structure on S admitting a parallel Hermitian endomorphism $A_{\rm c}$ (in other words the conclusion about the constant block in Proposition 4.1 now holds in the Kähler setting).

Summarizing, we see that the pair (g_Q,A_Q) admits the following local description.

PROPOSITION 4.2. – *Using the natural decomposition* $Q = U(x_1,\dots,x_\ell)\times S(y_1,\dots,y_{2k})$ *as in Proposition 4.1, we can write* g_Q *and* A_Q *as follows*

$$A_Q(x,y) = \begin{pmatrix} L(x) & 0 \\ 0 & A_c(y) \end{pmatrix} \quad \text{and} \quad g_Q(x,y) = \begin{pmatrix} h(x) & 0 \\ 0 & g_c(y)\cdot\chi_L(A_c), \end{pmatrix} \tag{4.3}$$

where

- (L,h) *is a compatible pair on* U *(in the projective sense) such that the eigenvalues* $\rho_1,\dots,\rho_\ell$ *of* L *are all distinct and* $\mathrm{d}\rho_i \neq 0$*. Moreover,* $\chi_L(t) = \det(t\cdot\mathrm{Id} - L)$ *denotes the characteristic polynomial of* L*;*
- (S, g_c, J_S) *is a Kähler manifold and* A_c *is a parallel Hermitian endomorphism on* S*.*

The metric $h = \sum_{i,j} B_{ij}\mathrm{d}x_i\mathrm{d}x_j$ can be rewritten in coordinates $\mu_1,\dots,\mu_\ell$

$$h = \sum_{i,j=1}^{\ell} B_{ij}\mathrm{d}x_i\mathrm{d}x_j = \sum_{\alpha,\beta=1}^{\ell} G_{\alpha\beta}\mathrm{d}\mu_\alpha\mathrm{d}\mu_\beta, \qquad B_{ij} = \sum_{\alpha,\beta=1}^{\ell} G_{\alpha\beta}\frac{\partial\mu_\alpha}{\partial x_i}\frac{\partial\mu_\beta}{\partial x_j}$$

as in Proposition 3.2. As we know from this proposition, the components G_{ij} must satisfy one additional condition, namely $\frac{\partial G_{ij}}{\partial\mu_k} = \frac{\partial G_{kj}}{\partial\mu_i}$. It turns out (see Proposition 4.4 and Lemma 4.5 below) that this property follows automatically from the compatibility of h and L. This means that we have no more restrictions onto the reduced pair g_Q and A_Q, and can now summarize the above discussion as follows.

PROPOSITION 4.3. – *Let* (M,g,J,ω) *be a Kähler* $2n$*-manifold and let* A *be a Hermitian solution to* (1.4)*. Then in the neighborhood of a regular point* $p\in M^0$*, where* A *has non-constant eigenvalues* $\rho_1,\dots,\rho_\ell$*, we can introduce a local coordinate system*

$$V(t_1,\dots,t_\ell)\times U(x_1,\dots,x_\ell)\times S(y_1,\dots,y_{2k})$$

in which g*,* ω *and* A *take the following form*

$$g = \sum_{\alpha,\beta=1}^{\ell} H_{\alpha\beta}\theta_\alpha\theta_\beta + \sum_{i,j=1}^{\ell} B_{ij}\mathrm{d}x_i\mathrm{d}x_j + \sum_{i=0}^{\ell}\mu_i\cdot(-1)^i g_c(A_c^{\ell-i}\cdot,\cdot) \tag{4.4}$$

$$\omega = \sum_{\alpha=1}^{\ell}\mathrm{d}\mu_\alpha\wedge\theta_\alpha + \sum_{i=0}^{\ell}\mu_i\cdot(-1)^i\omega_c(A_c^{\ell-i}\cdot,\cdot) \tag{4.5}$$

$$\begin{aligned} A = {} & \sum_{\alpha,\beta=1}^{\ell} M_\alpha^\beta(x)\,\theta_\beta\otimes\frac{\partial}{\partial t_\alpha} + \sum_{i,j=1}^{\ell} L_j^i(x)\,\mathrm{d}x_j\otimes\frac{\partial}{\partial x_i} \\ & + \sum_{p,q=1}^{2k}(A_c)_p^q\,\mathrm{d}y_p\otimes\left(\frac{\partial}{\partial y_q} - \sum_{i=1}^{\ell}\alpha_{iq}\frac{\partial}{\partial t_i}\right), \end{aligned} \tag{4.6}$$

where the ingredients in these formulas are as follows:

1. (g_c,ω_c) *is a Kähler structure and* $A_c = \sum_{p,q}(A_c)_p^q\,\mathrm{d}y_p\otimes\partial_{y_q}$ *is a parallel Hermitian endomorphism on* S*;*
2. $h = B_{ij}(x)\mathrm{d}x_i\mathrm{d}x_j$ *is a pseudo-Riemannian metric and* $L(x)$ *is an endomorphism on* U *forming a compatible pair (in the projective sense);*

3. *the eigenvalues* $\rho_1, \dots, \rho_\ell$ *of* L *are pairwise distinct and satisfy* $\mathrm{d}\rho_i \neq 0$ *on* U*; they are also different from the constant eigenvalues of* A_{c}*;*
4. μ_i *denote the elementary symmetric polynomials in* $\rho_1, \dots, \rho_\ell$*,* $i = 1, \dots, \ell$*, and we set* $\mu_0 = 1$*;*
5. $\theta_i = \mathrm{d}t_i + \alpha_i$*, where* $\alpha_i = \sum_q \alpha_{iq} \mathrm{d}y_q$ *is a 1-form on* S *satisfying* $\mathrm{d}\alpha_i = (-1)^i \omega_{\mathrm{c}}(A_{\mathrm{c}}^{\ell-i}\cdot, \cdot)$*;*
6. *and finally* $H_{\alpha\beta} = \sum_{i,j} B^{ij} \frac{\partial \mu_\alpha}{\partial x_i} \frac{\partial \mu_\beta}{\partial x_j}$*, where* B^{ij} *is the inverse of* B_{ij} *and* $M_\alpha^\beta = \sum_{i,j} L^i_j \frac{\partial \mu_\beta}{\partial x_i} \frac{\partial x_j}{\partial \mu_\alpha}$.

Proof. – The Formulas (4.4) and (4.5) follow from the discussion above. It remains to derive Formula (4.6) for A. First of all we note that the basis dual to the coframe θ_i, $\mathrm{d}x_j$, $\mathrm{d}y_q$ is given by

$$\frac{\partial}{\partial t_i}, \quad \frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial y_q} - \sum_{i=1}^{\ell} \alpha_{iq} \frac{\partial}{\partial t_i}.$$

Comparing Formula (4.3) for A_Q with Formula (4.6) for A, we see that the reduction of A given by (4.6) is indeed given by A_Q from Proposition 4.2. It remains to show how A acts on the Killing vector fields ∂_{t_i}. Formula (4.5) shows that $i_{\partial_{t_\beta}} \omega = -\mathrm{d}\mu_\beta$, hence, $\partial_{t_\beta} = J \operatorname{grad} \mu_\beta$. Using that A commutes with J and that L is h-selfadjoint, we obtain

$$A \frac{\partial}{\partial t_\beta} = JA(\operatorname{grad} \mu_\beta) = JL(\operatorname{grad}_h \mu_\beta) = \sum_{i,j,\alpha=1}^{\ell} L^i_j \frac{\partial \mu_\beta}{\partial x_i} \frac{\partial x_j}{\partial \mu_\alpha} \frac{\partial}{\partial t_\alpha} = \sum_{\alpha=1}^{\ell} M_\alpha^\beta \frac{\partial}{\partial t_\alpha},$$

which establishes Formula (4.6). □

Thus, we are led to the situation described in Example 3 and, therefore, the second part of Theorem 1.5 is proved.

The main ingredients in the above local formulas are the pair (h, L) on U and the triple $(g_{\mathrm{c}}, \omega_{\mathrm{c}}, A_{\mathrm{c}})$ on S. The 1-forms α_i on S are determined by $(\omega_{\mathrm{c}}, A_{\mathrm{c}})$ only up to the transformation $\alpha_i \longmapsto \alpha_i + \mathrm{d}f_i$ for arbitrary functions f_i on S. However, such functions define a fiber-preserving local transformation $f : M \to M$,

$$f(t, x, y) = (t_1 + f_1(y), \dots, t_\ell + f_\ell(y), x, y),$$

that fulfills $f^*\theta_i = \theta_i + \mathrm{d}f_i$, $f^*\mathrm{d}x_j = \mathrm{d}x_j$ and $f^*\mathrm{d}y_q = \mathrm{d}y_q$ and pulls back the objects in Proposition 4.3 written down w.r.t. θ_i to the corresponding objects written down w.r.t. $\tilde{\theta}_i = \theta_i + \mathrm{d}f_i$. All the other ingredients appearing in the formulas of Proposition 4.3 can be uniquely reconstructed from (h, L) and $(g_{\mathrm{c}}, \omega_{\mathrm{c}}, A_{\mathrm{c}})$. However, we do not know yet whether these ingredients can be arbitrarily chosen or should, perhaps, satisfy some additional restrictions which are not mentioned in Proposition 4.3. The next section shows that there are no more restrictions and (4.4), (4.5) and (4.6) can be used for the local description of c-compatible g and A. To that end, we only need to substitute into these formulas the local normal forms for (h, L) and $(g_{\mathrm{c}}, \omega_{\mathrm{c}}, A_{\mathrm{c}})$ which were previously found in [10] and [14] respectively.

4.2. Realization

The purpose of this section is to prove the following result which is equivalent to the first part of Theorem 1.5.

PROPOSITION 4.4. – *Let $h = \sum_{i,j=1}^{\ell} B_{ij}(x)\mathrm{d}x_i\mathrm{d}x_j$ be a pseudo-Riemannian metric and $L(x)$ an endomorphism on U forming a compatible pair (in the projective sense) and let $(g_{\mathrm{c}}, \omega_{\mathrm{c}})$ be a Kähler structure of arbitrary signature and A_{c} a parallel endomorphism on S. Suppose that the eigenvalues of L and A_{c} satisfy condition (3) from Proposition 4.3 and the 1-forms α_i on S are chosen as in condition (5). Then*

- *g and ω given by* (4.4) *and* (4.5) *define a Kähler structure on $V \times U \times S$;*
- *A given by* (4.6) *is Hermitian w.r.t. (g, ω) and satisfies* (1.4)*, in other words A and (g, ω) are c-compatible.*

Proof. – To verify that g and ω define a Kähler structure, we use Proposition 3.2. Formulas (4.4) and (4.5) are similar to (3.3) and (3.4) but we still need to verify some conditions. First of all, we can use $\mu_1, \ldots, \mu_\ell$ as local coordinates on U to rewrite the term $h = \sum_{i,j} B_{ij}\mathrm{d}x_i\mathrm{d}x_j$ as $\sum_{\alpha,\beta} G_{\alpha\beta}\mathrm{d}\mu_\alpha\mathrm{d}\mu_\beta$, where $B_{ij} = \sum_{\alpha,\beta} G_{\alpha\beta} \frac{\partial\mu_\alpha}{\partial x_i}\frac{\partial\mu_\beta}{\partial x_j}$ and then the matrices $H_{\alpha\beta}$ and $G_{\alpha\beta}$ are inverse to each other as required in Proposition 3.2. Next we need to check that $g_\mu = \sum_{i=0}^{\ell} \mu_i \cdot (-1)^i g_{\mathrm{c}}(A_{\mathrm{c}}^{\ell-i}\cdot,\cdot)$ and $\omega_\mu = \sum_{i=0}^{\ell} \mu_i \cdot (-1)^i \omega_{\mathrm{c}}(A_{\mathrm{c}}^{\ell-i}\cdot,\cdot)$ define a Kähler structure on S for any μ, but this condition immediately follows from the fact that $(g_{\mathrm{c}}, \omega_{\mathrm{c}})$ is Kähler and A_{c} is Hermitian and parallel with respect to it. Notice that the complex structure J_S is, by construction, the same for all (g_μ, ω_μ).

Less trivial is the fact that h is a Hessian metric in the coordinates $\mu_1, \ldots, \mu_\ell$, i.e., that $\partial_{\mu_i} G_{jk} = \partial_{\mu_j} G_{ik}$ holds for all i, j, k (condition (3) from Proposition 3.2). To prove it, we first notice that this condition is equivalent to the fact that the vector fields $\operatorname{grad}\mu_1, \ldots, \operatorname{grad}\mu_\ell$ commute. Indeed, $\partial_{\mu_i} G_{jk} = \partial_{\mu_j} G_{ik}$ means that the 1-forms $\beta_k = \sum_i G_{ik}\mathrm{d}\mu_i$ are all closed. Hence, the statement immediately follows from the observation that the forms $\beta_1, \ldots, \beta_\ell$ are dual to the vector fields $\operatorname{grad}\mu_1, \ldots, \operatorname{grad}\mu_\ell$, i.e., $\beta_k(\operatorname{grad}\mu_j) = \delta_{kj}$.

Thus, it remains to prove the following lemma (cf. Lemma 2.2 (6) which is a c-projective analogue of this statement).

LEMMA 4.5. – *Let h be a pseudo-Riemannian metric on $U \subset \mathbb{R}^\ell$ and L be an h-selfadjoint endomorphism compatible with h in the projective sense. Let*

$$\det(t \cdot \mathrm{Id} - L) = \sum_{i=0}^{\ell} (-1)^i \mu_i t^{\ell-i},$$

where the functions μ_i, $i = 1, \ldots, \ell$, are the elementary symmetric functions in the eigenvalues of L and $\mu_0 = 1$. Then,

$$[\operatorname{grad}\mu_i, \operatorname{grad}\mu_j] = 0 \textit{ for all } i, j.$$

Proof. – First of all, since the vector fields $\operatorname{grad}\mu_1, \dots, \operatorname{grad}\mu_\ell$ are constant linear combinations of the vector fields of the form $v_t = \operatorname{grad}\det(t \cdot \mathrm{Id} - L)$ for $t \in \mathbb{R}$ and vice versa, it suffices to prove that

$$[v_t, v_s] = 0 \tag{4.7}$$

for all $t, s \in \mathbb{R}$. Moreover, it suffices to prove (4.7) for t, s which do not belong to the spectrum of L locally in the neighborhood of a point. For $L : TM \to TM$ an arbitrary non-degenerate endomorphism and X an arbitrary vector field, recall the general formula

$$X(\det L) = (\det L)\mathrm{tr}(L^{-1}\nabla_X L).$$

In our case, L and therefore $L_s = s \cdot \mathrm{Id} - L$ satisfy (1.7), i.e.,

$$\nabla_X(s \cdot \mathrm{Id} - L) = -\nabla_X L = -X^\flat \otimes \Lambda - \Lambda^\flat \otimes X \tag{4.8}$$

holds for a certain vector field Λ. Defining $f_s = \det L_s$ and combining the previous two equations we obtain

$$X(f_s) = -2f_s h(X, L_s^{-1}\Lambda), \tag{4.9}$$

or equivalently, $v_s = \operatorname{grad} f_s = -2f_s L_s^{-1}\Lambda$. Note that this formula is meaningful and holds true even if s is in the spectrum of L. We calculate

$$\nabla_X v_s = 2f_s\left[h(X, L_s^{-1}\Lambda)L_s^{-1}\Lambda - h(\Lambda, L_s^{-1}\Lambda)L_s^{-1}X - L_s^{-1}\nabla_X\Lambda\right].$$

It is a well-known statement in projective geometry that the endomorphisms L and $\nabla\Lambda$ commute, see for example the discussion below Theorem 7 in [7]. Replacing X by v_t in the last equation and using $[L_t^{-1}, \nabla\Lambda] = 0$, we obtain

$$\nabla_{v_t} v_s = 4f_s f_t\left[-h(L_t^{-1}\Lambda, L_s^{-1}\Lambda)L_s^{-1}\Lambda + h(\Lambda, L_s^{-1}\Lambda)L_s^{-1}L_t^{-1}\Lambda + L_s^{-1}L_t^{-1}\nabla_\Lambda\Lambda\right].$$

Thus,

$$\begin{aligned}[v_t, v_s] &= \nabla_{v_t} v_s - \nabla_{v_s} v_t \\ &= 4f_s f_t\left[-h(\Lambda, L_s^{-1}L_t^{-1}\Lambda)(L_s^{-1} - L_t^{-1})\Lambda + h(\Lambda, (L_s^{-1} - L_t^{-1})\Lambda)L_s^{-1}L_t^{-1}\Lambda\right].\end{aligned}$$

Inserting the identity $L_s^{-1} - L_t^{-1} = (t-s)L_s^{-1}L_t^{-1}$ into the last equation, we obtain (4.7) as we claimed. □

Applying this lemma to h and L from Proposition 4.4, we get condition (3) from Proposition 3.2. Thus, now Proposition 3.2 implies that g and ω given by (4.4) and (4.5) indeed define a Kähler structure on $V \times U \times S$ which completes the proof of the first statement of Proposition 4.4.

It is easy to see that A is Hermitian w.r.t. (g, ω). It remains to show that A satisfies (1.4) and we will proceed as follows. Consider the Hermitian metric

$$\hat{g} = (\det_{\mathbb{C}} A)^{-1} g(A^{-1}\cdot, \cdot) \tag{4.10}$$

obtained from g and A by solving (1.2) w.r.t. $\hat{g}$. First, we show that $\hat{g}$ is a Kähler metric on $(V \times U \times S, J)$. Then we show that g and $\hat{g}$ are c-projectively equivalent. This implies that A satisfies (1.4) and we are done.

Proposition 4.6. – *The metric $\hat{g}$ is a Kähler metric on $(V \times U \times S, J)$.*

Proof. – We use the (local) formulas for g, ω and A from Proposition 4.3 and the notation introduced there. Since $\hat{g}$ is Hermitian w.r.t. J by construction, we only need to check that $\hat{\omega} = \hat{g}(J\cdot,\cdot) = (\det_{\mathbb{C}} A)^{-1}\omega(A^{-1}\cdot,\cdot)$ is closed.

Notice that $\det_{\mathbb{C}} A = c\cdot\mu_\ell$ for some constant c so that we may, without loss of generality, replace $\hat{\omega}$ by $\mu_\ell^{-1}\omega(A^{-1}\cdot,\cdot)$.

Then, by (4.5) and (4.6), we get

$$c\cdot\hat{\omega} = \sum_{i=1}^{\ell}\mu_\ell^{-1}\big(\mathrm{d}\mu_i\circ L^{-1}\big)\wedge\theta_i + \sum_{k=0}^{\ell}\frac{\mu_k}{\mu_\ell}\omega_k(A_{\mathrm{c}}^{-1}\cdot,\cdot),$$

where $\omega_k = (-1)^k\omega_{\mathrm{c}}(A_{\mathrm{c}}^{\ell-k}\cdot,\cdot)$.

The closeness of $\hat{\omega}$ now follows from two facts

— $\mu_\ell^{-1}\big(\mathrm{d}\mu_i\circ L^{-1}\big) = -\mathrm{d}\hat{\mu}_{\ell+1-i}$, where $\hat{\mu}_k$ denotes the kth elementary symmetric polynomial in the eigenvalues of L^{-1}, i.e., in $\rho_1^{-1},\dots,\rho_\ell^{-1}$. This is a general property of a compatible pair (h, L), see Lemma 4.7 below.

— $\sum_{k=0}^{\ell}\frac{\mu_k}{\mu_\ell}\omega_k(A_{\mathrm{c}}^{-1}\cdot,\cdot) = -\sum_{k=0}^{\ell}\hat{\mu}_{\ell-k}\omega_{k+1} = -\sum_{i=1}^{\ell}\hat{\mu}_{\ell+1-i}\omega_i - \omega_{\ell+1}$. This relation is straightforward.

Hence

$$c\cdot\hat{\omega} = -\sum_{i=1}^{\ell}\mathrm{d}\hat{\mu}_{\ell+1-i}\wedge\theta_i - \sum_{i=1}^{\ell}\hat{\mu}_{\ell+1-i}\omega_i - \omega_{\ell+1}$$

and the property $\mathrm{d}\hat{\omega} = 0$ becomes obvious, as ω_k's are all closed and $\mathrm{d}\theta_i = \omega_i$ by construction.

Thus, in order to complete the proof of Proposition 4.6 it remains to prove

LEMMA 4.7. – *Let h and L be compatible in the projective sense. Then the following relation holds*

$$\frac{1}{\det L}\big(\mathrm{d}\mu_i\circ L^{-1}\big) = -\mathrm{d}\hat{\mu}_{\ell+1-i}$$

for $i = 1,\dots,\ell$, where $\hat{\mu}_k$ is the kth symmetric polynomial in $\rho_1^{-1},\dots,\rho_\ell^{-1}$.

Proof. – Recall that the compatibility condition (1.7) implies that the Nijenhuis torsion of L vanishes (see for instance [8, Theorem 1]). Lemma 10 from [9] states that for such L the following formula holds:

$$\mathrm{d}\chi_L(t)\circ L - t\cdot\mathrm{d}\chi_L(t) = \chi_L(t)\cdot\mathrm{d}\operatorname{tr} L,$$

where $\chi_L(t) = \det(t\cdot\mathrm{Id} - L)$ is the characteristic polynomial of L. Let us multiply both sides of this formula by L^{-1}:

$$\mathrm{d}\chi_L(t) - t\cdot\mathrm{d}\chi_L(t)\circ L^{-1} = \chi_L(t)\cdot\mathrm{d}\operatorname{tr} L\circ L^{-1}.$$

Hence,

$$\mathrm{d}\chi_L(t)\circ L^{-1} = \frac{1}{t}\left(\mathrm{d}\chi_L(t) - \chi_L(t)\cdot\mathrm{d}\operatorname{tr} L\circ L^{-1}\right).$$

Using another nice formula $\mathrm{d}\operatorname{tr} L \circ L^{-1} = \mathrm{d}(\ln \det L)$, we get

$$\begin{aligned}\mathrm{d}\chi_L(t) \circ L^{-1} &= \frac{1}{t}\left(\mathrm{d}\chi_L(t) - \chi_L(t)\cdot \mathrm{d}(\ln\det L)\right) = \frac{1}{t}\frac{\det L \cdot \mathrm{d}\chi_L(t) - \chi_L(t)\cdot \mathrm{d}\det L}{\det L}\\ &= \frac{\det L}{t}\frac{\det L \cdot \mathrm{d}\chi_L(t) - \chi_L(t)\cdot \mathrm{d}\det L}{\det^2 L} = \frac{\det L}{t}\cdot \mathrm{d}\left(\frac{\chi_L(t)}{\det L}\right),\end{aligned}$$

or equivalently

$$\frac{1}{\det L}\mathrm{d}\chi_L(t)\circ L^{-1} = t^{-1}\mathrm{d}\left(\frac{\chi_L(t)}{\det L}\right),$$

which coincide with the desired relation if we take into account that $\chi_L(t) = \sum_{i=0}^{\ell}(-1)^i \mu_i t^{\ell-i}$ and consider t as a formal parameter. □

REMARK 4.1. – We can derive the formulas in Lemma 4.7 in an alternative way: by the same arguments as used in the proof of Lemma 2.2 (7), one easily derives the formula

$$\operatorname{grad}_{\hat h}\chi_{L^{-1}}(1/t) = \frac{(-1)^\ell}{t^{\ell-1}}\operatorname{grad}_h \chi_L(t), \tag{4.11}$$

where $\hat h$ is the metric given by (1.8). The only thing we used to derive (4.11) is that $\operatorname{grad}_h\rho_i$ is in the ρ_i-eigenspace of L for each eigenvalue ρ_i of L (and, of course, that each ρ_i is smooth with $\mathrm{d}\rho_i \neq 0$). The polynomial expression (4.11) in t resp. $1/t$ gives rise to equivalent equations on the coefficients. These equations are given by $\operatorname{grad}_h\mu_i = -\operatorname{grad}_{\hat h}\hat\mu_{\ell+1-i}$ (or, what is equivalent, $\operatorname{grad}_{\hat h}\hat\mu_i = -\operatorname{grad}_h\mu_{\ell+1-i}$) for all i. Taking into account Formula (1.8) for $\hat h$ and the fact that L is h-selfadjoint, one sees that the latter equations are just the gradient version of the formulas in Lemma 4.7.

Thus, $\hat g$ defined by (4.10) is a Kähler metric on $(V\times U\times S,\ J)$. □

Consider now the Kähler metrics $g, \hat g$ on $(V\times U\times S,\ J)$ with Levi-Civita connections $\nabla, \hat\nabla$ respectively. Let T be the $(1,2)$-tensor defined by

$$T(X,Y) = \hat\nabla_X Y - \nabla_X Y.$$

Since $\nabla, \hat\nabla$ are both torsion-free, T is symmetric in X, Y. Moreover, since both $\nabla, \hat\nabla$ preserve J, we have the symmetry

$$T(X, JY) = T(JX, Y) = JT(X,Y). \tag{4.12}$$

LEMMA 4.8. – *The tensor T satisfies*

$$T(X,Y) = \Phi(X)Y + \Phi(Y)X - \Phi(JX)JY - \Phi(JY)JX \tag{4.13}$$

for a certain 1*-form Φ on $V\times U\times S$.*

Proof. – First we recall that A preserves the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{Q} = \mathcal{V}^\perp$ so that $\mathcal{Q}$ does not change if we consider $\hat g$ instead of g. We will use the same symbol for a vector field on Q and its horizontal lift to $M = V\times U\times S$.

Denoting by $\mathrm{pr}_{\mathcal{Q}} : TM \to \mathcal{Q}$ the projection to the horizontal distribution, the O'Neill Formula (3.7) implies

$$\mathrm{pr}_{\mathcal{Q}}(T(X,Y)) = \hat\nabla^Q_X Y - \nabla^Q_X Y$$

for vector fields $X, Y \in \Gamma(TQ)$, where $\nabla^Q, \hat{\nabla}^Q$ are the Levi-Civita connections of the horizontal parts g_Q and $\hat{g}_Q$ of g and $\hat{g}$ respectively.

From Formula (4.10) we see that $\hat{g}_Q = c \cdot (\det A_Q)^{-1} g_Q(A_Q^{-1}\cdot, \cdot)$ for some constant c, where A_Q is the quotient of A and we used that $\det_{\mathbb{C}} A$ equals $\det A_Q$ up to multiplying with a constant. Comparing this formula for $\hat{g}_Q$ with (1.8) and noting that, by construction, g_Q and A_Q are compatible on Q, we see that $\hat{g}_Q$ is projectively equivalent to g_Q. Thus, we have that

$$\hat{\nabla}^Q_X Y - \nabla^Q_X Y = \Phi(X)Y + \Phi(Y)X \tag{4.14}$$

is satisfied for all $X, Y \in \Gamma(TQ)$ for a 1-form Φ on Q. Indeed, the fact that (4.14) is equivalent to $g_Q, \hat{g}_Q$ being projectively equivalent is a classical statement in projective geometry, see [26].

Using (4.14), we obtain that the horizontal part of T is given by

$$\mathrm{pr}_{\mathcal{Q}}(T(X, Y)) = \Phi(X)Y + \Phi(Y)X.$$

However, since $\mathcal{V}$ is spanned by the Killing vector fields K_i, any g-or $\hat{g}$-geodesic $\gamma(t)$ in M being initially tangent to $\mathcal{Q}$ remains tangent to it for all values of t. It follows that $\nabla_X X, \hat{\nabla}_X X \in \mathcal{Q}$ whenever $X \in \mathcal{Q}$. Then $T(X, X) \in \mathcal{Q}$ for all $X \in \mathcal{Q}$ and by polarization (recall that T is symmetric) we have $T(X, Y) \in \mathcal{Q}$ for all $X, Y \in \mathcal{Q}$. Thus, we obtain

$$T(X, Y) = \Phi(X)Y + \Phi(Y)X, \tag{4.15}$$

for all $X, Y \in \mathcal{Q}$. Since the form Φ in (4.14) is explicitly given by the formula

$$\Phi = -\frac{1}{2}\mathrm{d}\ \ln(\det L)$$

(which is a classical formula that can be obtained from (4.14) by contraction), we see that it vanishes upon insertion of vector fields that are contained in the generalized eigenspaces of A corresponding to constant eigenvalues. Thus, (4.15) establishes Formula (4.13) for vector fields tangent to $\mathcal{Q}$.

It remains to verify Equation (4.13) upon insertion of vertical vector fields JX, JY, where $X, Y \in J\mathcal{V} \subseteq \mathcal{Q}$. Using (4.12), we obtain

$$\begin{aligned} T(JX, JY) &= -T(X, Y) \overset{(4.15)}{=} -\Phi(X)Y - \Phi(Y)X \\ &= \underbrace{\Phi(JX)JY + \Phi(JY)JX}_{=0} - \Phi(JJX)JJY - \Phi(JJY)JJX, \end{aligned}$$

which establishes (4.13) evaluated on JX, JY. Further, for arbitrary Z tangent to $\mathcal{Q}$, we obtain

$$\begin{aligned} T(Z, JX) &= JT(Z, X) \overset{(4.15)}{=} \Phi(Z)JX + \Phi(X)JZ \\ &= \Phi(Z)JX + \underbrace{\Phi(JX)}_{=0} Z - \underbrace{\Phi(JZ)}_{=0} JJX - \Phi(JJX)JZ, \end{aligned}$$

establishing (4.13) when evaluated on Z, JX. Thus, we verified (4.13) on all possible combinations of tangent vectors and the claim follows. □

It is a classical statement in c-projective geometry, see for example [34, 43], and we used this fact already in the proof of Lemma 2.1 that two complex torsion-free connections $\nabla, \hat{\nabla}$ on a complex manifold (M, J) are c-projectively equivalent (i.e., their J-planar curves coincide) if and only if (4.13) is satisfied for a certain 1-form Φ. Lemma 4.8 then shows that $g, \hat{g}$ are c-projectively equivalent. This implies that $A = A(g, \hat{g})$ is a solution of Equation (1.4) and completes the proof of the second part of Proposition 4.4. □

4.3. Explicit formulas

In the preceding sections, we have proved Theorem 1.5 (see Propositions 4.3 and 4.4) which can be understood as an invariant version of Theorem 1.6. We are now going to derive the formulas from Example 5. Our starting point is Proposition 4.3. We will derive explicit formulas for all the objects that have been introduced there and thereby prove Theorem 1.6.

The compatible pair h, L can be described explicitly by using the results from [10]. The latter article contains explicit formulas for a compatible pair in the general pseudo-Riemannian case. In our case, there are no Jordan blocks (with non-constant eigenvalues) and the formulas become similar to the classical Levi-Civita theorem—the only modification being signs $\varepsilon_i = \pm 1$ for each non-constant real eigenvalue ρ_i (which allow us to "produce" an arbitrary signature) and the occurrence of complex eigenvalues. Let

$$E_{\mathrm{nc}} = \{\rho_1, \overline{\rho}_1, \dots, \rho_r, \overline{\rho}_r, \rho_{r+1}, \dots, \rho_{r+q}\}$$

denote the set of (non-constant) eigenvalues of L (r pairs of complex-conjugate eigenvalues and q real eigenvalues). Recall that the "gluing data" in [10, Theorem 1.3] takes the form of a 1-dimensional block

$$h_i = \varepsilon_i \mathrm{d}x_i^2, \;\; L_i = \rho_i(x_i)\partial_{x_i} \otimes \mathrm{d}x_i,$$

for a real eigenvalue ρ_i and (as follows from [10, Theorem 5] or [12, Theorem 2]) the form of a 2-dimensional block

$$h_i = \frac{1}{4}(\overline{\rho}_i(\overline{z}_i) - \rho_i(z_i))(\mathrm{d}z_i^2 - \mathrm{d}\overline{z}_i^2), \;\; L_i = \rho_i(z_i)\partial_{z_i} \otimes \mathrm{d}z_i + \overline{\rho}_i(\overline{z}_i)\partial_{\overline{z}_i} \otimes \mathrm{d}\overline{z}_i,$$

if $\rho_i, \overline{\rho}_i$ is a pair of complex conjugate eigenvalues, where z_j is a complex coordinate w.r.t. which ρ_j is a *holomorphic* function.

Thus, by [10, Theorem 1.3], we find local coordinates $z_1, \dots, z_r, x_{r+1}, \dots, x_{r+q}$ [(3)] (where the z_j are complex coordinates and the x_j are real coordinates) such that

$$\begin{aligned} h &= -\frac{1}{4}\sum_{i=1}^{r}\left(\Delta_i \mathrm{d}z_i^2 + c.c.\right) + \sum_{i=r+1}^{r+q} \Delta_i \varepsilon_i \mathrm{d}x_i^2, \\ L &= \sum_{i=1}^{r}(\rho_i \partial_{z_i} \otimes \mathrm{d}z_i + c.c.) + \sum_{i=r+1}^{r+q} \rho_i \partial_{x_i} \otimes \mathrm{d}x_i, \end{aligned} \tag{4.16}$$

(3) These coordinates should not be confused with the general coordinate system $x_1, \dots, x_\ell$ used, for instance, in Proposition 4.3.

where for $1 \le i \le r$, $\rho_i(z_i)$ is a holomorphic function of z_i, for $r+1 \le i \le r+q$, $\rho_i(x_i)$ only depends on x_i, "$c.c.$" denotes the complex conjugate of the preceding term and

$$\Delta_i = \prod_{\rho \in E_{\mathrm{nc}} \setminus \{\rho_i\}} (\rho_i - \rho).$$

The parts of g, ω and A in (1.16) and (1.18), which correspond to the "constant" block, are obtained from the expressions

$$\sum_{i=0}^{\ell} \mu_i \cdot (-1)^i g_{\mathrm{c}}(A_{\mathrm{c}}^{\ell-i} \cdot, \cdot), \quad \sum_{i=0}^{\ell} \mu_i \cdot (-1)^i \omega_{\mathrm{c}}(A_{\mathrm{c}}^{\ell-i} \cdot, \cdot) \text{ and } \sum_{p,q=1}^{2k} (A_{\mathrm{c}})_p^q \, \mathrm{d}y_p \otimes \left(\frac{\partial}{\partial y_q} - \sum_{i=1}^{\ell} \alpha_{iq} \frac{\partial}{\partial t_i} \right)$$

in the Formulas (4.4), (4.5) and (4.6) by decomposing $(g_{\mathrm{c}}, \omega_{\mathrm{c}})$ in the sense of de Rham–Wu [39, 44] according to the parallel distributions belonging to the generalized eigenspaces of A_{c}. Each component $(g_\gamma, \omega_\gamma, A_\gamma)$ of this decomposition, as we have already mentioned, can be described explicitly using the results of Boubel [14].

To establish the Formulas (1.16) and (1.18) for g, ω and A, it remains to find the formulas for the parts of g, ω and A in the direction tangent to the Killing vector fields. We will specialize the general coordinate system $x_1, \dots, x_\ell$ in Formula (4.4) by choosing $\mu_1, \dots, \mu_\ell$ as coordinates, compare also the Formulas (3.3)–(3.5) in Proposition 3.2. With this choice of coordinates, the matrix H_{ij} (inverse to G_{ij} defined by $h = G_{ij} \mathrm{d}\mu_i \mathrm{d}\mu_j$) is just given by

$$H_{ij} = g(K_i, K_j) = g(\mathrm{grad}_g \mu_i, \mathrm{grad}_g \mu_j) = h(\mathrm{grad}_h \mu_i, \mathrm{grad}_h \mu_j). \tag{4.17}$$

Let $\mu_i(\hat\rho)$ denote the ith elementary symmetric polynomial in the $\ell - 1$ variables $E_{\mathrm{nc}} \setminus \{\rho\}$. Writing $\mathrm{d}\mu_i = \sum_{\rho \in E_{\mathrm{nc}}} \mu_{i-1}(\hat\rho) \mathrm{d}\rho$ in the coordinates from Formula (4.16) we have

$$\mathrm{d}\mu_i = \sum_{s=1}^{r} \left(\mu_{i-1}(\hat\rho_s) \frac{\partial \rho_s}{\partial z_s} \mathrm{d}z_s + c.c. \right) + \sum_{s=r+1}^{r+q} \mu_{i-1}(\hat\rho_s) \frac{\partial \rho_s}{\partial x_s} \mathrm{d}x_s. \tag{4.18}$$

Using the Formula (4.16) for h together with (4.18), we obtain

$$\mathrm{grad}_h \, \mu_j = -4 \sum_{s=1}^{r} \left(\frac{\mu_{j-1}(\hat\rho_s)}{\Delta_s} \frac{\partial \rho_s}{\partial z_s} \frac{\partial}{\partial z_s} + c.c. \right) + \sum_{s=r+1}^{r+q} \varepsilon_s \frac{\mu_{j-1}(\hat\rho_s)}{\Delta_s} \frac{\partial \rho_s}{\partial x_s} \frac{\partial}{\partial x_s}. \tag{4.19}$$

From (4.17), (4.18) and (4.19), we obtain

(4.20)

$$H_{ij} = -4 \sum_{s=1}^{r} \left(\frac{\mu_{i-1}(\hat\rho_s) \mu_{j-1}(\hat\rho_s)}{\Delta_s} \left(\frac{\partial \rho_s}{\partial z_s} \right)^2 + c.c. \right) + \sum_{s=r+1}^{r+q} \varepsilon_s \frac{\mu_{i-1}(\hat\rho_s) \mu_{j-1}(\hat\rho_s)}{\Delta_s} \left(\frac{\partial \rho_s}{\partial x_s} \right)^2.$$

Inserting the Formulas (4.16) and (4.20) into (4.4), we obtain the Formula (1.16) for g. It remains to find the formula for A and the formulas for J acting on θ_i and $\mathrm{d}\rho_j$ respectively.

Taking the differential of the identity

$$\prod_{\rho \in E_{\mathrm{nc}}} (t - \rho) = \sum_{s=0}^{\ell} (-1)^s \mu_s t^{\ell - s}$$

and inserting $t = \rho_i$, we obtain

$$\mathrm{d}\rho_i = \frac{1}{\Delta_i} \sum_{s=1}^{\ell} (-1)^{s-1} \rho_i^{\ell - s} \mathrm{d}\mu_s.$$

By (3.5), we have

$$\mathrm{d}\rho_i \circ J = -\frac{1}{\Delta_i} \sum_{s,t=1}^{\ell} (-1)^{s-1} \rho_i^{\ell-s} H_{st} \theta_t.$$

Inserting (4.20) into this and applying standard Vandermonde identities (see the appendix of [1]), we obtain

$$\mathrm{d}z_i \circ J = 4\frac{1}{\Delta_i} \frac{\partial \rho_i}{\partial z_i} \sum_{j=1}^{\ell} \mu_{j-1}(\hat{\rho}_i)\theta_j \text{ for } 1 \le i \le r$$

and

$$\mathrm{d}x_i \circ J = -\frac{\varepsilon_i}{\Delta_i} \frac{\partial \rho_i}{\partial x_i} \sum_{j=1}^{\ell} \mu_{j-1}(\hat{\rho}_i)\theta_j \text{ for } r+1 \le i \le r+q.$$

Inverting these formulas by using Vandermonde identities shows the formula for $\theta_i \circ J$ expressed as a linear combination of the $\mathrm{d}z_i$ and $\mathrm{d}x_i$. This establishes the formula for J in (1.16).

Let us derive the remaining part of the Formula (1.18) for A. We have to show that $A\frac{\partial}{\partial t_i} = \mu_i \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_{i+1}}$ (where we put $\frac{\partial}{\partial t_{\ell+1}} = 0$). The formula for ω in Proposition 4.3 shows that the Killing vector fields $\frac{\partial}{\partial t_i}$ coincide with $K_i = J \operatorname{grad} \mu_i$. Thus, we need to show that

$$AK_i = \mu_i K_1 - K_{i+1} \text{ for all } i = 1, \ldots, \ell \tag{4.21}$$

(where we put $K_{\ell+1} = 0$), which immediately implies (1.18). As soon as (4.21) is derived, all formulas from Example 5 are established and Theorem 1.6 is proven.

Formula (4.21) is in fact the reformulation to our setting of [1, formula (58)] and it can be proven in the same way: let $v_t = \operatorname{grad}_h \chi_L(t)$, where as usual $\chi_L(t) = \det(t \cdot \mathrm{Id} - L) = \sum_{i=0}^{\ell} (-1)^i \mu_i t^{\ell-i}$ denotes the characteristic polynomial of L (in the terminology of Proposition 4.3). For abbreviation define $v_i = \operatorname{grad}_h \mu_i$ such that $K_i = Jv_i$ and $2\Lambda = v_1$. Recall that, using the compatibility of L and the metric h, we have derived the identity

$$(t \cdot \mathrm{Id} - L)v_t = -\det(t \cdot \mathrm{Id} - L)v_1 = -\sum_{i=0}^{\ell} (-1)^i \mu_i t^{\ell-i} v_1 \tag{4.22}$$

in the proof of Lemma 4.5. Inserting $v_t = \sum_{i=1}^{\ell} (-1)^i t^{\ell-i} v_i$ into the left-hand side of (4.22) and setting $v_{\ell+1} = 0$, we obtain $0 = \sum_{i=1}^{\ell} (-1)^i (v_{i+1} + Lv_i - \mu_i v_1) t^{\ell-i}$, hence, $Lv_i = \mu_i v_1 - v_{i+1}$. Of course, this holds also with L replaced by A and (4.21) follows after multiplying with J and using that A commutes with J.

5. Proof of the Yano-Obata conjecture (Theorem 1.1)

The main goal of this section is to prove Theorem 1.1. Simultaneously, we will give a proof of an important special case of the projective Lichnerowicz conjecture (Theorem 1.2), see Theorem 5.1.

Recall that the existence of a projective vector field v for a (pseudo)-Riemannian metric g implies the existence of an endomorphism A compatible with g. According to Proposition 4.1, in the neighborhood of a regular point, A naturally splits into two blocks A_{c} and L with constant and non-constant eigenvalues respectively.

If the non-constant block L (see Proposition 4.1) is diagonalisable [(4)], i.e., contains no Jordan blocks, then the proofs of Theorems 1.1 and 1.2 are almost identical. For that reason, parallel to the proof of the Yano-Obata conjecture, we will prove the following version of the (pseudo-Riemannian) projective Lichnerowicz conjecture:

THEOREM 5.1. – *Let M be a closed connected manifold with an indefinite metric g on it. Assume that (M, g) admits a projective vector field v and let A be an endomorphism compatible with g in the projective sense, $A \neq c \cdot \mathrm{Id}$. If there exists a regular point $p \in M^0$ at which the non-constant block of A is diagonalisable, then v is affine.*

In other words, this theorem says that in the absence of Jordan blocks with non-constant eigenvalues, the (pseudo-Riemannian) projective Lichnerowicz conjecture holds true, i.e., non-affine projective vector fields do not exist on compact manifolds with indefinite metrics. To complete the proof of this conjecture in full generality, it remains to show that Jordan blocks with non-constant eigenvalues are also "forbidden". For Lorentzian metrics, this will be done in Section 6 which is the final step of the proof of Theorem 1.2.

The proof of Theorem 5.1 is organized as a series of remarks: at any step of the proof of Theorem 1.1 we put a remark explaining how to change, if necessary, the proof in order to obtain a proof of an analogous step of Theorem 5.1. In particular, we use similar notations for the projective and c-projective cases.

5.1. Conventions and degree of mobility

Within the whole §5 (except Remarks 1–9 for Theorem 5.1 where we use similar notation in the projective setting) we assume that (M, g, J) is a closed connected Kähler manifold of (a priori) arbitrary signature and of dimension $2n \geq 4$ (though the case $2n = 4$ has been settled in [11]), and that v is a c-projective vector field which is not an affine vector field. We denote by Φ^v_t its flow, by definition the pullback of g w.r.t. Φ^v_t is a metric which is c-projectively equivalent to g.

We define the *degree of (c-projective) mobility* $D(g, J)$ of (g, J) as the dimension of the space of Hermitian solutions of Equation (1.4). We always have $1 \leq D(g, J) < \infty$ (cf. [16, 21]). If $D(g, J) = 1$, the flow of v acts by homotheties, since otherwise $\Phi^{*}_t g$ is non-proportional to g and hence $D(g, J) \geq 2$. Thus, this case is impossible since we assumed that v is not an affine vector field.

Now, in the case $D(g, J) \geq 3$, Theorem 1.1 follows from [21, Theorem 1], where it is proved that a closed connected Kähler manifold of arbitrary signature with $D(g, J) \geq 3$ is either $(\mathbb{C}P(n), c \cdot g_{\mathrm{FS}}, J_{\mathrm{standard}})$ for some $c \in \mathbb{R} \setminus \{0\}$ or any metric which is c-projectively equivalent to g is affinely equivalent to g. Thus, if $D(g, J) \geq 3$, we are done. In the remaining part of this section, we shall treat the case $D(g, J) = 2$.

(4) Recall that in the c-projective setting this condition is fulfilled automatically (see Lemma 2.2 (1)), whereas in the projective setting non-constant Jordan blocks are allowed.

Here is our first remark related to the proof of Theorem 5.1.

REMARK 1 FOR THEOREM 5.1. For a (pseudo)-Riemannian metric g, the *degree of (projective) mobility* $D(g)$ is the dimension of the vector space of g-selfadjoint solutions of (1.7). If $D(g) = 1$, every projective transformation is a homothety and is an affine transformation. If $D(g) \geq 3$ and g has indefinite signature, then by [30, Corollary 5.2] (which plays here the role analogous to the role of [21] in the Kähler setting) each projective transformation is an affine transformation. Therefore, in the rest of the proof of Theorem 1.1 we may (and will) assume that $D(g) = 2$.

5.2. Scheme of the proof

Let us outline the steps of the proof of Theorem 1.1 under the assumption $D(g, J) = 2$. In §5.3, we will derive the PDE's that describe the evolution of the c-compatible pair g and A along a projective vector field v.

In §5.4 we show that the equation for A can be reduced to one of three a priori possible canonical forms. By using these forms we show that A cannot have non-constant complex eigenvalues and moreover, the real eigenvalues are bounded only for one particular canonical form, namely $\mathcal{L}_v A = A(\mathrm{Id} - A)$. This equation will automatically imply that the constant eigenvalues of A, if they exist, are 0 and 1.

This simplifies the formulas in Theorem 1.6 considerably. In the local classification, g and A are in block-diagonal form. In §5.5, we will obtain a partial solution to the PDE system by only considering the part that corresponds to the block spanned by the gradients of the non-constant eigenvalues $\rho_1, \dots, \rho_\ell$ of A (that is the L-block from Theorem 1.5 and Example 3). The PDE system restricted to this block reduces to ordinary differential equations on a certain set of functions $F_1(\rho_1), \dots, F_\ell(\rho_\ell)$ and on the eigenvalues ρ_i. Thus, we obtain quite explicit formulas for A, g and v involving some yet unknown constants $a_1, \dots, a_\ell$ and $\mathcal{C}$ as parameters.

In §5.6, we will use these formulas to analyze the asymptotic properties of the scalar products $g(K_i, K_j)$ of the Killing vector fields and the eigenvalues of the curvature operator. We will conclude from this analysis that there cannot be more than one non-constant eigenvalue ρ (otherwise the eigenvalues of the curvature operator are unbounded for $t \to \pm\infty$ (which is impossible on a closed manifold) or $g(K_i, K_j)$ does not tend to zero for $t \to \pm\infty$ (as it should)).

Now we are left with a very specific form of the formulas in Theorem 1.6: there is only one non-constant eigenvalue ρ and at most two constant eigenvalues 0 and 1. In §5.7 we complete the proof of Theorem 1.1. We do this by deriving further restrictions on the constant $\mathcal{C}$ from above that appears as a parameter in the metric. Using results of [21] we are then able to conclude that the metric, up to a sign, is positive definite. This traces back the proof of Theorem 1.1 to the corresponding result [32] for positive signature.

REMARK 5.1. – As recalled in Introduction, a Riemannian version of Theorem 1.1 and its generalizations for complete manifolds and for discrete groups of transformations were proved before, in [32, 48, 16, 31]. However, the proofs in all these papers are visually very different from ours. Indeed, the papers [32, 48, 16, 31] do not use local description of c-projectively equivalent metrics or calculations in local coordinates at all. Instead, one studies the evolution of the sectional curvature or the norm of the c-projective Weyl curvature along the orbits of the c-projective group. One observes that they are either unbounded, which may not happen on a closed manifold M, or the metrics satisfy an additional equation which in view of results [21] implies that M is $\mathbb{C}P^n$ with the Fubini-Study metric on it.

This circle of ideas cannot be generalized for metrics of indefinite signature as in this case sectional curvature is usually unbounded even on closed manifolds, and vanishing of the norm of a tensor does not imply that the tensor is zero.

Of course, in our proof we do use certain ideas and results that have been developed and obtained before. As we explained in §5.1 above, a crucial role in our proof belongs to [21, Theorem 1]. Also the results and ideas of [32] related to c-projectively invariant form of (1.4) and special features of the degree of mobility 2 case, see §5.3, are very important for our proof; combining them with local description of c-projectively equivalent metrics we obtain a local description of (pseudo)Kähler metrics admitting essential c-projective vector fields, see §5.5. But then the analysis of such metrics requires completely new methods since those which were effective in the Riemannian situation simply do not work.

We would like to point out some of them, namely those which can be (and have been already) used for solving other problems in differential geometry, see e.g., [13] and [6]):

— Reduction from the c-projective to the projective setting that relates c-projectively equivalent (pseudo)-Kähler metrics with projectively equivalent metrics (with some special properties) and vice versa (Section 3).

— Analysis of the evolution of the eigenvalues of the Hermitian operator A associated with a pair of c-projectively equivalent metrics by (1.2) along a c-projective vector field in the complex domain which finally shows that A admits no complex eigenvalues in the case where the degree of mobility equals two (Section 5.4).

— Studying asymptotic properties of the scalar products $g(K_i, K_j)$ of the Killing vector fields (Section 5.6).

— Interpretation of the curvature tensor R of a Kähler metric as a linear operator acting on the unitary Lie algebra $\mathfrak{u}(g, J)$ and studying its algebraic properties in the case of c-projectively equivalent Kähler metrics. In particular, we show that R belongs to a very special class of the so-called sectional operators well-known and playing an important role in the theory of integrable systems on semisimple Lie algebras (see Appendix).

— Analysis of the eigenvalues of the curvature operator $R : \mathfrak{u}(g, J) \to \mathfrak{u}(g, J)$ based on the latter observation (Proposition A.2).

5.3. C-projectively invariant form of Equation (1.4) and special features of $D(g, J) = 2$

Consider the canonical line bundle $\mathcal{E} = \Lambda^{2n}T^*M$ over the Kähler manifold (M, g, J) (where $2n = \dim_{\mathbb{R}} M$). For a real number w, we define the line bundle $\mathcal{E}(w)$ whose transition functions are given by the transition functions of $\mathcal{E}$ to the power w. Note that M, as a complex manifold, has a canonical orientation and we can assume positivity of the transition functions of $\mathcal{E}$. For an arbitrary tensor bundle E over M, we can then define the "weighted version" $E(w) = E \otimes \mathcal{E}(w)$. Let $S^2_J TM$ denote the bundle of Hermitian contravariant

2-tensors and denote by ∇ the Levi-Civita connection of g. In the appendix of [32] it was shown that the PDE

$$\nabla_X \sigma = X \odot \Lambda_\sigma + JX \odot J\Lambda_\sigma, \quad X \in TM \tag{5.1}$$

on sections σ of $S^2_J TM(\frac{1}{n+1})$, is c-projectively invariant, that is, it does not depend on the choice of connection ∇ in the class $[\nabla]$ of c-projectively equivalent connections. The weighted vector field $\Lambda_\sigma \in \Gamma(TM(\frac{1}{n+1}))$ in (5.1) is the ∇-divergence of σ divided by $2n$ and $X \odot Y = X \otimes Y + Y \otimes X$ denotes the symmetric product. For details we refer the reader to the appendix of [32]. Let us denote by $\mathcal{A}([g], J) \subseteq S^2_J TM(\frac{1}{n+1})$ the space of solutions of (5.1). There is an isomorphism between the space $\mathcal{A}(g, J)$ of Hermitian solutions to (1.4) (i.e., the space of Hermitian endomorphism c-compatible with (g, J)) and $\mathcal{A}([g], J)$ given by

$$\varphi : \mathcal{A}([g], J) \longmapsto \mathcal{A}(g, J), \quad \varphi(\sigma) = \sigma \sigma_g^{-1},$$

where

$$\sigma_g = g^{-1} \otimes \mathrm{vol}_g^{\frac{1}{n+1}}$$

and vol_g is the volume form of g. Note that $\sigma_g^{-1} \in S^2_J T^*M(-\frac{1}{n+1})$, so in $\varphi(\sigma) = \sigma\sigma_g^{-1}$ the weight cancels out and we obtain an ordinary field of endomorphisms. As described in more detail in the appendix of [32], taking the Lie derivative $\mathcal{L}_v \sigma$ of a solution σ to (5.1) w.r.t. the c-projective vector field v yields again a solution to (5.1). Thus, we obtain a linear mapping

$$\mathcal{L}_v : \mathcal{A}([g], J) \longmapsto \mathcal{A}([g], J).$$

Under the assumption $D(g, J) = 2$, we can chose a basis $\sigma, \hat{\sigma}$ of $\mathcal{A}([g], J)$ and find the equations

$$\begin{aligned} \mathcal{L}_v \sigma &= \alpha\sigma + \beta\hat{\sigma}, \\ \mathcal{L}_v \hat{\sigma} &= \gamma\sigma + \delta\hat{\sigma} \end{aligned} \tag{5.2}$$

for certain real numbers $\alpha, \beta, \gamma, \delta$.

Using (5.2) we can easily derive the Lie derivatives of $A \in \mathcal{A}(g, J)$ and g along v.

PROPOSITION 5.2. – *Let v be a c-projective vector field for g and $A \in \mathcal{A}(g, J)$, $A \neq c \cdot \mathrm{Id}$. Then we have the equations*

$$\mathcal{L}_v A = -\beta A^2 + (\delta - \alpha) A + \gamma \mathrm{Id} \tag{5.3}$$

and

$$\mathcal{L}_v g = \left(-\frac{\beta}{2} \mathrm{tr}\, A - (n+1)\alpha \right) g - \beta g A, \tag{5.4}$$

for some constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Moreover, the restriction $\rho(t) = \rho(\Phi^v_t(p))$ of every eigenvalue ρ of A to an integral curve of v satisfies the ODE

$$\dot{\rho} = -\beta\rho^2 + (\delta - \alpha)\rho + \gamma. \tag{5.5}$$

Proof. – Take $\sigma = \sigma_g$ and $\hat{\sigma} = \varphi^{-1}(A) = A\sigma_g$ as a basis of $\mathcal{A}([g], J)$. Then by using (5.2), we get

$$\begin{aligned} \mathcal{L}_v A = \mathcal{L}_v(\hat{\sigma}\sigma_g^{-1}) &= (\gamma\sigma_g + \delta\hat{\sigma})\sigma_g^{-1} - \hat{\sigma}\sigma_g^{-1}(\alpha\sigma_g + \beta\hat{\sigma})\sigma_g^{-1} \\ &= \gamma \mathrm{Id} + \delta A - \alpha A - \beta A^2. \end{aligned}$$

This yields (5.3). Moreover, a straightforward calculation yields

$$\begin{aligned}\mathcal{L}_v g &= \mathcal{L}_v\left((\det g)^{\frac{1}{2(n+1)}}\,\sigma_g^{-1}\right) = \frac{1}{2(n+1)}\operatorname{tr}(g^{-1}\mathcal{L}_v g)g - g(\alpha\sigma_g + \beta\hat{\sigma})\sigma_g^{-1}\\ &= \left(\frac{1}{2(n+1)}\operatorname{tr}(g^{-1}\mathcal{L}_v g) - \alpha\right)g - \beta g A.\end{aligned}$$

Left-multiplying by g^{-1} and taking the trace of this equation gives us

$$\frac{1}{2(n+1)}\operatorname{tr}(g^{-1}\mathcal{L}_v g) = -\alpha n - \frac{\beta}{2}\operatorname{tr} A$$

and inserting this back into the equation for the Lie derivative of g, we obtain (5.4).

The remaining Formula (5.5) immediately follows from (5.3). □

Remark 2 for Theorem 5.1. The projective (and projectively invariant) analogue of (5.1) was obtained in [20]. Arguing as above we obtain the following version of Proposition 5.2:

Let v be a projective vector field for g and let A be compatible to g in the projective sense, $A \neq c \cdot \mathrm{Id}$. Then we have (5.3) *and* (5.5) *and the Lie derivative of g satisfies*

$$\mathcal{L}_v g = \big(-\beta \operatorname{tr} A - (n+1)\alpha\big)g - \beta g A, \quad n = \dim M,$$

i.e., in the first term on the right-hand side of (5.4) we simply need to replace $\frac{\beta}{2}$ by β.

5.4. Properties of the eigenvalues of A

The Equation (5.5) allows us to make several important conclusions about the eigenvalues of $A \in \mathcal{A}(g, J)$.

First of all we notice that the coefficient β in Equation (5.5) does not vanish. Indeed, otherwise this equation takes the form $\dot{\rho} = (\delta - \alpha)\rho + \gamma$ and its non-constant solutions are unbounded which is impossible due to compactness of M. Hence the eigenvalues of A are all constant which contradicts the assumption that v is not affine. Thus, $\beta \neq 0$.

Next, we see that the constant eigenvalues of A are solutions to the quadratic equation $0 = -\beta x^2 + (\delta - \alpha)x + \gamma$. Hence, there are at most two constant eigenvalues.

To simplify the further discussion, without loss of generality we may assume that the evolution of a non-constant eigenvalue ρ of A along v is given by one of the ODEs

$$\dot{\rho} = \rho^2 + 1, \quad \dot{\rho} = \rho(1-\rho) \text{ or } \dot{\rho} = \rho^2. \tag{5.6}$$

Indeed, the Equations (5.5) can be reduced to one of these canonical forms by rescaling v and replacing A by $c_1 A + c_2\mathrm{Id}$ for an appropriate choice of constants $c_1 \neq 0, c_2 \in \mathbb{R}$.

We now show that the eigenvalues of A cannot be complex.

Proposition 5.3. – *Let (M, g, J) be a closed connected Kähler manifold of degree of mobility $D(g, J) = 2$ and let v be a c-projective vector field which is not affine. Then all non-constant eigenvalues of $A \in \mathcal{A}(g, J)$ are real-valued.*

Proof. – If we allow ρ to be complex-valued, each of the above ODEs (5.6) should be considered as a system of two ODEs on the real and imaginary part of ρ. The phase portraits corresponding to these systems are shown in Figure 1. It can be seen from the pictures or shown directly using the ODEs (5.6), that a non-constant solution to one of these equations with imaginary part not identically zero is given by a circle in the complex plane.

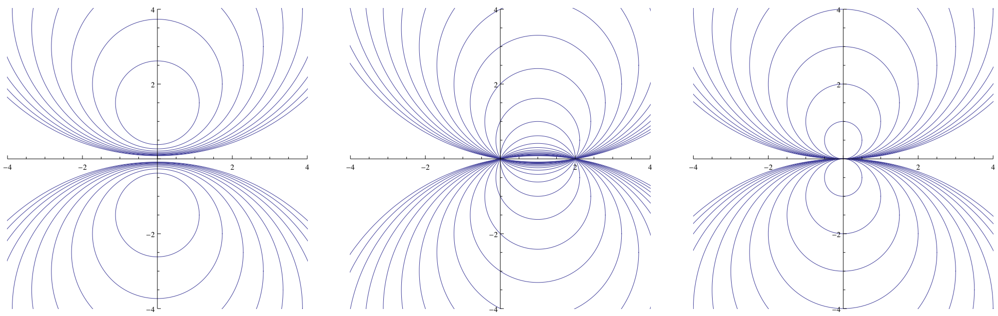

FIGURE 1. The phase portraits for the ODEs $\dot{\rho} = \rho^2 + 1$, $\dot{\rho} = \rho(1-\rho)$, $\dot{\rho} = \rho^2$ (from left to right).

As we see from Figure 1, the maximal value of an imaginary part of a complex eigenvalue, taken over all points of the manifold and all eigenvalues, is not equal to the imaginary part of a constant eigenvalue (for each case in (5.6), the constant eigenvalues are $\pm i$, 0 and 1 or 0 resp.). In particular, the derivative of this eigenvalue in the direction of the c-projective vector field is well-defined and non-zero. More precisely, the derivative of the imaginary part of ρ is zero whereas the derivative of the real part is not. We now show that for complex eigenvalues such a situation is impossible.

Basically this fact follows from our local description of c-projectively equivalent metrics (Theorem 1.6) which states that the complex eigenvalues are holomorphic in an appropriate local coordinate system. However, such a coordinate system exists only at generic points, so we need to modify this idea and in particular to take into account the fact that the eigenvalues cannot be considered as smooth functions at "collision" points where the multiplicities of the eigenvalues change.

Assume that at a point $p \in M$, a complex eigenvalue ρ of A, $\mathrm{Im}\,\rho(p) \neq 0$, has multiplicity k. It follows from standard facts that for small neighborhoods $U(p) \subseteq M$ of p and $U(\rho(p)) \subseteq \mathbb{C}$ of $\rho(p)$, A has precisely k eigenvalues $\rho_1, \ldots, \rho_k$ at each point of $U(p)$ contained in $U(\rho(p))$ and that the elementary symmetric functions in the variables $\rho_1, \ldots, \rho_k$ are smooth complex-valued functions on $U(p)$. In particular, the function $\rho_1 + \cdots + \rho_k$ is smooth in $U(p)$.

LEMMA 5.4. – *Let the differential of the imaginary part of $\rho_1 + \cdots + \rho_k$ vanish at p. Then*

$$\mathrm{d}(\rho_1 + \cdots + \rho_k) = 0.$$

Proof. – By contradiction, assume that $\mathrm{d}(\rho_1 + \cdots + \rho_k) \neq 0$. Consider the smooth endomorphism

$$\tilde{A} = (A - \rho_1 \cdot \mathrm{Id}) \cdots (A - \rho_k \cdot \mathrm{Id}) : T^*_{\mathbb{C}} U(p) \to T^*_{\mathbb{C}} U(p),$$

where $T^*_{\mathbb{C}}U(p)$ denotes the complexified cotangent bundle of $U(p)$. The kernel of $\tilde{A}$ defines a smooth complex k-dimensional distribution D in $T^*_{\mathbb{C}}U(p)$ whose value $D(p)$ at the point p coincides with the kernel of $(A - \rho(p) \cdot \mathrm{Id})^k$. The subspace $D(p)$ is therefore the generalized $\rho(p)$-eigenspace of A.

The differential of the smooth function $\rho_1 + \cdots + \rho_k$ at a generic point of $U(p)$ is $\mathrm{d}\rho_1 + \cdots + \mathrm{d}\rho_k$ and since $\mathrm{d}\rho_i \circ A = \rho_i \mathrm{d}\rho_i$ (because, by Lemma 2.2, grad ρ_i is an eigenvector with eigenvalue ρ_i of the g-selfadjoint endomorphism A), we obtain that $\mathrm{d}(\rho_1 + \cdots + \rho_k)$ is contained in D at every point of $U(p)$. In particular we have that at p, $\mathrm{d}(\rho_1 + \cdots + \rho_k)$ is contained in $D(p)$, i.e., it is a generalized eigenvector of A corresponding to $\rho(p)$. On the other hand, since $\mathrm{d}\big(\mathrm{Im}(\rho_1 + \cdots + \rho_k)\big)(p) = 0$, $\mathrm{d}(\rho_1 + \cdots + \rho_k)$ is a real 1-form at p. This contradicts the following simple fact from Linear Algebra: if ρ is a complex eigenvalue of a real linear endomorphism A, then the generalized $\rho(p)$-eigenspace of A contains no real vectors. □

Now take a point $p \in M$ such that the imaginary part of an eigenvalue ρ of multiplicity k at p takes its maximum in the sense that it is greater than or equal to the imaginary parts of all eigenvalues at all points of M and apply Lemma 5.4. The imaginary part of $\rho_1 + \cdots + \rho_k$ obviously satisfies $\mathrm{d}\,(\mathrm{Im} \sum \rho_i(p)) = 0$. On the other hand, the Lie derivative of $\sum \rho_i$ along v at the point p is $k \cdot \dot{\rho} \neq 0$, where $\dot{\rho}$ is defined by one of the Equations (5.6). Thus, $\mathrm{d}(\rho_1 + \cdots + \rho_k) \neq 0$ in contradiction with Lemma 5.4. This completes the proof. □

In fact, only the second equation of (5.6) is allowed. Indeed, since v is not affine, A must have at least one non-constant eigenvalue and, moreover, this eigenvalue has to be real according to Proposition 5.3. But it is easy to see that the equations $\dot{\rho} = \rho^2 + 1$ and $\dot{\rho} = \rho^2$ have no bounded real-valued non-constant solutions whereas $\rho(t)$ must be bounded due to compactness of M.

Thus, we are left with the case where the eigenvalues of A satisfy

$$v(\rho) = \rho(1 - \rho) \tag{5.7}$$

and we may summarize the above discussion in the following

PROPOSITION 5.5. – *Let (M, g, J) be a closed connected Kähler manifold of real dimension $2n \geq 4$ such that $D(g, J) = 2$ and let v be a c-projective vector field that is not affine. Then, after an appropriate rescaling of v, we can find $A \in \mathcal{A}(g, J)$ such that*

1. *the eigenvalues of A satisfy* (5.7)*;*
2. *the eigenvalues of A are all real;*
3. *A has at most two constant eigenvalues* 0 *and* 1.

REMARK 3 FOR THEOREM 5.1. The reduction of (5.5) to one of the canonical forms (5.6) is a simple general fact from the theory of ODE's. The statement of Proposition 5.3 was proved in [10, Theorem 1.11] even under more general assumptions: on a compact manifold M, each non-real eigenvalue of an endomorphism A compatible with g is necessarily constant.

The first and third canonical forms from (5.6) are impossible for the same reason: in these two cases non-constant real eigenvalues are not bounded (along v). Thus, the eigenvalues of A satisfy (5.7) and the statement of Proposition 5.5 remains true without any changes.

5.5. Explicit formulas for the non-constant block

In what follows, we will work with $A \in \mathscr{A}(g, J)$ and a c-projective vector field v as in Proposition 5.5. In particular, we assume that the non-constant eigenvalues $\rho_1, \dots, \rho_\ell$ of A are real-valued and the constant eigenvalues, 0 and 1, have multiplicities $m_0 \geq 0$ and $m_1 \geq 0$ respectively.

Under these assumptions, the PDE systems (5.3) and (5.4) take the form

$$\begin{aligned} \mathcal{L}_v A &= A(\mathrm{Id} - A), \\ \mathcal{L}_v g &= -gA - \left(\sum_{i=1}^{\ell} \rho_i + \mathcal{C}\right) g, \end{aligned} \tag{5.8}$$

where $\mathcal{C}$ is some constant (yet unknown and playing the role of an additional parameter).

We are going to evaluate the PDE system (5.8) component-wise. To make this more transparent, we notice that on the set of regular points M^0 there is a natural structure of two mutually orthogonal foliations. Recall that $\mathscr{J}$ denotes the integrable and totally geodesic distribution spanned by the commuting vector fields given by the Killing vector fields $K_1, \dots, K_\ell$ and the vector fields $JK_1, \dots, JK_\ell$. Now let $\mathscr{U}$ be the distribution generated by $\operatorname{grad} \rho_1, \dots, \operatorname{grad} \rho_\ell$ (or, equivalently, by $JK_1, \dots, JK_\ell$) so that $\mathscr{J} = \mathscr{U} \oplus \mathscr{V}$, where $\mathscr{V}$ is the distribution generated by $K_1, \dots, K_\ell$ and defined in the first part of the article. The leaves of the other distribution $\mathscr{U}^\perp$, orthogonal to $\mathscr{U}$, are just common level surfaces of the eigenvalues $\rho_1, \dots, \rho_\ell$. Note that both distributions are integrable: for $\mathscr{U}^\perp$ this holds by definition and for $\mathscr{U}$ it follows from the fact that $\mathscr{U}$ is generated by the commuting vector fields $JK_1, \dots, JK_\ell$.

The next statement summarizes some general properties of $\mathscr{U}$ which hold true for any metric g from Theorem 1.6.

Proposition 5.6. – *Let $M^0 \subset M$ be the set of regular points (in particular, the non-constant eigenvalues $\rho_1, \dots, \rho_\ell$ of A are all distinct and independent on M^0). Then on M^0 there is a structure of the integrable distribution $\mathscr{U}$ generated by* grad ρ_i $(i = 1, \dots, \ell)$ *with the following properties:*

1. *The leaves of $\mathscr{U}$ are totally geodesic.*
2. *The leaves of $\mathscr{U}^\perp$ are common level surfaces of $\rho_1, \dots, \rho_\ell$.*
3. *Let $\mathcal{L} \subset M^0$ be a leaf of $\mathscr{U}$, then $g|_{\mathcal{L}}$ and $A|_{\mathcal{L}}$ are compatible in the projective sense.*
4. *The non-constant eigenvalues $\rho_1, \dots, \rho_\ell$ can be considered as local coordinates on $\mathcal{L}$.*
5. *Locally the metric g can be written as*
$$g = \sum_{i=1}^{\ell} g_i(\vec{\rho}) \mathrm{d}\rho_i^2 + \sum_{\alpha,\beta} b_{\alpha\beta}(\rho, y) \mathrm{d}y_\alpha \mathrm{d}y_\beta,$$
where $\sum_{i=1}^{\ell} g_i(\vec{\rho}) \mathrm{d}\rho_i^2$ is $g|_{\mathcal{L}}$.
6. *The vector fields* grad ρ_i *on M and on $\mathcal{L}$ coincide (no matter which metric, g or $g|_{\mathcal{L}}$, is used to take the gradient) and therefore the quantities $g(\operatorname{grad} f_1(\vec{\rho}), \operatorname{grad} f_2(\vec{\rho}))$ do not depend on how we compute them (on M or on $\mathcal{L}$).*

7. *The vector field* $\Lambda = \frac{1}{4}\mathrm{grad}\,(\mathrm{tr}\,A)$ *is the same on* $\mathcal{L}$ *and* M. *The tangent space of* $\mathcal{L}$ *is invariant with respect to the endomorphism* $\nabla\Lambda$, *moreover, the restriction of* $\nabla\Lambda$ *on* $\mathcal{L}$ *coincides with* $\nabla|_{\mathcal{L}}\Lambda$ *computed on* $\mathcal{L}$. *This implies in particular that the eigenvalues of* $\nabla|_{\mathcal{L}}\Lambda$ *are some of the eigenvalues of* $\nabla\Lambda$.
8. *The leaves of* $\mathcal{U}$ *are (locally) isometric.*

Proof. – In view of $\nabla J = 0$ and the fact that K_i, JK_i are holomorphic, (1) follows immediately from Lemma 2.2 (8).

(2) is clear from the definition.

(3) follows immediately from (1) combined with Equation (1.4) (compare also Lemma 3.3).

(4) follows from the assumption that $\rho_1, \dots, \rho_\ell$ are independent.

(5) is immediate from the local classification Theorem 1.6 and (6) follows directly from (5). The first statement of (7) follows from (6) and the formula $\Lambda = \frac{1}{2}\sum_{i=1}^{\ell}\mathrm{grad}\,\rho_i$. The remaining statements of (7) then follow directly from (1).

The last statement (8) can be seen from the formula for g in (5). □

Remark 4 for Theorem 5.1. In the projective setting, the definition of the set M^0 of regular points remains essentially the same. We say that $p \in M$ is regular if: 1) the algebraic type of A does not change in a (sufficiently small) neighborhood U of p, 2) each eigenvalue ρ is either constant on U (and then ρ equals either 0 or 1) or, if ρ is not constant on U, then $\mathrm{d}\rho(p) \neq 0$. Recall that in view of (5.7):

$$\rho(p) \neq 0 \text{ or } 1 \quad \Rightarrow \quad \mathrm{d}\rho(p) \neq 0.$$

Clearly, M^0 is open and everywhere dense on M. However, a priori M^0 might contain several components related to different algebraic types. In particular, the number ℓ of non-constant eigenvalues may be different for different components of M^0. We continue to work with one of the components of M^0 (for simplicity, we will still denote this component by M^0). Since we want to prove Theorem 5.1 by contradiction, we assume from now on that there is a regular point (or, which is the same, one of the connected components of M^0) where the non-constant part of A is diagonalisable (over $\mathbb{R}$), i.e., with no Jordan blocks. Recall from the formulas in [10, Theorems 1.3 and 1.5] that in this case each non-constant eigenvalue has multiplicity one.

Summarizing we have an open subset $M^0 \subset M$ with the following properties:

1. at each point $p \in M^0$, the endomorphism A has ℓ non-constant real eigenvalues $\rho_1 < \dots < \rho_\ell$, each of multiplicity one;
2. $\mathrm{d}\rho_i \neq 0$ everywhere on M^0;
3. the “constant part” of A has some fixed algebraic type with at most two eigenvalues 0 and 1 of multiplicity m_0 and m_1 respectively.

In particular, according to [10], locally in the neighborhood of every point $p \in M^0$, we can choose a coordinate system $\rho_1, \dots, \rho_\ell, y_1, \dots, y_s$ (cf. Proposition 4.1) in which both g and A split:

$$g = \begin{pmatrix} h(\rho) & 0 \\ 0 & g_{\mathrm{c}}(y)\cdot\chi_L\big(A_{\mathrm{c}}(y)\big)\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} L(\rho) & 0 \\ 0 & A_{\mathrm{c}}(y)\end{pmatrix}.$$

Here $L = \operatorname{diag}(\rho_1, \dots, \rho_\ell)$, the metric h and the endomorphism L are compatible in the projective sense, A_c is parallel w.r.t. g_c and $\chi_L(t) = (t - \rho_1)(t - \rho_2) \cdots (t - \rho_\ell)$ is the characteristic polynomial of L. This decomposition into "constant" and "non-constant" blocks can naturally be reformulated in terms of two orthogonal foliations $\mathcal{U}$ and $\mathcal{U}^\perp$. Proposition 5.6 remains true without any changes.

Also notice that (5.8) holds without any change in spite of the fact that in (5.4) we should replace $\frac{\beta}{2}$ by β. This happens because $\frac{1}{2}$ is compensated by the factor 2 in the formula for the trace in the c-projective setting where $\operatorname{tr} A = 2\sum \rho_i + \text{const}$ (whereas in the real case, $\operatorname{tr} A = \sum \rho_i + \text{const}$).

The further analysis deals with the restrictions of g and v to leaves of $\mathcal{U}$. These restrictions are exactly the same for the projective and c-projective cases and, until §5.7, the proof for both cases will be the same. The difference which appears in §5.7 will be clearly described.

We now want to derive explicit formulas for the restrictions of g and v to leaves of $\mathcal{U}$. This is sufficient for many discussions since some of the globally defined objects derived from g and A only depend on the level sets of the non-constant eigenvalues, see §5.6 below.

We first notice that according to Theorem 1.6, at regular points $p \in M^0$ we have $\rho_i = \rho_i(x_i)$ and moreover $\frac{\partial \rho_i}{\partial x_i} \neq 0$. This means, in particular, that we can use ρ_i as local coordinates instead of x_i. This change of variables will simplify further computations.

Let $v = v_1 + v_2$ be the decomposition of the c-projective vector field v w.r.t. $TM = \mathcal{U} \oplus \mathcal{U}^\perp$. According to Proposition 5.6, there are certain functions v_1^i such that v can be written as

$$v = \sum_{i=1}^{\ell} v_1^i \frac{\partial}{\partial \rho_i} + v_2. \tag{5.9}$$

Since each eigenvalue ρ_i satisfies (5.7) and is constant in the direction of $\mathcal{U}^\perp$, we immediately see that

$$v(\rho_i) = v_1^i \frac{\partial \rho_i}{\partial \rho_i} = \rho_i(1 - \rho_i),$$

that is, $v_1^i = \rho_i(1 - \rho_i)$.

From Theorem 1.6, we know that in our simplified situation (no complex non-constant eigenvalues), g and gA can be written in the form (after the above change of variables $x_i \leftrightarrow \rho_i$),

$$\begin{aligned} g &= \sum_{i=1}^{\ell} \frac{\Delta_i}{F_i} \mathrm{d}\rho_i^2 + \sum_{i,j=1}^{\ell} H_{ij}\theta_i\theta_j + g_c(\chi_{nc}(A_c)\cdot, \cdot), \\ gA &= \sum_{i=1}^{\ell} \rho_i \frac{\Delta_i}{F_i} \mathrm{d}\rho_i^2 + \sum_{i,j=1}^{\ell} \tilde{H}_{ij}\theta_i\theta_j + g_c(\chi_{nc}(A_c)A_c\cdot, \cdot), \end{aligned} \tag{5.10}$$

where $\Delta_i = \prod_{j \neq i}(\rho_i - \rho_j)$, $F_i = \varepsilon_i \left(\frac{\partial \rho_i}{\partial x_i}\right)^2$, the functions H_{ij}, $\tilde{H}_{ij}$ only depend on $\rho_1, \dots, \rho_\ell$ and $\chi_{nc}(t) = \prod_{i=1}^{\ell}(t - \rho_i)$.

PROPOSITION 5.7. – *Locally, the metric g and the c-projective vector field v in the coordinates $\rho_1, \dots, \rho_\ell$ are given by the formulas*

$$g = \sum_{i=1}^{\ell} \frac{\Delta_i}{F_i} \mathrm{d}\rho_i^2 + \cdots, \qquad v = \sum_{i=1}^{\ell} \rho_i(1-\rho_i)\frac{\partial}{\partial \rho_i} + \cdots, \tag{5.11}$$

where $\Delta_i = \prod_{j\neq i}(\rho_i - \rho_j)$ and

$$F_i(t) = a_i(1-t)^{-\mathcal{C}} t^{1+\ell+\mathcal{C}} \tag{5.12}$$

for some real constants a_i and $\mathcal{C}$.

Note that the term $g - \sum_{i=1}^{\ell} \frac{\Delta_i}{F_i}\mathrm{d}\rho_i^2$ in Formula (5.11), which is not written down explicitly, coincides with $\sum_{\alpha,\beta} b_{ij}(\rho, y) dy_\alpha dy_\beta$ from part (5) of Proposition 5.6. The expression $v - \sum_{i=1}^{\ell} \rho_i(1-\rho_i)\frac{\partial}{\partial\rho_i}$ is just the projection of v onto $\mathcal{U}^\perp$.

Proof. – It easily follows from the explicit form of g given by (5.10) that the second equation of (5.8) implies

$$\mathcal{L}_{v_1}\left(\sum_{i=1}^{\ell} \frac{\Delta_i}{F_i}\mathrm{d}\rho_i^2\right) = -\sum_{i=1}^{\ell}\left(\rho_i + \sum_{j=1}^{\ell}\rho_j + \mathcal{C}\right)\frac{\Delta_i}{F_i}\mathrm{d}\rho_i^2, \tag{5.13}$$

where $v_1 = \sum \rho_j(1-\rho_j)\frac{\partial}{\partial\rho_j}$. In other words, the first part of g can be differentiated along v independently of the remaining terms.

From this equation we can easily derive the explicit formulas for F_i. To that end, we compute the left hand side of (5.13):

$$\begin{aligned}
\mathcal{L}_{v_1}\left(\sum_{i=1}^{\ell}\frac{\Delta_i}{F_i}\mathrm{d}\rho_i^2\right) &= \sum_{i=1}^{\ell}\sum_{j\neq i}\rho_j(1-\rho_j)\frac{\partial}{\partial\rho_j}\left(\frac{\Delta_i}{F_i}\right)\mathrm{d}\rho_i^2 \\
&\quad + \sum_{i=1}^{\ell}\rho_i(1-\rho_i)\frac{\partial}{\partial\rho_i}\left(\frac{\Delta_i}{F_i}\right)\mathrm{d}\rho_i^2 + \sum_{i=1}^{\ell}\frac{\Delta_i}{F_i}L_{v_1}\mathrm{d}\rho_i^2 \\
&= \sum_{i=1}^{\ell}\sum_{j\neq i}\frac{\rho_j(1-\rho_j)}{\rho_j-\rho_i}\cdot\frac{\Delta_i}{F_i}\mathrm{d}\rho_i^2 \\
&\quad + \sum_{i=1}^{\ell}\sum_{j\neq i}\frac{\rho_i(1-\rho_i)}{\rho_i-\rho_j}\cdot\frac{\Delta_i}{F_i}\mathrm{d}\rho_i^2 - \sum_{i=1}^{\ell}\rho_i(1-\rho_i)\frac{\partial F_i}{\partial\rho_i}\frac{\Delta_i}{F_i^2}\mathrm{d}\rho_i^2 \\
&\quad + 2\sum_{i=1}^{\ell}\frac{\Delta_i}{F_i}\frac{\partial}{\partial\rho_i}\big(\rho_i(1-\rho_i)\big)\mathrm{d}\rho_i^2 \\
&= \sum_{i=1}^{\ell}\sum_{j\neq i}\left(\frac{\rho_j(1-\rho_j)}{\rho_j-\rho_i} + \frac{\rho_i(1-\rho_i)}{\rho_i-\rho_j}\right)\cdot\frac{\Delta_i}{F_i}\mathrm{d}\rho_i^2 \\
&\quad - \sum_{i=1}^{\ell}\rho_i(1-\rho_i)\frac{\partial F_i}{\partial\rho_i}\frac{\Delta_i}{F_i^2}\mathrm{d}\rho_i^2 + 2\sum_{i=1}^{\ell}\frac{\Delta_i}{F_i}(1-2\rho_i)\mathrm{d}\rho_i^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\ell}\left(\sum_{j\neq i}(1-\rho_i-\rho_j)-\rho_i(1-\rho_i)\frac{\partial \ln F_i}{\partial \rho_i}+2(1-2\rho_i)\right)\frac{\Delta_i}{F_i}\mathrm{d}\rho_i^2\\
&= \sum_{i=1}^{\ell}\left(\ell-1-(\ell-2)\rho_i-\sum_{j=1}^{\ell}\rho_j-\rho_i(1-\rho_i)\frac{\partial \ln F_i}{\partial \rho_i}+2(1-2\rho_i)\right)\frac{\Delta_i}{F_i}\mathrm{d}\rho_i^2\\
&= \sum_{i=1}^{\ell}\left(\ell+1-(\ell+2)\rho_i-\sum_{j=1}^{\ell}\rho_j-\rho_i(1-\rho_i)\frac{\partial \ln F_i}{\partial \rho_i}\right)\frac{\Delta_i}{F_i}\mathrm{d}\rho_i^2.
\end{aligned}
$$

Comparing this expression with the right hand side of (5.13), we obtain a simple ODE on F_i:

$$
\ell+1-(\ell+2)\rho_i-\sum_{j=1}^{\ell}\rho_j-\rho_i(1-\rho_i)\frac{\partial \ln F_i}{\partial \rho_i} = -\left(\rho_i+\sum_{j=1}^{\ell}\rho_j+\mathcal{C}\right),
$$

which after simplification becomes

$$
\frac{\partial \ln F_i}{\partial \rho_i} = \frac{\ell+1+\mathcal{C}-(\ell+1)\rho_i}{\rho_i(1-\rho_i)} = \frac{\mathcal{C}}{1-\rho_i}+\frac{1+\ell+\mathcal{C}}{\rho_i}
$$

and we get

$$
F_i = a_i(1-\rho_i)^{-\mathcal{C}}\rho_i^{1+\ell+\mathcal{C}},
$$

as required. □

In the proof of Proposition 5.7, we have actually shown that for $D(g,J)=2$, not only g and A restricted to integral leaves $\mathcal{L}$ of $\mathcal{U}$ are compatible in the projective sense, but also the projection v_1 of v to $\mathcal{L}$ is a projective vector field for $h = g|_{\mathcal{L}}$. Indeed, the second equation of (5.8) we used in the construction implies the second equation of (5.14) which is equivalent to the property of v_1 to be a projective vector field for h. Moreover, we already know that $v_1 = v_1(\rho)$ depends only on ρ which is equivalent to the fact that v preserves the distribution $\mathcal{U}^{\perp}$. Similarly, we have that $v_2 = v_2(y)$ (in the notation of Proposition 5.6) for the component of v tangent to $\mathcal{U}^{\perp}$ or, equivalently, that v preserves $\mathcal{U}$. To see this, note that we have $[v, K_i](\rho_j) = 0$ for all $i, j = 1, \dots, \ell$ (since $K_i(\rho_j) = 0$ and $v(\rho_j) = \rho_j(1-\rho_j)$), hence, $[v, K_i] \in \mathcal{U}^{\perp}$. Since v is holomorphic and, by the first equation in (5.8), preserves the generalized eigenspaces of A, we obtain that $[v, JK_i] \in \mathcal{U}$ for all $i = 1, \dots, \ell$, hence, v preserves $\mathcal{U}$. We summarize this discussion in the following

Corollary 5.8. – *Let v be a c-projective vector field and $D(g,J) = 2$. Consider the natural decomposition of v associated with the distributions $\mathcal{U}$ and $\mathcal{U}^{\perp}$:*

$$
v = v_1 + v_2, \quad v_1 \in \mathcal{U}, v_2 \in \mathcal{U}^{\perp}.
$$

Let $\mathcal{L}$ denote an integral leaf of $\mathcal{U}$ and denote by $h = g|_{\mathcal{L}}$ and $L = A|_{\mathcal{L}}$ the restrictions of g resp. A to $\mathcal{L}$. Then

1. *The vector field v preserves both distributions $\mathcal{U}$ and $\mathcal{U}^{\perp}$, that is, $v_1 = v_1(\rho)$ and $v_2 = v_2(y)$ (in the notation of Proposition 5.6).*

2. *The projection v_1 of v onto $\mathcal{L}$ is a projective vector field for h and the Equations* (5.8) *can be naturally restricted onto $\mathcal{L}$, namely we have*

$$\begin{aligned} \mathcal{L}_{v_1} L &= L(\mathrm{Id} - L), \\ \mathcal{L}_{v_1} h &= -hL - \big(\sum_{i=1}^{\ell} \rho_i + \mathcal{C}\big)h. \end{aligned} \tag{5.14}$$

5.6. There is only one non-constant eigenvalue

The goal of this subsection is to prove

PROPOSITION 5.9. – *Let $A \in \mathcal{A}(g, J)$ and the assumptions be as in Proposition 5.5. Then A cannot have more than one non-constant eigenvalue.*

The idea of the proof is based on the analysis of geometric properties of the metric g given by (5.11) or, more precisely, of the restriction $h = g_{|_{\mathcal{L}}}$. By construction these explicit formulas for g are local, but we will show that they make sense for all admissible values of ρ_i.

PROPOSITION 5.10. – *Consider the domain $U = \{0 < \rho_1 < \cdots < \rho_\ell < 1\}$ with the metric*

$$\sum_{i=1}^{\ell} \frac{\Delta_i}{F_i} \mathrm{d}\rho_i^2$$

(compare (5.11)*). There is a natural isometric embedding of $\phi : U \to M$ (as a maximal leaf of the totally geodesic foliation $\mathcal{U}$).*

Proof. – Locally, ϕ is defined in a very natural way. We choose a particular leaf $\mathcal{L}$ with ρ_i as local coordinates and say that $\phi(\rho_1, \ldots, \rho_\ell)$ is the point on $\mathcal{L}$ with the same coordinates $\rho_1, \ldots, \rho_\ell$. We start with a certain point $a_0 \in U$ and then extend this map as long as we can. The map ϕ so obtained is obviously an isometry. We need to show that such a prolongation can be made to any point of U.

The argument is standard. Consider a smooth curve $a(t)$ with $a(0) = a_0 \in U$ and $a(1) = a_1 \in U$, and choose $T_0 \in [0, 1]$ to be the supremum of those $T \in [0, 1]$ for which the extension along the curve $a(t)$, $t \in [0, T]$, is well defined. Take the image $\phi(a(t))$, $t \in [0, T_0)$. Since M is compact, we can find a limit point p of $\phi(a(t))$ as $t \to T_0$. By continuity, the eigenvalues of $A(p)$ coincide with the coordinates of $a(T_0)$ in U, i.e., $0 < \rho_1 < \cdots < \rho_\ell < 1$. But this condition guarantees that $p \in M^0$, i.e., in the neighborhood of p, the foliation $\mathcal{U}$ is defined. This obviously implies that $p \in \mathcal{L}$, moreover $p = \phi(a(T_0))$ and we can extend ϕ to some neighborhood of $a(T_0) \in U$. In particular, T_0 cannot be an interior point of [0, 1]. In other words, the prolongation of ϕ along $a(t)$ is well defined for all $t \in [0, 1]$. $\square$

Consider the function $f : U \to \mathbb{R}$:

$$f(\vec{\rho}) = h\big(\mathrm{grad} \sum_{i=1}^{\ell} \rho_i, \mathrm{grad} \sum_{i=1}^{\ell} \rho_i\big), \tag{5.15}$$

where h is the metric on U defined explicitly by (the first term of g in) (5.11).

According to Proposition 5.10, we can naturally identify the domain

$$U = \{0 < \rho_1 < \cdots < \rho_\ell < 1\}$$

with a leaf $\mathcal{L}$ of $\mathcal{U}$. Moreover, the function f can be considered (up to a constant multiple) as the restriction to $\mathcal{L}$ of the function $g(\operatorname{grad}\operatorname{tr} A, \operatorname{grad}\operatorname{tr} A)$ which is globally defined and smooth on M (see part (6) of Proposition 5.6). This immediately implies certain conditions on $f(\vec\rho)$.

PROPOSITION 5.11. – *The function $f(\vec\rho)$ must be bounded on U. Moreover, we have $f(\vec\rho) \to 0$ as $(\rho_1,\dots,\rho_\ell) \to (0,\dots,0)$ or $(\rho_1,\dots,\rho_\ell) \to (1,\dots,1)$.*

Proof. – The first claim follows from the compactness of M. Since $\operatorname{tr} A$ takes its minimum resp. maximum value at the limit points $(0,\dots,0)$ resp. $(1,\dots,1)$, we obtain that $\operatorname{grad}\operatorname{tr} A$ tends to zero if $(\rho_1,\dots,\rho_\ell)$ tends to $(0,\dots,0)$ or $(1,\dots,1)$. The proves the proposition. □

These conditions will give us some further restrictions on g. It is straightforward to compute the function $f(\vec\rho)$ explicitly using (5.11):

$$f(\vec\rho) = \sum_{i=1}^{\ell} \frac{F_i}{\Delta_i} = \sum_{i=1}^{\ell} \frac{a_i(1-\rho_i)^{-C}\rho_i^{1+\ell+C}}{\prod_{j\neq i}(\rho_i-\rho_j)}. \tag{5.16}$$

To study the limiting behavior of such functions we use the following

LEMMA 5.12. – 1. *Let $k_1,\dots,k_\ell$ be functions of one variable. Then the function*

$$f = \sum_{i=1}^{\ell} \frac{k_i(\rho_i)}{\prod_{j\neq i}(\rho_i-\rho_j)},$$

defined on the domain $\rho_1 < \dots < \rho_\ell$, is equal to the quotient of determinants

$$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_\ell \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{\ell-2} & \rho_2^{\ell-2} & \cdots & \rho_\ell^{\ell-2} \\ k_1(\rho_1) & k_2(\rho_2) & \cdots & k_\ell(\rho_\ell) \end{pmatrix} \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_\ell \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{\ell-2} & \rho_2^{\ell-2} & \cdots & \rho_\ell^{\ell-2} \\ \rho_1^{\ell-1} & \rho_2^{\ell-1} & \cdots & \rho_\ell^{\ell-1} \end{pmatrix}^{-1}$$

2. *If f is bounded on the domain $\rho_1 < \dots < \rho_\ell$, then $k_1 = \dots = k_\ell$.*
3. *We have $f(\rho_1,\dots,\rho_\ell) \equiv 0$ if and only if $k_i(\rho_i) = p(\rho_i)$, where $p(\rho_i)$ is a polynomial in ρ_i (independent of i) of degree $\leq \ell - 2$.*

Proof. – The proof of (1) follows from standard manipulations for calculating determinants and by applying the formula for the determinant of the Vandermonde matrix.

To prove (2), consider for instance the limit $\rho_1 \to \rho_2$ under which the Vandermonde determinant in the denominator of the expression for f tends to zero. Since f is bounded, the determinant in the numerator must also tend to zero which implies that k_1 and k_2 are equal.

Part (3) follows immediately from the formula in part (1) and the vanishing of the determinant in the numerator. □

We apply part (2) of Lemma 5.12 to the function $f(\vec\rho)$ from (5.15) written in the form (5.16) to obtain

COROLLARY 5.13. – *The parameters a_i's in* (5.11), (5.12) *are all equal. In other words, the functions F_i from* (5.11) *coincide and take the form*

$$F_i(t) = F(t) = a(1-t)^{-\mathcal{C}} t^{1+\ell+\mathcal{C}}.$$

To find restrictions on the constant $\mathcal{C}$ and the number ℓ of non-constant eigenvalues we use another interesting property of functions of the form (5.16).

LEMMA 5.14. – *Let*

$$f(\vec{\rho}) = \sum_{i=1}^{\ell} \frac{k(\rho_i)}{\prod_{j\neq i}(\rho_i - \rho_j)},$$

where $\vec{\rho} = (\rho_1, \dots, \rho_\ell)$. If k is smooth in the neighborhood of $x \in \mathbb{R}$, then

$$\lim_{\vec{\rho}\to\vec{x}} f(\vec{\rho}) = \frac{1}{(\ell-1)!} k^{(\ell-1)}(x),$$

where we define $\vec{x} = (x, \dots, x)$.

Proof. – In the neighborhood of x, we can write k in the form

$$k(t) = \sum_{j=0}^{\ell-1} \frac{1}{j!} k^{(j)}(x)(t-x)^j + \mathcal{O}((t-x)^\ell).$$

Inserting this into the formula for f, we obtain

$$f(\vec{\rho}) = \sum_{j=0}^{\ell-1} \frac{1}{j!} k^{(j)}(x) \sum_{i=1}^{\ell} \frac{(\rho_i - x)^j}{\prod_{r\neq i}(\rho_i - \rho_r)} + \sum_{i=1}^{\ell} \frac{\mathcal{O}((\rho_i - x)^\ell)}{\prod_{j\neq i}(\rho_i - \rho_j)}.$$

Applying Lemma 5.12 (3) to the functions $k_i(\rho_i)$ of the form $(\rho_i - x)^j$, we obtain

$$f(\vec{\rho}) = \frac{1}{(\ell-1)!} k^{(\ell-1)}(x) + \sum_{i=1}^{\ell} \frac{\mathcal{O}((\rho_i - x)^\ell)}{\prod_{j\neq i}(\rho_i - \rho_j)}.$$

The claim now follows by taking the limit $\vec{\rho} \to \vec{x}$. □

The function $F(t)$ from Corollary 5.13 is clearly smooth at each point $t \in (0,1)$. Applying Lemma 5.14, we obtain that $\lim_{\vec{\rho}\to\vec{t}} f(\vec{\rho}) = F^{(\ell-1)}(t)/(\ell-1)!$ holds for the function $f(\vec{\rho}) = \sum_{i=1}^{\ell} F(\rho_i)/\Delta_i$ from (5.15). The limiting behavior $\lim_{\vec{\rho}\to\vec{t}} f(\vec{\rho}) \to 0$ for $t \to 1$ and $t \to 0$ (see Proposition 5.11) then implies the inequalities

$$-\mathcal{C} > \ell - 1 \text{ and } 1 + \ell + \mathcal{C} > \ell - 1.$$

Equivalently we have

$$-2 < \mathcal{C} < 1 - \ell.$$

This inequality is not fulfilled for $\ell \geq 3$. Thus, we have

COROLLARY 5.15. – *The number ℓ of non-constant eigenvalues is either* 1 *or* 2. *Moreover, we have*

$$-2 < \mathcal{C} < -1 \ \textit{for } \ell = 2 \qquad \textit{and} \qquad -2 < \mathcal{C} < 0 \ \textit{for } \ell = 1.$$

Now let us show that also $\ell = 2$ contradicts our assumptions. If $\ell = 2$, then the metric $h = g|_{\mathcal{L}}$ (see Proposition 5.7) takes the form

$$(\rho_1 - \rho_2)\left(\frac{\mathrm{d}\rho_1^2}{F(\rho_1)} - \frac{\mathrm{d}\rho_2^2}{F(\rho_2)}\right), \tag{5.17}$$

where $F(t) = a(1-t)^{-\mathcal{C}}t^{3+\mathcal{C}}$ and $-2 < \mathcal{C} < -1$. Without loss of generality we may assume that $a = 1$.

Let us compute one special eigenvalue of the curvature operator of the metric g on M by using Proposition A.2, see appendix. We know that each eigenvector of A (e.g., $\operatorname{grad}\rho_i$) is at the same time an eigenvector of $\nabla\Lambda$ (Lemma A.1 and Remark A.1). In particular, we must have

$$\nabla_{\operatorname{grad}\rho_i}\Lambda = m_i \operatorname{grad}\rho_i \quad \text{for } i = 1, 2,$$

where the functions m_1, m_2 are the eigenvalues of the endomorphism $\nabla\Lambda$ (of course, this observation is not limited to the case $\ell = 2$). Then according to Proposition A.2 and Remark A.2, the function

$$\lambda = \frac{m_1 - m_2}{\rho_1 - \rho_2}$$

is an eigenvalue of the curvature operator.

PROPOSITION 5.16. – *The eigenvalue λ of the curvature operator is given (up to multiplication with a non-zero constant) by the formula*

$$\lambda = \frac{(\rho_1 - \rho_2)(F'(\rho_1) + F'(\rho_2)) + 2(F(\rho_2) - F(\rho_1))}{4(\rho_1 - \rho_2)^3}, \tag{5.18}$$

where $F(t) = (1-t)^{-\mathcal{C}}t^{3+\mathcal{C}}$.

Proof. – We have $\Lambda = \frac{1}{2}(\operatorname{grad}\rho_1 + \operatorname{grad}\rho_2)$, hence,

$$\nabla_\Lambda\Lambda = \frac{1}{2}\nabla_{\operatorname{grad}\rho_1}\Lambda + \frac{1}{2}\nabla_{\operatorname{grad}\rho_2}\Lambda = \frac{1}{2}(m_1\operatorname{grad}\rho_1 + m_2\operatorname{grad}\rho_2).$$

Dualizing this (and using that $\nabla\Lambda$ is g-selfadjoint), we obtain

$$\mathrm{d}g(\Lambda, \Lambda) = m_1\mathrm{d}\rho_1 + m_2\mathrm{d}\rho_2.$$

Thus, to find the formulas for m_1 and m_2 it remains to calculate the differential of $g(\Lambda, \Lambda)$.

By (5.17) we have

$$g(\Lambda, \Lambda) = \frac{1}{4}(|\operatorname{grad}\rho_1|^2 + |\operatorname{grad}\rho_2|^2) = \frac{1}{4}\frac{F_1(\rho_1) - F_2(\rho_2)}{\rho_1 - \rho_2}.$$

The functions $F_i(t)$ are given by Formula (5.12) (with $\ell = 2$) and by Corollary 5.13, they are all equal. Then,

$$\begin{aligned}\mathrm{d}g(\Lambda, \Lambda) = \frac{1}{4(\rho_1 - \rho_2)^2}\Big[&((\rho_1 - \rho_2)F'(\rho_1) + F(\rho_2) - F(\rho_1))\mathrm{d}\rho_1 \\ &- ((\rho_1 - \rho_2)F'(\rho_2) + F(\rho_2) - F(\rho_1))\mathrm{d}\rho_2\Big]\end{aligned}$$

and we obtain

$$m_1 = \frac{(\rho_1 - \rho_2)F'(\rho_1) + F(\rho_2) - F(\rho_1)}{4(\rho_1 - \rho_2)^2}$$

and

$$m_2 = -\frac{(\rho_1 - \rho_2)F'(\rho_2) + F(\rho_2) - F(\rho_1)}{4(\rho_1 - \rho_2)^2}.$$

Thus, $\lambda = (m_1 - m_2)/(\rho_1 - \rho_2)$ is given by Formula (5.18) as we claimed. □

From Proposition 5.6, we also know that this number λ can be computed for the restriction h of g to the totally geodesic leaves of the distribution $\mathcal{U}$, i.e., for the metric (5.17). Notice that (5.18) coincides with the usual formula for the scalar curvature of h. Being an eigenvalue of the curvature operator of g, the function λ must be bounded on $U = \{0 < \rho_1 < \rho_2 < 1\}$.

LEMMA 5.17. – *If F is smooth at $x \in \mathbb{R}$, then the function $\lambda(\rho_1, \rho_2)$ given by* (5.18) *is bounded in the neighborhood of the point* (x, x). *Moreover,*

$$\lim_{(\rho_1,\rho_2)\to(x,x)} \lambda(\rho_1, \rho_2) = \frac{1}{24}F'''(x).$$

Conversely, if $\lim_{t\to x} F'''(t) = \infty$, *then λ is not bounded as* $(\rho_1, \rho_2) \to (x, x)$.

Proof. – Setting $y = \rho_1$ and $x = \rho_2$ in Formula (5.18) for λ and inserting the Taylor expansion at the point x of $F(y)$ considered as a function of y, we obtain

$$\begin{aligned}
\lim_{y\to x} \lambda(y, x) &= \lim_{y\to x} \frac{(y - x)(F'(y) + F'(x)) + 2(F(x) - F(y))}{4(y - x)^3} \\
&= \lim_{y\to x} \frac{1}{4(y - x)^3}\Big((y - x)(2F'(x) + F''(x)(y - x) + \frac{1}{2}F'''(x)(y - x)^2) \\
&\quad - 2(F'(x)(y - x) + \frac{1}{2}F''(x)(y - x)^2 + \frac{1}{6}F'''(x)(y - x)^3) + \mathcal{O}((y - x)^4)\Big) \\
&= \lim_{y\to x} \frac{\frac{1}{6}F'''(x)(y - x)^3 + \mathcal{O}((y - x)^4)}{4(y - x)^3} = \frac{1}{24}F'''(x).
\end{aligned}$$

□

Now it is easy to see that the condition that $F'''(t)$ is bounded as $t \to 0$ and $t \to 1$ for $F(t) = (1 - t)^{-\mathcal{C}}t^{3+\mathcal{C}}$ can only be fulfilled for $\mathcal{C} = 0, -1, -2, -3$ (by the way, in this case λ is constant). But in our case, $-2 < \mathcal{C} < -1$ so that λ goes to infinity either for $(\rho_1, \rho_2) \to (1, 1)$ or $(\rho_1, \rho_2) \to (0, 0)$.

Thus, we conclude that $\ell = 2$ is forbidden and the only remaining case is $\ell = 1$. This completes the proof of Proposition 5.9.

REMARK 5 FOR THEOREM 5.1. In the proof of Proposition 5.16, we used Proposition A.2 to derive a formula for one of the eigenvalues of the curvature operator. An analogue of Proposition A.2 holds in the non-Kähler case too (see [6, Proposition 6]) and the proof remains essentially the same. Thus, Proposition 5.16 and Lemma 5.17 remain unchanged and we obtain that the number of non-constant eigenvalues ℓ is 1 (in the neighborhood of a regular point).

5.7. Proof of Theorem 1.1 when there is only one non-constant eigenvalue

We deal with the PDE system (5.8) which, in the case one single non-constant eigenvalue ρ of A, takes the form

$$\begin{aligned} \mathcal{L}_v A &= A(\mathrm{Id} - A), \\ \mathcal{L}_v g &= -gA - (\rho + \mathcal{C})\, g. \end{aligned} \tag{5.19}$$

Recall from Proposition 5.6 that we have two mutually orthogonal integrable distributions $\mathcal{U}$ and $\mathcal{U}^\perp$ on M, the first one $\mathcal{U}$ being 1-dimensional and totally geodesic and the metric takes the following matrix form

$$g = \begin{pmatrix} g_1(\rho) & 0 \\ 0 & g_2(\rho, y) \end{pmatrix}$$

w.r.t. the orthogonal decomposition $TM = \mathcal{U} \oplus \mathcal{U}^\perp$. We also have a c-projective vector field v preserving both $\mathcal{U}$ and $\mathcal{U}^\perp$, so that $v = v_1(\rho) + v_2(y)$, where v_1 and v_2 are the components of v w.r.t. $\mathcal{U}$ resp. $\mathcal{U}^\perp$ (see also Corollary 5.8). Hence

$$\mathcal{L}_v g = \begin{pmatrix} \mathcal{L}_{v_1} g_1 & 0 \\ 0 & \mathcal{L}_{v_2} g_2 + \mathcal{D}_{v_1} g_2 \end{pmatrix}, \tag{5.20}$$

where $\mathcal{D}_{v_1}$ means that we differentiate each term of the matrix g_2 along v_1.

Our goal is to analyze how the volume form of the metric g_2, defined on the leaves of $\mathcal{U}^\perp$, is changing under the flow generated by v_2. In other words, we want to compute the coefficient $f(\rho, y)$ in the formula $\mathcal{L}_{v_2}\mathrm{vol}_{g_2} = f(\rho, y)\cdot \mathrm{vol}_{g_2}$. We will show that this coefficient is constant. Namely,

Proposition 5.18. – *We have* $\mathcal{L}_{v_2}\mathrm{vol}_{g_2} = (-\mathcal{C} - 1)(m_1 + m_0 + 1)\mathrm{vol}_{g_2}$.

Proof. – By (5.19), we have $\mathcal{L}_v g = -g\cdot\big(A + (\rho + \mathcal{C})\mathrm{Id}\big)$. Since A admits a natural splitting $\left(\begin{smallmatrix} \rho & 0 \\ 0 & A_2 \end{smallmatrix}\right)$ w.r.t. $\mathcal{U}$ and $\mathcal{U}^\perp$, we get (using (5.11))

$$\mathcal{L}_v g = \begin{pmatrix} -\frac{1}{F}(2\rho + \mathcal{C}) & 0 \\ 0 & -g_2\big(A_2 + (\rho + \mathcal{C})\mathrm{Id}\big) \end{pmatrix}.$$

Hence, comparing with (5.20), we obtain $\mathcal{L}_{v_2} g_2 = -g_2\big(A_2 + (\rho + \mathcal{C})\mathrm{Id}\big) - \mathcal{D}_{v_1} g_2$.

We now use the following general formula that explains the relation between $\mathcal{L}_{v_2} g_2$ and $\mathcal{L}_{v_2}\mathrm{vol}_{g_2}$:

$$\mathcal{L}_{v_2}\mathrm{vol}_{g_2} = f \cdot \mathrm{vol}_{g_2}, \quad \text{where } f = \frac{1}{2}\mathrm{tr}\,(g_2^{-1}\mathcal{L}_{v_2} g_2).$$

Hence in our case

$$f = -\frac{1}{2}\,\mathrm{tr}\,\big(A_2 + (\rho + \mathcal{C})\mathrm{Id} + g_2^{-1}\mathcal{D}_{v_1} g_2\big).$$

Let us compute $\mathrm{tr}\,(g_2^{-1}\mathcal{D}_{v_1} g_2)$. In our case $v_1 = \rho(1 - \rho)\frac{\partial}{\partial\rho}$ and the metric g_2 has the form

$$F(\rho)\theta^2 + g_{\mathrm{c}}\big((A_{\mathrm{c}} - \rho\cdot\mathrm{Id})\cdot, \cdot\big),$$

hence, $\mathcal{D}_{v_1} g_2 = \rho(1-\rho)F'(\rho)\theta^2 - \rho(1-\rho)g_c(\cdot,\cdot)$. We may think of g_2 as a block diagonal form, then $\mathcal{D}_{v_1} g_2$ is block-diagonal too so that we can write it as $D_{v_1} g_2 = g_2 C$, where C is an endomorphism with the matrix

$$C = \begin{pmatrix} \rho(1-\rho)\frac{F'(\rho)}{F(\rho)} & 0 \\ 0 & -\rho(1-\rho)(A_c-\rho)^{-1} \end{pmatrix}.$$

Therefore we have the following formula:

$$\operatorname{tr} g_2^{-1} \mathcal{D}_{v_1} g_2 = \operatorname{tr} C = \rho(1-\rho)\frac{\partial \ln F}{\partial \rho} - \rho(1-\rho)\operatorname{tr}(A_c-\rho)^{-1}.$$

The first term can be easily computed from the explicit formula for F. The second term is easy to find as we know that the eigenvalues of A_c are 0 and 1 with multiplicities $2m_0$ and $2m_1$ respectively. We get

$$\begin{aligned}\operatorname{tr} g_2^{-1} \mathcal{D}_{v_1} g_2 &= (2+\mathcal{C}-2\rho) - \rho(1-\rho)\big(-2m_0\rho^{-1} + 2m_1(1-\rho)^{-1}\big)\\ &= 2+\mathcal{C}+2m_0 - \rho(2m_0+2m_1+2).\end{aligned}$$

The matrix of A_2 is known, namely $A_2 = \left(\begin{smallmatrix} \rho & 0 \\ 0 & A_c\end{smallmatrix}\right)$. So $\operatorname{tr} A_2 = \rho + \operatorname{tr} A_c = \rho + 2m_1$. Thus, finally

$$\begin{aligned} f &= -\frac{1}{2}\big(\rho + 2m_1 + (\rho+\mathcal{C})(2m_1+2m_0+1) + 2 + \mathcal{C} + 2m_0 - \rho(2m_0+2m_1+2)\big)\\ &= (-\mathcal{C}-1)(m_0+m_1+1),\end{aligned}$$

as claimed. □

This proposition immediately implies that $\mathcal{C} = -1$. Indeed, consider $M_\rho = \{\rho = c\}$ where $c \in (0,1)$. Then M_ρ is a compact smooth submanifold of M which entirely belongs to the set of regular points M^0 and, therefore, the formula from Proposition 5.18 holds for the volume form on M_ρ as a whole. However, due to compactness of M_ρ, this is impossible unless $-\mathcal{C}-1=0$. Thus, we have completely reconstructed the "non-constant" block of the metric g and now we can rewrite g as follows:

$$g = F(\rho)^{-1}d\rho^2 + F(\rho)\theta^2 + g_c\big((A_c-\rho\,\mathrm{Id})\cdot,\cdot\big), \tag{5.21}$$

where $F(\rho) = -4B(1-\rho)\rho$ and $B \neq 0$ is some constant (the notation $-4B$ for the constant factor is chosen to emphasize the relationship with some formulas from [21] that are used at the final stage of our proof).

REMARK 6 FOR THEOREM 5.1. The proof of and the formula in Proposition 5.18 slightly change because now the terms with θ do not appear. Here is the modified version in the projective setting:

PROPOSITION 5.19. – *We have* $\mathcal{L}_{v_2}\mathrm{vol}_{g_2} = \frac{1}{2}(-\mathcal{C}-1)(m_1+m_0)\mathrm{vol}_{g_2}$.

Proof. – By (5.19), we have $\mathcal{L}_v g = -g\cdot\big(A+(\rho+\mathcal{C})\mathrm{Id}\big)$. Since A admits a natural splitting $\left(\begin{smallmatrix} \rho & 0 \\ 0 & A_2\end{smallmatrix}\right)$ w.r.t. $\mathcal{U}$ and $\mathcal{U}^\perp$, we get

$$\mathcal{L}_v g = \begin{pmatrix} -\frac{1}{F}(2\rho+\mathcal{C}) & 0 \\ 0 & -g_2\big(A_2+(\rho+\mathcal{C})\mathrm{Id}\big)\end{pmatrix}.$$

Hence, comparing with (5.20), we obtain $\mathcal{L}_{v_2} g_2 = -g_2\big(A_2 + (\rho + \mathcal{C})\mathrm{Id}\big) - \mathcal{D}_{v_1} g_2$. Using

$$\mathcal{L}_{v_2}\mathrm{vol}_{g_2} = f \cdot \mathrm{vol}_{g_2}, \quad \text{where } f = \frac{1}{2}\mathrm{tr}\,(g_2^{-1}\mathcal{L}_{v_2} g_2),$$

we obtain in our case

$$f = -\frac{1}{2}\,\mathrm{tr}\,(A_2 + (\rho + \mathcal{C})\mathrm{Id} + g_2^{-1}\mathcal{D}_{v_1} g_2).$$

Let us compute $\mathrm{tr}\,(g_2^{-1}\mathcal{D}_{v_1} g_2)$. In our case $v_1 = \rho(1-\rho)\frac{\partial}{\partial\rho}$ and the metric g_2 has the form $g_2(\cdot,\cdot) = g_{\mathrm{c}}\big((A_{\mathrm{c}} - \rho\cdot\mathrm{Id})\cdot,\cdot\big)$, hence,

$$\mathcal{D}_{v_1} g_2 = -\rho(1-\rho) g_{\mathrm{c}}(\cdot,\cdot).$$

Thus we can write $\mathcal{D}_{v_1} g_2$ as $D_{v_1} g_2 = g_2 C$, where C is an endomorphism with the matrix

$$C = -\rho(1-\rho)(A_{\mathrm{c}} - \rho)^{-1}.$$

Therefore we have the following formula:

$$\mathrm{tr}\ g_2^{-1}\mathcal{D}_{v_1} g_2 = \mathrm{tr}\ C = -\rho(1-\rho)\mathrm{tr}\,(A_{\mathrm{c}} - \rho)^{-1}.$$

This quantity is easy to find as we know the eigenvalues of A_{c} are 0 and 1 with multiplicities m_0 and m_1 respectively. We get

$$\mathrm{tr}\ g_2^{-1}\mathcal{D}_{v_1} g_2 = -\rho(1-\rho)\big(-m_0\rho^{-1} + m_1(1-\rho)^{-1}\big) = m_0(1-\rho) - m_1\rho.$$

The matrix of A_2 coincides with A_{c} so that $\mathrm{tr}\,A_2 = \mathrm{tr}\,A_{\mathrm{c}} = m_1$. Thus, finally

$$f = -\frac{1}{2}\big(m_1 + (\rho + \mathcal{C})(m_1 + m_0) + m_0(1-\rho) - m_1\rho\big) = \frac{1}{2}(-\mathcal{C} - 1)(m_0 + m_1),$$

as claimed. □

The conclusion from this proposition remains unchanged: $\mathcal{C} = -1$ as required. We use the fact that the leaves of $\mathcal{U}^{\perp}$ are smooth closed submanifolds of the form $\{\rho = c\}$ which are entirely located in the set of regular points. Since the leaves of $\mathcal{U}^{\perp}$ are common levels of non-constant eigenvalues, the latter follows from the fact (used already several times) that $0 < \rho < 1$ implies $\mathrm{d}\rho(p) \neq 0$. Hence we have the following formula for the metric (this is (5.21) with the term with θ removed):

$$g = F(\rho)^{-1}\mathrm{d}\rho^2 + g_{\mathrm{c}}\big((A_{\mathrm{c}} - \rho\,\mathrm{Id})\cdot,\cdot\big), \tag{5.22}$$

where $F(\rho) = -4B(1-\rho)\rho$, $B \neq 0$ is some constant and A_{c} is parallel w.r.t. g_{c}.

Denote by E_0 resp. E_1 the generalized eigenspaces of A corresponding to the eigenvalues 0 resp. 1. Let $L_0 = A|_{E_0}$ and $L_1 = A|_{E_1}$ denote the restrictions of A to these subspaces. We start with describing $\nabla\Lambda$ restricted to E_0 resp. E_1 explicitly as a matrix function of L_0 resp. L_1. In fact, it is a general statement (see Remark A.1 in the appendix), that $\nabla\Lambda$ and A commute and, moreover, at each point $\nabla\Lambda$ can be written as a function (or even polynomial) of A.

Lemma 5.20. – *At each regular point, we have*

$$\nabla\Lambda|_{E_0\oplus E_1} = -g(\Lambda,\Lambda)(A - \rho\,\mathrm{Id})^{-1}|_{E_0\oplus E_1} = B(1-\rho)\rho(A - \rho\,\mathrm{Id})^{-1}|_{E_0\oplus E_1}. \tag{5.23}$$

Proof. – In the special case of only one non-constant eigenvalue ρ, $\Lambda = \frac{1}{2}\mathrm{grad}\,\rho$ itself is an eigenvector field of A corresponding to the eigenvalue ρ. Hence, we can apply (2.4) with X replaced by Λ. Then (2.4) becomes

$$(A - \rho\,\mathrm{Id})\nabla_Y \Lambda = -g(\Lambda, \Lambda)Y,$$

for any tangent vector $Y \in E_0 \oplus E_1$, or equivalently, if we take into account that $E_0 \oplus E_1$ is invariant under $A - \rho \cdot \mathrm{Id}$ and $(A - \rho \cdot \mathrm{Id})|_{E_0 \oplus E_1}$ is invertible:

$$\nabla\Lambda|_{E_0\oplus E_1} = -g(\Lambda, \Lambda)(A - \rho\,\mathrm{Id})^{-1}|_{E_0\oplus E_1}.$$

It remains to notice that (5.21) implies $g(\Lambda, \Lambda) = \frac{1}{4}g(\mathrm{grad}\,\rho, \mathrm{grad}\,\rho) = B(1-\rho)\rho$, as stated. ☐

Lemma 5.21. – *At each regular point, we have $A|_{E_0} = L_0 = 0$ and $A|_{E_1} = L_1 = \mathrm{Id}$.*

Proof. – Our statement is equivalent to the absence of non-trivial Jordan blocks corresponding to the constant eigenvalues 0 and 1. By contradiction, assume that non-trivial Jordan blocks exist and apply Proposition A.2 to compute the eigenvalues of the curvature operator of g related to these blocks.

Formula (5.23) expresses $\nabla\Lambda|_{E_0\oplus E_1}$ (pointwise) as a matrix function $f(A|_{E_0\oplus E_1})$, where $f(t) = B(1-\rho)\rho(t-\rho)^{-1}$. Then according to Proposition A.2 and Remark A.2 from appendix, we conclude that if L_0 (resp. L_1) has a non-trivial Jordan block, then

$$4f'(0) = -4B\frac{1-\rho}{\rho} \qquad \left(\text{resp. } 4f'(1) = -4B\frac{\rho}{1-\rho}\right)$$

is an eigenvalue of the curvature operator R of (M, g, J). When restricted to a non-constant integral curve of v, this eigenvalue goes to infinity for $t \to -\infty$ resp. $t \to +\infty$ which contradicts the boundedness of the eigenvalues of R. Thus, we conclude $L_0 = 0$ and $L_1 = \mathrm{Id}$. ☐

Remark 7 for Theorem 5.1. Lemma 5.20 uses Formula (2.4) which, in the pseudo-Riemannian case takes the form:

$$(A - \rho\,\mathrm{Id})\nabla_Y X = \mathrm{d}\rho(Y)X - g(X, Y)\Lambda - g(X, \Lambda)Y$$

where $\Lambda = \frac{1}{2}\mathrm{grad}\,\mathrm{tr}\,A = \frac{1}{2}\mathrm{grad}\,\rho$, X is a vector field satisfying $(A - \rho\,\mathrm{Id})X = 0$, i.e., a ρ-eigenvector field and Y is an arbitrary vector field.

Notice that in this case Λ satisfies the condition for X, so in this formula we may set $X = \Lambda$. Then we get

$$(A - \rho\,\mathrm{Id})\nabla_Y \Lambda = 2g(\Lambda, Y)\Lambda - g(\Lambda, \Lambda)Y - g(Y, \Lambda)\Lambda = g(\Lambda, Y)\Lambda - g(\Lambda, \Lambda)Y,$$

and if we assume that $Y \in E_0 \oplus E_1$ and take into account that Λ is orthogonal to $E_0 \oplus E_1$, we obtain the same formula as in the Kähler case:

$$(A - \rho\,\mathrm{Id})\nabla_Y \Lambda = -g(\Lambda, \Lambda)Y.$$

The rest of the proof does not change and therefore, in the projective setting, Lemmas 5.20 and 5.21 remain unchanged.

LEMMA 5.22. – *At every point, we have*

$$\nabla\Lambda = \mu \operatorname{Id} + BA \tag{5.24}$$

for the constant B and a function $\mu = B(\rho - 1)$. Moreover,

$$\nabla\mu = 2B\Lambda^\flat. \tag{5.25}$$

Proof. – We will first prove the lemma near a regular point. We know that the eigenspaces of A are invariant under $\nabla\Lambda$ and from Lemma 5.20, we know the formula for the restriction of $\nabla\Lambda$ onto $E_0 \oplus E_1$. Taking into account Lemma 5.21 we obtain

$$\nabla\Lambda|_{E_0} = B(1-\rho)\rho(-\rho \operatorname{Id})^{-1} = B(\rho-1)\operatorname{Id} = (\mu \operatorname{Id} + BA)|_{E_0},$$

and

$$\nabla\Lambda|_{E_1} = B(1-\rho)\rho(\operatorname{Id} - \rho \operatorname{Id})^{-1} = B\rho \operatorname{Id} = (\mu \operatorname{Id} + BA)|_{E_1},$$

Thus, it remains to verify the formula for the restriction of $\nabla\Lambda$ onto the two-dimensional ρ-eigenspace E_ρ.

Notice that $\Lambda \in E_\rho$ and moreover Λ is the tangent vector to the one-dimensional totally geodesic distribution $\mathcal{U}$. In other words, we can consider Λ as a tangent vector to a geodesic $\gamma(t)$. Hence, $(\nabla\Lambda)\Lambda = \nabla_\Lambda\Lambda = f\Lambda$ where $f = \frac{1}{2}\frac{\Lambda(g(\Lambda,\Lambda))}{g(\Lambda,\Lambda)}$.

Using $\Lambda = \frac{1}{2}\operatorname{grad}\rho$ and the explicit Formula (5.21) for g, we obtain

$$f = B(2\rho - 1).$$

The other ρ-eigenvector is $J\Lambda$. Since Λ and $J\Lambda$ commute, we get

$$(\nabla\Lambda)J\Lambda = \nabla_{J\Lambda}\Lambda = \nabla_\Lambda J\Lambda = J\nabla_\Lambda\Lambda = J\big(B(2\rho-1)\Lambda\big) = B(2\rho-1)J\Lambda.$$

Thus,

$$\nabla\Lambda|_{E_\rho} = B(2\rho-1)\operatorname{Id} = (\mu\operatorname{Id} + BA)|_{E_\rho}.$$

Since at regular points TM decomposes as $TM = E_\rho \oplus E_0 \oplus E_1$, we have verified Equation (5.24) in the neighborhood of every point of a dense and open subset of M for a local constant B and a locally defined function $\mu = B(\rho - 1)$. Taking the derivative of this function (and using $d\rho = 2\Lambda^\flat$) shows that it satisfies (5.25). It was proven in [21, §2.5] that having the Equations (5.24) and (5.25) satisfied in the neighborhood of almost every point for locally defined constant B and function μ, the constant B is the same for each such neighborhood, hence, is globally defined, and therefore μ is also globally defined. This completes the proof of the lemma. □

REMARK 8 FOR THEOREM 5.1. The statement of Lemma 5.22 remains unchanged. The proof of Lemma 5.22 changes slightly, as the ρ-eigenspace is one-dimensional (not two-dimensional as it was in the c-projective setting). This makes the proof shorter as we do not need to consider the second eigenvector $J\Lambda$. The projective analogue of [21, §2.5] is [23, §2.3.4].

Now we can prove Theorem 1.1. We have shown that the existence of a non-affine c-projective vector field on a closed connected Kähler manifold (M, g, J) of arbitrary signature implies the existence of a solution (A, Λ, μ) of the system

$$\begin{aligned}\nabla_X A &= X^\flat \otimes \Lambda + \Lambda^\flat \otimes X + (JX)^\flat \otimes J\Lambda + (J\Lambda)^\flat \otimes JX,\\ \nabla\Lambda &= \mu\,\mathrm{Id} + BA,\\ \nabla\mu &= 2B\Lambda^\flat,\end{aligned}$$

for a certain constant B. This solution is non-trivial in the sense that Λ is not identically zero. By [21, Remark 12], for a certain constant $c \neq 0$, $(M, c \cdot g, J)$ is isometric to $(\mathbb{C}P^n, g_{\mathrm{FS}}, J_{\mathrm{standard}})$ as we claimed. Theorem 1.1 is proved.

REMARK 9 FOR THEOREM 5.1. The existence of (L, Λ, μ) satisfying (1.7) and (5.24) immediately implies that the (non-constant) function $\alpha = \mathrm{tr}(L)$ satisfies the equation

$$\nabla^3\alpha(X,Y,Z) - B\cdot\big(2(\nabla\alpha\otimes g)(X,Y,Z) + (\nabla\alpha\otimes g)(Y,X,Z) + (\nabla\alpha\otimes g)(Z,X,Y)\big) = 0, \tag{5.26}$$

see e.g., [23, Corollary 4]. By [30, Theorem 1], the metric $-Bg$ is positive definite. This contradiction proves Theorem 5.1.

6. Final step of the proof of Theorem 1.2: the case of a non-constant Jordan block

In §5, the Lichnerowicz conjecture was proved under the additional assumption that the non-constant part of A is diagonalisable.

In the case where the endomorphism A has a non-trivial Jordan block with a non-constant eigenvalue, the scheme of the proof remains essentially the same but we need to modify some steps accordingly. In fact, the proof becomes much easier because the presence of a Jordan block, as we shall see below, immediately leads to unboundedness of one special eigenvalue of the curvature operator.

The first steps of the proof do not use the algebraic type of A and, therefore, they remain the same as in Sections 5.3 and 5.4. Namely, we may assume that the degree of mobility of g equals 2 and g, A and v satisfy the equations

$$\begin{aligned}\mathcal{L}_v A &= -A^2 + A,\\ \mathcal{L}_v g &= -gA - (\mathrm{tr}\,A + \mathcal{C}')g.\end{aligned} \tag{6.1}$$

These equations coincide with (5.8) though we need to replace $\sum \rho_i$ by $\mathrm{tr}\,A$ as some non-constant eigenvalues may now have multiplicity > 1, see also our comment on (5.8) in Remark 4 for Theorem 5.1.

All the eigenvalues of A are real and satisfy the equation $\mathcal{L}_v\rho = -\rho^2 + \rho$, in particular, there are at most two constant eigenvalues, 0 and 1, and, due to boundedness of the non-constant eigenvalues, they satisfy $0 \le \rho_i \le 1$.

The definition of the set M^0 of regular points also remains the same but the algebraic type of A changes. Recall that in general M^0 may consist of several connected components with different algebraic types, so we continue working with one of them (we still denote it by M^0 and refer to it as the set of regular points). In the Lorentzian case, two sizes of Jordan blocks for g-selfadjoint endomorphisms are allowed, 2×2 and 3×3. Thus, to complete the proof of Theorem 1.2 we need to consider two additional types of regular points $p \in M^0$.

Namely, below we assume that the endomorphism A has ℓ non-constant distinct real eigenvalues $\rho_1, \dots, \rho_\ell$ and, possibly, two constant eigenvalues 0 and 1 of multiplicity m_0 and m_1 respectively. The first eigenvalue ρ_1 has multiplicity 2 or 3, and the endomorphism A "contains" a single 2×2 or resp. 3×3 Jordan ρ_1-block. On M^0, the eigenvalues ρ_i's are ordered:

(6.2)

$$0 < \rho_2 < \cdots < \rho_\ell < 1 \quad \text{and } \rho_1 \text{ belongs to one of the intervals } (0, \rho_2), (\rho_2, \rho_3), \dots, (\rho_\ell, 1).$$

Due to the existence of the projective vector field v, the above conditions automatically imply that $d\rho_i \neq 0$ on M^0.

In the neighborhood of any point $p \in M^0$ we can now, following [10], find a canonical coordinate system and reduce g and A to a normal form. In general, this normal form contains arbitrary functions $F_i(\rho_i)$. The existence of a projective vector field v on M^0, satisfying (6.1) allows us to reconstruct these functions (as well as the components of v) almost uniquely, i.e., up to a finite number of arbitrary constants of integration (cf. Proposition 5.7). This "reconstruction" can be done by a straightforward computation as in Proposition 5.7, but since now we deal with a more complicated situation involving Jordan blocks, we prefer to use the following general statement which explains how to split (6.1) in a block-wise manner. This statement is a direct corollary of the splitting construction from [9].

THEOREM 6.1. – *Let (h, L) be a compatible pair on M which splits into two blocks in the sense of* [9, 10]*, i.e., there exist a local coordinate system x, y with $x = (x_1, \dots, x_{n_1})$ and $y = (y_1, \dots, y_{n_2})$ and compatible pairs $\big(h_1(x), L_1(x)\big)$ and $\big(h_2(y), L_2(y)\big)$ such that*

$$h(x, y) = \begin{pmatrix} h_1(x)\chi_{L_2}(L_1(x)) & 0 \\ 0 & h_2(y)\chi_{L_1}(L_2(y)) \end{pmatrix}, \quad L(x, y) = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix},$$

where $\chi_{L_1}(\cdot)$ and $\chi_{L_2}(\cdot)$ denote the characteristic polynomials of the blocks L_1 and L_2 respectively. Let v be a projective vector field for g satisfying the equations

$$\begin{aligned} \mathcal{L}_v L &= -L^2 + L, \\ \mathcal{L}_v h &= -hL - (\operatorname{tr} L + C)h. \end{aligned} \tag{6.3}$$

Then the vector field v and Equations (6.3) *also split as follows:*

$$v(x, y) = v_1(x) + v_2(y),$$

where v_i denote the natural projections of v on the x-- and y--subspaces and (6.3) *is equivalent to*

$$\begin{aligned} \mathcal{L}_{v_1} L_1 &= -L_1^2 + L_1, \qquad & \mathcal{L}_{v_1} h_1 &= (n_2 - 1)h_1 L_1 - (\operatorname{tr} L_1 + C + n_2)h_1, \\ \mathcal{L}_{v_2} L_2 &= -L_2^2 + L_2, \qquad & \mathcal{L}_{v_2} h_2 &= (n_1 - 1)h_2 L_2 - (\operatorname{tr} L_2 + C + n_1)h_2. \end{aligned}$$

Proof. – The first part $v(x, y) = v_1(x) + v_2(y)$ follows from the fact that v preserves the invariant x-and y-subspaces of L (see also [9, Lemma 3]). The latter can be seen from the equation $\mathcal{L}_v L = -L^2 + L$. The formulas for $\mathcal{L}_{v_1} L_1$ and $\mathcal{L}_{v_2} L_2$ are straightforward. After this we can differentiate h as follows:

$$\mathcal{L}_{v_1+v_2} \begin{pmatrix} h_1\chi_{L_2}(L_1) & 0 \\ 0 & h_2\chi_{L_1}(L_2) \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{v_1}\big(h_1\chi_{L_2}(L_1)\big) + h_1(\mathcal{D}_{v_2}\chi_{L_2})(L_1) & 0 \\ 0 & \mathcal{L}_{v_2}\big(h_2\chi_{L_1}(L_2)\big) + h_2(\mathcal{D}_{v_1}\chi_{L_1})(L_2) \end{pmatrix},$$

where $\mathcal{D}_{v_i}\chi_{L_i}$ means that we differentiate each coefficient of the characteristic polynomial χ_{L_i} along v_i in the usual sense. We can rewrite the right-hand side as [(5)]

$$\begin{aligned}\mathcal{L}_{v_1}\big(h_1\chi_{L_2}(L_1)\big) + h_1(\mathcal{D}_{v_2}\chi_{L_2})(L_1) = (\mathcal{L}_{v_1}h_1)\chi_{L_2}(L_1)\\ + h_1\chi_{L_2}(L_1)\mathcal{L}_{v_1}\big(\ln(\chi_{L_2}(L_1))\big)\\ + h_1\chi_{L_2}(L_1)\mathcal{D}_{v_2}\big(\ln(\chi_{L_2}(L_1))\big).\end{aligned}$$

On the other hand, from (6.3) we know that this expression equals

$$h_1\chi_{L_2}(L_1)(-L_1 - \operatorname{tr} L{\cdot}\mathrm{Id} - \mathcal{C}\cdot\mathrm{Id}).$$

Multiplying by $\chi_{L_2}(L_1)^{-1}$ we get

$$\mathcal{L}_{v_1}h_1 = -h_1\Big(\mathcal{L}_{v_1}\big(\ln(\chi_{L_2}(L_1))\big) + \mathcal{D}_{v_2}\big(\ln(\chi_{L_2}(L_1))\big) + L_1 + \operatorname{tr} L{\cdot}\mathrm{Id} + \mathcal{C}{\cdot}\mathrm{Id}\Big).$$

To evaluate this further, let $\lambda_1,\dots,\lambda_{n_2}$ denote the eigenvalues of L_2, i.e., the roots of χ_{L_2} (some of them may coincide). Then,

$$\begin{aligned}\mathcal{L}_{v_1}\big(\ln(\chi_{L_2}(L_1))\big) &= \mathcal{L}_{v_1}\left(\sum_{i=1}^{n_2}\ln(L_1-\lambda_i{\cdot}\mathrm{Id})\right)\\ &= \sum_{i=1}^{n_2}(L_1-\lambda_i{\cdot}\mathrm{Id})^{-1}\mathcal{L}_{v_1}L_1 = \sum_{i=1}^{n_2}(L_1-\lambda_i{\cdot}\mathrm{Id})^{-1}(-L_1^2+L_1)\\ \mathcal{D}_{v_2}\big(\ln(\chi_{L_2}(L_1))\big) &= \mathcal{D}_{v_2}\left(\sum_{i=1}^{n_2}\ln(L_1-\lambda_i{\cdot}\mathrm{Id})\right)\\ &= -\sum_{i=1}^{n_2}(L_1-\lambda_i\cdot\mathrm{Id})^{-1}\mathcal{D}_{v_2}\lambda_i = -\sum_{i=1}^{n_2}(L_1-\lambda_i{\cdot}\mathrm{Id})^{-1}(-\lambda_i^2+\lambda_i).\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{L}_{v_1}\big(\ln(\chi_{L_2}(L_1))\big) + \mathcal{D}_{v_2}\big(\ln(\chi_{L_2}(L_1))\big) &= \sum_{i=1}^{n_2}(L_1-\lambda_i\cdot\mathrm{Id})^{-1}\big(-L_1^2+L_1+\lambda_i^2\cdot\mathrm{Id}-\lambda_i\cdot\mathrm{Id}\big)\\ &= \sum_{i=1}^{n_2}(\mathrm{Id}-\lambda_i\cdot\mathrm{Id}-L_1) = n_2\cdot\mathrm{Id} - \operatorname{tr} L_2\cdot\mathrm{Id} - n_2L_1.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\mathcal{L}_{v_1}h_1 &= -h_1\,(n_2{\cdot}\mathrm{Id} - \operatorname{tr} L_2{\cdot}\mathrm{Id} - n_2L_1 + L_1 + \operatorname{tr} L{\cdot}\mathrm{Id} + \mathcal{C}{\cdot}\mathrm{Id})\\ &= -h_1\big((1-n_2)L_1 + (\operatorname{tr} L_1 + \mathcal{C} + n_2)\mathrm{Id}\big),\end{aligned}$$

as we claimed. The calculations for $\mathcal{L}_{v_2}h_2$ are analogous. □

Let us apply the theorem by specifying the blocks into which we want to split:

(5) To make some expressions shorter, we are using the formula $\mathcal{L}_v(\ln B) = B^{-1}\mathcal{L}_v B$ which, in the matrix case, holds true if B and $\mathcal{L}_v B$ commute. In our case this condition is fulfilled for $B = L_1$ and, consequently, for $B = \chi_{L_2}(L_1)$ since $\mathcal{L}_{v_1}L_1 = -L_1^2 + L_1$.

COROLLARY 6.2. – *Let (h, L) be a compatible pair that splits into two blocks of compatible pairs (h_1, L_1), (h_2, L_2) of dimensions n_1 and n_2 respectively as in Theorem 6.1. Suppose v is a projective vector field such that h, L and v satisfy* (6.3).

1. Case of a trivial 1×1 Jordan ρ-block. *Let (h_1, L_1) be given by*

$$h_1 = \frac{1}{F(\rho)}\mathrm{d}\rho^2,\quad L_1 = \rho\frac{\partial}{\partial\rho}\otimes \mathrm{d}\rho,$$

w.r.t. a coordinate ρ. Then $v_1 = \rho(1-\rho)\frac{\partial}{\partial\rho}$ and

$$F = a(1-\rho)^{-\mathcal{C}}\rho^{n+1+\mathcal{C}}. \tag{6.4}$$

2. Case of a 2×2 Jordan ρ-block. *Let (h_1, L_1) be given by*

$$h_1 = \begin{pmatrix} 0 & F(\rho)+x \\ F(\rho)+x & 0 \end{pmatrix} \text{ and } L_1 = \begin{pmatrix} \rho & F(\rho)+x \\ 0 & \rho \end{pmatrix} \tag{6.5}$$

w.r.t. coordinates x, ρ. Then $v_1 = G(x,\rho)\frac{\partial}{\partial x} + \rho(1-\rho)\frac{\partial}{\partial\rho}$, where

$$G(x,\rho) = \frac{1}{2}((n_2-1)\rho - 1 - \mathcal{C} - n_2)x + G_1(\rho)$$

and $F(\rho), G_1(\rho)$ satisfy the ODE system

$$\begin{aligned} F' &= \tfrac{1}{\rho(1-\rho)}\left(\tfrac{1}{2}((n_2-1)\rho - 1 - \mathcal{C} - n_2)F - G_1\right), \\ G_1' &= \tfrac{1}{2}(n_2-1)F. \end{aligned} \tag{6.6}$$

3. Case of a 3×3 Jordan ρ-block. *Let (h_1, L_1) be given by*

$$h_1 = \begin{pmatrix} 0 & 0 & F(\rho)+2x_2 \\ 0 & 1 & x_1 \\ F(\rho)+2x_2 & x_1 & x_1^2 \end{pmatrix} \text{ and } L_1 = \begin{pmatrix} \rho & 1 & x_1 \\ 0 & \rho & F(\rho)+2x_2 \\ 0 & 0 & \rho \end{pmatrix} \tag{6.7}$$

in coordinates x_1, x_2, ρ. Then,

$$v_1 = G(x_1,x_2,\rho)\frac{\partial}{\partial x_1} + H(x_2,\rho)\frac{\partial}{\partial x_2} + \rho(1-\rho)\frac{\partial}{\partial\rho},$$

where

$$G(x_1,x_2,\rho) = -\frac{1}{2}(\mathcal{C} + n_2 + 2 - n_2\rho)x_1 + \frac{1}{2}n_2x_2 + G_1(\rho)$$

and

$$H(x_2,\rho) = -\frac{1}{2}(\mathcal{C} + n_2 + (4-n_2)\rho)x_2 + H_1(\rho)$$

and $F(\rho), H_1(\rho), G_1(\rho)$ satisfy the ODE system

$$\begin{aligned} F' &= -\tfrac{1}{\rho(1-\rho)}\left(\tfrac{1}{2}(\mathcal{C} + n_2 + (4-n_2)\rho)F + 2H_1\right), \\ H_1' &= \tfrac{1}{2}(n_2-2)F - G_1, \\ G_1' &= 0. \end{aligned} \tag{6.8}$$

REMARK 6.1. – Formulas (6.5) and (6.7) are exactly the normal forms obtained in [10] for compatible pairs h and L in the case where L is conjugate to a 2×2 resp. 3×3 Jordan block with a non-constant eigenvalue. Notice that these formulas are only meaningful at those points where $F + x \neq 0$ (resp. $F + 2x_2 \neq 0$).

REMARK 6.2. – In part (1) of Corollary 6.2, we actually re-derived the Formulas (5.12) for the components of v and the functions F_i parametrizing the metric

$$h = g|_{\mathcal{L}} = \sum_{i=1}^{\ell} \frac{\Delta_i}{F_i} \mathrm{d}\rho_i^2$$

obtained from the metric g by restricting it to leaves $\mathcal{L}$ of the distribution $\mathcal{U}$.

Proof of Corollary 6.2. – Theorem 6.1 shows that h_1, L_1 and v_1 have to satisfy the equations

$$\mathcal{L}_{v_1} L_1 = -L_1^2 + L_1 \text{ and } \mathcal{L}_{v_1} h_1 = (n_2 - 1) h_1 L_1 - (\operatorname{tr} L_1 + \mathcal{C} + n_2) h_1, \tag{6.9}$$

where $n_2 = n - n_1$ is the dimension of the block (h_2, L_2) of (h, L) complementary to (h_1, L_1).

(1) From the first equation in (6.9), it follows immediately that $v = \rho(1-\rho)\partial_\rho$. This solves the first equation identically. It is straightforward to check that the second equation in (6.9) with n_2 replaced by $n - 1$ is equivalent to the ODE

$$\frac{\mathrm{d}F}{F} = \frac{n + 1 + \mathcal{C} - (n+1)\rho}{\rho(1-\rho)} \mathrm{d}\rho.$$

The solution to this ODE is (6.4) as we claimed.

(2) Since the first equation in (6.9) implies that v_1 preserves invariant subspaces of L_1, we can suppose that

$$v_1 = G(x, \rho)\partial_x + \rho(1-\rho)\partial_\rho$$

for a certain function $G(x, \rho)$. Using this, together with the explicit formulas for h_1 and L_1, we see that the first equation in (6.9) is equivalent to

$$\rho(1-\rho)F' + G - (F + x)\partial_x G = 0 \tag{6.10}$$

whilst the second equation in (6.9) is equivalent to the equations

$$\rho(1-\rho)F' + G + (F + x)(\partial_x G + 1 + \mathcal{C} + n_2 + (1 - n_2)\rho) = 0 \tag{6.11}$$

and

$$2\partial_\rho G = (n_2 - 1)(F + x). \tag{6.12}$$

Substracting (6.10) from (6.11) and dividing by $F + x$ yields

$$2\partial_x G + 1 + \mathcal{C} + n_2 + (1 - n_2)\rho = 0. \tag{6.13}$$

This shows that G must be of the form

$$G(x, \rho) = \frac{1}{2}((n_2 - 1)\rho - 1 - \mathcal{C} - n_2)x + G_1(\rho),$$

as we claimed. Inserting this into (6.10) (now equivalent to (6.11)) and (6.12), we obtain (6.6) after rearranging terms.

(3) Again, since v_1 preserves invariant subspaces of L_1, we have

$$v_1 = G(x_1, x_2, \rho)\frac{\partial}{\partial x_1} + H(x_2, \rho)\frac{\partial}{\partial x_2} + \rho(1-\rho)\frac{\partial}{\partial \rho}$$

for certain functions $G(x_1, x_2, \rho)$ and $H(x_2, \rho)$. A straightforward calculation gives that the first equation in (6.9) is equivalent to the equations

$$-1 + 2\rho + \partial_{x_2} H - \partial_{x_1} G = 0, \tag{6.14}$$

$$2x_2 + F + G + \partial_\rho H - (F + 2x_2)\partial_{x_2} G - x_1 \partial_{x_1} G = 0 \tag{6.15}$$

and

$$2H + \rho(1-\rho)F' - (F + 2x_2)\partial_{x_2} H = 0, \tag{6.16}$$

whilst the second equation in (6.9) is equivalent to the equations

$$2H + \rho(1-\rho)F' + (F + 2x_2)(1 + \mathcal{C} + n_2 + (2 - n_2)\rho + \partial_{x_1} G) = 0, \tag{6.17}$$

$$\mathcal{C} + n_2 + (4 - n_2)\rho + 2\partial_{x_2} H = 0, \tag{6.18}$$

(6.19)

$$(1 + \mathcal{C} + n_2 + (2 - n_2)\rho + \partial_{x_2} H)x_1 + G + \partial_\rho H - (F + 2x_2)(n_2 - 1 - \partial_{x_2} G) = 0$$

and

$$(2 + \mathcal{C} + n_2 - n_2\rho)x_1^2 + 2(G + \partial_\rho H)x_1 - 2(F + 2x_2)((n_2 - 1)x_1 - \partial_\rho G) = 0. \tag{6.20}$$

Subtracting (6.16) from (6.17) and dividing by $F + 2x_2$, we obtain

$$1 + \mathcal{C} + n_2 + (2 - n_2)\rho + \partial_{x_1} G + \partial_{x_2} H = 0.$$

Hence, using (6.14),

$$\partial_{x_1} G = \frac{1}{2}(n_2\rho - 2 - \mathcal{C} - n_2) \text{ and } \partial_{x_2} H = \frac{1}{2}((n_2 - 4)\rho - \mathcal{C} - n_2)$$

or, in other words,

$$G(x_1, x_2, \rho) = \frac{1}{2}(n_2\rho - 2 - \mathcal{C} - n_2)x_1 + \tilde{G}(x_2, \rho)$$

$$\text{and } H(x_2, \rho) = \frac{1}{2}((n_2 - 4)\rho - \mathcal{C} - n_2)x_2 + H_1(\rho)$$

for certain functions $\tilde{G}(x_2, \rho)$, $H_1(\rho)$. Inserting this back into our PDE system (6.14)–(6.20) we obtain that (6.9) is equivalent to the equations

$$\frac{1}{2}n_2 x_2 + F + \tilde{G} + H_1' - (F + 2x_2)\partial_{x_2}\tilde{G} = 0, \tag{6.21}$$

$$(\mathcal{C} + n_2 + (4 - n_2)\rho)F + 4H_1 + 2\rho(1-\rho)F' = 0, \tag{6.22}$$

$$-\frac{3}{2}n_2 x_2 + \tilde{G} + H_1' + (F + 2x_2)\partial_{x_2}\tilde{G} + (1 - n_2)F = 0 \tag{6.23}$$

and

$$(-n_2 x_2 - (n_2 - 2)F + 2\tilde{G} + 2H_1')x_1 + 2(F + 2x_2)\partial_\rho \tilde{G} = 0. \tag{6.24}$$

Substracting (6.21) from (6.23) and dividing by $F + 2x_2$ gives $-n_2 + 2\partial_{x_2}\tilde{G} = 0$ and we see that

$$\tilde{G}(x_2, \rho) = \frac{1}{2}n_2 x_2 + G_1(\rho).$$

Inserting this formula for $\tilde{G}$ into (6.21)–(6.24), we obtain that (6.9) is equivalent to the equations

$$\begin{aligned}\frac{1}{2}(2-n_2)F + G_1 + H_1' &= 0,\\ (\mathcal{C} + n_2 + (4-n_2)\rho)F + 4H_1 + 2\rho(1-\rho)F' &= 0,\\ 4x_2G_1' + ((2-n_2)x_1 + 2G_1')F + 2x_1(G_1 + H_1') &= 0.\end{aligned}$$

The first two of these equations give the first two equations in (6.8). Multiplying the first equation by $2x_1$ and subtracting it from the third gives $2(F + 2x_2)G_1' = 0$, hence, $G_1' = 0$ as we claimed. □

Now we are ready to describe the local structure of g, A and v in the case where A "contains" a 2×2 or 3×3 Jordan block.

PROPOSITION 6.3. – 1. *Let $p \in M^0$ be a regular point and A contain a 2×2 Jordan block with a non-constant eigenvalue $\rho = \rho_1$. Then in the neighborhood of p there exists a local coordinate system $x,\ \rho_1,\dots,\rho_\ell,\ y_1,\dots,y_N$, $N = m_0 + m_1$, in which g and A take the following form:*

$$A = \begin{pmatrix} L(x,\vec\rho) & 0\\ 0 & A_c(y)\end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} h(x,\vec\rho) & 0\\ 0 & g_c(y)\cdot\chi_L(A_c(y))\end{pmatrix}, \tag{6.25}$$

where

$$L = \begin{pmatrix} L_1(x,\rho_1) & 0 & \cdots & 0\\ 0 & \rho_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \rho_\ell\end{pmatrix},\quad h = \begin{pmatrix} h_1(x,\rho_1)\cdot\Delta_1(L_1) & 0 & \cdots & 0\\ 0 & F_2^{-1}(\rho_2)\cdot\Delta_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & F_\ell^{-1}(\rho_\ell)\cdot\Delta_\ell\end{pmatrix} \tag{6.26}$$

and the ingredients in these matrices are as follows:

- *L_1 and h_1 are defined by* (6.5) *with $\rho=\rho_1$,*
- *$\Delta_1(\cdot)$ is the polynomial of the form $\Delta_1(t) = \Pi_{j=2}^{\ell}(t-\rho_j)$,*
- *$F_i(\rho_i) = a_i(1-\rho_i)^{-\mathcal{C}}\rho_i^{\ell+2+\mathcal{C}}$,*
- *$\Delta_i = (\rho_i-\rho_1)^2\Pi_{j=2,j\neq i}^{\ell}(\rho_i-\rho_j)$, $i = 2,\dots,\ell$,*
- *$A_c(y)$ is selfadjoint and parallel w.r.t. $g_c(y)$.*

Furthermore,

$$v = G(x,\rho_1)\frac{\partial}{\partial x} + \sum_{i=1}^{\ell}\rho_i(1-\rho_i)\frac{\partial}{\partial\rho_i} + \cdots,$$

with $G(x,\rho_1)$ as in Corollary 6.2 (2), with $n_2 = \ell - 1$, $\rho_1 = \rho$.

2. *Similarly, let $p \in M^0$ be a regular point and A contain a 3×3 Jordan block with a non-constant eigenvalue $\rho = \rho_1$. Then in the neighborhood of $p \in M^0$ we can choose local coordinates $x_1, x_2,\ \rho_1,\dots,\rho_\ell,\ y_1,\dots,y_N$ such that A and g are given by* (6.25), (6.26) *where $x = (x_1, x_2)$ and the other ingredients are as follows:*

— *the* 3×3 *blocks* $L_1(x_1,x_2,\rho_1)$ *and* $h_1(x_1,x_2,\rho_1)$ *are defined by* (6.7) *with* $\rho_1=\rho$,

— $\Delta_1(\cdot)$ *is the polynomial of the form* $\Delta_1(t)=\Pi_{j=2}^{\ell}(t-\rho_j)$,

— $F_i(\rho_i){=}a_i(1-\rho_i)^{-\mathcal{C}}\rho_i^{\ell+3+\mathcal{C}}$, $i=2,\dots,\ell$,

— $\Delta_i{=}(\rho_i-\rho_1)^3\Pi_{j=2,j\neq i}^{\ell}(\rho_i-\rho_j)$, $i=2,\dots,\ell$,

— $A_{\mathrm{c}}(y)$ *is selfadjoint and parallel w.r.t.* $g_{\mathrm{c}}(y)$.

Furthermore,

$$v=G(x_1,x_2,\rho_1)\frac{\partial}{\partial x_1}+H(x_2,\rho_1)\frac{\partial}{\partial x_2}+\sum_{i=1}^{\ell}\rho_i(1-\rho_i)\frac{\partial}{\partial\rho_i}+\cdots,$$

with $G(x_1,x_2,\rho_1)$ *and* $H(x_2,\rho)$ *as in Corollary 6.2 (3), with* $n_2=\ell-1$, $\rho_1=\rho$.

Proof. – The Formulas (6.25) and (6.26) (with $F(\rho), F_2(\rho_2),\dots,F_\ell(\rho_\ell)$ being arbitrary functional parameters) are just a reformulation of the main result of [10] in the case where A has the algebraic type described above.

In our situation we have, in addition, a projective vector field v satisfying (6.1). Consider the natural decomposition of v that corresponds to the splitting (6.25) of g, A into "constant" and "non-constant" blocks: $v=v_{\mathrm{nc}}(x,\vec{\rho})+v_{\mathrm{c}}(y)$.

It is easy to see (cf. (5.13), (5.14)) that (6.1) can be rewritten for the non-constant block without any change, i.e.,

$$\mathcal{L}_{v_{\mathrm{nc}}}L=-L^2+L,\quad \mathcal{L}_{v_{\mathrm{nc}}}h=-hL-(\operatorname{tr}L+\mathcal{C})h.$$

Here $\operatorname{tr}A+\mathcal{C}'=\operatorname{tr}L+\mathcal{C}$ and $\mathcal{C}=\mathcal{C}'+m_1$ where m_1 is the multiplicity of the constant eigenvalue 1 or, which is the same, $m_1=\operatorname{tr}A_{\mathrm{c}}$.

After this remark, Proposition 6.3 follows immediately by applying Theorem 6.1 and Corollary 6.2 (for h, L and v_{nc} but not v !) to reconstruct the functions $F(\rho), F_2(\rho_2),\dots,F_\ell(\rho_\ell)$ as well as the components of v_{nc} (the components of $v_{\mathrm{c}}(y)$ are not important for our purposes and we ignore them, in Proposition 6.3 they are denoted by "$\cdots$"). ☐

Partitioning local coordinates into two groups $x,\rho_1,\dots,\rho_\ell$ and $y_1,\dots,y_N$ determines two natural integrable distributions $\mathcal{U}$ and $\mathcal{U}^{\perp}$ on M^0 similar to those from §5.5. All geometric properties of the corresponding foliations listed in Proposition 5.6 still hold with one little amendment that $\dim\mathcal{U}=\ell+1$ or $\ell+2$ (but not ℓ as before) so that now we should think of x as an additional coordinate to ρ_i's.

The next statement is an analogue of Proposition 5.10. Consider the domain $U\subset\mathbb{R}^{\ell+1}(x,\rho_1,\dots,\rho_\ell)$ in the case of a $2{\times}2$ Jordan block (resp. $U\subset\mathbb{R}^{\ell+2}(x_1,x_2,\rho_1,\dots,\rho_\ell)$ in the case of a $3{\times}3$ Jordan block) on which the above local Formulas (6.26) for h are naturally defined. More precisely, U is defined by the inequalities (6.2) for the ρ_i's and the additional coordinates x and x_2 satisfy $F(\rho_1)+x\neq 0$ for a 2×2-block and resp. $F_1(\rho_1)+2x_2\neq 0$ for a 3×3 block, see Remark 6.1.

Proposition 6.4. – *There is a natural isometric immersion* $\phi : U\to M$ *(as a leaf of the totally geodesic foliation determined by* $\mathcal{U}$*). In other words, the above Formulas* (6.25) *and* (6.26) *have global meaning on* M *for all admissible values of coordinates.*

Proof. – The idea of the proof is similar to that of Proposition 5.10. We start with a certain point $p \in M^0$ and locally identify the leaf of $\mathcal{U}$ through p with U by using a canonical coordinate system in its neighborhood constructed in Proposition 6.3.

After this we use prolongation along a path as in the proof of Proposition 5.10. We need to justify several facts which guarantee that such a prolongation is always possible, namely, that the limit point of the curve $\phi(a(t))$ as $t \to T_0$ (we use the same notation as in Proposition 5.10) exists, is unique and lies in M^0, the set of regular points. First, we will ensure that for any sequence $t_1 < t_2 < \cdots < t_i \cdots \overset{i\to\infty}{\to} T_0$ such that $\phi(a(t_i))$ converges, we have $\lim_{i\to\infty} \phi(a(t_i)) \in M^0$. This will follow from Lemma 6.5 below.

Since ϕ preserves the eigenvalues of L, the multiplicities of the eigenvalues remain unchanged and the inequalities (6.2) hold at the limit point. The condition $d\rho_i \neq 0$ is fulfilled automatically and we only need to check that the Jordan block "survives" at the limit point. A priori under continuous deformations the Jordan block may split into smaller blocks and we need to show that this event may not happen under our assumptions.

To prove this fact we use the following algebraic lemma.

LEMMA 6.5. – *Let h be a non-degenerate bilinear form of Lorentzian signature and L be an h-selfadjoint endomorphism. Assume that L has a single real eigenvalue ρ and e_1 is a ρ-eigenvector of L. Then in dimension* 2 *and* 3 *we have respectively:*

1. *For any canonical basis e_1, e_2 (i.e., such that $h = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or, equivalently, $h(e_i, e_j) = \delta_{i,3-j}$), the matrix of L has the form*
$$L = \begin{pmatrix} \rho & \alpha \\ 0 & \rho \end{pmatrix}.$$
Moreover, α does not depend on the choice of e_2, and can be computed from the following formula $\alpha = \mathrm{vol}_h\big(e_2, (L - \rho\,\mathrm{Id})e_2\big)$.

2. *For any canonical basis e_1, e_2, e_3 (i.e., such that $h = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ or, equivalently, $h(e_i, e_j) = \delta_{i,4-j}$), the matrix of L has the form*
$$L = \begin{pmatrix} \rho & \alpha & \beta \\ 0 & \rho & \alpha \\ 0 & 0 & \rho \end{pmatrix}.$$
Moreover, α does not depend on the choice of e_2 and e_3, and can be computed from the following formula $\alpha = \mathrm{vol}_h\big(e_3, (L - \rho\,\mathrm{Id})e_3, (L - \rho\,\mathrm{Id})^2 e_3\big)^{\frac{1}{3}}$.

Proof. – The proof is straightforward and we only give some comments for $\dim = 3$. The first statement follows immediately from two facts:

1. L is h-selfadjoint and therefore the matrices of L and h satisfy $L^\top h = hL$;
2. the first column of L is $(\rho, 0, 0)^\top$.

The formula for α is obvious in the basis e_1, e_2, e_3. We now check that this formula is independent of the choice of e_2 and e_3. Let e_1, e_2', e_3' be another canonical basis. Then $e_3' = a_1 e_1 + a_2 e_2 + a_3 e_3$, but $h(e_1, e_3) = h(e_1, e_3') = 1$ immediately implies that $a_3 = 1$. Hence, using the explicit formula for L, we can easily conclude that the additional terms $a_1 e_1 + a_2 e_2$ do not contribute. Indeed,

$$\begin{aligned}
\mathrm{vol}_h(e_3', &(L-\rho\,\mathrm{Id})e_3', (L-\rho\,\mathrm{Id})^2 e_3') = \\
&\mathrm{vol}_h(e_3 + a_1 e_1 + a_2 e_2, (L-\rho\,\mathrm{Id})e_3 + a_2(L-\rho\,\mathrm{Id})e_2, (L-\rho\,\mathrm{Id})^2 e_3) = \\
&\mathrm{vol}_h(e_3 + a_1 e_1 + a_2 e_2, \beta e_1 + \alpha e_2, \alpha^2 e_1) = \alpha^3 \mathrm{vol}_g(e_3, e_2, e_1) = \alpha^3. \qquad \square
\end{aligned}$$

This lemma gives us a simple method to recognize if L has a Jordan block of maximal size or not.

Let us return to the proof of Proposition 6.4. Since M is closed, any sequence has a convergent subsequence. We take a sequence $t_1 < t_2 < \cdots < t_i \cdots \overset{i\to\infty}{\to} T_0$ and assume without loss of generality that $\phi(a(t_i))$ converges to a certain point which we denote by p. Let us show that p is regular, i.e., the Jordan block "survives". We know that the eigenvalue ρ_1 of the Jordan block is a smooth function on M^0. Moreover, the vector field $e_1 = \mathrm{grad}\,\rho_1$ does not vanish and is an eigenvector of the ρ_1-block. Notice that these conditions hold not only on M^0, but also on a slightly bigger set $\widetilde{M}^0$ ($M^0 \subset \widetilde{M}^0$) which can be characterized by the property that the multiplicities of eigenvalues are fixed but the algebraic type of L is allowed to change, namely, the Jordan ρ_1-block may split into smaller ρ_1-blocks. Notice that the natural splitting into blocks corresponding to the eigenvalues of g makes sense on $\widetilde{M}^0$, so we can work with each block separately. An important additional fact, we are going to use, is that ϕ, by construction, preserves $\mathrm{grad}\,\rho_1$.

We use Lemma 6.5 to verify that the parameter α in the matrix of $L(p)$ does not vanish. Since both L and e_1 are smooth, we have by continuity $\alpha = \lim_{t_i\to T_0} \alpha(t_i)$, where $\alpha(t_i)$ is computed at the point $\phi(a(t_i))$ (w.r.t. to $e_1 = \mathrm{grad}\,\rho$). But since ϕ is an isometry whenever it is well defined, then the limit can be computed on U. Since all the points of U are regular by construction, we have $\lim_{t_i\to T_0}\alpha(t_i) \neq 0$ as required. Thus, the limit point p of the sequence $\phi(a(t_i))$ is regular.

We continue the proof for $U \subset \mathbb{R}^{2+\ell}$, i.e., in the case of 3×3 Jordan block; the case of 2×2 block is similar. In some neighborhood $V(p) \subset M^0$, we can choose a canonical coordinate system $(x_1, x_2, \rho_1, \rho_2, \ldots, \rho_\ell, y_1, \ldots, y_N)$ as in Proposition 6.3 adapted to the orthogonal integrable distributions $\mathcal{U}$ and $\mathcal{U}^\perp$. A similar canonical coordinate system $x_1, x_2, \rho_1, \rho_2, \ldots, \rho_\ell$ can be chosen on U in the neighborhood $U(a(T_0))$ of $a(T_0)$.

In both cases, ρ_i are defined intrinsically as being eigenvalues of L. According to [10, Remark 1.8], the other two coordinates x_1, x_2 are defined up to shift $x_i \mapsto x_i + c_i(\rho_1)$ (with $c_i(\rho_1)$ explicitly given in [10, Remark 1.8]) so that they become unique if we "fix the origin" by setting $x_1(q) = 0$, $x_2(q) = 0$ for a chosen point q.

Now choose $t' = t_k$ such that $a(t') \in U(a(T_0))$ and $\phi\big(a(t')\big) \in V(p)$ and "shift" the coordinate systems introduced in $V(p)$ and $U\big(a(T_0)\big)$ to make them *centered* at $\phi\big(a(t')\big)$ and $a(t')$ respectively as just explained. In terms of these coordinate systems, the prolongation of ϕ along $a(t)$ is defined simply by

$$a(t) = \big(x_1(t), x_2(t), \rho_1(t), \ldots, \rho_\ell(t)\big) \mapsto \phi\big(a(t)\big) = \big(x_1(t), x_2(t), \rho_1(t), \ldots, \rho_\ell(t), \hat{y}_1, \ldots, \hat{y}_N\big),$$

where $\hat{y}_i$ are constant, i.e., $\phi\big(a(t)\big)$ always remains on the same $\mathcal{U}$-leaf. This formula makes sense as long as $a(t)|_{[t',T_0]}$ remains within the coordinate neighborhood $U\big(a(T_0)\big)$ and $\phi\big(a(t)|_{[t',T_0]}\big)$ remains within the coordinate neighborhood $V(p)$. We can easily guarantee these conditions by an appropriate choice of $t' = t_k$ making $a(t')$ very close to $a(T_0)$ and $\phi\big(a(t')\big)$ very close to p.

Thus, the prolongation along $a(t)$ from $t = t'$ up to $t = T_0$ (and even a bit further if $T_0 < 1$) is well defined by the above formula. In particular, $\lim_{t\to T_0} \phi(a(t))$ exists and therefore coincides with $p = \phi\big(a(T_0)\big)$ so that p belongs to the same $\mathcal{U}$-leaf as the one we started with. This completes the proof. □

Now to prove that Jordan blocks with non-constant eigenvalues cannot appear on compact manifolds, we compute one special eigenvalue of the curvature operator of the metric g given by the formulas from Proposition 6.3. For this computation we use the following real analogue of Proposition A.2, which can be proved in a similar way.

PROPOSITION 6.6 ([6]). – *Let g and L be compatible in the projective sense and Λ be as in* (1.7). *Let $\nabla\Lambda = f(L)$ at some point $p \in M$, where $f(\cdot)$ is a polynomial (or, more generally, an analytic function) and suppose $L(p)$ has a non-trivial Jordan ρ-block. Then one of the eigenvalues of the curvature operator of g at the point p takes the form*

$$f'(\rho).$$

This number can be computed for our metric g (equivalently, for h given by (6.26)) A straightforward calculation shows the following:

PROPOSITION 6.7. – 1. *For a 2×2 Jordan ρ_1-block, we have*

$$f'(\rho_1) = -\frac{F_1'(\rho_1)}{(F_1(\rho_1)+x)^3\prod_{i\geq 2}(\rho_1-\rho_i)} + \sum_{i\geq 2}\frac{F_i(\rho_i)}{4(\rho_i-\rho_1)^4\prod_{j\notin\{1,i\}}(\rho_i-\rho_j)}.$$

2. *For a 3×3 Jordan ρ_1-block, we have*

$$f'(\rho_1) = -\frac{3}{4(F_1(\rho_1)+2x_2)^2\prod_{i\geq 2}(\rho_1-\rho_i)} + \sum_{i\geq 2}\frac{F_i(\rho_i)}{4(\rho_i-\rho_1)^5\prod_{j\notin\{1,i\}}(\rho_i-\rho_j)}.$$

These formulas immediately imply that the quantity $f'(\rho_1)$ (which is some special eigenvalue of the curvature operator of g) is unbounded on M^0. Indeed, x and x_2 may vary independently of the other coordinates and, in particular, we may fix the values of all ρ_i's and then vary x (resp. x_2) so that $F_1(\rho_1)+x$ (resp. $F_1(\rho_1)+2x_2$) tends to 0 and therefore $f'(\rho_1) \to \infty$, which is impossible due to compactness of M. Thus, Jordan blocks with non-constant eigenvalues may not occur in our situation and this conclusion completes the proof of Theorem 1.2.

Appendix

Eigenvalues of the curvature operator

In what follows, we consider a real vector space V with a complex structure J and an inner product g (not necessarily positive definite) such that $g(Ju, Jv) = g(u, v)$. Such a triple (V, g, J) will be referred to as a *pseudo-Hermitian vector space*. We use the symbol

$$\mathfrak{u}(g, J) = \{X \in \mathfrak{gl}(V) : [X, J] = 0 \text{ and } g(Xu, v) = -g(u, Xv)\}$$

to denote the space (Lie algebra) of skew-Hermitian endomorphisms on V.

Let us first reformulate the integrability condition for Equation (1.4) in a way adapted to the Lie theory. Recall that the Riemann curvature operator (at a point $x \in M$) can be understood as a map $R : T_xM \otimes T_xM \to \mathfrak{so}(g)$, $R(u, v) = \nabla_u\nabla_v - \nabla_v\nabla_u - \nabla_{[u,v]}$. Taking into account the fact that we are dealing with a Kähler manifold and using the symmetries of the curvature tensor of a Kähler metric, we can also think of R as an operator defined on the unitary Lie algebra (we still use the same notation)

$$R : \mathfrak{u}(g, J) \to \mathfrak{u}(g, J)$$

by setting $R(u, v) = \frac{1}{4}R(u \wedge_J v)$, where

$$u \wedge_J v = u^\flat \otimes v - v^\flat \otimes u + (Ju)^\flat \otimes Jv - (Jv)^\flat \otimes Ju \in \mathfrak{u}(g, J) \tag{A.1}$$

and $u^\flat = g(u, \cdot)$ denotes the metric dual of u.

LEMMA A.1. – *Let (M, g, J) be a Kähler manifold of arbitrary signature, $A \in \mathscr{A}(g, J)$ be a Hermitian solution of* (1.4) *and* $\Lambda = \frac{1}{4}\mathrm{grad}(\mathrm{tr}\, A)$. *Then the curvature operator $R : \mathfrak{u}(g, J) \to \mathfrak{u}(g, J)$ satisfies the relation*

$$[R(X), A] = 4[X, \nabla\Lambda] \textit{ for all } X \in \mathfrak{u}(g, J). \tag{A.2}$$

Proof. – The Ricci identity applied to an arbitrary field of endomorphisms A reads

$$\nabla_u\nabla_v A - \nabla_v\nabla_u A - \nabla_{[u,v]}A = [R(u, v), A]$$

for any vector fields u, v. Let now $A \in \mathscr{A}(g, J)$ be a Hermitian solution of (1.4). Since $\nabla\Lambda$ is g-selfadjoint and J-linear, i.e., Hermitian too, we have

$$\begin{aligned}\nabla_u\nabla_v A - \nabla_v\nabla_u A - \nabla_{[u,v]}A = {} & v^\flat \otimes \nabla_u\Lambda + (\nabla_u\Lambda)^\flat \otimes v - u^\flat \otimes \nabla_v\Lambda - (\nabla_v\Lambda)^\flat \otimes u \\ & + (Jv)^\flat \otimes \nabla_{Ju}\Lambda + (\nabla_{Ju}\Lambda)^\flat \otimes Jv \\ & - (Ju)^\flat \otimes \nabla_{Jv}\Lambda - (\nabla_{Jv}\Lambda)^\flat \otimes Ju = [X, \nabla\Lambda],\end{aligned}$$

where $X = u \wedge_J v$. This proves Formula (A.2) for elements $X \in \mathfrak{u}(g, J)$ of the form $u \wedge_J v$ and the claim follows from the fact that all skew-Hermitian endomorphisms are sums of such elements. □

REMARK A.1. – If Formula (A.2) holds for an operator $R : \mathfrak{u}(g, J) \to \mathfrak{u}(g, J)$ and Hermitian endomorphisms A and $\nabla\Lambda$ then, in fact, $\nabla\Lambda$ can be presented in the form $\nabla\Lambda = p(A)$ for some polynomial $p(\cdot)$ with real coefficients. To show this, take an arbitrary J-complex matrix Y and consider the following algebraic relations:

$$\mathrm{tr}\left(X \cdot [\nabla\Lambda, Y]\right) = \mathrm{tr}\left(Y \cdot [X, \nabla\Lambda]\right) = \frac{1}{4}\mathrm{tr}\left(Y \cdot [R(X), A]\right) = \frac{1}{4}\mathrm{tr}\left(R(X) \cdot [A, Y]\right),$$

where $X \in \mathfrak{u}(g, J)$ and tr denotes the complex trace. Since $\mathfrak{u}(g, J)$ spans $\mathfrak{gl}(T_x M, J)$ in the complex sense, we conclude that $[\nabla\Lambda, Y] = 0$ for any Y commuting with A. It is a well-known algebraic fact that in this case $\nabla\Lambda$ can be written as a polynomial of A. Moreover, as both A and $\nabla\Lambda$ are Hermitian, this polynomial must be real, i.e., with real coefficients.

Proposition A.2 below together with Formula (A.2) allows us to calculate eigenvalues of the curvature operator in terms of the eigenvalues of A and $\nabla\Lambda$. For the main concepts of the proof of this proposition and for the relation to sectional operators in the theory of integrable systems compare also with [7, §3] and [6, 5].

Proposition A.2. – *Let (V, g, J) be a pseudo-Hermitian vector space and let $A : V \to V$ be a Hermitian endomorphism. Suppose an operator $R : \mathfrak{u}(g, J) \to \mathfrak{u}(g, J)$ satisfies*

$$[R(X), A] = [X, B] \text{ for all } X \in \mathfrak{u}(g, J), \tag{A.3}$$

where $B = p(A)$ and $p(\cdot)$ is a polynomial with real coefficients. Then we have the following:

1. *For all real eigenvalues $\lambda_i \neq \lambda_j$ of A,*
$$\frac{p(\lambda_i) - p(\lambda_j)}{\lambda_i - \lambda_j}$$
is an eigenvalue of R.
2. *If A has a non-trivial λ_i-Jordan block, $\lambda_i \in \mathbb{R}$, then $p'(\lambda_i)$ is an eigenvalue of R.*

Remark A.2. – The first item of Proposition A.2 can be understood in a slightly different way. Notice that $B = p(A)$ implies that each eigenvector of A with an eigenvalue λ_i is, at the same time, an eigenvector of B with the eigenvalue $m_i = p(\lambda_i)$. Hence the formula for the eigenvalue of R from item (1) can be rewritten as $\frac{m_i - m_j}{\lambda_i - \lambda_j}$ so that we do not actually need to find $p(\cdot)$ explicitly; it is sufficient to know the eigenvalues m_i of B corresponding to λ_i.

The second item of Proposition A.2 can also be modified by using the following simple fact from Linear Algebra. Let λ be an eigenvalue of an endomorphism A having a non-trivial λ-Jordan block. Let $p(\cdot)$ and $q(\cdot)$ be two polynomials (or even more generally, analytic functions) such that $p(A) = q(A)$, then $p(\lambda) = q(\lambda)$ and $p'(\lambda) = q'(\lambda)$. It follows from this statement that the polynomial p in the second item of Proposition A.2 can be replaced by any other function q satisfying $p(A|_{V_{\lambda_i}}) = q(A|_{V_{\lambda_i}})$, where V_{λ_i} denotes the generalized λ_i-eigenspace of A.

Proof of Proposition A.2. – We start with some general considerations regarding Formula (A.3). We view this formula as an equation on R for fixed A and $B = p(A)$. Suppose $R_1, R_2 : \mathfrak{u}(g, J) \to \mathfrak{u}(g, J)$ are two solutions of (A.3). Then,

$$[R_1(X) - R_2(X), A] = 0 \text{ for all } X \in \mathfrak{u}(g, J),$$

that is, $R_1 - R_2$ takes values in the Lie algebra

$$\mathfrak{g}_A = \{X \in \mathfrak{u}(g, J) : [X, A] = 0\},$$

the centralizer of A in $\mathfrak{u}(g, J)$. Thus, any solution R of (A.3) is unique up to adding an operator $\mathfrak{u}(g, J) \to \mathfrak{g}_A$. Moreover, an operator R satisfying (A.3) preserves the centralizer $\mathfrak{g}_A$.

Indeed, since $B = p(A)$ is a polynomial in A, we have $[X, B] = 0$ for all $X \in \mathfrak{g}_A$. Then (A.3) implies $[R(X), A] = 0$ for $X \in \mathfrak{g}_A$, showing

$$R(\mathfrak{g}_A) \subseteq \mathfrak{g}_A,$$

as we claimed. For any solution R of (A.3), we may therefore consider the induced operator

$$\tilde{R} : \mathfrak{u}(g, J)/\mathfrak{g}_A \to \mathfrak{u}(g, J)/\mathfrak{g}_A$$

on the quotient space. It is a general fact that eigenvalues of the quotient operator $\tilde{R}$ are eigenvalues of the original operator R. On the other hand, we have just seen that the quotient map $\tilde{R}$ is the same for all solutions R of (A.3). We will use these facts by working with the quotient map $\tilde{R}_0$ coming from a special solution R_0 of (A.3) defined by

$$R_0 = \frac{\mathrm{d}}{\mathrm{d}t} p(A + tX)|_{t=0}.$$

This is indeed a solution of (A.3) as follows immediately from differentiating the identity $[p(A + tX), A + tX] = 0$ at $t = 0$. By definition, if $p(t) = \sum_{k=0}^{m} a_k t^k$, then

$$R_0(X) = \sum_{k=1}^{m} a_k \sum_{p+q=k-1} A^p X A^q.$$

Hence for a generating element $u \wedge_J v$, we obtain

$$R_0(u \wedge_J v) = \sum_{k=1}^{m} a_k \sum_{p+q=k-1} A^p u \wedge_J A^q v. \tag{A.4}$$

We are now in the position to prove Proposition A.2. First, we show that R_0 has eigenvalues as given in part (1) and (2) of the proposition:

(1) Suppose u and v are eigenvectors of A for real eigenvalues λ_i and λ_j respectively, $\lambda_i \neq \lambda_j$. Then (A.4) becomes equal to

$$R_0(u \wedge_J v) = \left(\sum_{k=1}^{m} a_k \sum_{r=0}^{k-1} \lambda_i^r \lambda_j^{k-1-r}\right) u \wedge_J v = \left(\sum_{k=1}^{m} a_k \frac{\lambda_i^k - \lambda_j^k}{\lambda_i - \lambda_j}\right) u \wedge_J v = \frac{p(\lambda_i) - p(\lambda_j)}{\lambda_i - \lambda_j} u \wedge_J v.$$

Hence,

$$\frac{p(\lambda_i) - p(\lambda_j)}{\lambda_i - \lambda_j}$$

is an eigenvalue of R_0 with eigenvector $u \wedge_J v$.

(2) Let us first argue, that without loss of generality we can suppose that a fixed real eigenvalue λ_i of A is equal to zero. Indeed, using $\tilde{A} = A - \lambda_i \cdot \mathrm{Id}$ instead of A in (A.3), the Equation (A.3) holds for R_0, $\tilde{A}$ and the same $B = \tilde{p}(\tilde{A})$ for another polynomial $\tilde{p}(t) = \sum_{k=0}^{m} \tilde{a}_k t^k$ that is equal to $p(t + \lambda_i)$. Since $\tilde{p}'(0) = p'(\lambda_i)$, we may assume that the fixed eigenvalue λ_i under consideration is equal to zero. Suppose A has a non-trivial λ_i-Jordan block and denote by V_{λ_i} the generalized λ_i-eigenspace. Let $u \in V_{\lambda_i}$ be an eigenvector of A, i.e., $Au = 0$, and $v \in V_{\lambda_i}$ satisfy $Av = u$. Then (A.4) becomes

$$R_0(u \wedge_J v) = \sum_{k=1}^{m} a_k (u \wedge_J A^{k-1} v) = a_1 u \wedge_J v.$$

Thus, a_1 is an eigenvalue of R_0 with eigenvector $u \wedge_J v$. Since $a_1 = p'(0)$, the eigenvalue is as in part (2) of Proposition A.2.

To summarize, we have shown that R_0 has eigenvalues as given in part (1) and (2) of the proposition. It remains to show that these eigenvalues are also eigenvalues for the quotient map $\tilde{R}_0$. Since for an arbitrary operator R solving (A.3) we have $\tilde{R} = \tilde{R}_0$, we then obtain that $\tilde{R}$ and hence R, has eigenvalues as in part (1) and (2) of the proposition.

It is straightforward to show that for any operator $\varphi : V \to V$ with a φ-invariant subspace $U \subseteq V$, an eigenvalue λ of φ is also an eigenvalue of the quotient map

$$\tilde{\varphi} : V/U \to V/U$$

if and only if the generalized λ-eigenspace of φ is not contained in U (although, it may have a non-trivial intersection with U).

To complete the proof of Proposition A.2, it therefore suffices to show the following statements, each of which proves one of the parts of the proposition:

(1) $u \wedge_J v \notin \mathfrak{g}_A$ for eigenvectors u and v of A corresponding to real eigenvalues λ_i and λ_j respectively, $\lambda_i \neq \lambda_j$.

(2) $u \wedge_J v \notin \mathfrak{g}_A$ for an eigenvector u of A corresponding to a real eigenvalue λ_i and a vector $v \in V_{\lambda_i}$ such that $Av = u + \lambda_i v$.

Introducing the notation

$$u \odot_J v = u^\flat \otimes v + v^\flat \otimes u + (Ju)^\flat \otimes Jv + (Jv)^\flat \otimes Ju,$$

we have $[u \wedge_J v, A] = Au \odot_J v - u \odot_J Av$. Thus, for case (1) we obtain

$$[u \wedge_J v, A] = (\lambda_i - \lambda_j) u \odot_J v,$$

which is non-zero, hence, $u \wedge_J v \notin \mathfrak{g}_A$. For case (2) we obtain

$$[u \wedge_J v, A] = \lambda_i u \odot_J v - u \odot_J (u + \lambda_i v) = -u \odot_J u,$$

which is non-zero, hence, $u \wedge_J v \notin \mathfrak{g}_A$. This finishes the proof of the proposition. ☐

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ARITHMETIC AMPLENESS AND AN ARITHMETIC BERTINI THEOREM

BY François CHARLES

ABSTRACT. – Let $\mathcal{X}$ be a projective arithmetic variety of dimension at least 2. If $\overline{\mathcal{L}}$ is an ample hermitian line bundle on $\mathcal{X}$, we prove that the proportion of those effective sections σ of $\overline{\mathcal{L}}^{\otimes n}$ such that the divisor of σ on $\mathcal{X}$ is irreducible tends to 1 as n tends to ∞. We prove variants of this statement for schemes mapping to such an $\mathcal{X}$.

On the way to these results, we discuss some general properties of arithmetic ampleness, including restriction theorems, and upper bounds for the number of effective sections of hermitian line bundles on arithmetic varieties.

RÉSUMÉ. – Soit $\mathcal{X}$ une variété arithmétique projective de dimension au moins 2, et soit $\overline{\mathcal{L}}$ un fibré hermitien sur $\mathcal{X}$. Si $\overline{\mathcal{L}}$ est ample, on démontre que la proportion des sections effectives de $\overline{\mathcal{L}}^{n}$ qui définissent un diviseur irréductible sur $\mathcal{X}$ tend vers 1 quand n tend vers ∞. On démontre également des variantes de ce résultat pour des schémas admettant un morphisme vers $\mathcal{X}$.

On prouve par ailleurs un certain nombre de propriétés générales de l'amplitude arithmétique, autour notamment de théorèmes de restriction et d'estimées pour le nombre de sections effectives de fibrés en droites hermitiens.

1. Introduction

1.1. Bertini theorems over fields

Let k be an infinite field, and let X be an irreducible variety over k with dimension at least 2. Given an embedding of X in some projective space over k, the classical Bertini theorem [23, Theorem 3.3.1] shows, in its simplest form, that infinitely many hyperplane sections of X are irreducible.

In the case where k is finite, the Bertini theorem can fail, since the finitely many hyperplane sections of X can all be reducible. As was first explained in [26] in the setting of smoothness theorems, this phenomenon can be dealt with by replacing hyperplane sections with ample hypersurfaces of higher degree. We can state the main result of [11]—see Theorem 1.6 in [11] and the discussion in the proof of Theorem 6.1 below—as follows: let k be a finite field, let

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X be a projective variety over k and let L be an ample line bundle on X. Let Y be an integral scheme of finite type over k together with a morphism $f : Y \to X$. Assume that the image of f has dimension at least 2. If Z is a subscheme of Y, write Z_{horiz} for the union of those irreducible components of Z that do not map to a closed point of X. Then the set

$$\mathcal{P} = \Big\{\sigma \in \bigcup_{n>0} H^0(X, L^{\otimes n}),\ \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}\Big\}$$

has density 1, in the sense that

$$\lim_{n\to\infty} \frac{|\mathcal{P} \cap H^0(X, L^{\otimes n})|}{|H^0(X, L^{\otimes n})|} = 1.$$

Here if S is a set, we denote by $|S|$ its cardinality. When Y is a subscheme of X, we can disregard the horizontality subscript.

1.2. The arithmetic case

In this paper, we deal with an arithmetic version of Bertini theorems as above. Let $\mathcal{X}$ be an arithmetic variety, that is, an integral scheme which is separated, flat of finite type over $\mathrm{Spec}\,\mathbb{Z}$. Assume that $\mathcal{X}$ is projective, and let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{X}$. As is well known, sections of $\mathcal{L}$ over $\mathcal{X}$ are not the analogue of global sections of a line bundle over a projective variety over a field. Indeed, it is more natural to consider a hermitian line bundle $\overline{\mathcal{L}}$ with underlying line bundle $\mathcal{L}$ and consider the sets

$$H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$$

of sections with norm at most 1 everywhere. We discuss ampleness for hermitian line bundles in Section 2, which we refer to for definitions.

Given finite sets $(X_n)_{n>0}$, and a subset $\mathcal{P}$ of $\bigcup_{n>0} X_n$, say that $\mathcal{P}$ has density ρ if the following equality holds:

$$\lim_{n\to\infty} \frac{|\mathcal{P} \cap X_n|}{|X_n|} = \rho.$$

The main result of this paper is the following arithmetic Bertini theorem. Again, given a morphism of schemes $f : Y \to X$, and if Z is a subscheme of Y, we denote by Z_{horiz} the union of those irreducible components of Z that do not map to a closed point of X. If $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ is a hermitian vector bundle and δ is a real number, write $\overline{\mathcal{M}}(\delta)$ for the hermitian vector bundle $(\mathcal{M}, e^{-\delta}||.||)$. Write $||\sigma||_\infty$ for the sup norm of a section of a hermitian vector bundle.

THEOREM 1.1. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let $\mathcal{Y}$ be an integral scheme of finite type over* $\mathrm{Spec}\,\mathbb{Z}$ *together with a morphism $f : \mathcal{Y} \to \mathcal{X}$ which is generically smooth over its image. Assume that the image of $\mathcal{Y}$ has dimension at least* 2. *Let ε be a positive real number. Then the set*

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n}),\ ||\sigma_{|f(\mathcal{Y}(\mathbb{C}))}||_\infty \leq 1 \text{ and } \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}\Big\}$$

has density 1 *in* $\big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n}),\ ||\sigma_{|f(\mathcal{Y}(\mathbb{C}))}||_\infty \leq 1\big\}$.

Recall that by definition, the condition $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n})$ means

$$||\sigma||_\infty \leq \varepsilon^n.$$

REMARK 1.2. – A Weil divisor is said to be irreducible if it comes from an integral codimension 1 subscheme.

REMARK 1.3. – The hypothesis that f is generically smooth over its image is necessary: when f is the Frobenius morphism of a fiber $\mathcal{X}_p$, all $\operatorname{div}(f^*\sigma)$ have components with multiplicities divisible by p. Of course, it holds when $\mathcal{Y}$ is flat over $\operatorname{Spec}\mathbb{Z}$. Without this hypothesis on f, the conclusion is only that the support of $\operatorname{div}(f^*\sigma)$ is irreducible for a density 1 set of σ.

An important special case of the theorem deals with the special case where f is dominant. In this case, generic smoothness is automatic.

THEOREM 1.4. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let $\mathcal{Y}$ be an integral scheme of finite type over* $\operatorname{Spec}\mathbb{Z}$ *together with a morphism $f : \mathcal{Y} \to \mathcal{X}$. Assume that the image of $\mathcal{Y}$ has dimension at least* 2 *and f is dominant. Then the set*

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}),\ \operatorname{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}\Big\}$$

has density 1 *in* $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

REMARK 1.5. – We will prove Theorem 1.1 as a consequence of Theorem 1.4. However, the latter is a special case of the former. Indeed, with the notation of Theorem 1.1, when f is dominant, if $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(\varepsilon)^{\otimes n}$, then the condition

$$||\sigma|_{f(\mathcal{Y}(\mathbb{C}))}||_\infty \leq 1$$

is equivalent to

$$||\sigma||_\infty \leq 1,$$

i.e., $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

The case where $\mathcal{Y} = \mathcal{X}$ is particularly significant. We state it independently below. Most of this paper will be devoted to its proof, and we will prove 1.1 and 1.4 as consequences.

THEOREM 1.6. – *Let $\mathcal{X}$ be a projective arithmetic variety of dimension at least* 2, *and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then the set*

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}),\ \operatorname{div}(\sigma) \text{ is irreducible}\Big\}$$

has density 1 *in* $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

Theorem 1.6 is stronger than the Bertini irreducibility theorem of [11, Theorem 1.1], as we explain in Section 3. Note however that we use the results of [11] in our proofs.

In Theorem 1.6, the case where $\mathcal{X}$ has dimension at least 3—that is, relative dimension at least 2 over $\operatorname{Spec}\mathbb{Z}$—is easier. Indeed, if p is a large enough prime number, we can apply the Bertini irreducibility theorems over finite fields to the reduction of $\mathcal{X}$ modulo p, which with moderate work is enough to prove the theorem. However, when $\mathcal{X}$ is an arithmetic surface, Theorem 1.6 is genuinely different from its finite field counterpart. Note that the hardest case of the main result of [11] is the surface case as well.

Theorem 1.1 should be compared to the Hilbert irreducibility theorem, which implies, if $\mathcal{L}$ is very ample on the generic fiber of $\mathcal{X}$ and $\mathcal{Y}$ is flat over $\mathbb{Z}$, the existence of many sections σ of $\mathcal{L}$ such that the generic fiber of $\operatorname{div}(f^*\sigma)$ is irreducible. However, the Hilbert irreducibility theorem does not guarantee that these sections have small norm. To our knowledge, Theorem 1.1 does not imply the Hilbert irreducibility theorem, nor does it follow from it.

1.3. Previous results and applications

Arithmetic Bertini theorems, in the setting of both general arithmetic geometry and Arakelov geometry, have appeared in the literature. In [26], Poonen is able to prove a Bertini regularity theorem for ample line bundles on regular quasi-projective schemes over $\operatorname{Spec}\mathbb{Z}$ under the abc conjecture and technical assumptions. The statement does not involve hermitian metrics but still involves a form of density.

In [24], Moriwaki proves a Bertini theorem showing the existence of at least one effective section of large powers of an arithmetically ample line bundle that defines a generically smooth divisor—this was reproved and generalized in [19]. As a byproduct of our discussion of arithmetic ampleness in Section 2 and Poonen's result over finite fields, we will give a short proof of a more precise version of this result.

THEOREM 1.7. – *Let $\mathcal{X}$ be a projective arithmetic variety with smooth generic fiber, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then the set*

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}),\ \operatorname{div}(\sigma)_{\mathbb{Q}} \text{ is smooth}\Big\}$$

has density 1 *in* $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

Of course, this result can be combined with Theorem 1.6 if $\mathcal{X}$ has dimension at least 2.

Weaker Bertini theorems over rings of integers in number fields have been used in higher class-field theory, under the form of the Bloch-Raskind-Kerz approximation lemma proved in [5, 28, 35, 21]—see [33, Lemme 5.2] for a discussion. These results can be obtained easily as a special case of our Corollary 3.6 (or its variant corresponding to Theorem 1.1 for Wiesend's version)—this corollary allows us furthermore to control the cohomology class of the irreducible subvarieties involved.

An arithmetic Bertini theorem has been proved by Autissier in [2, 3]. Counts of irreducible divisors on arithmetic varieties have been provided by many authors, starting with Faltings in [14].

Our Bertini theorem is expected to be used in its precise form in upcoming work of Hrushovski on the model theory of number fields. We hope to discuss its applications to both general Arakelov geometry and number-theoretic problems in future work.

1.4. Strategy of the proofs

The starting point of our proof is that, as in [11], the Bertini irreducibility theorem is susceptible to a counting approach: to show that most divisors are irreducible, simply bound the number of the reducible ones.

To carry on this approach, we need to translate in Arakelov geometry results from classical geometry. The two key results in that respect are the study of restriction maps for powers of ample hermitian line bundles we prove in 2.3 and bounds for sections of hermitian line bundles on surfaces in 4.2. We hope that these results have independent interest.

Even with these tools at our disposal, we are not able to adapt the methods of [11], for two reasons. First, the error terms in the various estimates we deal with (including arithmetic Hilbert-Samuel) are big enough that we need a more involved sieving technique than in [11] involving the analysis of simultaneous restriction of sections modulo infinite families of subschemes. Second, given a section of a hermitian line bundle with reducible divisor on a regular arithmetic surface, we need to construct a corresponding decomposition of the hermitian line bundle, which involves constructing suitable metrics. The relevant analysis is dealt with in 4.1 and can only be applied when suitable geometric bounds hold. To get a hold of the geometry, we need a careful analysis dealing with infinite families of curves over finite fields—coming from the reduction of our given arithmetic surface modulo many primes. This is the content of 5.2.

1.5. Notation and definitions

If S is a set, we denote by $|S|$ the cardinal of S.

If $\mathcal{X}$ is a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$, we denote by $\mathcal{X}^{\mathrm{an}}$ the associated complex analytic space.

By an arithmetic variety, we mean an integral scheme which is flat of finite type over $\operatorname{Spec}\mathbb{Z}$. A projective arithmetic variety of dimension 2 is an arithmetic surface. If $\mathcal{X}$ is a scheme over $\operatorname{Spec}\mathbb{Z}$ and if p is a prime number, we will denote by $\mathcal{X}_p$ the reduction of $\mathcal{X}$ modulo p. If $f : X \to Y$ is a morphism of noetherian schemes, we say that an irreducible component of X is vertical if its image is a closed point of Y, and horizontal if not. We denote by X_{horiz} the union of the horizontal components of X.

Let X be a complex analytic space. A hermitian vector bundle $\overline{M} = (M, ||.||)$ is a pair consisting of a vector bundle M on X and a hermitian metric on the restriction of M to the reduced subspace X_{red}. We require furthermore for the metric to be smooth, i.e., if X is of pure dimension d, given any holomorphic map from the unit disk D^d in $\mathbb{C}^d$ to X, the metric pulled-back from X to D^d is smooth.

Let $\mathcal{X}$ be a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. A hermitian vector bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$ is a pair $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ where $\mathcal{M}$ is a vector bundle on $\mathcal{X}$ and $||.||$ is a smooth metric on the restriction of $\mathcal{M}$ to the complex space $\mathcal{X}(\mathbb{C})$. If $\overline{\mathcal{M}}$ is a hermitian vector bundle over a scheme $\mathcal{X}$ of finite type over $\mathbb{Z}$, we will denote by $\mathcal{M}$ the underlying vector bundle. Note that if the generic fiber $\mathcal{X}_{\mathbb{Q}}$ is empty, i.e., if $\mathcal{X}$ is vertical, a hermitian vector bundle on $\mathcal{X}$ is nothing but a vector bundle. If $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ is a hermitian vector bundle and δ is a real number, write $\overline{\mathcal{M}}(\delta)$ for the hermitian vector bundle $(\mathcal{M}, e^{-\delta}||.||)$.

Let $\overline{\mathcal{M}}$ be a hermitian vector bundle on a proper scheme $\mathcal{X}$ over $\mathbb{Z}$. If σ is a section of $\mathcal{M}$ on $\mathcal{X}$, we will often denote by $||\sigma||_\infty$ the sup norm of σ, that is, σ is the supremum of the $||\sigma(x)||$ as x runs through all complex points of x. We will call $||\sigma||_\infty$ the norm of σ.

If $||\sigma||_\infty \leq 1$ (resp. $||\sigma||_\infty < 1$), we say that σ is effective (resp. strictly effective). We denote by $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}})$ the set of effective sections of $\overline{\mathcal{L}}$, and write

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}}) = \log |H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}})|.$$

If $\mathcal{X}_{\mathbb{Q}}$ is generically reduced, then $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}})$ is finite. Note again that if $\mathcal{X}_{\mathbb{Q}}$ is empty, then

$$H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}}) = H^0(\mathcal{X}, \mathcal{M}).$$

We say that a hermitian line bundle on $\mathcal{X}$ is effective if it has a regular effective section.

1.6. Outline of the paper

Section 2 is devoted to a general discussion of arithmetic ampleness. After setting definitions, we recall aspects of the arithmetic Hilbert-Samuel theorem, taking care of error terms. We then prove a number of results concerning the image of restriction maps for sections of large powers of ample hermitian line bundles.

In the short Section 3, we make use of the previous section to discuss consequences and variants of the main theorems. We prove Theorem 1.7.

In Section 4, we gather general results—both analytic and arithmetic—dealing with hermitian line bundles on Riemann surfaces and arithmetic surfaces. We prove norm estimates in the spirit of [10], and we prove a basic upper bound on the number of effective sections for positive line bundles—in some sense—on arithmetic surfaces, making use of the θ-invariants of Bost, as well as a result on the effective cone of arithmetic surfaces.

Section 5 is devoted to the proof of Theorem 1.6, and Section 6 to the remaining theorems of the introduction.

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2. Some results on arithmetic ampleness

In this section, we gather some results on ample hermitian line bundles on arithmetic varieties. Most results are well-known and can be found in a similar form in the literature. Aside from a precise statement regarding error terms in the arithmetic Hilbert-Samuel theorem, our main original contribution consists in the results of 2.3 that deals with restriction maps for sections of ample line bundles.

2.1. Definitions and basic properties

We recall basic properties of arithmetic ampleness as used in the work of Zhang [39].

DEFINITION 2.1. – *Let X be a complex analytic space, and let $\overline{L} = (L, ||.||)$ be a hermitian line bundle on X. We say that $\overline{L}$ is* semipositive *if for any open subset U of X, and any section s of $\overline{\mathcal{L}}$ on U, the function $-\log ||s||^2$ is plurisubharmonic on U.*

REMARK 2.2. – Since for any holomorphic function f, the function $-\log|f|^2$ is harmonic, it is readily checked that $\overline{L}$ is *semipositive* if X admits a covering by open subsets U such that there exists a nowhere vanishing section s of L on U such that the function $-\log||s||^2$ is plurisubharmonic on U. In particular, semipositivity is a local property on X.

DEFINITION 2.3. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be a hermitian line bundle on $\mathcal{X}$. We say that $\overline{\mathcal{L}}$ is* ample *if $\mathcal{L}$ is ample, $\overline{\mathcal{L}}$ is semipositive on $\mathcal{X}^{\mathrm{an}}$ and for any large enough integer n, there exists a basis of $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ consisting of strictly effective sections.*

PROPOSITION 2.4. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ be a hermitian vector bundle on $\mathcal{X}$, and let $\mathcal{F}$ be a coherent subsheaf of $\mathcal{M}$. Then there exists a positive real number ε such that for any large enough integer n, there exists a basis of $H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{F}) \subset H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{M})$ consisting of sections with norm at most $e^{-n\varepsilon}$.*

Proof. – Since $\mathcal{L}$ is relatively ample, for any large enough integers a and b, the morphism

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes a}) \otimes H^0(\mathcal{X}, \mathcal{L}^{\otimes b} \otimes \mathcal{F}) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes a+b} \otimes \mathcal{F})$$

is surjective. As a consequence, for any two large enough integers a and b, and any positive integer n, the morphism

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes a})^{\otimes n} \otimes H^0(\mathcal{X}, \mathcal{L}^{\otimes b} \otimes \mathcal{F}) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes an+b} \otimes \mathcal{F})$$

is surjective.

Choose a large enough so that the space $H^0(\mathcal{X}, \mathcal{L}^{\otimes a})$ has a basis consisting of sections with norm at most α for some $\alpha < 1$. Choose $b_1, \ldots, b_a$ large enough integers that form a complete residue system modulo l. We can assume that the maps

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes a})^{\otimes n} \otimes H^0(\mathcal{X}, \mathcal{L}^{\otimes b_i} \otimes \mathcal{F}) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes an+b_i} \otimes \mathcal{F})$$

are surjective for all positive integer n and all i between 1 and a. Now choose bases for the spaces $H^0(\mathcal{X}, \mathcal{L}^{\otimes b_i} \otimes \mathcal{F})$ as i varies between 1 and a, and let β be an upper bound for the norm of any element of these bases. Taking products of elements of these bases, we find a subspace of full rank in $H^0(\mathcal{X}, \mathcal{L}^{\otimes an+b_i} \otimes \mathcal{F})$ which has a basis whose elements have norm at most $\alpha^n\beta$. By [38, Lemma 1.7], this implies that $H^0(\mathcal{X}, \mathcal{L}^{\otimes an+b_i} \otimes \mathcal{F})$ has a basis whose elements have norm at most $r\alpha^n\beta$, where r is the rank of $H^0(\mathcal{X}, \mathcal{L}^{\otimes an+b_i} \otimes \mathcal{F})$.

The theory of Hilbert polynomials shows that the rank of $H^0(\mathcal{X}, \mathcal{L}^{an+b_i})$ is bounded above by a polynomial in $an + b_i$. Since $\alpha < 1$, the number $r\alpha^n\beta$ decreases exponentially as $an+b_i$ grows, which shows the result since any integer can be written as $an+b_i$ for some i and n. □

Corollary 2.5. – *Let $\mathcal{X}$ be a projective arithmetic variety. Let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$ and let $\overline{\mathcal{M}}$ be a hermitian line bundle on $\mathcal{X}$. Then for any large enough integer n, the hermitian line bundle $\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}$ is ample if and only if it is semipositive.*

Proof. – Since $\mathcal{L}$ is relatively ample, for any large enough integer n, the line bundle $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is relatively ample and the morphisms

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{M})^{\otimes m} \to H^0(\mathcal{X}, (\mathcal{L}^{\otimes n} \otimes \mathcal{M})^{\otimes m})$$

are surjective for any positive integer m.

For large enough n, Proposition 2.4 guarantees that there is a basis for $H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{M})$ consisting of sections with norm at most $e^{-n\varepsilon}$ for some positive number ε. As a consequence, we can find a subspace of full rank in $H^0(\mathcal{X}, (\mathcal{L}^{\otimes n} \otimes \mathcal{M})^{\otimes m})$ with a basis consisting of sections with norm at most $e^{-mn\varepsilon}$. By [38, Lemma 7.1], this implies that $H^0(\mathcal{X}, (\mathcal{L}^{\otimes n} \otimes \mathcal{M})^{\otimes m})$ itself has a basis whose elements have norm at most $re^{-mn\varepsilon}$, where r is the rank of $H^0(\mathcal{X}, (\mathcal{L}^{\otimes n} \otimes \mathcal{M})^{\otimes m})$. Since again r is bounded above by a polynomial in mn, this shows the result. □

Corollary 2.6. – *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of projective arithmetic varieties, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then there exists a positive real number ε such that for any large enough integer n, there exists a basis of $H^0(\mathcal{Y}, f^*\mathcal{L}^{\otimes n})$ consisting of sections with norm bounded above by $e^{-n\varepsilon}$.*

Proof. – By the projection formula, for any integer k, we have a canonical isomorphism

$$H^0(\mathcal{Y}, f^*\mathcal{L}^{\otimes k}) \simeq H^0(\mathcal{X}, \mathcal{L}^{\otimes k} \otimes f_*\mathcal{O}_{\mathcal{Y}}).$$

Since $\mathcal{L}$ is relatively ample, for any two large enough integers n and k, the map

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \otimes H^0(\mathcal{X}, \mathcal{L}^{\otimes k} \otimes f_*\mathcal{O}_{\mathcal{Y}}) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes (n+k)} \otimes f_*\mathcal{O}_{\mathcal{Y}})$$

is surjective, which means that the natural map

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \otimes H^0(\mathcal{Y}, f^*\mathcal{L}^{\otimes k}) \to H^0(\mathcal{Y}, f^*\mathcal{L}^{\otimes (n+k)})$$

is surjective.

Fix a large enough integer k for the previous assumption to hold. By Proposition 2.4, the space $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ admits a basis consisting of elements with norm decreasing exponentially with n, which shows that the same property holds for $H^0(\mathcal{Y}, f^*\mathcal{L}^{\otimes (n+k)})$. □

Corollary 2.7. – *Let f be a finite morphism between projective arithmetic varieties. The pullback of an ample hermitian line bundle by f is ample.*

Proof. – By the previous results, we only have to show that if $f : X \to Y$ is a finite map between complex projective varieties, and if $\overline{L}$ is a semipositive hermitian line bundle on Y, then $f^*\overline{L}$ is semipositive.

Let U be an open subset of Y on which L is trivial, and let s be a nowhere vanishing section of L on U. Then f^*s is a nowhere vanishing section of f^*L on $f^{-1}(U)$, and the function

$$-\log ||f^*s||^2 = (-\log ||s||^2) \circ f$$

is plurisubharmonic on $f^{-1}(U)$, being the composition of a holomorphic function and a plurisubharmonic function. Remark 2.2 shows that $f^*\overline{L}$ is semipositive. □

Let n be a positive integer, and consider the complex projective space $\mathbb{CP}^n$. The line bundle $\mathcal{O}(1)$ on $\mathbb{CP}^n$ is endowed with the *Fubini-Study metric* $||.||$ defined as follows. Let x be a point of $\mathbb{CP}^n$ with homogeneous coordinates $[x_0 : \cdots : x_n]$. The fiber of $\mathcal{O}(1)$ at x may be identified with linear forms $\mathbb{C}(x_0, \ldots, x_1) \to \mathbb{C}$. Endow the line $\mathbb{C}(x_0, \ldots, x_1)$ with the norm induced by the standard hermitian norm on $\mathbb{C}^{n+1}$. Then the Fubini-Study metric on $\mathcal{O}(1)$ is the one that corresponds to the operator norm on linear forms.

The following is the basic example of an ample hermitian line bundle.

Proposition 2.8. – *Let n be a positive integer, and let $\overline{\mathcal{O}(1)}$ be the hermitian line bundle on $\mathbb{P}^n_{\mathbb{Z}}$ corresponding to the line bundle $\mathcal{O}(1)$ endowed with the Fubini-Study metric. Then for any $\varepsilon > 0$, the hermitian line bundle $\overline{\mathcal{O}(1)}(\varepsilon)$ is ample on $\mathbb{P}^n_{\mathbb{Z}}$.*

Proof. – The line bundle $\mathcal{O}(1)$ is ample on $\mathbb{P}^n_{\mathbb{Z}}$, and the Fubini-Study metric is known to have positive curvature.

Let $X_0^{d_0} \cdots X_n^{d_n}$ be a monomial of total degree $d > 0$, seen as a section of $\mathcal{O}(d)$. With respect to the Fubini-Study metric, if x is a point of $\mathbb{CP}^n$ with homogeneous coordinates $[x_0 : \cdots : x_n]$, we have

$$||X_0^{d_0} \cdots X_n^{d_n}(x)|| = \frac{|x_0^{d_0} \cdots x_n^{d_n}|}{(|x_0|^2 + \cdots + |x_n|^2)^{d/2}} \leq 1.$$

This shows that $H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(d))$ has a basis consisting of sections of norm bounded above by 1, and proves the result. □

The following follows immediately from Proposition 2.8 and Corollary 2.7.

Corollary 2.9. – *Let $\mathcal{X}$ be an arithmetic variety, and let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{X}$. Then there exists a metric $||.||$ on $\mathcal{L}_{\mathbb{C}}$, invariant under complex conjugation, such that the hermitian line bundle $(\mathcal{L}, ||.||)$ is ample.*

Proof. – Some positive power $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$ is the pullback of the line bundle $\mathcal{O}(1)$ on some projective space. By Proposition 2.8 and Corollary 2.7, the pullback of the Fubini-Study metric, scaled by $e^{-\varepsilon}$ for some $\varepsilon > 0$, to $\mathcal{L}^{\otimes n}$ gives $\mathcal{L}^{\otimes n}$ the structure of an ample hermitian line bundle.

Endow $\mathcal{L}$ with the hermitian metric $||.||$ whose n-th power is the metric above. We get a hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, ||.||)$ such that $\overline{\mathcal{L}}^{\otimes n}$ is ample. This implies that $\overline{\mathcal{L}}$ is ample. □

2.2. Arithmetic Hilbert-Samuel

We turn to the arithmetic Hilbert-Samuel theorem, giving an estimate for $h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, where $\overline{\mathcal{L}}$ is ample and n is large. This has been proved by Gillet-Soulé in [17, Theorem 8 and Theorem 9] and extended by [39, Theorem (1.4)], see also [1] and [6]. In later arguments, we will need an estimate for the error term in the arithmetic Hilbert-Samuel theorem. In the case where the generic fiber of $\mathcal{X}$ is smooth, such an estimate follows from the work of Gillet-Soulé and Bismut-Vasserot. The general case does not seem to be worked out. However, for arithmetic surfaces, an argument of Vojta gives us enough information for our later needs.

We start with a proposition relating the Hilbert-Samuel function of a hermitian line bundle and its pullback under a birational morphism.

Proposition 2.10. – *Let $f : \mathcal{Y} \to \mathcal{X}$ be a birational morphism of projective arithmetic varieties, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then there exists a positive integer k and a positive real number C such that for any integer n and any hermitian vector bundle $\overline{\mathcal{M}}$ on $\mathcal{X}$, the following equality holds:*

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) \leq h^0_{\mathrm{Ar}}(\mathcal{Y}, f^*(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})) \leq h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes (n+k)} \otimes \overline{\mathcal{M}}(C)).$$

Proof. – Pullback of sections induces an injective map

$$f^* : H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) \to H^0_{\mathrm{Ar}}(\mathcal{Y}, f^*(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})),$$

which proves the first inequality.

The coherent sheaf $\mathcal{H}\wr\Updownarrow(f_*\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{X}})$ is non-zero. As a consequence, there exists a positive integer k such that the sheaf

$$\mathcal{H}\wr\Updownarrow(f_*\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{X}}) \otimes \mathcal{L}^{\otimes k} = \mathcal{H}\wr\Updownarrow(f_*\mathcal{O}_{\mathcal{Y}}, \mathcal{L}^{\otimes k})$$

has a nonzero global section ϕ. Since f is birational, the morphism

$$\phi : f_*\mathcal{O}_{\mathcal{Y}} \to \mathcal{L}^{\otimes k}$$

is injective. If U is an open subset of the compact complex manifold $\mathcal{X}(\mathbb{C})$ and n is an integer, we endow the spaces

$$H^0(U, \mathcal{L}^{\otimes n} \otimes \mathcal{M} \otimes f_*\mathcal{O}_{\mathcal{Y}}) = H^0(f^{-1}(U), f^*\mathcal{L}^{\otimes n} \otimes \mathcal{M})$$

and

$$H^0(U, \mathcal{L}^{\otimes (n+k)} \otimes \mathcal{M})$$

with the sup norm—which might take the value ∞. Since $\mathcal{X}(\mathbb{C})$ is compact, we can find a constant C such that the maps

$$\phi_U : H^0(U, f_*\mathcal{O}_{\mathcal{Y}}) \to \mathcal{H}^0(U, \mathcal{L}^{\otimes k})$$

all have norm bounded above by e^C. As a consequence, all the maps

$$\mathrm{Id} \otimes \phi_U : H^0(U, \mathcal{L}^{\otimes n} \otimes \mathcal{M} \otimes f_*\mathcal{O}_{\mathcal{Y}}) \to H^0(U, \mathcal{L}^{\otimes (n+k)} \otimes \mathcal{M})$$

have norm bounded above by e^C as well, and the induced map

$$H^0(\mathcal{Y}, f^*(\mathcal{L}^{\otimes n} \otimes \mathcal{M})) \to H^0(\mathcal{X}, \mathcal{L}^{\otimes (n+k)} \otimes \mathcal{M})$$

has norm bounded above by e^C. Since this map is injective, we have an injection

$$H^0_{\mathrm{Ar}}(\mathcal{Y}, f^*(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})) \to H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes (n+k)} \otimes \overline{\mathcal{M}}(C)),$$

which shows the second inequality. □

We may now state some forms of the arithmetic-Hilbert-Samuel theorem. For the purposes of this paper, the key statement is (iii). We will need the more precise estimate on the error term it provides compared to (i).

THEOREM 2.11. – *Let $\mathcal{X}$ be a projective arithmetic variety of dimension d, let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$, and let $\overline{\mathcal{M}}$ be a hermitian vector bundle of rank r.*

(i) *As n tends to ∞, we have*

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) = \frac{r}{d!}\overline{\mathcal{L}}^d n^d + o(n^d);$$

(ii) *if $\mathcal{X}_{\mathbb{Q}}$ is smooth and the curvature form of $\overline{\mathcal{L}}$ is positive, then*

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) = \frac{r}{d!}\overline{\mathcal{L}}^d n^d + O(n^{d-1}\log n)$$

as n tends to ∞;

(iii) *if $d = 2$, then*

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) \geq \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + O(n\log n)$$

as n tends to ∞.

Proof. – The first statement can be found in [36, Corollary 2.7(1)]. It is a consequence of the extension by Zhang in [39, Theorem (1.4)] of the arithmetic Hilbert-Samuel theorem of Gillet-Soulé of [17, Theorem 8], together with [16, Theorem 1].

Let us prove the second statement. Choose a Kähler metric on $\mathcal{X}(\mathbb{C})$, assumed to be invariant under complex conjugation, and write $\chi_{L^2}(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})$ (resp. $\chi_{sup}(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})$) for the logarithm of the covolume of $H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})$ for the associated L^2 norm (resp. for the sup norm). Then by [17, Theorem 8], we have

$$\chi_{L^2}(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) = \frac{r}{d!}\overline{\mathcal{L}}^d n^d + O(n^{d-1}\log n).$$

By the Gromov inequality as in for instance [36, Corollary 2.7(2)], this implies

$$\chi_{sup}(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) = \frac{r}{d!}\overline{\mathcal{L}}^d n^d + O(n^{d-1}\log n).$$

Consider the lattice $\Lambda = H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{M})$, endowed with the sup norm. Then since Λ is generated by elements of norm strictly smaller than 1, the dual of Λ does not contain any nonzero element of norm smaller than 1. Furthermore, the geometric version of the Hilbert-Samuel theorem shows that the rank of Λ has the form $O(n^{d-1})$. By [16, Theorem 1], we get

$$\left|h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) - \chi_{sup}(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}})\right| = O(n^{d-1}\log n),$$

which proves the desired result.

We now prove the last statement. Let $f : \mathcal{Y} \to \mathcal{X}$ be the normalization of $\mathcal{X}$, so that f is birational, finite, and $\mathcal{Y}$ has smooth generic fiber.

Since f is finite, the line bundle $f^*\mathcal{L}$ is ample. Let $\overline{\mathcal{L}}'$ be $f^*\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}'$ be $f^*\overline{\mathcal{M}}$. Choose a Kähler metric on $\mathcal{Y}(\mathbb{C})$, assumed to be invariant under complex conjugation, and again write

$\chi_{L^2}(\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}')$ for the logarithm of the covolume of $H^0(\mathcal{Y}, \mathcal{L}'^{\otimes n} \otimes \mathcal{M}')$ for the associated L^2 norm.

By Proposition 2.10, we can find a constant C and an integer k such that for any integer n greater or equal to k, we have

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) \geq h^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}'^{\otimes n-k} \otimes \overline{\mathcal{M}}'(-C)\big). \tag{2.1}$$

Applying the arithmetic Riemann-Roch theorem for n large enough so that the higher cohomology groups of $\mathcal{L}'^{\otimes n} \otimes \mathcal{M}'$ vanish, we get the following equality:

$$\chi_{L^2}\big(\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)\big) - \frac{1}{2}T_n = \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + O(n),$$

where by T_n we denote the analytic torsion of the hermitian vector bundle $\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'$. The equality above is proved via the computations of [17, Theorem 8], or [15, Théorème 1]. In contrast with the usual setting of Hilbert-Samuel, note that the curvature form of $\overline{\mathcal{L}}'$ might not be positive everywhere, so that we cannot apply the estimates of [4] for T_n. However, since the dimension of $\mathcal{X}$ is 2, we have

$$T_n = \zeta'_{1,n}(0),$$

where ζ_1 is the zeta function of the Laplace operator acting on forms of type $(0,1)$ with values in $\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)$. We can use an estimate of Vojta to control the analytic torsion T_n. Indeed, by [34, Proposition 2.7.6], we have

$$\zeta'_{1,n}(0) \geq -Kn\log n$$

for some constant K, so that

$$\chi_{L^2}\big(\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)\big) = \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + \frac{1}{2}T_n + O(n) \geq \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + O(n\log n). \tag{2.2}$$

Combining as above Gromov's inequality, [16, Theorem 1], Corollary 2.6 and the geometric version of Hilbert-Samuel, we can write

$$\big|h^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)\big) - \chi_{L^2}\big(\overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)\big)\big| = O(n\log n),$$

which together with (2.2) gives the estimate

$$h^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}'^{\otimes n} \otimes \overline{\mathcal{M}}'(-C)\big) \geq \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + O(n\log n).$$

From (2.1), we finally obtain

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}) \geq \frac{r}{2}\overline{\mathcal{L}}^2 (n-k)^2 + O(n\log n) \geq \frac{r}{2}\overline{\mathcal{L}}^2 n^2 + O(n\log n). \qquad \square$$

2.3. Restriction of sections

Let k be a field, and let X be a projective variety over k. Let $\mathcal{L}$ be an ample line bundle on X. If Y is any closed subscheme of X, consider the restriction maps

$$\phi_n : H^0(X, \mathcal{L}^{\otimes n}) \to H^0(Y, \mathcal{L}|_Y^{\otimes n}).$$

The map ϕ_n is surjective if n is large enough and, obviously, there are bijections between the different fibers of ϕ_n when it is surjective.

In this section, we give arithmetic analogues of these results, looking at H^0_{Ar} instead of H^0—this is Theorem 2.17. Furthermore, we show in Theorem 2.21 that the lower bound on the dimension of the image of the restriction map can be given to be independent of Y.

In the following, let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. If n is an integer, we denote by Λ_n the space $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ endowed with the sup norm, and we write r_n for its rank. If R is a nonnegative real number, let $B_n(R)$ be the closed ball of radius R in Λ_n. In particular, we have

$$B_n(1) = H^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}).$$

Let $B_n(R)_{\mathbb{R}}$ be the closed ball of radius R in $\Lambda_n \otimes \mathbb{R}$. Let Vol denote the volume with respect to the unique translation-invariant measure on $\Lambda_n \otimes \mathbb{R}$ for which $\mathrm{Vol}(B_n(1)_{\mathbb{R}}) = 1$.

If $\mathcal{I}$ is a quasi-coherent sheaf of ideals on $\mathcal{X}$, we write $\Lambda_n^{\mathcal{I}}$ for the subgroup $H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{I})$ of $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$, endowed with the induced norm. We write $r_n^{\mathcal{I}}$, $B_n(R)^{\mathcal{I}}$, $B_n(R)_{\mathbb{R}}^{\mathcal{I}}$, $\mathrm{Vol}^{\mathcal{I}}$ for the corresponding objects.

We gather a few results regarding point counting in the lattices $\Lambda_n^{\mathcal{I}}$. The following is a basic estimate.

Lemma 2.12. – *Let η be a positive real number. Let C be any real number. Then, as n goes to infinity, we have, for any positive R,*

$$\mathrm{Vol}^{\mathcal{I}}\big(B_n(R + Ce^{-n\eta})_{\mathbb{R}}^{\mathcal{I}}\big) = R^{r_n^{\mathcal{I}}}\big(1 + CR^{-1}r_n^{\mathcal{I}}e^{-n\eta} + o(R^{-1}r_n^{\mathcal{I}}e^{-n\eta})\big),$$

where the implied constant in o depends only on C.

Proof. – We write

$$\begin{aligned}\mathrm{Vol}^{\mathcal{I}}\big(B_n(R + Ce^{-n\eta})_{\mathbb{R}}^{\mathcal{I}}\big) &= (R + Ce^{-n\eta})^{r_n^{\mathcal{I}}}\\ &= R^{r_n^{\mathcal{I}}} \exp\big(r_n^{\mathcal{I}} \log(1 + CR^{-1}e^{-n\eta})\big)\\ &= R^{r_n^{\mathcal{I}}} \exp\big(CR^{-1}r_n^{\mathcal{I}}e^{-n\eta} + o(R^{-1}r_n^{\mathcal{I}}e^{-n\eta})\big)\\ &= R^{r_n^{\mathcal{I}}}\big(1 + CR^{-1}r_n^{\mathcal{I}}e^{-n\eta} + o(R^{-1}r_n^{\mathcal{I}}e^{-n\eta})\big).\end{aligned}$$ □

Fix $\mathcal{I}$ as above. Let n be a large enough integer. By Proposition 2.4, we can find a positive number $\varepsilon^{\mathcal{I}}$, independent of n, and a basis $\sigma_1, \dots, \sigma_{r_n^{\mathcal{I}}}$ of $\Lambda_n^{\mathcal{I}}$ such that $||\sigma_i||_\infty \leq e^{-n\varepsilon^{\mathcal{I}}}$ for all $i \in \{1, \dots, r_n^{\mathcal{I}}\}$. Consider the fundamental domain

$$D_n^{\mathcal{I}} = \Big\{\sum_{i=1}^{r_n^{\mathcal{I}}} \lambda_i \sigma_i | \forall i \in \{1, \dots, n\},\ 0 \leq \lambda_i < 1\Big\}. \tag{2.3}$$

Proposition 2.13. – *Let α be a positive number with $0 < \alpha < 1$. As n tends to ∞, we have, for any $R > e^{-n^\alpha}$,*

$$|B_n(R)^{\mathcal{I}}|\mathrm{Vol}^{\mathcal{I}}(D_n^{\mathcal{I}}) \sim R^{r_n^{\mathcal{I}}}.$$

Proof. – Let n be a large enough integer. As σ runs through the elements of $\Lambda_n^{\mathcal{I}}$, the sets $\sigma + D_n^{\mathcal{I}}$ are pairwise disjoint and cover $\Lambda_n^{\mathcal{I}} \otimes \mathbb{R}$. Furthermore, the diameter of $D_n^{\mathcal{I}}$ is bounded above by $r_n^{\mathcal{I}}e^{-n\varepsilon^{\mathcal{I}}}$. As a consequence, if σ is any element of $\Lambda_n^{\mathcal{I}}$, then

$$\sigma + D_n^{\mathcal{I}} \subset B_n(||\sigma||_\infty + r_n^{\mathcal{I}}e^{-n\varepsilon^{\mathcal{I}}})_{\mathbb{R}}^{\mathcal{I}}$$

and

$$(\sigma + D_n^{\mathcal{I}}) \cap B_n(||\sigma||_\infty - r_n^{\mathcal{I}}e^{-n\varepsilon^{\mathcal{I}}})_{\mathbb{R}}^{\mathcal{I}} = \emptyset.$$

As a consequence, we have

$$\mathrm{Vol}^{\mathcal{I}}\big(B_n(R - r_n^{\mathcal{I}} e^{-n\varepsilon^{\mathcal{I}}})_{\mathbb{R}}^{\mathcal{I}}\big) \leq |B_n(r)^{\mathcal{I}}| \mathrm{Vol}^{\mathcal{I}}(D_n^{\mathcal{I}}) \leq \mathrm{Vol}^{\mathcal{I}}\big(B_n(R + r_n^{\mathcal{I}} e^{-n\varepsilon^{\mathcal{I}}})_{\mathbb{R}}^{\mathcal{I}}\big).$$

By Riemann-Roch, the rank $r_n^{\mathcal{I}}$ grows at most polynomially in n. As a consequence, $R^{-1} r_n^{\mathcal{I}} e^{-n\epsilon^{\mathcal{I}}}$ goes to 0 as n goes to infinity, and Lemma 2.12 shows that both the left and right terms are equivalent to $R^{r_n^{\mathcal{I}}}$ as n goes to infinity. □

Proposition 2.14. – *Let α and η be positive real numbers with $0 < \alpha < 1$. Let C be any real number. Then, as n tends to ∞, there exists a positive real number η' such that we have, for any positive $R > e^{-n^\alpha}$,*

$$\frac{\big||B_n(R + Ce^{-n\eta})^{\mathcal{I}}| - |B_n(R)^{\mathcal{I}}|\big|}{|B_n(R)^{\mathcal{I}}|} = O(e^{-n\eta'}),$$

where the implied constants depend on α, C and η.

Proof. – We assume that C is positive. The case where C is negative (or zero) can be treated by the same computations.

Let η' be a positive number strictly smaller than both $\varepsilon^{\mathcal{I}}$ and η. Since the $\sigma + D_n^{\mathcal{I}}$ are pairwise disjoint as σ runs through the elements of $\Lambda_n^{\mathcal{I}}$, we get, for large enough n

$$\begin{aligned}
\big(|B_n(R + Ce^{-n\eta})^{\mathcal{I}}| &- |B_n(R)^{\mathcal{I}}|\big)\mathrm{Vol}^{\mathcal{I}}(D_n^{\mathcal{I}}) \\
&\leq \mathrm{Vol}^{\mathcal{I}}(B_n(R + Ce^{-n\eta} + r_n^{\mathcal{I}} e^{-n\varepsilon^{\mathcal{I}}})_{\mathbb{R}}^{\mathcal{I}}) - \mathrm{Vol}^{\mathcal{I}}(B_n(R - r_n^{\mathcal{I}} e^{-n\varepsilon^{\mathcal{I}}})_{\mathbb{R}}^{\mathcal{I}}) \\
&\leq \mathrm{Vol}^{\mathcal{I}}(B_n(R + e^{-n\eta'})_{\mathbb{R}}^{\mathcal{I}}) - \mathrm{Vol}^{\mathcal{I}}(B_n(R - e^{-n\eta'})_{\mathbb{R}}^{\mathcal{I}}) \\
&\sim 2R^{r_n^{\mathcal{I}}-1} r_n^{\mathcal{I}} e^{-n\eta'},
\end{aligned}$$

where in the last line we applied Lemma 2.12, using that $R^{-1} r_n^{\mathcal{I}} e^{-n\eta'}$ tends to 0 as n tends to ∞.

Putting the previous estimate together with Proposition 2.13 and replacing η' with a smaller positive number, we get the desired result. □

The following is a first step in controlling restriction maps.

Proposition 2.15. – *Let α be a positive number with $0 < \alpha < 1$. There exists a positive constant η such that for any large enough integer n, if N is any positive integer with $N < e^{n^\alpha}$, then the following statements hold:*

(i) *the map $\phi_{n,N} : H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) \to \Lambda_n/N\Lambda_n$ is surjective;*

(ii) *for any two s, s' in $\Lambda_n/N\Lambda_n$, we have*

$$\frac{\big||\phi_{n,N}^{-1}(s)| - |\phi_{n,N}^{-1}(s')|\big|}{|\phi_{n,N}^{-1}(s)|} \leq e^{-n\eta}.$$

Proof. – Let r_n be the rank of Λ_n. Let n be a positive integer, which will be chosen large enough, and let N be an integer bounded above by e^{n^α}.

By Proposition 2.4, we can find a positive number ε, independent of n, and a basis $\sigma_1, \ldots, \sigma_{r_n}$ of Λ_n such that $||\sigma_i||_\infty \leq e^{-n\varepsilon}$ for all $i \in \{1, \ldots, r_n\}$. Any element of $\Lambda_n/N\Lambda_n$ is the image of an element of Λ_n of the form

$$\sigma = \lambda_1\sigma_1 + \cdots + \lambda_{r_n}\sigma_{r_n},$$

where the λ_i are integers between 0 and $N-1$. We have

$$||\sigma||_\infty < N r_n e^{-n\varepsilon} \leq r_n e^{n^\alpha - n\varepsilon}.$$

We know that r_n is a polynomial in n for large enough n and $\alpha < 1$ by assumption, so that any σ as above is strictly effective for large enough n. This shows that the map $\phi_{n,N}$ is surjective and proves (i).

We now proceed to the proof of (ii). Let n be a large enough integer. By the discussion above, we can find a positive real number ε' such that for any large enough integer n, and any s in $\Lambda_n/N\Lambda_n$, there exists an element σ_0 in Λ_n with $||\sigma_0||_\infty \leq e^{-n\varepsilon'}$ that restricts to s. We have

$$\phi_{n,N}^{-1}(s) = \{\sigma_0 + N\sigma | \sigma \in \Lambda_n, ||\sigma_0 + N\sigma||_\infty \leq 1\},$$

so that, up to replacing ε' by a smaller positive number

$$|B_n(1/N - e^{-n\varepsilon'})| \leq |\phi_{n,N}^{-1}(s)| \leq |B_n(1/N + e^{-n\varepsilon'})|$$

and

$$\big| |\phi_{n,N}^{-1}(s)| - |\phi_{n,N}^{-1}(s')| \big| \leq |B_n(1/N + e^{-n\varepsilon'})| - |B_n(1/N - e^{-n\varepsilon'})| \tag{2.4}$$

for any two s, s' in $\Lambda_n/N\Lambda_n$. We conclude by applying Proposition 2.14. □

We need a variant of a theorem of Bost.

Proposition 2.16. – *Let X be a reduced complex analytic space, Y a closed reduced subspace of X, L an ample line bundle on X and $||.||$ a semipositive smooth metric on L. Then for any $\varepsilon > 0$, any large enough integer n, and any $s \in H^0(Y, L_{|Y}^{\otimes n})$, we can find $\sigma \in H^0(X, L^{\otimes n})$ such that $\sigma_{|Y} = s$ and*

$$||\sigma||_\infty \leq e^{\varepsilon n} ||s||_\infty.$$

Proof. – If the metric $||.||$ is positive, then this is the content of [8, Theorem A.1] [(1)]. Since L is ample, it admits a positive hermitian metric, so that we can find a smooth function $\phi : X \to \mathbb{R}$ such that $||.||e^{-\phi}$ is positive. Since $||.||$ is semipositive, the metric $||.||e^{-\delta\phi}$ is positive for any $\delta > 0$.

Let ε be a positive real number, and choose $\delta > 0$ such that

$$\forall x \in X, |\delta\phi(x)| \leq \varepsilon.$$

Apply [8, Theorem A.1] to the line bundle L with the positive metric $||.||e^{-\delta\phi}$: if n is large enough, and $s \in H^0(Y, L^{\otimes n})$, we can find $\sigma \in H^0(X, L^{\otimes n})$ such that $\sigma_{|Y} = s$ and

$$||\sigma||_\infty \leq e^{3\varepsilon n} ||s||_\infty.$$

This shows the result. □

The following is a key property of ample line bundles.

(1) The only assumption necessary in [8] is that $\mathcal{Y}_\mathbb{C}$ and $\mathcal{X}_\mathbb{C}$ are reduced, see [27] for a statement that makes this explicit.

Theorem 2.17. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample line bundle on $\mathcal{X}$. Let $\mathcal{Y}$ be a closed subscheme of $\mathcal{X}$, such that $\mathcal{Y}_{\mathbb{Q}}$ is reduced. If n is a positive integer, let*

$$\phi_n : H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \to H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})$$

be the restriction map. For any positive ε, define

$$\Lambda_n^{\varepsilon} = H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) \cap \phi_n^{-1}\big(H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})\big),$$

that is, Λ_n^{ε} is the space of effective sections σ of $\overline{\mathcal{L}}^{\otimes n}$ such that the restriction of σ to $\mathcal{Y}$ has norm at most $e^{-n\varepsilon}$. Write $\psi_n := (\phi_n)|_{\Lambda_n^{\varepsilon}}$. Then the following statements hold:

(i) *for any large enough integer n, the restriction map*

$$\psi_n : \Lambda_n^{\varepsilon} \to H^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n}\big)$$

is surjective;

(ii) *there exists a positive constant η such that for any large enough integer n, and any two s, s' in $H^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n}\big)$, we have*

$$\frac{\big||\phi_n^{-1}(s)| - |\phi_n^{-1}(s')|\big|}{|\phi_n^{-1}(s)|} \leq e^{-n\eta};$$

(iii) *η being chosen as above, for any $s \in H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$, we have*

$$\left|\phi_n^{-1}(s) - \frac{|\Lambda_n^{\varepsilon}|}{|H^0_{\mathrm{Ar}}\big(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)^{\otimes n}\big)|}\right| \leq e^{-n\eta}|\phi_n^{-1}(s)|.$$

Proof. – Fix $\varepsilon > 0$. The group

$$\{\sigma \in H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n}), ||\sigma_{\mathbb{C}}|| = 0\}$$

is the torsion subgroup of $H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$, which we denote by $H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})_{\mathrm{tor}}$—note that this group does not depend on ε nor the hermitian metric. This is a finite group. Let N be a positive integer with

$$N H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})_{\mathrm{tor}} = 0.$$

Assume n is large enough. The restriction map

$$\phi_n : \Lambda_n = H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \to H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})$$

is surjective since $\mathcal{L}$ is relatively ample. The map

$$\Lambda_n / N\Lambda_n \to H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})_{\mathrm{tor}}$$

is well-defined and surjective as well. Applying Proposition 2.15, this shows that the image of ψ_n contains $H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})_{\mathrm{tor}}$.

Let s be an element of $H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$. Let ε' be a real number with $0 < \varepsilon' < \varepsilon$. Apply Proposition 2.16 to the closed subspace $\mathcal{Y}_{\mathbb{C}}$ of $\mathcal{X}_{\mathbb{C}}$. If n is large, we can find a section σ of $\mathcal{L}^{\otimes n}$ on $\mathcal{X}_{\mathbb{C}}$ with $||\sigma||_{\infty} \leq e^{-n\varepsilon'}$ and $\sigma|_{\mathcal{Y}_{\mathbb{C}}} = s_{\mathbb{C}}$. Up to replacing σ with $\sigma + \overline{\sigma}$, and making ε' smaller, we may assume that σ is a section of $\mathcal{L}^{\otimes n}$ over $\mathcal{X}_{\mathbb{R}}$, that is,

$$\sigma \in B_n(e^{-n\varepsilon'})_{\mathbb{R}}.$$

Let $\mathcal{I}$ be the ideal of $\mathcal{Y}$ in $\mathcal{X}$. The kernel of the—surjective when n is large enough—restriction map

$$\phi_n : \Lambda_n \to H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})$$

is $\Lambda_n^{\mathcal{I}}$. Let σ' be an element of Λ_n mapping to s. Then $\sigma \in (\Lambda_n^{\mathcal{I}})_{\mathbb{R}} + \sigma'$.

The fundamental domain $D_n^{\mathcal{I}}$ defined in (2.3) has diameter bounded above by $r_n e^{-n\varepsilon^{\mathcal{I}}}$—note that $r^n \geq r_n^{\mathcal{I}}$. In particular, we can find $\sigma'' \in \Lambda_n^{\mathcal{I}} + \sigma'$ with

$$||\sigma'' - \sigma||_\infty \leq r_n e^{-n\varepsilon^{\mathcal{I}}},$$

so that

$$||\sigma''||_\infty \leq e^{-n\varepsilon'} + r_n e^{-n\varepsilon^{\mathcal{I}}} < 1$$

for large enough n. We have $\psi_n(\sigma)_{\mathbb{C}} = s_{\mathbb{C}}$, i.e., $\psi_n(\sigma) - \sigma$ is torsion. This shows that the image of ψ_n maps surjectively onto the quotient of $H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$ by $H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})_{\mathrm{tor}}$. Since we showed above that it contains $H^0_{\mathrm{Ar}}(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})_{\mathrm{tor}}$, this proves that ψ_n is surjective.

Apply statement (i) after replacing $\overline{\mathcal{L}}$ with $\overline{\mathcal{L}}(-\delta)$, where $\delta > 0$ is chosen small enough so that $\overline{\mathcal{L}}(-\delta)$ is ample. Then if $\varepsilon > \delta$ and n is large enough, for any $s \in H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$, we can find $\sigma_0 \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}(-\delta)^{\otimes n})$ that restricts to s.

To prove (ii), we argue as in Proposition 2.15. Let s and σ_0 be as above. Then

$$\psi_n^{-1}(s) = \{\sigma_0 + \sigma | \sigma \in \Lambda_n^{\mathcal{I}}, ||\sigma_0 + \sigma||_\infty \leq 1\}$$

and

$$|B_n(1 - e^{-n\delta})^{\mathcal{I}}| \leq |\psi_n^{-1}(s)| \leq B_n(1 + e^{-n\delta})^{\mathcal{I}}.$$

Using Proposition 2.14 again, this proves (ii).

To prove (iii), write

$$|\Lambda_n^\varepsilon| = \sum_{s \in H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})} |\psi_n^{-1}(s)|,$$

so that for any large enough n and any $s \in H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$, we have

$$\big||\Lambda_n^\varepsilon| - |\psi_n^{-1}(s)|\,|H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})|\big| \leq e^{-n\eta}\,|\psi_n^{-1}(s)|\;|H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})|. \qquad \square$$

We keep the notation of the theorem.

COROLLARY 2.18. – *Let E be a subset of $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$. Set*

$$E' := \Big\{\sigma \in \bigcup_{n>0} \Lambda_n^\varepsilon, \sigma|_{\mathcal{Y}} \in E\Big\}.$$

For any $0 \leq \rho \leq 1$, the set E has density ρ in $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)|_{\mathcal{Y}}^{\otimes n})$ if and only if E' has density ρ in $\bigcup_{n>0} \Lambda_n^\varepsilon$.

Proof. – For any positive integer n, define

$$E_n := E \cap H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)^{\otimes n}_{|\mathcal{Y}}),\ E'_n := E' \cap H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) = E' \cap \Lambda^{\varepsilon}_n.$$

Denoting by ψ_n the restriction maps as before, we can write

$$|E'_n| = \sum_{s \in E_n} |\psi_n^{-1}(s)|.$$

Summing the estimate of Theorem 2.17, (iii) over all $s \in E_n$ for large enough n, we can find a positive constant η such that, for large enough n,

$$\Big| |E'_n| - \frac{|E_n|}{|H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)^{\otimes n}_{|\mathcal{Y}})|} |\Lambda^{\varepsilon}_n| \Big| \leq e^{-n\eta} |E'_n|$$

and, dividing by $|\Lambda^{\varepsilon}_n| \leq |E'_n|$,

$$\Big| \frac{|E'_n|}{|\Lambda^{\varepsilon}_n|} - \frac{|E_n|}{|H^0_{\mathrm{Ar}}(\mathcal{Y}, \overline{\mathcal{L}}(-\varepsilon)^{\otimes n}_{|\mathcal{Y}})|} \Big| \leq e^{-n\eta}.$$

Letting n tend to ∞ gives us the result we were looking for. □

As a special case of the theorem, we get the following.

Corollary 2.19. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample line bundle on $\mathcal{X}$. Let Y be a closed subscheme of $\mathcal{X}$ lying over $\mathbb{Z}/N\mathbb{Z}$ for some positive integer N. Then for any large enough integer n, the restriction map*

$$\phi_n : H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) \to H^0(Y, \mathcal{L}^{\otimes n}_{|Y})$$

is surjective and there exists a positive constant η such that for any $s \in H^0(Y, \mathcal{L}^{\otimes n})$, we have

$$\Big| \phi_n^{-1}(s) - \frac{|H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})|}{|H^0(Y, \mathcal{L}^{\otimes n}_{|Y})|} \Big| \leq e^{-n\eta} |\phi_n^{-1}(s)|.$$

We now turn to uniform lower bounds on the image of restriction maps. We first deal with a geometric result.

Proposition 2.20. – *Let S be a noetherian scheme, and let X be a projective scheme over S. Let $\mathcal{L}$ be a line bundle on X, relatively ample over S. Let d be a positive integer. Then there exists an integer N and a positive constant C such that for any point s of S, any closed subscheme Y of X_s of dimension d, and any $n \geq N$, the image of the restriction map*

$$H^0(X_s, \mathcal{L}^{\otimes n}) \to H^0(Y, \mathcal{L}^{\otimes n}_{|Y})$$

has dimension at least Cn^d.

Proof. – Since S is noetherian, we can find an integer M such that for any point s of S and any integer $n \geq M$, the restriction of $\mathcal{L}^{\otimes n}$ to X_s is very ample.

Let s be a point of S, and let Y be a closed subscheme of X_s of positive dimension d. Let k be an infinite field containing the residue field of s, and write X_k for the base change of X_s to k.

Since $\mathcal{L}^{\otimes n}$ is very ample on X_k and k is infinite, we can find a $d+1$-dimensional subspace $V \subset H^0(X, \mathcal{L}^{\otimes n})$ such that the restriction to Y of the rational map

$$\phi : X \dashrightarrow \mathbb{P}(V^*)$$

is dominant. Let $\sigma_0, \dots, \sigma_d$ be a basis of V, and let H_∞ be the divisor $\operatorname{div}(\sigma_0)$. Identify the subspace of $\mathbb{P}(V^*)$ defined by $\sigma_0 \neq 0$ to the standard affine space $\mathbb{A}^d_k$ with coordinates $x_1, \dots, x_d$. Then the map ϕ is defined outside H_∞—as certainly the base locus of V is contained in H_∞, and maps onto $\mathbb{A}^d_k$.

For any positive integer r and any integer $n \geq (r+1)M$, the line bundle $\mathcal{L}^{\otimes n}(-rH_\infty) \simeq \mathcal{L}^{\otimes n - rM}$ is very ample. In particular, we can find a section σ of $\mathcal{L}^{\otimes n}$ that vanishes to the order r along H_∞, but does not vanish on Y.

Let $P \in k[x_1, \dots, x_d]$ be a polynomial of degree at most r, considered as a morphism $\mathbb{A}^d_k \to \mathbb{A}^1_k$. Since σ vanishes to the order r along H_∞, the section $(P \circ \phi)\sigma$ of $\mathcal{L}^{\otimes n}$, which is a priori defined only outside H_0, defines a global section of $\mathcal{L}^{\otimes n}$. Because σ does not vanish on Y, the restrictions $(P \circ \phi)\sigma|_Y$ are linearly independent as sections of $L^{\otimes n}|_Y$ as P varies. In particular, the image of the restriction map

$$H^0(X, \mathcal{L}^{\otimes n}) \to H^0(Y, \mathcal{L}|_Y^{\otimes n})$$

has dimension at least equal to the dimension of the space of polynomials of degree at most r in $x_1, \dots, x_d$, so that it has dimension at least

$$\binom{r+d}{d} = \frac{1}{d!} r^d + O(r^{d-1})$$

for any r with $r + 1 \leq n/M$. This proves the result. □

THEOREM 2.21. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. If $\mathcal{Y}$ is a subscheme of $\mathcal{X}$, let*

$$\phi_{n,\mathcal{Y}} : H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \to H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})$$

be the restriction map.

There exists an integer N and a positive real number η such that for any $n \geq N$ and any closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ of dimension $d > 0$, we have

$$\frac{|\operatorname{Ker}(\phi_{n,\mathcal{Y}}) \cap H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})|}{|H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})|} = O(e^{-n^d \eta}),$$

where the implied constant depends on $\mathcal{X}$ and $\overline{\mathcal{L}}$, but not on $\mathcal{Y}$.

Proof. – We only have to consider those $\mathcal{Y}$ that are irreducible. Let us first assume that $\mathcal{Y}$ is flat over $\operatorname{Spec} \mathbb{Z}$. Let H_n be the kernel of the restriction map

$$\Lambda_n = H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \to H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n}),$$

and let k_n be the corank of H_n, i.e., the rank of the image of the restriction map. By Proposition 2.20 applied to $\mathcal{X}_\mathbb{Q}$ and $\mathcal{Y}_\mathbb{Q}$, there exists a positive constant C, independent of $\mathcal{Y}$, such that if n is larger than some integer N, independent of $\mathcal{Y}$, then

$$k_n \geq C n^{d-1}. \tag{2.5}$$

Up to enlarging N, Proposition 2.4 allows us to assume that for $n \geq N$, Λ_n has a basis consisting of elements with norm at most $e^{-n\varepsilon}$ for some $\varepsilon > 0$. For $n \geq N$, we can find elements $\sigma_1, \dots, \sigma_{k_n}$ of Λ_n that are linearly independent in Λ_n/H_n and satisfy $||\sigma_i||_\infty \leq e^{-n\varepsilon}$ for $i \in \{1, \dots, k_n\}$.

Let η be a positive number smaller than ε. Then for any $\sigma \in H_n \cap B_n(1)_{\mathbb{R}}$ and any integers $\lambda_1, \dots, \lambda_{k_n}$ with $|\lambda_i| \leq e^{n\eta}$ for $i \in \{1, \dots, k_n\}$, we have

$$||\sigma + \sum_{i=1}^{k_n} \lambda_i \sigma_i||_\infty \leq 1 + k_n e^{-n(\varepsilon-\eta)}.$$

Furthermore, as σ runs through the elements of H_n, and $\lambda_1, \dots, \lambda_{k_n}$ run through the integers, the $\sigma + \sum_{i=1}^{k_n} \lambda_i \sigma_i$ are pairwise distinct. As a consequence, we have

$$e^{nk_n\eta}|H \cap B_n(1)_{\mathbb{R}}| \leq \left|B_n(1 + k_n e^{-n(\varepsilon-\eta)})\right|.$$

Applying Proposition 2.14 and noting that k_n is bounded above by r_n, which is a polynomial in n, we get

$$\frac{|B_n(1) \cap H_n|}{|B_n(1)|} = O(e^{-nk_n\eta}).$$

Together with (2.5), this shows the required estimate.

Now assume that $\mathcal{Y}$ is not flat over $\mathbb{Z}$. Since $\mathcal{Y}$ is irreducible, it lies over a closed point p of Spec $\mathbb{Z}$. By Proposition 2.20, we can find an integer N and a constant C, independent of $\mathcal{Y}$ and p, such that for any $n \geq N$, the kernel of the restriction map

$$H^0(\mathcal{X}_p, \mathcal{L}^{\otimes n}) \to H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})$$

has codimension at least Cn^d as a vector space over $\mathbb{F}_p$. Let k_n be this codimension. Then

$$k_n \geq Cn^d. \tag{2.6}$$

Again, by Proposition 2.4, up to enlarging N, we can find a positive number ε, depending only on $\mathcal{X}$ and $\overline{\mathcal{L}}$, such that for any $n \geq N$, there exist sections $\sigma_1, \dots, \sigma_{k_n}$ of $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ with $||\sigma_i||_\infty \leq e^{-n\varepsilon}$ for all $i \in \{1, \dots, k_n\}$, such that the images of $\sigma_1, \dots, \sigma_{k_n}$ in $H^0(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}}^{\otimes n})$ are linearly independent over $\mathbb{F}_p$.

Let H_n be the kernel of the restriction map $\phi_{n,\mathcal{Y}}$. If σ is an element of H_n, and if $\lambda_1, \dots, \lambda_{k_n}$ are integers running through $\{0, \dots, p-1\}$, then $\sigma + \lambda_1\sigma_1 + \dots + \lambda_{k_n}\sigma_{k_n}$ belongs to H_n if and only if all the λ_i vanish. Furthermore, the elements $\sigma + \lambda_1\sigma_1 + \dots + \lambda_{k_n}\sigma_{k_n}$ are pairwise disjoint. As a consequence, considering only those λ_i that are 0 or 1, we have

$$2^{k_n}|H_n \cap H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})| \leq |B_n(1 + k_n e^{-n\varepsilon})|.$$

Again, applying Proposition 2.14 and noting that k_n is bounded above by r_n, which is a polynomial in n, we get

$$\frac{|B_n(1) \cap H_n|}{|B_n(1)|} = O(e^{-k_n\eta}).$$

Together with (2.6), this shows the required estimate. □

3. Variants and consequences

3.1. The irreducibility theorem over finite fields

The arithmetic Bertini theorems we prove are stronger than their finite fields counterparts. Since the latter are already known, we give only an example to illustrate how one can deduce them.

PROPOSITION 3.1. – *Assume Theorem 1.1. Let k be a finite field, and let X be an irreducible projective variety over k of dimension at least 2. Let L be a very ample line bundle on X. Then the set*

$$\{\sigma \in \bigcup_{n>0} H^0(X, L^{\otimes n}),\ \mathrm{div}(\sigma) \text{ is irreducible}\}$$

has density 1 in $\bigcup_{n>0} H^0(X, L^{\otimes n})$.

Proof. – Since L is very ample, we can find a positive integer N and a closed embedding $i : X \to \mathbb{P}^N_k$ such that $L = i^*\mathcal{O}(1)$. Apply Theorem 1.1 to the composition

$$f : X \to \mathbb{P}^N_k \to \mathbb{P}^N_{\mathcal{O}_K},$$

where K is a number field together with a finite prime $\mathfrak{p}$ such that $\mathcal{O}_K/\mathfrak{p} = k$, and the line bundle $\overline{\mathcal{L}} = \overline{\mathcal{O}_{\mathbb{P}^N_{\mathcal{O}_K}}(1)}(\varepsilon)$, $\varepsilon > 0$, endowed with the Fubini-Study metric scaled by $e^{-\varepsilon}$. The hermitian line bundle $\overline{\mathcal{L}}$ is the pullback of $\overline{\mathcal{O}(1)}$ by the finite map $\mathbb{P}^N_{\mathcal{O}_K} \to \mathbb{P}^N_{\mathbb{Z}}$. By Proposition 2.8 and Corollary 2.7, $\overline{\mathcal{L}}$ is ample.

Since $f(X)$ is supported over a closed point of $\operatorname{Spec}\mathbb{Z}$, Theorem 1.1 guarantees that the set

$$\Big\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathbb{P}^N_{\mathcal{O}_K}, \overline{\mathcal{L}}^{\otimes n}),\ \mathrm{div}(\sigma|_X) \text{ is irreducible}\Big\}$$

has density 1 in $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathbb{P}^N_{\mathcal{O}_K}, \overline{\mathcal{L}}^{\otimes n})$. By Corollary 2.18, the theorem holds. □

Note that since on a scheme X defined over a finite field, every line bundle is a hermitian line bundle, and every section is effective, we can remove the flatness assumptions on the theorems of the introduction and have uniform statements that cover both the results of this paper and those of [11].

3.2. Generic smoothness

We first state the Bertini smoothness theorem of Poonen [26] in the form we need—see [13] for the proof of this version.

THEOREM 3.2. – *Let X be a smooth projective variety over a finite field k, and let L be an ample line bundle on X. Then the density of those $\sigma \in \bigcup_{n>0} H^0(X, L^{\otimes n})$ such that $\mathrm{div}(\sigma)$ is smooth is equal to $\zeta_X\big(1 + \dim(X)\big)^{-1}$, where ζ_X is the zeta function of X.*

Applying the above result together with the restriction results of Corollary 2.18, we find the following.

Proposition 3.3. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let p be a prime number such that $\mathcal{X}_p$ is smooth over $\mathbb{F}_p$. Then the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\mathrm{div}(\sigma_{|\mathcal{X}_p})$ is smooth is equal to $\zeta_p(\dim(\mathcal{X}))^{-1}$, where ζ_p is the zeta function of $\mathcal{X}_p$.*

Proof. – We apply Corollary 2.18 to the subspace $\mathcal{Y} = \mathcal{X}_p$ of $\mathcal{X}$ and the subset E of $\bigcup_{n>0} H^0(\mathcal{X}_p, \mathcal{L}_{|\mathcal{X}_p}^{\otimes n})$ consisting of sections with smooth divisor. Theorem 3.2 shows that E has density $\zeta_p(1+\dim(\mathcal{X}_p))^{-1} = \zeta_p(\dim(\mathcal{X}))^{-1}$, which implies the result. □

We may prove Theorem 1.7.

Proof of Theorem 1.7. – In the situation of the theorem, we know that $\mathcal{X}_p$ is smooth for all large enough p. Furthermore, denoting again the zeta function of $\mathcal{X}_p$ by ζ_p, we have

$$\lim_{p\to\infty} \zeta_p(x) = 1$$

for any $x > 1$ by [30, 1.3]. This shows that the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that there exists p with $\mathcal{X}_p$ smooth and $\mathrm{div}(\sigma_{|\mathcal{X}_p})$ smooth is equal to 1. For any such σ, the divisor $\mathrm{div}(\sigma)_{\mathbb{Q}}$ is smooth, which proves the result. □

3.3. Irreducibility theorems with local conditions

We can give variants of the irreducibility theorems with conditions at prescribed subschemes. For an easier formulation, we give them in the setting of Theorem 1.6.

Proposition 3.4. – *Let $\mathcal{X}$ be a projective arithmetic variety, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let Z_1 be a finite subscheme of $\mathcal{X}$, and let Z_2 be a positive-dimensional subscheme of $\mathcal{X}$. Choose a trivialization $\phi : \mathcal{L}_{|Z_1} \simeq \mathcal{O}_{Z_1}$, and let T be a subset of $H^0(Z_1, \mathcal{O}_{Z_1})$. Then the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\sigma_{|Z_1}$ belongs to T (under the trivialization ϕ) and σ does not vanish identically on any component of Z_2 is equal to*

$$\frac{|T|}{|H^0(Z_1, \mathcal{O}_{Z_1})|}.$$

Proof. – By Corollary 2.18, the density of those σ such that $\sigma_{|Z_1}$ belongs to T is indeed $\frac{|T|}{|H^0(Z_1,\mathcal{O}_{Z_1})|}$. On the other hand, Theorem 2.21 ensures that the density of those σ that do not vanish identically on any component of Z_2 is equal to 1. □

Given Theorem 1.6—proven in the last section of this paper—and Theorem 1.7, we find the two following results.

Corollary 3.5. – *Let $\mathcal{X}$ be a projective arithmetic variety of dimension at least 2, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let Z_1 be a finite subscheme of $\mathcal{X}$, and let Z_2 be a positive-dimensional subscheme of $\mathcal{X}$. Choose a trivialization $\phi : \mathcal{L}_{|Z_1} \simeq \mathcal{O}_{Z_1}$, and let T be a subset of $H^0(Z_1, \mathcal{O}_{Z_1})$. Then the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that the following conditions hold:*

(i) *$\sigma_{|Z_1}$ belongs to T (under the trivialization ϕ);*

(ii) *σ does not vanish identically on any component of Z_2;*

(iii) $\mathrm{div}(\sigma)$ *is irreducible,*

is equal to

$$\frac{|T|}{|H^0(Z_1, \mathcal{O}_{Z_1})|}.$$

COROLLARY 3.6. – *Let $\mathcal{X}$ be a projective arithmetic variety with smooth generic fiber, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let Z_1 be a finite subscheme of $\mathcal{X}$, and let Z_2 be a positive-dimensional subscheme of $\mathcal{X}$. Choose a trivialization $\phi : \mathcal{L}|_{Z_1} \simeq \mathcal{O}_{Z_1}$, and let T be a subset of $H^0(Z_1, \mathcal{O}_{Z_1})$. Then the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that the following conditions hold:*

(i) *$\sigma|_{Z_1}$ belongs to T (under the trivialization ϕ);*

(ii) *σ does not vanish identically on any component of Z_2;*

(iii) $\mathrm{div}(\sigma)_{\mathbb{Q}}$ *is smooth,*

is equal to

$$\frac{|T|}{|H^0(Z_1, \mathcal{O}_{Z_1})|}.$$

4. Preliminary estimates

This section gathers preliminary material on hermitian line bundles on arithmetic surfaces, which will be used in the proof of Theorem 1.6. In 4.1, we give lower bounds for the norm of products of sections of hermitian line bundles. In 4.2, we give an upper bound for the number of effective sections of a hermitian line bundle in terms of its degree with respect to a positive enough hermitian line bundle. Such a result is closely related to the effective bounds of [37]. Our proof is better expressed in terms of the θ-invariants of Bost [9], which we only consider in a finite-dimensional setting. In 4.3, we give an estimate for the number of effective hermitian line bundles satisfying certain boundedness properties.

4.1. Norm estimates for sections of hermitian line bundles

Let X be a compact connected Riemann surface. Let ω be a real semipositive 2-form of type $(1,1)$ on X with

$$\int_X \omega = 1.$$

Define

$$d^c = \frac{1}{2i\pi}(\partial - \overline{\partial}),$$

so that

$$dd^c = \frac{i}{\pi}\partial\overline{\partial}.$$

Let s be a section of a hermitian line bundle on X. In what follows, we will write $||s||$ for the function $P \mapsto ||s(P)||$.

Let $\overline{L} = (L, ||.||)$ be a hermitian line bundle on X. If s is a nonzero section of L, the Lelong-Poincaré formula gives us the equality of currents

$$-dd^c \log ||s|| = c_1(\overline{L}) - \delta_D,$$

where $c_1(\overline{L})$ is the curvature form of $\overline{L}$, D is the divisor of s and δ_D is the current of integration along D.

Define, following [10, (1.4.8)]

$$||s||_0 = \exp\Big(\int_X \log ||s||\omega\Big).$$

Since $\int_X \omega = 1$, the following inequality holds:

$$||s||_0 \leq ||s||_\infty.$$

Say that $\overline{L}$ is *ω-admissible*, or admissible for short, if $c_1(\overline{L})$ is proportional to ω. If $\overline{L}$ is admissible, then the Gauss-Bonnet formula shows

$$c_1(\overline{L}) = (\deg L)\omega.$$

Let M be any line bundle on X. By the $\partial\overline{\partial}$ lemma, we can find a hermitian metric $||.||$ on M such that the hermitian line bundle $(M, ||.||)$ is admissible. Given a nonzero global section s of M, there exists a unique such metric such that $||s||_0 = 1$.

If D is an effective divisor on X, let σ_D be the section of $\mathcal{O}(D)$ that is the image of 1 under the natural morphism $\mathcal{O}_X \to \mathcal{O}(D)$. The discussion above shows that there exists a unique admissible hermitian line bundle $\overline{\mathcal{O}(D)} = (\mathcal{O}(D), ||.||)$ on X such that $||\sigma_D||_0 = 1$. Of course, if D_1 and D_2 are effective divisors, we have

$$\overline{\mathcal{O}(D_1 + D_2)} = \overline{\mathcal{O}(D_1)} \otimes \overline{\mathcal{O}(D_2)}.$$

The functions σ_P satisfy basic uniformities in the point P of X which are readily proved by the following argument using Green functions.

Proposition 4.1. – *Endow X with a Riemannian metric with induced geodesic distance d. Then there exist positive constants C, C' and η such that the following inequalities hold:*

(i) $\forall P \in X,\ ||\sigma_P||_\infty \leq C$;

(ii) $\forall (P, Q) \in X \times X,\ \sigma_P(Q) \geq \min(C'd(P, Q), \eta)$.

Proof. – Let $\Delta \subset X \times X$ be the diagonal. Let α be a real closed form of type $(1, 1)$ on $X \times X$ of the form

$$\alpha = p_1^*\omega + p_2^*\omega + \sum_{i \in I} p_1^*\beta_i \wedge p_2^*\gamma_i,$$

where p_1 and p_2 are the two projections from $X \times X$ to X and the β_i (resp. γ_i) are 1-forms on X. Choose the β_i and γ_i so that α is symmetric with respect to the involution of $X \times X$ that exchanges the two factors, and that it is cohomologous to the class of the diagonal Δ in the de Rham cohomology of $X \times X$. By the $\partial\overline{\partial}$ lemma, we can find a hermitian metric on the line bundle $\mathcal{O}(\Delta)$ with curvature form α. For any P in X, this hermitian metric induces a hermitian metric on $\mathcal{O}(P)$ by restriction to $\{P\} \times X$.

Let σ_Δ be the global section of $\mathcal{O}(\Delta)$ corresponding to the constant function 1. For any P in X, write τ_P for the section of $\mathcal{O}(P)$.

$$\tau_P : Q \mapsto \sigma_\Delta(P, Q).$$

Then

$$-dd^c \log ||\tau_P|| = \alpha|_{\{P\}\times X} - \delta_P = \omega - \delta_P,$$

which shows that the metric on $\mathcal{O}(P)$ coming from that on $\mathcal{O}(\Delta)$ differs from the canonical one defined above by a homothety. In particular, we can find a continuous function $X \to \mathbb{R}_+^*,\ P \mapsto \lambda(P)$ such that

$$\forall (P, Q) \in X \times X, ||\sigma_P(Q)|| = \lambda(P)||\sigma_\Delta(P, Q)||.$$

Since $(P, Q) \mapsto \sigma_\Delta(P, Q)$ is a smooth section of $\mathcal{O}(\Delta)$ that vanishes with the order 1 along Δ, this shows the result [(2)]. □

We will make use of the uniformity above to prove inequalities between norms. The following is a variant of [10, Corollary 1.4.3].

Proposition 4.2. – *Let $\overline{L} = (L, ||.||)$ be an admissible hermitian line bundle on X. Let P be a point of X, and let s be a section of L. Then*

$$||s(P)|| \le ||s||_0\, ||\sigma_P||_\infty^{\deg L}.$$

In particular, there exists a positive constant C_1 such that

$$||s||_\infty \le C_1^{\deg L}||s||_0.$$

Proof. – We can assume that s is nonzero. Let D be the divisor of s. Define

$$g = -\log ||s||$$

and

$$g_P = -\log ||\sigma_P||.$$

By Lelong-Poincaré, we have

$$dd^c g = (\deg L)\omega - \delta_D$$

and

$$dd^c g_P = \omega - \delta_P.$$

The Stokes formula

$$\int_X g\, dd^c g_P = \int_X g_P\, dd^c g$$

gives us

$$-\log ||s||_0 + \log ||s(P)|| = -\deg L \log ||\sigma_P||_0 + \log ||\sigma_P(D)|| = \log ||\sigma_P(D)||,$$

where, if $D = \sum_i n_i P_i$, we wrote

$$||\sigma_P(D)|| = \prod_i ||\sigma(P_i)^{n_i}||.$$

(2) Actually, a straightforward computation shows that $||\tau_P||_0$ is a constant function of P, so that λ is constant.

Since the degree of D is equal to the degree of L, we get the first inequality. The second one follows from the first and Proposition 4.1. □

LEMMA 4.3. – *Let* $\overline{L} = (L, ||.||)$ *be an admissible hermitian line bundle on* X *with positive degree. Then for any section* s *of* L*, and any* P *in* X*, the following inequality holds:*

$$||s||_\infty \leq C_2(\deg L)||s\sigma_P||_\infty,$$

where C_2 *is a positive constant depending only on* X *and* ω*.*

Proof. – Let B be the ball $\{z \in \mathbb{C}|\,|z| < 3\}$. Let $(U_i)_{i\in I}$ be a finite cover of X by open subsets such that there exist biholomorphic functions

$$f_i : B \to U_i$$

and assume that X is covered by the $f_i(\{z \in \mathbb{C}|\,|z| < 1\})$. For all $i \in I$, choose a smooth function $\phi_i : B \to \mathbb{R}$ such that $-dd^c\phi_i = f_i^*\omega$.

Since $\overline{L}$ is admissible, the curvature form of $\overline{L}$ is $(\deg L)\omega$ and for any $i \in I$, we can find an isomorphism of hermitian line bundles on B

$$f_i^*\overline{L} \simeq (\mathcal{O}, e^{-\lambda\phi_i}),$$

where $|.|$ is the standard absolute value and $\lambda = \deg L$.

Choose an element $i \in I$, and a complex number z with $|z| < 1$ such that

$$|f_i^*s(z)|e^{-\lambda\phi_i(z)} = ||s||_\infty,$$

where $f_i^*s(z)$ is considered as a complex number via the isomorphism above. By Proposition 4.1, we can find positive constants $\varepsilon < \lambda$ and η depending only on X and ω such that either $|\sigma_P(z)| \geq \frac{\eta}{\lambda}$ or,

$$\forall z' \in \mathbb{C}, |z' - z| = \frac{\varepsilon}{\lambda} \implies |\sigma_P(z')| \geq \frac{\eta}{\lambda}.$$

If $\sigma_P(z) \geq \frac{\eta}{\lambda}$, then

$$||s\sigma_P||_\infty \geq ||s\sigma_P(f_i(z))|| \geq \frac{\eta}{\lambda}||s(f_i(z))|| = \frac{\eta}{\deg L}||s||_\infty.$$

Assume on the contrary $\sigma_P(z) < \frac{\eta}{\lambda}$. By the maximum principle, we can find a complex number z' with $|z' - z| = \frac{\varepsilon}{\lambda}$ and $|f_i^*s(z')| \geq |f_i^*s(z)|$. In particular, $|z'| < 2$ and

$$||s\sigma_P||_\infty \geq ||s\sigma_P(f_i(z'))|| \geq \frac{\eta}{\lambda}||s||_\infty e^{-\lambda(\phi_i(z)-\phi_i(z'))} \geq e^{-C}\frac{\eta}{\deg L}||s||_\infty,$$

where C is an upper bound for the differential of the ϕ_i on the ball $\{z \in \mathbb{C}|\,|z| < 2\}$ as i varies through the finite set I. □

PROPOSITION 4.4. – *Let* $\overline{L} = (L, ||.||)$ *and* $\overline{M} = (M, ||.||)$ *be two admissible hermitian line bundles on* X*. Then for any two sections* s *and* σ *of* L *and* M *respectively, the following inequality holds:*

$$||s||_\infty||\sigma||_0 \leq (C_2(\deg L + \deg M))^{\deg M}||s\sigma||_\infty,$$

where C_2 *is a positive constant depending only on* X *and* ω*.*

Proof. – Let D be the divisor of σ. Then M is isomorphic to $\mathcal{O}(D)$, and the hermitian line bundles $\overline{M}$ and $\overline{\mathcal{O}(D)}$, as well as the sections σ and σ_D differ by a homothethy. Since the inequality we want to prove is invariant under scaling, we can assume that $\overline{M} = \overline{\mathcal{O}(D)}$ and $\sigma = \sigma_D$. If $D = \sum_i n_i P_i$, we have

$$\overline{\mathcal{O}(D)} = \bigotimes_i \overline{\mathcal{O}(P_i)}^{\otimes n_i}$$

and

$$\sigma_D = \Pi_i \sigma_{P_i}^{n_i},$$

so that the result follows from successive applications of Lemma 4.3. □

4.2. An upper bound for the number of sections

Let $\mathcal{X}$ be a projective arithmetic variety with smooth generic fiber. Choose a Kähler form on $\mathcal{X}(\mathbb{C})$ which is invariant under complex conjugation and has volume 1. If $\overline{\mathcal{L}}$ is a hermitian line bundle on $\mathcal{X}$, we write $h^0_\theta(\mathcal{X}, \overline{\mathcal{L}})$ for $h^0_\theta(H^0_{L^2}(\mathcal{X}, \overline{\mathcal{L}}))$, where the hermitian vector bundle $H^0_{L^2}(\mathcal{X}, \overline{\mathcal{L}})$ over $\operatorname{Spec}\mathbb{Z}$ is endowed with the L^2 norm induced by the Kähler metric on $\mathcal{X}$.

We will need a comparison result between the sup norm and the L^2 norm on the space of sections of hermitian line bundles, which we will obtain through a minor generalization of Gromov's lemma [17, Lemma 30]. We follow the proof of Gillet-Soulé and start with a local result.

In the following, if z is an element of $\mathbb{C}^d$, we write $z_1, \ldots, z_d \in \mathbb{C}$ for its coordinate, and, for any $k \in \{1, \ldots, d\}$, we write $z_k = x_k + i y_k$, where x_k and y_k are real.

LEMMA 4.5. – *Let d be a positive integer, and let B be the open ball $\{z \in \mathbb{C}^d \,|\, |z| < 3\}$ in $\mathbb{C}^d$. Let ϕ be a real-valued smooth function on B, and let g be a smooth positive function on B. Then there exists a positive constant C depending only on ϕ and g such that for any real number $\lambda \geq 1$, any holomorphic function f on B and any w in B with $|w| < 1$,*

$$\int \cdots \int_{|z-w|<1} |f(z)|^2 e^{-2\lambda\phi(z)} g(z) dx_1 \cdots dy_d \geq C |f(w)|^2 e^{-2\lambda\phi(w)} \lambda^{-2d}.$$

Proof. – If λ is an integer, the inequality is the "local statement" proved in the beginning of the proof of [17, Lemma 30].

To prove our result, after adding a negative constant to ϕ, we can assume that ϕ is negative on the ball $|z| < 2$. Let C' be a lower bound for the values of ϕ on the ball $|z| < 2$. If $\lambda > 1$ is arbitrary, write $\lambda = n + r$, with $0 \leq r < 1$. Then

$$e^{-2\lambda\phi(z)} = e^{-2n\phi(z)} e^{-2r\phi(z)} \geq e^{-2n\phi(z)}$$

for any z with $|z| < 2$, so that

$$\begin{aligned}\int \cdots \int_{|z-w|<1} |f(z)|^2 e^{-2\lambda\phi(z)} g(z) dx_1 \cdots dy_d &\geq \int \cdots \int_{|z-w|<1} |f(z)|^2 e^{-2n\phi(z)} g(z) dx_1 \cdots dy_d \\ &\geq C |f(w)|^2 e^{-2n\phi(w)} n^{-2d} \\ &\geq C |f(w)|^2 e^{-2\lambda\phi(w)} e^{2r\phi(w)} \lambda^{-2d} \\ &\geq C e^{2C'} |f(w)|^2 e^{-2\lambda\phi(w)} \lambda^{-2d}.\end{aligned}$$

Replacing C with $Ce^{2C'}$, we get the result. □

PROPOSITION 4.6. – *Let X be a compact connected riemannian complex manifold of dimension d, let ω be a real form of type $(1,1)$ on X. Then there exists a positive constant C such that for any hermitian line bundle $\overline{\mathcal{L}}$ on X with positive degree and curvature form $\lambda\omega$ with $|\lambda| > 1$, and any section s of $\mathcal{L}$ over X, we have*

$$||s||_{L^2} \geq C|\lambda|^{-d}||s||_\infty,$$

where $||s||_{L^2}$ denotes the L^2 norm of s with respect to the given metric on X.

In particular, if $d = 1$, there exists a positive constant C' such that for any hermitian line bundle $\overline{\mathcal{L}}$ with curvature form proportional to ω and positive degree, and any section s of $\mathcal{L}$, we have

$$||s||_{L^2} \geq C'(\deg \mathcal{L})^{-1}||s||_\infty.$$

Proof. – As above, let B be the open ball $\{z \in \mathbb{C}^d \,|\, |z| < 3\}$ in $\mathbb{C}^d$. Let $(U_i)_{i\in I}$ be a finite cover of X by open subsets such that there exists biholomorphic functions

$$f_i : B \to U_i$$

and assume that X is covered by the $f_i(\{z \in \mathbb{C}^d \,|\, |z| < 1\})$. For any $i \in I$, we can find a positive smooth function g_i such that the pullback of the standard metric of X to B by f_i is $g_i\, dx_1 \cdots dy_d$.

For all $i \in I$, choose a function ϕ_i on B such that $-dd^c\phi_i = f_i^*\omega$. Let $\overline{\mathcal{L}}$ be a hermitian line bundle with curvature form $\lambda\omega$ for some real number λ with $|\lambda| > 1$. Then, for any $i \in I$, we can fix an isomorphism of hermitian line bundles

$$f_i^*\overline{\mathcal{L}} \simeq (\mathcal{O}_B, e^{-\lambda\phi_i}|.|),$$

where $|.|$ is the standard absolute value. Applying Lemma 4.5 (up to replacing ϕ_i by $-\phi_i$ if λ is negative), we can find a positive constant K, independent of $\overline{\mathcal{L}}$, such that, given any section s of $\overline{\mathcal{L}}$, for any $i \in I$ and any w in B with $|w| < 1$, we have

$$\int |_{z-w|<1} |f_i^*s(z)|^2 e^{-\lambda\phi_i(z)} g(z) dx_1 \cdots dy_d \geq K|f_i^*s(w)|^2 e^{-\lambda\phi_i(w)}|\lambda|^{-2d},$$

where we consider f_i^*s as a holomorphic function via the local trivializations of $\mathcal{L}$. This inequality means

$$\int |_{z-w|<1} ||s(f_i(z))||^2 g(z) dx_1 \cdots dy_d \geq K|\lambda|^{-2d}||s(f_i(w))||^2,$$

so that

$$||s||_{L^2}^2 \geq \int |_{z-w|<1} ||s(f_i(z))||^2 g(z) dx_1 \cdots dy_d \geq K|\lambda|^{-2d}||s(f_i(w))||^2$$

for any w, which proves the first result.

The second result is a consequence of the first one and the Gauss-Bonnet formula. □

Given a real form ω of type $(1,1)$, write $\widehat{\mathrm{Pic}}_\omega(\mathcal{X})$ for the group of ω-admissible hermitian line bundles on $\mathcal{X}$, that is, hermitian line bundles whose curvature form is proportional to ω.

PROPOSITION 4.7. – *Let $\mathcal{X}$ be a regular projective arithmetic surface. Choose a Kähler form on $\mathcal{X}(\mathbb{C})$ which is invariant under complex conjugation, and let $\overline{\mathcal{B}}$ be a hermitian line bundle on $\mathcal{X}$. Let ω be a real form of type $(1,1)$ on $\mathcal{X}(\mathbb{C})$ with $\int_{\mathcal{X}(\mathbb{C})}\omega \neq 0$. Assume that the following conditions hold:*

(i) *Some positive power of $\overline{\mathcal{B}}$ is effective;*

(ii) $\overline{\mathcal{B}}.\overline{\mathcal{B}} > 0$*;*

(iii) *If $\overline{\mathcal{M}}$ is an effective hermitian line bundle on $\mathcal{X}$, then $\overline{\mathcal{B}}.\overline{\mathcal{M}} \geq 0$.*

Then for any effective $\overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}_\omega(\mathcal{X})$, we have

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}}) \leq \frac{(\overline{\mathcal{B}}.\overline{\mathcal{M}})^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(\overline{\mathcal{M}}.\overline{\mathcal{B}}\log(1+\overline{\mathcal{M}}.\overline{\mathcal{B}})) + O(\deg\mathcal{M}_{\mathbb{Q}}\log(1+\deg\mathcal{M}_{\mathbb{Q}})) + O(1),$$

where the implied constants depend on $\mathcal{X}$, $\overline{\mathcal{B}}$ and ω, but not on $\overline{\mathcal{M}}$.

REMARK 4.8. – Using the precise computations of [9, Chapter 3], it would be possible to make the implied constants above effective.

REMARK 4.9. – If $\overline{\mathcal{M}}$ is effective, then both $\overline{\mathcal{B}}.\overline{\mathcal{M}}$ and $\deg\mathcal{M}_{\mathbb{Q}}$ are nonnegative.

Proof. – Let $\overline{\mathcal{M}}$ be an effective, ω-admissible, hermitian line bundle. If $\mathcal{M}_{\mathbb{Q}}$ has degree zero, then the curvature form of $\overline{\mathcal{M}}$ vanishes, so that $\overline{\mathcal{M}}$ is isomorphic to $\overline{\mathcal{O}}_{\mathcal{X}}$ and the inequality of the proposition holds. We can assume that the degree of $\mathcal{M}_{\mathbb{Q}}$ is positive. Let us write d for the degree of $\mathcal{M}_{\mathbb{Q}}$.

After replacing $\overline{\mathcal{B}}$ by a positive power, we can assume that $\overline{\mathcal{B}}$ is effective. Let σ be a nonzero effective section of $\overline{\mathcal{B}}$ with divisor D. We have an exact sequence of line bundles

$$0 \to \mathcal{M}\otimes\mathcal{B}^{\otimes -1} \to \mathcal{M} \to \mathcal{M}_{|D} \to 0,$$

in which the first map is multiplication by σ and the second one is restriction of sections. Taking global sections, we get an exact sequence

$$0 \to H^0(\mathcal{X}, \mathcal{M}\otimes\mathcal{B}^{\otimes -1}) \to H^0(\mathcal{X},\mathcal{M}) \to H^0(D, \mathcal{M}_{|D}).$$

The map of lattices

$$i : H^0_{L^2}(\mathcal{X}, \overline{\mathcal{M}}\otimes\overline{\mathcal{B}}^{\otimes -1}) \to H^0_{L^2}(\mathcal{X},\overline{\mathcal{M}})$$

is the multiplication by the section σ, whose sup norm is bounded above by 1, so the operator norm of i is bounded above by 1.

Endow $H^0(D, \overline{\mathcal{M}}_{|D})$ with the L^2 norm

$$||t||^2_{L^2} = \sum_{z\in D(\mathbb{C})} ||t(z)||^2$$

for $t \in H^0(D, \overline{\mathcal{M}}_{|D})$. Then for any section t of $\mathcal{M}$ over D, we have

$$||t||^2_\infty \geq \frac{1}{\deg D_{\mathbb{Q}}}||t||^2_{L^2}.$$

If s is a global section of $\overline{\mathcal{M}}$ on $\mathcal{X}$, then certainly we have, for the sup norms

$$||s||_\infty \geq ||s_{|D}||_\infty$$

and consequently

$$||s||_\infty \geq \frac{1}{\deg D_{\mathbb{Q}}} ||s|_D||^2_{L^2}.$$

By Proposition 4.6, with s as above, we have

$$||s||_{L^2} \geq C\, d^{-1} ||s||_\infty,$$

where we recall that d is the degree of $\mathcal{M}_{\mathbb{Q}}$ and C is a positive constant independent of $\overline{\mathcal{M}}$. We obtain

$$||s||_{L^2} \geq \frac{C}{\deg D_{\mathbb{Q}}} d^{-1} ||s|_D||^2_{L^2}.$$

In other words, the operator norm of the map of lattices

$$r : H^0_{L^2}(\mathcal{X}, \overline{\mathcal{M}}) \to H^0_{L^2}(D, \overline{\mathcal{M}}|_D),$$

given by restricting sections to D is bounded above by $C'd$, where C' is a positive constant independent of $\overline{\mathcal{M}}$. In other words, the induced map of lattices

$$H^0_{L^2}(\mathcal{X}, \overline{\mathcal{M}}) \to H^0_{L^2}(D, \overline{\mathcal{M}}|_D)(\log C' + \log d)$$

has norm at most 1—here if Λ is a lattice and δ a real number, we write $\Lambda(\delta)$ for the lattice Λ with the metric scaled by $e^{-\delta}$. Note that from [9, Corollary 3.3.5, (2)], we have

$$h^0_\theta(H^0_{L^2}(D, \overline{\mathcal{M}}|_D)(\log C' + \log d)) \leq h^0_\theta(D, \overline{\mathcal{M}}|_D) + \deg D_{\mathbb{Q}}(\log C' + \log d).$$

From the monotonicity and the subadditivity of θ-invariants proved in [9, Proposition 3.3.2, Proposition 3.8.1], we get

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}}) \leq h^0_\theta(\mathcal{X}, \overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -1}) + h^0_\theta(D, \overline{\mathcal{M}}|_D) + O(\log d) + O(1), \tag{4.1}$$

where the implied constants are independent of $\overline{\mathcal{M}}$.

By [9, Proposition 3.7.1, Proposition 3.7.2], we have [3]

$$h^0_\theta(D, \overline{\mathcal{M}}|_D) \leq \max(\deg \overline{\mathcal{M}}|_D, 0) + O(1) \leq \overline{\mathcal{M}}.\overline{\mathcal{B}} + O(1)$$

since we assumed that $\overline{\mathcal{M}}.\overline{\mathcal{B}} \geq 0$ and since D is the zero locus of an effective section of $\overline{\mathcal{B}}$. Together with (4.1), we obtain

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}}) \leq h^0_\theta(\mathcal{X}, \overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -1}) + \overline{\mathcal{M}}.\overline{\mathcal{B}} + O(\log d) + O(1). \tag{4.2}$$

Now let m be the smallest integer such that $m\overline{\mathcal{B}}.\overline{\mathcal{B}} > \overline{\mathcal{M}}.\overline{\mathcal{B}}$, so that

$$m \leq \lfloor \overline{\mathcal{M}}.\overline{\mathcal{B}} / \overline{\mathcal{B}}.\overline{\mathcal{B}} \rfloor + 1.$$

Applying the argument above inductively to $\overline{\mathcal{L}}.\overline{\mathcal{B}}^{\otimes -i}$ as i runs from 0 to $m-1$, we get

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}}) \leq h^0_\theta(\mathcal{X}, \overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -m}) + \frac{(\overline{\mathcal{M}}.\overline{\mathcal{B}})^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(\overline{\mathcal{M}}.\overline{\mathcal{B}} \log d) + O(\overline{\mathcal{M}}.\overline{\mathcal{B}}) + O(1). \tag{4.3}$$

By construction, $\overline{\mathcal{B}}.(\overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -m}) < 0$, so that condition (iii) ensures that $\overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -m}$ is not effective. By [9, Corollary 4.1.2], we get

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}} \otimes \overline{\mathcal{B}}^{\otimes -m}) \leq O(d \log d) + O(1)$$

[3] In [9], hermitian vector bundles are only considered over the ring of integers of number fields. However, this assumption is irrelevant, and can be removed by considering the pullback of $\overline{\mathcal{M}}|_D$ to the normalization of D.

since the rank of $H^0(\mathcal{X}, \mathcal{M} \otimes \mathcal{B}^{\otimes -m})$ is certainly bounded above by $O(d)$.

Finally, we have

$$h^0_\theta(\mathcal{X}, \overline{\mathcal{M}}) \leq \frac{(\overline{\mathcal{M}}.\overline{\mathcal{B}})^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(\overline{\mathcal{M}}.\overline{\mathcal{B}} \log d) + O(d \log d) + O(\overline{\mathcal{M}}.\overline{\mathcal{B}}) + O(1), \tag{4.4}$$

which shows the result. □

Corollary 4.10. – *Let $\mathcal{X}$ be a regular projective arithmetic surface, and let $\overline{\mathcal{B}}$ be a hermitian line bundle on $\mathcal{X}$. Let ω be a real form of type $(1,1)$ on $\mathcal{X}(\mathbb{C})$ with $\int_{\mathcal{X}(\mathbb{C})} \omega \neq 0$. Assume that the following conditions hold:*

(i) *Some positive power of $\overline{\mathcal{B}}$ is effective;*

(ii) $\overline{\mathcal{B}}.\overline{\mathcal{B}} > 0$*;*

(iii) *If $\overline{\mathcal{M}}$ is an effective hermitian line bundle on $\mathcal{X}$, then $\overline{\mathcal{B}}.\overline{\mathcal{M}} \geq 0$.*

Then for any effective $\overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}_\omega(\mathcal{X})$, we have

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}}) \leq \frac{(\overline{\mathcal{B}}.\overline{\mathcal{M}})^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(\overline{\mathcal{M}}.\overline{\mathcal{B}} \log(\overline{\mathcal{M}}.\overline{\mathcal{B}})) + O(\deg \mathcal{M}_{\mathbb{Q}} \log(\deg \mathcal{M}_{\mathbb{Q}})) + O(1),$$

where the implied constants depend on $\mathcal{X}$ and $\overline{\mathcal{B}}$, but not on $\overline{\mathcal{M}}$.

Proof. – From Proposition 4.7 and [9, Theorem 4.1.1], we find that the inequality holds if one replaces $h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}})$ with $h^0_{\mathrm{Ar},L^2}(\mathcal{X}, \overline{\mathcal{M}})$—this expression being defined as the logarithm of the number of sections of $\overline{\mathcal{M}}$ with L^2 norm bounded above by 1. Choosing the Kähler form on $\mathcal{X}$ to have volume 1, we have

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}}) \leq h^0_{\mathrm{Ar},L^2}(\mathcal{X}, \overline{\mathcal{M}}),$$

which finishes the proof. □

Remark 4.11. – In [37, Theorem A], Yuan and Zhang prove an explicit upper bound for $h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{M}})$ from which one can deduce—via log-concavity of volumes—special cases of our inequality.

4.3. An upper bound for the number of effective hermitian line bundles

Lemma 4.12. – *Let $\mathcal{X}$ be a projective arithmetic surface, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. If $\overline{\mathcal{M}}$ is an effective hermitian line bundle on $\mathcal{X}$ which is not isomorphic to $\overline{\mathcal{O}}_{\mathcal{X}}$, then*

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} > 0.$$

Proof. – Let s be an effective section of $\overline{\mathcal{M}}$, and let D be the divisor of s. Then by the formula [12, (6.3.2)], we have

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} = h_{\overline{\mathcal{L}}}(D) - \int_{\mathcal{X}(\mathbb{C})} \log ||s_{\mathbb{C}}|| c_1(\overline{\mathcal{L}}),$$

where $h_{\overline{\mathcal{L}}}$ denotes the height with respect to D. The first term is nonnegative since $\overline{\mathcal{L}}$ is ample, and vanishes if and only if $D = 0$. Since s is effective and the curvature form of $\overline{\mathcal{L}}$ is semipositive—and positive on a Zariski-dense open subset of $\mathcal{X}(\mathbb{C})$ as $\mathcal{L}_{\mathbb{C}}$ is ample—the second term is nonnegative as well, and vanishes if and only if the norm of s is identically 1.

As a consequence, for $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ to vanish, it is necessary for $\overline{\mathcal{M}}$ to have a nowhere vanishing section of norm identically 1, i.e., to be isomorphic to $\overline{\mathcal{O}}_{\mathcal{X}}$. □

PROPOSITION 4.13. – *Let $\mathcal{X}$ be a projective arithmetic surface, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Let ω be a semipositive real form of type $(1,1)$ on $\mathcal{X}(\mathbb{C})$ with $\int_{\mathcal{X}(\mathbb{C})}\omega \neq 0$. Let N be a subgroup of the group $\widehat{\mathrm{Pic}}_{\omega}(\mathcal{X})$ of ω-admissible hermitian line bundles on $\mathcal{X}$. Assume that the intersection of N with $\mathrm{Ker}(\widehat{\mathrm{Pic}}_{\omega}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})) \simeq \mathbb{R}$ is discrete. Then N is a group of finite type. Let ρ be the rank of N, and let N_{eff} denote the subspace of N consisting of effective line bundles. As n tends to ∞, we have*

$$\left|\{\overline{\mathcal{M}} \in N_{\mathrm{eff}} | \, \overline{\mathcal{L}}.\overline{\mathcal{M}} \leq n\}\right| = O(n^{\rho}).$$

Proof. – The abelian group $\mathrm{Pic}(\mathcal{X})$ is finitely generated by [29]—see [20] for a modern proof—so that the image of N in $\mathrm{Pic}(\mathcal{X})$ is a group of finite type. Since the intersection of N with $\mathrm{Ker}(\widehat{\mathrm{Pic}}_{\omega}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}))$ is discrete, it is of finite type as well, which proves that N is a group of finite type.

The linear form on N

$$\overline{\mathcal{M}} \mapsto \overline{\mathcal{L}}.\overline{\mathcal{M}}$$

extends to a linear form on $N_{\mathbb{R}} := N \otimes \mathbb{R}$ which we still denote by

$$\alpha \mapsto \overline{\mathcal{L}}.\alpha.$$

Let $\overline{N}_{\mathrm{eff}}$ be the closure of N_{eff} in $N \otimes \mathbb{R}$. Lemma 4.12 shows that the linear form above is nonnegative on N_{eff}, so it is nonnegative on $\overline{N}_{\mathrm{eff}}$.

Our assumption on N guarantees that the first chern class map

$$c_1 : \widehat{\mathrm{Pic}}_{\omega}(\mathcal{X}) \to \widehat{\mathrm{CH}}^1(\mathcal{X})$$

extends to an injection

$$c_{1,\mathbb{R}} : N \otimes \mathbb{R} \to \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}),$$

where $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ is the arithmetic Chow group with real coefficients defined in [7, 5.5]. Indeed, we have an exact sequence

$$0 \to (N \cap \mathrm{Ker}(\widehat{\mathrm{Pic}}_{\omega}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}))) \otimes \mathbb{R} \to N \otimes \mathbb{R} \to \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{R},$$

and the first term can be identified with $\mathbb{R}$ by assumption.

By the Hodge index theorem of Faltings [14] and Hriljac [18] as stated in [7, Theorem 5.5, (2)], the intersection pairing on $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ is non-degenerate: it has signature $(+,-,-,\ldots)$. Since ω is semipositive and $\int_{\mathcal{X}(\mathbb{C})}\omega \neq 0$, there exists an ample line bundle $\overline{\mathcal{H}}$ in $\widehat{\mathrm{Pic}}_{\omega}(\mathcal{X})$. Then $\overline{\mathcal{H}}.\overline{\mathcal{H}} > 0$, so that the intersection pairing on $\widehat{\mathrm{Pic}}_{\omega}(\mathcal{X})$ is non-degenerate as well.

In particular, if x is a nonzero element of $\overline{N}_{\mathrm{eff}}$, we can find a hermitian line bundle $\overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}_{\omega}(\mathcal{X})$ with $\overline{\mathcal{M}}.x < 0$. If n is a large enough integer, Corollary 2.5 shows that $\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}$ is ample, so that the discussion above guarantees the inequality

$$(\overline{\mathcal{L}}^{\otimes n} \otimes \overline{\mathcal{M}}).x = n\overline{\mathcal{L}}.x + \overline{\mathcal{M}}.x \geq 0.$$

This shows that $\overline{\mathcal{L}}.x$ is positive.

The linear form $x \mapsto \overline{\mathcal{L}}.x$ is positive on the complement of the origin in the closed cone $\overline{N}_{\mathrm{eff}}$. As a consequence, the number of integral points x of $\overline{N}_{\mathrm{eff}}$ with $\overline{\mathcal{L}}.x \leq n$ is bounded above by a quantity of the form $O(n^{\rho})$, where ρ is the rank of N. □

5. Irreducible ample divisors on arithmetic surfaces

5.1. Setup and an easy estimate

In this section, we prove Theorem 1.6 for arithmetic surfaces.

Let $f : \mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ be a projective arithmetic surface, and $\overline{\mathcal{L}}$ an ample line bundle on $\mathcal{X}$. If n is a large enough integer, we want to give an upper bound for the number of sections of $\overline{\mathcal{L}}^{\otimes n}$ that define a divisor which is not irreducible. We will give three different bounds that depend on the geometry and the arithmetic of the irreducible components of that divisor.

In the statement below, $\mathcal{X}$ is not assumed to be regular, but heights are still well-defined, see [39, (1.2)].

Proposition 5.1. – *Let α be a real number with $0 < \alpha < \frac{1}{2}$. If n is an integer, the proportion of those elements s of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that vanish on some Weil divisor D of $\mathcal{X}$ with $h_{\overline{\mathcal{L}}}(D) \leq n^{\alpha}$ goes to zero as n goes to infinity.*

Proof. – Assume that n is large enough. By [25, Theorem B], the number of divisors D on $\mathcal{X}$ with $h_{\overline{\mathcal{L}}}(D) \leq n^{\alpha}$ is bounded above by $e^{Cn^{2\alpha}}$ for some positive constant C. By Theorem 2.21, we can find positive constants C' and η such that for any D as above, the proportion of those elements s of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that vanish on D is bounded above by $C'e^{-n\eta}$.

As a consequence, the proportion of those s that vanish on any D with $h_{\overline{\mathcal{L}}}(\mathcal{X}) \leq n^{\alpha}$ is bounded above by

$$C'e^{Cn^{2\alpha}-n\eta},$$

which goes to zero as n goes to infinity. □

5.2. Degree bounds and reduction modulo p

Let $f : \mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ be as above. We want to investigate irreducible divisors on the fibers of f above closed points and derive global consequences. Our goal here is to prove Proposition 5.6.

Since $\mathcal{X}$ is reduced, we can find a non-empty open subset S of $\operatorname{Spec} \mathbb{Z}$ such that the restriction $f_S : \mathcal{X}_S \to S$ has reduced fibers.

Let r be the number of irreducible components of the geometric generic fiber of f. Up to shrinking S, we may assume that if $\overline{s}$ is any geometric point of S, then the number of irreducible components of $\mathcal{X}_{\overline{s}}$ is exactly r. Since $\mathcal{X}_s$ is reduced by assumption, this is equivalent to the fact that the specialization map induces a bijection between the components of $\mathcal{X}_{\overline{\mathbb{Q}}}$ and those of $\mathcal{X}_{\overline{s}}$.

The degree of $\mathcal{L}_{\mathbb{Q}}$ equals rd, where d is the degree of the restriction of $\mathcal{L}$ to a component of $\mathcal{X}_{\overline{\mathbb{Q}}}$. Write $\mathcal{L}_p$ for the restriction of $\mathcal{L}$ to $\mathcal{X}_p$.

If X is a reduced scheme, and C is an irreducible component of X, we will always consider C as a closed subscheme of X, endowed with its reduced structure.

LEMMA 5.2. – *Let C be an integral projective curve over a perfect field, with arithmetic genus $p_a(C)$. Let $\mathcal{L}$ be a line bundle on C. Then*

$$h^0(C,\mathcal{L}) \geq 1 - p_a(C) + \deg(\mathcal{L})$$

and equality holds if the degree of $\mathcal{L}$ is strictly bigger than $p_a(C)$.

Proof. – The first statement follows directly from the Riemann-Roch theorem. To prove the second one, consider the normalization $\pi : \widetilde{C} \to C$ of C. Then $\widetilde{C}$ is smooth over the base field k, and its genus is bounded above by the arithmetic genus $p_a(C)$ of C.

Since C is reduced, it is Cohen-Macaulay, so that the dualizing sheaf $\omega_{C/k}$ of C is Cohen-Macaulay by [32, Tag 0BS2]. In particular, it is torsion-free, so that the morphism $\pi^*\omega_{C/k} \to \omega_{\widetilde{C}/k}$ is injective. Now assume that the degree of $\mathcal{L}$ is strictly bigger than $p_a(C)$. In particular, we have

$$\deg(\pi^*\mathcal{L}) > \deg(\omega_{\widetilde{C}/k})$$

and

$$h^1(C,\mathcal{L}) = h^0(C,\mathcal{L}^\vee \otimes_{\mathcal{O}_C} \omega_{C/k}) \leq h^0(\widetilde{C},\pi^*\mathcal{L}^\vee \otimes_{\mathcal{O}_{\widetilde{C}}} \omega_{\widetilde{C}/k}) = 0.$$

By Riemann-Roch, we have

$$h^0(C,\mathcal{L}) = \chi(\mathcal{L}) = 1 - p_a(C) + \deg(\mathcal{L}). \qquad \square$$

LEMMA 5.3. – *Let p be a prime number corresponding to a point in S, and let $\overline{\mathbb{F}_p}$ be an algebraic closure of $\mathbb{F}_p$. If C is an irreducible component of $\mathcal{X}_p$, let r_C be the number of irreducible components of $C_{\overline{\mathbb{F}_p}}$, and if k is a positive integer, let $N_k(C)$ be the number of irreducible divisors of degree k on C. Then the following holds as k tends to ∞:*

$$\left|N_{r_C k}(C) - \frac{1}{k}p^{r_C k}\right| = O(p^{\frac{r_C k}{2}}),$$

where the implied constants only depend on $f_S : \mathcal{X}_S \to S$.

Proof. – The r_C irreducible components of $C_{\overline{\mathbb{F}_p}}$ are all defined over $\mathbb{F}_{p^{r_C}}$, and they form a single orbit under Galois. Denote them by $C_1, \ldots, C_{r_C}$. The Lang-Weil estimates of [22] give us the inequality, for any positive integer k:

$$\left||C_1(\mathbb{F}_{p^{r_C k}})| - p^{r_C k}\right| = O(p^{\frac{r_C k}{2}}),$$

where the implied constants only depend on the degree of an embedding of C_1 into some projective space—in particular, it only depends on f_S. As a consequence, if M_k is the number of elements in $C_1(\mathbb{F}_{p^{r_C k}})$ with residue field exactly $\mathbb{F}_{p^{r_C k}}$, we have:

$$\left|M_k - p^{r_C k}\right| \leq \textstyle\sum_{i|k, i\neq k} p^{r_C i} + O(\sum_{i|k} p^{\frac{r_C i}{2}}) = O(kp^{\frac{r_C k}{2}}).$$

Now assume that $r_C k$ is strictly larger than the degree of the residue field of any singular point of C—this degree can be bounded independently of C as f_S is generically smooth. Irreducible divisors of degree $r_C k$ on C are in one-to-one correspondence irreducible divisors of degree k on $C_1/\mathbb{F}_{p^{r_C}}$, which in turn are in one-to-one correspondence with Galois orbits over $\mathbb{F}_{p^{r_C}}$ of elements of $C_1(\mathbb{F}_{p^{k r_C}})$ with residue field exactly $\mathbb{F}_{p^{r_C k}}$. As a consequence, we have

$$N_k(C) = \frac{1}{k}M_k,$$

which proves the lemma. $\square$

LEMMA 5.4. – *There exists a positive integer N with the following property: for any prime number p corresponding to a point in S, any irreducible component C of $\mathcal{X}_p$, and any $n \geq N$, the restriction map*

$$H^0(\mathcal{X}_p, \mathcal{L}_p^{\otimes n}) \to H^0(C, \mathcal{L}_p^{\otimes n})$$

is surjective.

Proof. – Since the result certainly holds if N is allowed to depend on p by general vanishing results for ample line bundle, we may replace S by any nonempty open subset, which we will do along the proof.

Choose a finite flat map $S' \to S$ such that the irreducible components of the generic fiber of $\mathcal{X}_{S'} \to S$ are geometrically ireducible. In particular, our assumption on s guarantees that the irreducible components of the fiber of $\mathcal{X}_{S'} \to S'$ over any closed point s' are geometrically irreducible, and are the intersection of an irreducible component of $\mathcal{X}_{S'}$ with $\mathcal{X}_{s'}$.

Let s' be a point of S' over p, and let $C_{s'}$ be the union of irreducible components of $\mathcal{X}_{s'}$ corresponding to C. Up to shrinking S, we may assume that $C_{s'}$, as a reduced scheme, is the intersection of $\mathcal{X}_{s'}$ and some union $\mathcal{C}$ of irreducible components of $\mathcal{X}_{S'}$. Let $\mathcal{I}_{\mathcal{C}}$ be the sheaf of ideals on $\mathcal{X}$ defining $\mathcal{C}$. Then the sheaf of ideals defining $C_{s'}$ is $\mathcal{I}_{\mathcal{C}} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{O}_{\mathcal{X}_{s'}}$. Note that there are only finitely many possibilities for $\mathcal{C}$.

Let k be a positive integer such that $\mathcal{L}^{\otimes k}$ has a nonzero section. Up to shrinking S, we may assume that this section does not vanish along any component of a fiber of $\mathcal{X}_{S'} \to S'$. Consider the map

$$\pi : \mathcal{X}_{S'} \to S'$$

If n is large enough and since $\mathcal{L}$ is relatively ample, relative vanishing guarantees that the coherent sheaf on S

$$R^1\pi_*(\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{\mathcal{C}})$$

is zero. Pick a positive N once and for all such that the vanishing above holds for $n = N, \ldots, N + k - 1$. Then after shrinking S once again, we may assume that the vanishing above implies

$$H^1(\mathcal{X}_{s'}, \mathcal{L}^{\otimes N+i} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'}) = 0$$

for $i = 0, \ldots, k - 1$.

Now since $\mathcal{L}^{\otimes k}$ has a nonzero section over $\mathcal{X}_{s'}$, we have an exact sequence, for any integer n,

$$0 \to \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'} \to \mathcal{L}^{\otimes n+k} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'} \to \mathcal{K} \to 0,$$

where $\mathcal{K}$ is a coherent sheaf supported on a zero-dimensional subscheme of $\mathcal{X}_{s'}$. In particular, the map

$$H^1(\mathcal{X}_{s'}, \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'}) \to H^1(\mathcal{X}_{s'}, \mathcal{L}^{\otimes n+k} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'})$$

is onto and the right-hand term vanishes as soon as the left-hand one does. Finally, we have found N, independent of C and p, such that for all $n \geq N$, we have

$$H^1(\mathcal{X}_{s'}, \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_{\mathcal{X}_{S'}}} \mathcal{I}_{C'}) = 0,$$

which implies that the map

$$H^0(\mathcal{X}_p, \mathcal{L}_p^{\otimes n}) \to H^0(C, \mathcal{L}_p^{\otimes n})$$

is surjective. □

Proposition 5.5. – *Let p be a prime number corresponding to a point in S, and let $\mathcal{L}_p$ be the restriction of $\mathcal{L}$ to $\mathcal{X}_p$. Let C be an irreducible component of $\mathcal{X}_p$, and let r_C be the number of irreducible components of $C_{\overline{\mathbb{F}_p}}$.*

Let β be a real number with $0 < \beta < 1$. There exist positive constants A and B, depending only on β and $\mathcal{X}_S \to S$ but not on p, such that for any $n \geq A$, the proportion of those sections s of $H^0(\mathcal{X}_p, \mathcal{L}_p^{\otimes n})$ that do not vanish identically on C and such that div(s) *has an irreducible component of degree at least $r_C(nd - n^\beta)$ lying on C is at least $Bn^{\beta-1}$.*

Proof. – Our assumption on p guarantees that C is reduced. The degree of $\mathcal{L}_p$ on C equals $r_C d$. Let n be a large enough positive integer. Let k be an integer such that $nr_C d \geq r_C k$. Let D be an irreducible divisor of degree $r_C k$ on C. Then the number of sections of $\mathcal{L}_p^{\otimes n}$ over C that vanish on D is equal to the number of sections of $\mathcal{L}_p^{\otimes n}(-D)$ over C, which, according to Lemma 5.2, is bounded below by

$$p^{1-p_a(C)+nr_C d - r_C k}.$$

Assume that $r_C k > \frac{1}{2} n r_C d$. Then a nonzero section of $\mathcal{L}_p^{\otimes n}$ over C vanishes on at most one irreducible divisor of degree $r_C k$. Applying Lemma 5.3, it follows that the number of nonzero sections of $\mathcal{L}_p^{\otimes n}$ over C that vanish on some irreducible divisor of degree $r_C k$ is bounded below by

$$\frac{1}{k} p^{1-p_a(C)+nr_C d}(1 - O(p^{-\frac{1}{2} r_C k})) - \frac{1}{k} p^{r_C k},$$

the last term taking care of the zero section being counted multiple times.

Assume now that

$$r_C k \leq n r_C d - p_a(C).$$

Then the term above is bounded below by

$$\frac{1}{2k} p^{1-p_a(C)+nr_C d}$$

for large enough n.

Summing over all those k such that $r_C k \geq n r_C d - r_C n^\beta$, we find that the number of those elements s of $H^0(\mathcal{X}_p, \mathcal{L}_p^{\otimes n})$ such that div(s) has an irreducible component of degree at least $nr_C d - n^\beta$ is at least

$$n^\beta \frac{1}{2nd} p^{1-p_a(C)+nr_C d}(1 + o(1)).$$

as n goes to infinity, the implied constants depending only on β, $p_a(C)$ and the ones occurring in Lemma 5.3. Since $p_a(C)$ is the genus of some reunion of irreducible components of the geometric generic fiber of $\mathcal{X}$, the implied constants only depend on β and $\mathcal{X}$.

By Lemma 5.2, if $nr_C d > p_a(C)$, we have

$$h^0(C, \mathcal{L}_p^{\otimes n}) = 1 - p_a(C) + nr_C d.$$

This shows that the proportion of those sections s of $\mathcal{L}_p^{\otimes n}$ over C such that div(s) has an irreducible component of degree at least $nr_C d - r_C n^\beta$ is at least $Bn^{\beta-1}$ for some constant B as in the statement of the proposition.

By Lemma 5.4, after choosing n large enough, this implies the desired statement. □

We can now prove the main result of 5.2.

PROPOSITION 5.6. – *In the situation of 5.1, let β be a real number with $0 < \beta < 1$. Then the proportion of those elements σ of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\operatorname{div}(\sigma)_{\mathbb{Q}}$ has an irreducible component on $\mathcal{X}_{\mathbb{Q}}$ of degree at least $n \deg \mathcal{L}_{\mathbb{Q}} - rn^{\beta}$ goes to 1 as n goes to infinity.*

Proof. – Let γ be a real number with $1 - \beta < \gamma < 1$. Let n be a large enough integer. Letting t be the largest integer smaller than n^{γ}, let $p_1, \ldots, p_t$ be the t smallest primes corresponding to points of S, and let N be their product. By the prime number theorem, we have $p_i \sim i \log i$ as $t \to \infty$, so that $p_i \leq 2i \log i$ for large enough i, and, when t is large enough:

$$N \leq (2t \log t)^t = O(e^{n^{\gamma'}}), \tag{5.1}$$

where γ' is any real number with $\gamma < \gamma' < 1$.

Write Λ_n for $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ and let $\mathcal{X}_N \to \operatorname{Spec} \mathbb{Z}/N\mathbb{Z}$ be the reduction of $\mathcal{X}$ modulo N. The exact sequence defining $\mathcal{X}_N$ is

$$0 \to N\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}_N} \to 0,$$

hence the exact sequence

$$0 \to \Lambda_n/N\Lambda_n \to H^0(\mathcal{X}_N, \mathcal{L}^{\otimes n}) \to H^1(\mathcal{X}, \mathcal{L}^{\otimes n})[N] \to 0.$$

If n is large enough, then $H^1(\mathcal{X}, \mathcal{L}^{\otimes n}) = 0$ and we have

$$H^0(\mathcal{X}_N, \mathcal{L}^{\otimes n}) = \Lambda_n/N\Lambda_n. \tag{5.2}$$

The scheme $\mathcal{X}_N$ is the disjoint union of the $\mathcal{X}_{p_i}$, $1 \leq i \leq t$. As a consequence, we have

$$H^0(\mathcal{X}_N, \mathcal{L}^{\otimes n}) = \prod_{1 \leq i \leq t} H^0(\mathcal{X}_{p_i}, \mathcal{L}^{\otimes n}).$$

Given a prime number p that corresponds to a point of S, let E_p be the subset of $H^0(\mathcal{X}_p, \mathcal{L}^{\otimes n})$ described by Proposition 5.5: E_p is the set of sections s of $H^0(\mathcal{X}_p, \mathcal{L}^{\otimes n})$ such that there exists an irreducible component C of $\mathcal{X}_p$, such that $C_{\overline{\mathbb{F}_p}}$ has r_C irreducible components, the restriction of s to C is not identically zero and vanishes along an irreducible divisor D_p of degree at least $r_C(nd - n^{\beta})$.

By Proposition 5.5, if n is greater than A, the proportion of those elements s of $H^0(\mathcal{X}_N, \mathcal{L}^{\otimes n})$ such that s does not project to E_{p_i} for any $i \in \{1, \ldots, t\}$ is bounded above by

$$(1 - Bn^{\beta-1})^t \geq (1 - Bn^{\beta-1})^{n^{\gamma}} = \exp(-Bn^{\gamma+\beta-1} + o(n^{\gamma+\beta-1})) = o(1)$$

since $\gamma + \beta - 1 > 0$, so that as n goes to infinity, the proportion of those elements of $H^0(\mathcal{X}_N, \mathcal{L}^{\otimes n})$ that project to at least one of the E_{p_i} goes to 1.

By Proposition 2.15 which we may apply thanks to (5.1), and by (5.2), the proportion of those elements of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that restrict to E_{p_i} for some $i \in \{1, \ldots, t\}$ goes to 1 as n goes to infinity. We claim that these elements satisfy the condition of the proposition we are proving.

Let p be a prime number that corresponds to a point of S. Let σ be a section of $\mathcal{L}^{\otimes n}$ over $\mathcal{X}$ such that the restriction of σ to $\mathcal{X}_p$ belongs to E_p. Let C be a component of $\mathcal{X}_p$ such that $C_{\overline{\mathbb{F}_p}}$ has r_C irreducible components, and let D_p be an irreducible divisor of degree at least $r_C(nd - n^{\beta})$ on C such that σ vanishes on D_p.

We can find an irreducible component D of $\operatorname{div}(\sigma)$ with $D_{|\mathcal{X}_p} = D_p$. If n is large enough, we can assume that no component of $(D_p)_{\overline{\mathbb{F}_p}}$ lies on two distinct irreducible components of $C_{\overline{\mathbb{F}_p}}$—it is enough to require that n is large enough compared to the degree of the residue fields of the intersection points of any two components of $C_{\overline{\mathbb{F}_p}}$. As a consequence, the degree of the restriction of D_p to any of the r_C components of $C_{\overline{\mathbb{F}_p}}$ is at least $nd - n^\beta$. Since $\mathcal{X}_{\mathbb{Q}}$ is irreducible, the degree of the restriction of $D_{\mathbb{Q}}$ to any component of $\mathcal{X}_{\overline{\mathbb{Q}}}$ is at least $nd - n^\beta$ as well, so that the degree of $D_{\mathbb{Q}}$ is at least

$$r(nd - n^\beta) = n \deg \mathcal{L}_{\mathbb{Q}} - rn^\beta.$$

This is what we needed to prove. □

5.3. End of the proof

We can finish the proof of Theorem 1.6 in the case where $\mathcal{X}$ is an arithmetic surface. We will state this intermediate result in Proposition 5.10 below. The strategy follows roughly the outline of the proof of [11, Proposition 4.1] which deals with the corresponding result over finite fields.

Let $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a resolution of singularities of $\mathcal{X}$. Recall that we denoted by r the number of irreducible components of $\mathcal{X}_{\mathbb{C}}$. Then the complex curve $\widetilde{\mathcal{X}}_{\mathbb{C}}$ is the disjoint union of r smooth, connected components.

Define $\overline{\mathcal{B}} = \pi^*\overline{\mathcal{L}}$. Let ω be the first Chern class of $\overline{\mathcal{B}}$. Then ω is semipositive. We say that a hermitian line bundle on $\widetilde{\mathcal{X}}$ is admissible if it is ω-admissible, and we write $\widehat{\operatorname{Pic}}_\omega(\widetilde{\mathcal{X}})$ for the group of isomorphism classes of ω-admissible hermitian line bundles on $\widetilde{\mathcal{X}}$.

We have an exact sequence

$$0 \to \mathbb{R} \to \widehat{\operatorname{Pic}}_\omega(\widetilde{\mathcal{X}}) \to \operatorname{Pic}(\widetilde{\mathcal{X}}) \to 0.$$

We fix once and for all a subgroup N of $\widehat{\operatorname{Pic}}_\omega(\widetilde{\mathcal{X}})$ such that the following conditions hold:

(i) N is a group of finite type;

(ii) N surjects onto $\operatorname{Pic}(\widetilde{\mathcal{X}})$ and contains the class of $\overline{\mathcal{B}}$;

(iii) $N \cap \operatorname{Ker}(\widehat{\operatorname{Pic}}_\omega(\widetilde{\mathcal{X}}) \to \operatorname{Pic}(\widetilde{\mathcal{X}}))$ has rank 1.

Such a group N certainly exists since $\operatorname{Pic}(\widetilde{\mathcal{X}})$ is a group of finite type. Note that these conditions mean that N is a discrete cocompact subgroup of $\widehat{\operatorname{Pic}}_\omega(\widetilde{\mathcal{X}})$. In particular, there exists a positive constant C such that for any admissible hermitian line bundle $\overline{\mathcal{M}} = (\mathcal{M}, ||.||)$ on $\widetilde{\mathcal{X}}$, there exists a hermitian metric $||.||'$ on $\mathcal{M}$ such that $(\mathcal{M}, ||.||')$ belongs to N and the norms $||.||$ and $||.||'$ satisfy the inequality

$$C^{-1}||.|| \leq ||.||' \leq C||.||. \tag{5.3}$$

The following result is classical in the geometric setting: big divisors are (rationally) the sum of ample divisors and effective divisors.

LEMMA 5.7. – *The hermitian line bundle $\overline{\mathcal{B}}$ satisfies the conditions of Proposition 4.7. Furthermore, there exists a positive integer k, and line bundles $\overline{\mathcal{A}}$ and $\overline{\mathcal{E}}$ on $\widetilde{\mathcal{X}}$ which are ample and effective respectively, such that*

$$\overline{\mathcal{B}}^{\otimes k} \simeq \overline{\mathcal{A}} \otimes \overline{\mathcal{E}}.$$

Proof. – Since $\overline{\mathcal{L}}$ is ample, some power of $\overline{\mathcal{L}}$ is effective, and so is the same power of $\overline{\mathcal{B}}$. We also have $\overline{\mathcal{B}}.\overline{\mathcal{B}} = \overline{\mathcal{L}}.\overline{\mathcal{L}} > 0$. Finally, let $\overline{\mathcal{M}}$ be an effective line bundle on $\widetilde{\mathcal{X}}$, let s be a nonzero effective section of $\overline{\mathcal{M}}$, and let D be the divisor of s. Then

$$\overline{\mathcal{B}}.\overline{\mathcal{M}} = h_{\overline{\mathcal{B}}}(D) - \int_{\widetilde{\mathcal{X}}(\mathbb{C})} \log ||s_{\mathbb{C}}|| \pi^* c_1(\overline{\mathcal{L}}).$$

Considering an effective section of some power of $\overline{\mathcal{B}}$ that does not vanish along any component of D—which exists since large powers of $\overline{\mathcal{L}}$ are generated by their effective sections—we see that first term is nonnegative. The second one is nonnegative as well since $c_1(\overline{\mathcal{L}})$ is semipositive on $\mathcal{X}$. This shows the first statement of the proposition.

Let $\overline{\mathcal{A}}$ be an effective ample line bundle on $\widetilde{\mathcal{X}}$, let σ be a section of $\overline{\mathcal{A}}$ and let H be the divisor of σ. Let H' be the schematic image $\pi(H)$. Since $\mathcal{L}$ is ample, we can find an integer k_1 and a nonzero section s_1 of $\mathcal{L}^{\otimes k_1}$ that vanishes on H'. We can write

$$\pi^* s_1 = \sigma\sigma_1,$$

where σ_1 is a section of $\mathcal{B}^{\otimes k_1} \otimes \mathcal{A}^{\otimes -1}$. Choose a large enough integer k_2, and let s_2 be a nonzero section of $\mathcal{L}^{\otimes k_2}$ with small enough norm. Writing $\sigma_2 = \pi^* s_2$, we have

$$\pi^*(s_1 s_2) = \sigma\sigma_1\sigma_2,$$

and $\sigma_1\sigma_2$ is an effective section of the hermitian line bundle $\overline{\mathcal{B}}^{\otimes(k_1+k_2)} \otimes \overline{\mathcal{A}}^{\otimes -1}$, which proves the result. □

Let α and β be real numbers with $0 < \beta < \alpha < \frac{1}{2}$. If n is a positive integer, let H'_n be the subset of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ consisting of those effective sections σ of $\overline{\mathcal{L}}^{\otimes n}$ such that:

(i) σ does not vanish on any Weil divisor D of $\mathcal{X}$ with $h_{\overline{\mathcal{L}}}(D) \leq n^{\alpha}$;

(ii) there exists an irreducible component D of $\mathrm{div}(\sigma)$ such that

$$\deg(D_{\mathbb{Q}}) \geq n \deg \mathcal{L}_{\mathbb{Q}} - r n^{\beta}.$$

Use Lemma 5.7 to find a positive integer k with

$$\overline{\mathcal{B}}^{\otimes k} \simeq \overline{\mathcal{A}} \otimes \overline{\mathcal{E}},$$

where $\overline{\mathcal{A}}$ is ample and $\overline{\mathcal{E}}$ is effective.

LEMMA 5.8. – *The set* $\bigcup_{n>0} H'_n$ *has density* 1 *in* $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$.

Proof. – This is a direct consequence of Proposition 5.1 and Proposition 5.6. □

LEMMA 5.9. – *Let* δ, γ *be any real numbers with* $0 < \beta < \gamma < \delta < \alpha$. *Let n be a large enough integer, and let σ be an element of H'_n such that* $\mathrm{div}(\sigma)$ *is not irreducible. Then we can find hermitian line bundles $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ on $\widetilde{\mathcal{X}}$, and sections*

$$\sigma_i \in H^0(\widetilde{\mathcal{X}}, \mathcal{L}_i),$$

$i = 1, 2$, *with the following properties:*

(i) $\overline{\mathcal{L}}_1$ *and* $\overline{\mathcal{L}}_2$ *belong to* N;

(ii) $||\sigma_i||_{\infty} \leq e^{n^{\gamma}}, i = 1, 2$;

(iii) $n^{\delta} \leq \overline{\mathcal{L}}_1.\overline{\mathcal{B}} \leq n\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^{\delta}$;

(iv) $\overline{\mathcal{L}}_1.\overline{\mathcal{A}} \leq kn\,\overline{\mathcal{B}}.\overline{\mathcal{B}}$;

(v) $\overline{\mathcal{L}}_1 \otimes \overline{\mathcal{L}}_2 \simeq \overline{\mathcal{B}}^{\otimes n}$;

(vi) *up to the isomorphism above,* $\sigma_1\sigma_2 = \sigma$.

Proof. – Let D be the divisor of $\pi^*\sigma$. Since the divisor of σ is not irreducible, D is not irreducible either and we can write

$$D = D_1 + D_2,$$

where the D_i are nonzero effective divisors on the regular scheme $\widetilde{\mathcal{X}}$ such that both Weil divisors $\pi_*(D_1)$ and $\pi_*(D_2)$ are nonzero. Since $\mathrm{div}(\sigma)$ has an irreducible component of generic degree bounded below by $n \deg \mathcal{L}_{\mathbb{Q}} - rn^\beta$, and since

$$\deg D_{1,\mathbb{Q}} + \deg D_{2,\mathbb{Q}} = n \deg \mathcal{L}_{\mathbb{Q}}, \tag{5.4}$$

we can assume, up to exchanging D_1 and D_2,

$$n \deg \mathcal{L}_{\mathbb{Q}} - rn^\beta \leq \deg D_{1,\mathbb{Q}} \leq n \deg \mathcal{L}_{\mathbb{Q}}, \tag{5.5}$$

$$0 \leq \deg D_{2,\mathbb{Q}} \leq rn^\beta. \tag{5.6}$$

We can also assume that no component of D_1 is contracted by the morphism $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$— simply by replacing D_2 by the sum of D_2 and all those contracted components of D_1, which are all supported above closed points of $\mathrm{Spec}\,\mathbb{Z}$. Let $\mathcal{L}_i$ be the line bundle $\mathcal{O}_{\widetilde{\mathcal{X}}}(D_i)$ for $i = 1, 2$. Then we can identify $\mathcal{L}_1 \otimes \mathcal{L}_2$ with $\mathcal{B}^{\otimes n}$, and we can find sections σ_i of $\mathcal{L}_i$ with $\mathrm{div}(\sigma_i) = D_i$ such that $\sigma = \sigma_1\sigma_2$.

Recall that we defined ω as $c_1(\overline{\mathcal{B}})$. We consider the norms $||.||_0$ with respect to ω. Consider the unique hermitian metric $||.||_{\sigma_2}$ on $\mathcal{L}_2$ which is admissible with respect to ω, scaled so that

$$||\sigma_2||_{\sigma_2,0} = 1.$$

By (5.3), we can find a metric $||.||$ on $\mathcal{L}_2$ such that $\overline{\mathcal{L}}_2 := (\mathcal{L}_2, ||.||)$ belongs to N and

$$C^{-1} \leq ||\sigma_2||_0 \leq C. \tag{5.7}$$

Endow $\mathcal{L}_1$ with the unique hermitian metric such that

$$\overline{\mathcal{B}}^{\otimes n} = \overline{\mathcal{L}}_1 \otimes \overline{\mathcal{L}}_2$$

as hermitian line bundles on $\widetilde{\mathcal{X}}$, where we write $\overline{\mathcal{L}}_1$ for the induced hermitian line bundles. Since $\overline{\mathcal{B}}$ belongs to N by assumption, so do $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$. This makes sure that conditions (i), (v) and (vi) of the lemma are satisfied.

Since $||\sigma||_\infty \leq 1$, we have

$$||\sigma_2||_0 \leq C||\sigma_1|||_0||\sigma_2||_0 = C||\sigma||_0 \leq C. \tag{5.8}$$

The inequalities (5.4), (5.6) imply, via Proposition 4.4 the following estimate, since $||\sigma_2||_0 \geq C^{-1}$ and $||\sigma||_\infty \leq 1$:

$$||\sigma_1||_\infty \leq C^{-1}(nC_2 \deg \mathcal{L}_{\mathbb{Q}})^{rn\beta}$$

for some constant C_2 depending only on $\mathcal{X}$ and $\overline{\mathcal{L}}$. Similarly, (5.6) and Proposition 4.2 give us, for some constant C_2 depending only on $\mathcal{X}$ and $\overline{\mathcal{L}}$:

$$||\sigma_2||_\infty \leq CC_1^{rn^\beta}.$$

For any $\gamma > \beta$, and any n large enough, this ensures that condition (ii) is satisfied.

We now turn to condition (iii). For $i = 1, 2$, choose a nonzero effective section s_i of some power $\overline{\mathcal{L}}^{\otimes \ell}$ of $\overline{\mathcal{L}}$ such that the divisor of $\pi^* s_i$ has no common component with D_i. Computing the height $h_{\overline{\mathcal{B}}^{\otimes \ell}}(D_i)$ using the section $\pi^* s_i$ of $\overline{\mathcal{B}}^{\otimes \ell}$, we get:

$$h_{\overline{\mathcal{B}}^{\otimes \ell}}(D_i) = h_{\overline{\mathcal{L}}^{\otimes \ell}}(\pi_*(D_i)).$$

and

$$h_{\overline{\mathcal{B}}^{\otimes \ell}}(D_i) = \ell\, h_{\overline{\mathcal{L}}^{\otimes \ell}}(\pi_*(D_i)) \geq \ell n^{\alpha}.$$

Write

$$\ell \overline{\mathcal{L}}_i.\overline{\mathcal{B}} = h_{\overline{\mathcal{B}}^{\otimes \ell}}(D_i) - \ell \int_{\mathcal{X}(\mathbb{C})} \log ||\sigma_i|| \omega \geq \ell n^{\alpha} - \ell \int_{\mathcal{X}(\mathbb{C})} \log ||\sigma_i|| \omega$$

and use $\log ||\sigma_i||_{\infty} \leq n^{\gamma}$. We find

$$\overline{\mathcal{L}}_i.\overline{\mathcal{B}} \geq n^{\alpha} - n^{\gamma} \deg \mathcal{L}_{\mathbb{Q}} \geq n^{\delta} \tag{5.9}$$

for any large enough n since $\delta, \gamma < \alpha$. Since

$$\overline{\mathcal{L}}_1.\overline{\mathcal{B}} + \overline{\mathcal{L}}_2.\overline{\mathcal{B}} = n \overline{\mathcal{B}}.\overline{\mathcal{B}},$$

this proves that (iii) holds.

Let us prove condition (iv). Since $\overline{\mathcal{B}}^{\otimes k}$ is isomorphic to $\overline{\mathcal{A}} \otimes \overline{\mathcal{E}}$, we have

$$\overline{\mathcal{L}}_i.\overline{\mathcal{A}} = k \overline{\mathcal{L}}_i.\overline{\mathcal{B}} - \overline{\mathcal{L}}_i.\overline{\mathcal{E}}$$

for $i = 1, 2$, so that

$$\overline{\mathcal{L}}_1.\overline{\mathcal{A}} = k \overline{\mathcal{L}}_1.\overline{\mathcal{B}} - \overline{\mathcal{L}}_1.\overline{\mathcal{E}} = kn \overline{\mathcal{B}}.\overline{\mathcal{B}} - k \overline{\mathcal{L}}_2.\overline{\mathcal{B}} - \overline{\mathcal{L}}_1.\overline{\mathcal{E}}. \tag{5.10}$$

Let τ be a nonzero effective section of $\overline{\mathcal{E}}$, with divisor D_{τ}. Then we have

$$\overline{\mathcal{L}}_1.\overline{\mathcal{E}} = h_{\overline{\mathcal{L}}_1}(D_{\tau}) - \int_{\widetilde{\mathcal{X}}(\mathbb{C})} \log ||\tau|| c_1(\overline{\mathcal{L}}_1).$$

Since the degree of $\overline{\mathcal{L}}_1$ is nonnegative, the form $c_1(\overline{\mathcal{L}}_1)$ is a nonnegative multiple of ω, and since τ is effective, we have

$$-\int_{\widetilde{\mathcal{X}}(\mathbb{C})} \log ||\tau|| c_1(\overline{\mathcal{L}}_1) \geq 0.$$

By assumption, no component of the divisor D_1 of σ_1 is contracted by the resolution π. Furthermore, the definition of the set H'_n guarantees that if C is any component of D_1, then the height of $\pi_*(C)$ with respect to $\overline{\mathcal{L}}$ is bounded below by n^{α}. This implies that if n is large enough, the divisors D_1 and D_{τ} have no component in common, so that

$$h_{\overline{\mathcal{L}}_1}(D_{\tau}) \geq -\deg D_{\tau,\mathbb{Q}} \log ||\sigma_1|| \geq -\deg D_{\tau,\mathbb{Q}} n^{\gamma}$$

and, as a consequence,

$$\overline{\mathcal{L}}_1.\overline{\mathcal{E}} \geq -n^{\gamma} \deg \mathcal{E}_{\mathbb{Q}}. \tag{5.11}$$

Putting the inequalities (5.11) and (5.9) together with (5.10), we obtain

$$\overline{\mathcal{L}}_1.\overline{\mathcal{A}} \leq kn\, \overline{\mathcal{B}}.\overline{\mathcal{B}} + n^{\gamma} \deg \mathcal{E}_{\mathbb{Q}} - k\, n^{\delta}.$$

Since $\overline{\mathcal{L}}$ is ample, $\overline{\mathcal{B}}.\overline{\mathcal{B}} = \overline{\mathcal{L}}.\overline{\mathcal{L}}$ is positive, and since $\gamma < \delta$, this shows that condition (iv) of the lemma is satisfied as soon as n is large enough. □

We can finally prove the key result of this paper via a counting argument.

Proposition 5.10. – *Let $\mathcal{X}$ be an integral projective arithmetic surface, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then the set*

$$\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}),\ \mathrm{div}(\sigma) \text{ is irreducible}\}$$

has density 1.

Proof. – Choose δ and γ with $\beta < \gamma < \delta < \alpha$. Lemma 5.8 shows that the set $\bigcup_{n>0} H'_n$ has density 1 in $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, so that we only have to prove that the set of those σ in $\bigcup_{n>0} H'_n$ with reducible divisor has density 0 in $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$. Let Z_n be this set.

Let n be large enough so that Lemma 5.9 applies. To any σ in Z_n, we can associate hermitian line bundles $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$, together with respective sections σ_1 and σ_2, so that the conditions $(i)-(vi)$ of the lemma hold. Since $\sigma = \sigma_1\sigma_2$, the data of the $\overline{\mathcal{L}}_i$ and σ_i for $i = 1, 2$ determine σ.

We will give an upper bound for the number of elements σ in Z_n by estimating the number of possible $\overline{\mathcal{L}}_i$ and σ_i. In other words, we will count the number of triples $(\overline{\mathcal{L}}_1, \sigma_1, \sigma_2)$, where $\overline{\mathcal{L}}_1$ is a hermitian line bundle on $\widetilde{\mathcal{X}}$, σ_1 is a section of $\mathcal{L}_1$, and, setting $\overline{\mathcal{L}}_2 := \overline{\mathcal{B}}^{\otimes n} \otimes \overline{\mathcal{L}}_1^{\otimes -1}$, σ_2 is a section of $\overline{\mathcal{L}}_2$, so that

(i) $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ belong to N;

(ii) $||\sigma_i|| \leq e^{n^\gamma}, i = 1, 2$;

(iii) $n^\delta \leq \overline{\mathcal{L}}_1.\overline{\mathcal{B}} \leq n\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^\delta$;

(iv) $\overline{\mathcal{L}}_1.\overline{\mathcal{A}} \leq k\, n\, \overline{\mathcal{B}}.\overline{\mathcal{B}}$.

Below, when using the O notations, implied constants only depend on $\widetilde{\mathcal{X}} \to \mathcal{X}, \overline{\mathcal{L}}, \overline{\mathcal{A}}, \alpha, \beta, \delta, \gamma$.

Let $\overline{\mathcal{L}}_1$ be a hermitian line bundle as above, and write $i := \overline{\mathcal{L}}_1.\overline{\mathcal{B}}$, so that

$$n^\delta \leq i \leq n\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^\delta.$$

We want to bound the number of sections of $\mathcal{L}_1$ that have norm at most e^{n^γ}, that is, the number of effective sections of $\overline{\mathcal{L}}_1(n^\gamma)$. First remark that $\deg \mathcal{L}_{1,\mathbb{Q}} \leq n \deg \mathcal{B}_{\mathbb{Q}}$ as the degree of $\mathcal{L}_{1,\mathbb{Q}}$ and $\mathcal{L}_{2,\mathbb{Q}}$ are both nonnegative and have sum $n \deg \mathcal{B}_{\mathbb{Q}}$. Furthermore, we have

$$\overline{\mathcal{L}}_1(n^\gamma).\overline{\mathcal{B}} = i + n^\gamma \overline{\mathcal{O}}_{\widetilde{\mathcal{X}}}(1).\overline{\mathcal{B}} = O(n)$$

since $\gamma < \delta < 1$.

Corollary 4.10 gives us

$$h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_1(n^\gamma)) \leq \frac{(i + Kn^\gamma)^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(n \log n), \tag{5.12}$$

where K is the constant $\overline{\mathcal{O}}_{\widetilde{\mathcal{X}}}(1).\overline{\mathcal{B}}$.

Similarly, we have

$$h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_2(n^\gamma)) \leq \frac{(n\overline{\mathcal{B}}.\overline{\mathcal{B}} - i + Kn^\gamma)^2}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(n \log n). \tag{5.13}$$

Adding (5.12) and (5.13), we find, recalling that $0 < \gamma < 1$:

$$\begin{aligned} h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_1(n^\gamma)) + h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_2(n^\gamma)) &\leq \frac{1}{2} n^2 \overline{\mathcal{B}}.\overline{\mathcal{B}} - \frac{2i(n\overline{\mathcal{B}}.\overline{\mathcal{B}} - i)}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} \\ &\quad + \frac{2K^2 n^{2\gamma} + 2K\overline{\mathcal{B}}.\overline{\mathcal{B}} n^{1+\gamma}}{2\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(n \log n) \\ &\leq \frac{1}{2} n^2 \overline{\mathcal{B}}.\overline{\mathcal{B}} - \frac{i(n\overline{\mathcal{B}}.\overline{\mathcal{B}} - i)}{\overline{\mathcal{B}}.\overline{\mathcal{B}}} + O(n^{1+\gamma}). \end{aligned}$$

Since $n^\delta \leq i \leq n\overline{\mathcal{B}}.\overline{\mathcal{B}} - n^\delta$, we have

$$\frac{i(n\overline{\mathcal{B}}.\overline{\mathcal{B}} - i)}{\overline{\mathcal{B}}.\overline{\mathcal{B}}} \geq n^{1+\delta} - \frac{1}{\overline{\mathcal{B}}.\overline{\mathcal{B}}} n^{2\delta}$$

and, since $2\delta < 1 < 1 + \gamma$,

$$h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_1(n^\gamma)) + h^0_{\mathrm{Ar}}(\widetilde{\mathcal{X}}, \overline{\mathcal{L}}_2(n^\gamma)) \leq \frac{1}{2} n^2 \overline{\mathcal{B}}.\overline{\mathcal{B}} - n^{1+\delta} + O(n^{1+\gamma}).$$

We now count the number of possible $\overline{\mathcal{L}}_1$. Let $t > 0$ be such that $\overline{\mathcal{O}}(t)$ belong to N. Let $k(n)$ be the smallest positive integer such that $k(n)t \geq n^\gamma$. Then the hermitian line bundle $\overline{\mathcal{L}}_1(k(n)t)$ is effective, belongs to N, and we have

$$\overline{\mathcal{L}}_1(k(n)t).\overline{\mathcal{A}} = O(n)$$

since $\gamma < 1$. As a consequence of Proposition 4.13, this shows that the number of possible $\overline{\mathcal{L}}_1$—or equivalently, $\overline{\mathcal{L}}_1(k(n)t)$—appearing in the triples above is bounded by $O(n^\rho)$, where ρ is the rank of N.

The estimates above show that we have the following inequality:

$$\log |Z_n| \leq O(\rho \log n) + \frac{1}{2} n^2 \overline{\mathcal{B}}.\overline{\mathcal{B}} - n^{1+\delta} + O(n^{1+\gamma}) = \frac{1}{2} n^2 \overline{\mathcal{B}}.\overline{\mathcal{B}} - n^{1+\delta} + O(n^{1+\gamma}).$$

However, Theorem 2.11, (iii) shows that we have

$$h^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) \geq \frac{1}{2} n^2 \overline{\mathcal{L}}.\overline{\mathcal{L}} + O(n \log n) = \frac{1}{2} n^2 \overline{\mathcal{B}}.\overline{\mathcal{B}} + O(n \log n).$$

Since $\delta > \gamma$, these two inequalities prove that $\bigcup_{n>0} Z_n$ has density 0 in $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, which proves the proposition. □

6. Proofs of the main results

The goal of this section is to give a proof of Theorem 1.1. We will deduce it from its special case Theorem 1.6

6.1. Proof of Theorem 1.6

We first state the Bertini irreducibility theorem of [11] in the form that we will need.

THEOREM 6.1. – *Let k be a finite field, and let X be a projective variety over k. Let L be an ample line bundle over X. Let Y be an integral scheme of finite type over k, and let $f : Y \to X$ be a morphism which is generically smooth onto its image. Assume that the dimension of the closure of $f(Y)$ is at least 2. Then the set of those $\sigma \in \bigcup_{n>0} H^0(X, \mathcal{L}^{\otimes n})$ such that $\mathrm{div}(f^*\sigma)_{\mathrm{horiz}}$ is an irreducible Cartier divisor has density 1.*

Proof. – This is almost a special case of [11, Theorem 1.6]. There, the result is given when X is a projective space and $L = \mathcal{O}(1)$. This means that—unfortunately—[11] can formally only be applied to the situation where L is very ample. However, the proofs of [11] apply with no change when projective space is replaced by an arbitrary projective scheme with a distinguished ample line bundle.

A second difference between our statement and that of [11, Corollary 1.4] is that we claim that we can require $\mathrm{div}(f^*\sigma)$ to be irreducible as a Cartier divisor: the underlying scheme is irreducible and has no multiple component, whereas the statement in [11] only states irreducibility.

The fact that for a density 1 of σ, the divisor $\mathrm{div}(\sigma)$ has no multiple component follows from arguments in [11]. Indeed, since Y is reduced and k is perfect, there is a dense open subset U of Y that is smooth over k and such that $f_{|_U}$ is smooth onto its image. By [11, Lemma 3.3], for a density 1 of sections σ, all the components of $\mathrm{div}(f^*\sigma)_{\mathrm{horiz}}$ intersect U, and by [11, Lemma 3.5], for a density 1 of σ, the intersection $\mathrm{div}(f^*\sigma) \cap U$ is smooth outside a finite number of points, so that it does not have any multiple component. □

LEMMA 6.2. – *Let $\mathcal{X}$ be a projective arithmetic variety of dimension at least 2, and let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$. Then the set*

$$\left\{\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}), \mathrm{div}(\sigma) \text{ has no vertical component}\right\}$$

has density 1.

Proof. – If $\mathcal{X}$ is an arithmetic surface, the result follows from Proposition 5.10. Let d be the relative dimension of $\mathcal{X}$ over $\mathrm{Spec}\,\mathbb{Z}$, and assume that $d \geq 2$.

Apply Theorem 2.21 where Y runs through the irreducible components of the fibers of $\mathcal{X}$ over closed points of $\mathrm{Spec}\,\mathbb{Z}$. Since these components have dimension d, we find that for any small enough $\varepsilon > 0$, the proportion of these elements σ of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\mathrm{div}(\sigma)$ has a vertical component over some prime p with $p \leq \exp(\varepsilon n^2)$ is bounded above by a quantity of the form

$$O(\exp(\varepsilon n^2 - \eta n^d)) = o(1),$$

as n goes to infinity.

We now show that for most $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, $\mathrm{div}(\sigma)$ does not have any vertical component above a large prime.

Let $C \subset \mathcal{X}$ be a closed arithmetic curve, flat over $\mathrm{Spec}\,\mathbb{Z}$, such that for any large enough prime p, the intersection of C with any irreducible component of the fiber $\mathcal{X}_p$ of $\mathcal{X}$ above p is nonempty. Let n be a positive integer, and let σ be an element of $H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$. If $\mathrm{div}(\sigma)$

does not contain C, and if it has a vertical component above a prime p, then $\mathrm{div}(\sigma)$ and C intersect at a point above p, so that

$$n h_{\overline{\mathcal{L}}}(C) = h_{\overline{\mathcal{L}}^{\otimes n}}(C) \geq \log p.$$

In particular, for such a σ, we have $p \leq \exp(n h_{\overline{\mathcal{L}}}(C))$.

By Theorem 2.21, the proportion of those $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that vanish on C tends to 0 as n tends to infinity. In particular, the proportion of those $\sigma \in H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\mathrm{div}(\sigma)$ has a vertical component above a prime $p > \exp(n h_{\overline{\mathcal{L}}}(C))$ goes to 0 as n goes to infinity.

Together with the above estimate, this shows the result. □

Proof of Theorem 1.6. – If $\mathcal{X}$ is an arithmetic surface, then the result was proved in Proposition 5.10. Assume that $\mathcal{X}$ has dimension at least 3. Let p be a prime number large enough so that $\mathcal{X}_p$ is reduced, and specialization indices a bijection between the irreducible components of $\mathcal{X}_{\overline{\mathbb{Q}}}$ and those of $\mathcal{X}_{\overline{\mathbb{F}_p}}$. Let $\mathcal{X}_{0,p}$ be an irreducible component of $\mathcal{X}_p$, endowed with the reduced structure.

Let n be a positive integer, and let σ be a global section of $\mathcal{L}^{\otimes n}$. If D is a horizontal component of $\mathrm{div}(\sigma)$, then D intersects all components of $\mathcal{X}_{\overline{\mathbb{Q}}}$, so that D intersects $\mathcal{X}_{0,p}$. This shows that for any section σ of $\mathcal{L}^{\otimes n}$, if $\mathrm{div}(\sigma|_{\mathcal{X}_{0,p}})$ is irreducible as a Weil divisor, then $\mathrm{div}(\sigma)$ has a single component that is flat over $\mathbb{Z}$.

Now we have the following results:

(i) the density of those $\sigma_p \in \bigcup_{n>0} H^0(\mathcal{X}_{0,p}, \mathcal{L}^{\otimes n})$ such that $\mathrm{div}(\sigma_p)$ is an irreducible Cartier divisor is 1;

(ii) the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\mathrm{div}(\sigma)$ does not have a vertical component is 1.

Indeed, (i) follows from Theorem 6.1 with $X = Y$, and (ii) is Lemma 6.2. By the discussion above, if σ satisfies (i) and (ii), then $\mathrm{div}(\sigma)$ is irreducible. Finally, Corollary 2.18 shows that the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that the restriction of σ to $\mathcal{X}_{0,p}$ satisfies (i) is 1. This proves the result. □

6.2. Proof of Theorem 1.1

In this section, we deduce Theorem 1.4 from Theorem 1.6, following the arguments of [11, Section 5]. We then prove Theorem 1.1 as a consequence.

In the following, fix a projective arithmetic variety $\mathcal{X}$, together with an ample hermitian line bundle $\overline{\mathcal{L}}$.

LEMMA 6.3. – *Let $\mathcal{Y}$ be an irreducible scheme of finite type over* Spec $\mathbb{Z}$*, together with a morphism $f : \mathcal{Y} \to \mathcal{X}$. Let U be an open dense subscheme of $\mathcal{Y}$. Then for all σ in a density* 1 *subset of $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, we have the equivalence*

$$\mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible} \Leftrightarrow (\mathrm{div}(f^*\sigma) \cap U)_{\mathrm{horiz}} \text{ is irreducible.}$$

Proof. – This is analogous to [11, Lemma 3.3]. The implication

$$\operatorname{div}(f^*\sigma)_{\text{horiz}} \text{ is irreducible} \implies (\operatorname{div}(f^*\sigma) \cap U)_{\text{horiz}} \text{ is irreducible}$$

always holds. We prove the reverse implication.

Let D be an irreducible component of $\mathcal{Y} \setminus U$ whose image under f is positive-dimensional—meaning by definition that D is a component of $(\mathcal{Y} \setminus U)_{\text{horiz}}$. By Theorem 2.21, the density of those $\sigma \in \bigcup_{n>0} H^0_{\text{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that vanish identically on $f(D)$ is zero.

Now assume that σ does not vanish identically along any component of $(\mathcal{Y} \setminus U)_{\text{horiz}}$—this is a condition satisfied by a density 1 set of sections by the paragraph above. Then any horizontal component of $\operatorname{div}(f^*\sigma)_{\text{horiz}}$ meets U, which implies that the Zariski closure of $(\operatorname{div}(f^*\sigma) \cap U)_{\text{horiz}}$ is $\operatorname{div}(f^*\sigma)_{\text{horiz}}$.

In particular, for those σ, the implication

$$(\operatorname{div}(f^*\sigma) \cap U)_{\text{horiz}} \text{ is irreducible} \implies \operatorname{div}(f^*\sigma)_{\text{horiz}} \text{ is irreducible}$$

holds. □

Lemma 6.4. – *Let $\mathcal{Y}$ and $\mathcal{Z}$ be two irreducible schemes that are flat, of finite type over* $\operatorname{Spec} \mathbb{Z}$. *Let*

$$\pi : \mathcal{Y} \to \mathcal{Z}$$

be a finite étale morphism, and let

$$\psi : \mathcal{Z} \to \mathcal{X}$$

be a morphism that has relative dimension s at all points of $\mathcal{Z}$. Assume that the dimension of the closure of $\psi(\mathcal{Z})$ in $\mathcal{X}$ is at least 2. *Then for all σ in a density* 1 *subset of* $\bigcup_{n>0} H^0_{\text{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, *we have the implication*

$$\operatorname{div}(\psi^*\sigma) \text{ is irreducible} \implies \operatorname{div}(\pi^*\psi^*\sigma) \text{ is irreducible.}$$

Proof. – We follow the argument of [11, Lemma 5.1]. Irreducibility is more difficult to achieve if we replace $\mathcal{Y}$ by a finite cover. As a consequence, we may assume that π is a Galois étale cover. Let G be the corresponding Galois group. Let m be the dimension of $\overline{\psi(\mathcal{Z})}$.

If z is a closed point of $\mathcal{Z}$, let $|z|$ be the cardinality of the residue field of z and let F_z denote the conjugacy class in G associated to the Frobenius. We claim that for a density 1 set of σ, the conjugacy classes F_z cover all conjugacy classes of G as z runs through the closed points of $\operatorname{div}(\psi^*\sigma)$.

Indeed, let C be such a conjugacy class. Let U be a normal, dense affine open subset of $\mathcal{Z}$. By the Chebotarev density theorem of [31, Theorem 9.11] applied to $\pi^{-1}(U) \to U$, the number of closed points z of U with $|z| \leq t$ and $F_z = C$ is equivalent to

$$\frac{|C|}{|G|} \frac{t^{s+m}}{(s+m)\log t}$$

as t tends to ∞. Let $E_{C,t}$ be the set of those z.

By the Lang-Weil estimates, since the fibers of ψ have all dimension s, the number of points z with $|z| \leq t$ in a given fiber of ψ above a closed point is bounded above by a quantity of the form

$$\alpha t^s,$$

for some positive α, so that $|\psi(E_{C,t})|$ is bounded below by a quantity of the form

$$\beta\frac{t^m}{\log t}$$

for some positive β. Note that if $x \in \psi(E_{C,t})$, then $|x| \leq t$.

Fix t large enough. Theorem 2.17 shows that the density of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ that do not vanish on any element of $\psi(E_{C,t})$ is equal to

$$\Pi_{x\in\psi(E_{C,t})}(1-|x|^{-1}) \leq (1-t^{-1})^{\beta t^m/\log t} = \exp\Big(-\beta\frac{t^{m-1}}{\log t}(1+o(1))\Big),$$

which tends to zero as t tends to ∞ since $m \geq 2$. As a consequence, the density of those σ such that $\psi^*\sigma$ vanishes at a closed point z with $F_z = C$ is 1, which proves the claim.

Now let $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$ such that $\mathrm{div}(\psi^*\sigma)$ is irreducible and contains closed points z such that the F_z cover all conjugacy classes of G. Then $\pi^{-1}\mathrm{div}(\psi^*\sigma) = \mathrm{div}(\pi^*\psi^*\sigma)$ is irreducible. This proves the lemma. □

Proof of Theorem 1.4. – We follow the argument of [11, Lemma 5.2]. By Lemma 6.3, we can replace $\mathcal{Y}$ by any dense open subscheme. As a consequence, we can assume that f factors as

$$\mathcal{Y} \xrightarrow{\pi} \mathcal{Z} \xrightarrow{\psi} \mathcal{X},$$

where π is finite étale, $\mathcal{Z}$ is an open subset of some affine space $\mathbb{A}^s_{\mathcal{X}}$ and ψ is the projection onto $\mathcal{X}$—indeed, the function field of $\mathcal{Y}$ is a finite separable extension of a purely transcendental extension of the function field of $\mathcal{X}$.

By Lemma 6.3 and Lemma 6.4, for σ in a density 1 subset of $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, the implication

$$\mathrm{div}(\sigma) \text{ is irreducible } \Longrightarrow \mathrm{div}(f^*\sigma)_{\mathrm{horiz}} \text{ is irreducible}$$

holds. By Theorem 1.6, the divisor $\mathrm{div}(\sigma)$ is irreducible for σ in a density 1 subset of $\bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})$, which proves the result. □

Proof of Theorem 1.1. – We first assume that $\mathcal{Y}$ is not flat over $\operatorname{Spec}\mathbb{Z}$. Then $f : \mathcal{Y} \to \mathcal{X}$ factors as

$$\mathcal{Y} \xrightarrow{f_p} \mathcal{X}_p \longrightarrow \mathcal{X}$$

for some prime number p. By Theorem 6.1, the density of those $s \in \bigcup_{n>0} H^0(\mathcal{X}_p, \mathcal{L}^{\otimes n})$ such that $\mathrm{div}(f_p^*s)_{\mathrm{horiz}}$ is irreducible is equal to 1. Applying Corollary 2.18 to $\overline{\mathcal{L}}(\varepsilon)$ proves the theorem.

We now assume that $\mathcal{Y}$ is flat over $\operatorname{Spec}\mathbb{Z}$. Let $\mathcal{Y}'$ be the Zariski closure of $f(\mathcal{Y})$ in $\mathcal{X}$. Then $\mathcal{Y}'$ is a projective arithmetic variety, and the restriction of $\overline{\mathcal{L}}$ to $\mathcal{Y}'$ is ample by Corollary 2.7. Furthermore, the map $f_{\mathcal{Y}'} : \mathcal{Y} \to \mathcal{Y}'$ is dominant by assumption. Theorem 1.4 guarantees that the density of the set E consisting of those $\sigma \in \bigcup_{n>0} H^0_{\mathrm{Ar}}(\mathcal{Y}', \overline{\mathcal{L}}^{\otimes n})$ such that $\mathrm{div}(f^*_{\mathcal{Y}'}\sigma)_{\mathrm{horiz}}$ is irreducible is equal to 1. Applying Corollary 2.18 to $\overline{\mathcal{L}}(\varepsilon)$ proves the theorem. □

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FINITENESS OF SUPERELLIPTIC CURVES WITH CM JACOBIANS

BY KE CHEN, XIN LU AND KANG ZUO

ABSTRACT. – This paper proves the Coleman conjecture for superelliptic curves: there are, up to isomorphism, at most finitely many superelliptic curves whose Jacobians are CM abelian varieties, as long as these curves are of genus at least 8. Here superelliptic curves are smooth projective curves over $\mathbb{C}$ admitting affine equations of the form $y^n = \zeta(x)$ with ζ a separable polynomial. The proof is reduced to the geometry of superelliptic Torelli locus $\mathcal{TS}_g$ in the Siegel modular variety $\mathcal{A}_g$: we establish the generic exclusion from $\mathcal{TS}_g$ of any special subvariety of dimension > 0 in $\mathcal{A}_g$ for $g \geq 8$, and the stability properties of Higgs bundles associated to surface fibrations play a crucial role in our study.

RÉSUMÉ. – Dans ce travail on montre la conjecture de Coleman pour les courbes super-elliptiques: l'ensemble des classes d'isomorphismes des courbes super-elliptiques dont les jacobiennes sont à multiplication complexe comme variétés abéliennes est au plus fini, lorsque ces courbes sont de genre au moins 8. Par courbes super-elliptiques on comprend les courbes projectives lisses sur $\mathbb{C}$ admettant une équation affine sous la forme $y^n = \zeta(x)$ où ζ est un polynôme séparable. La démonstration se réduit à la géométrie du lieu de Torelli super-elliptique $\mathcal{TS}_g$ dans la variété modulaire de Siegel $\mathcal{A}_g$: on montre qu'aucune sous-variété spéciale de dimension > 0 dans $\mathcal{A}_g$ n'est contenue génériquement dans $\mathcal{TS}_g$ pour $g \geq 8$, et un rôle crucial dans nos études est joué par les propriétés de stabilité des fibrés de Higgs associés aux fibrations des surfaces algébriques.

1. Introduction

This paper is dedicated to the Coleman conjecture for the superelliptic curves.

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1.1. Coleman conjecture

We start with the Coleman conjecture in its original form:

CONJECTURE 1.1 (Coleman). – *Up to isomorphism, there are at most finitely many smooth projective curves whose Jacobians are abelian varieties with complex multiplication, as long as these curves are of sufficiently high genus.*

The conjecture was made in the 1980s, cf. [7], and no precise bound on the genus was given. We can reformulate the conjecture in terms of geometry of moduli spaces. Recall that the Torelli morphism in genus $g \geq 2$ is

$$j : \mathcal{M}_g \to \mathcal{A}_g,\ [C] \mapsto [\mathrm{Jac}(C)],$$

where

(1) $\mathcal{A}_g$ is the moduli scheme of principally polarized abelian varieties of dimension g, with suitable level structure to insure the representability of the moduli functor;
(2) $\mathcal{M}_g$ is the moduli scheme of smooth projective curves of genus g, with similar constraints on the level structure induced from (1);
(3) j sends the isomorphism class of a curve $[C]$ to the isomorphism class of its Jacobian variety $[\mathrm{Jac}(C)]$, which is functorial because $\mathrm{Jac}(C)$ is nothing but the neutral component of the Picard scheme Pic°_C, and thus j is well-defined as a morphism between moduli schemes.

In this setting the Coleman conjecture amounts to the finiteness of CM points inside the schematic image $\mathrm{Im} j \subset \mathcal{A}_g$, where by CM points we mean points in $\mathcal{A}_g$ parametrizing abelian varieties with complex multiplication. It is well-known that $\mathrm{Im} j$ is a locally closed subscheme in $\mathcal{A}_g$ of dimension $3g-3$, and it is referred to as the open Torelli locus $\mathcal{T}_g^\circ$. Its closure is the Torelli locus $\mathcal{T}_g$. Since $\mathcal{T}_g^\circ$ is dense in $\mathcal{A}_g$ for $g = 2, 3$, it suffices to study Coleman's conjecture for $g \geq 4$.

The following equivalent form of Conjecture 1.1 is more convenient to work with:

CONJECTURE 1.2 (Coleman-Oort). – *When the integer g is sufficiently large, any special subvariety $S \subset \mathcal{A}_g$ of dimension > 0 is NOT* generically contained *in $\mathcal{T}_g$.*

Here a closed subvariety $S \subset \mathcal{A}_g$ is said to be *generically contained* in $\mathcal{T}_g$ if S is contained in $\mathcal{T}_g$ and that the intersection $S \cap \mathcal{T}_g^\circ$ is Zariski dense inside S; and special subvarieties in $\mathcal{A}_g$ are (geometrically) connected closed subvarieties parametrizing abelian varieties with "additional Hodge symmetry", see section 2 as well as [5] for details. Special subvarieties of dimension zero inside $\mathcal{A}_g$ are the same as CM points, and any special subvariety always contains a Zariski-dense subset of CM points. The equivalence of Conjecture 1.2 and Conjecture 1.1 is an immediate consequence of the following theorem, proved by Tsimerman in [37]:

THEOREM 1.3 (André-Oort conjecture for $\mathcal{A}_g$). – *Let Σ be an infinite subset of CM points in $\mathcal{A}_g$. Then the Zariski closure of Σ equals a finite union of special subvarieties.*

Roughly speaking, a special subvariety in $\mathcal{A}_g$ supports a universal family of abelian varieties with prescribed Hodge classes, and Conjecture 1.2 predicts that such a family cannot be deduced generically from a universal family of Jacobians (supported by a subvariety inside $\mathcal{T}_g^\circ$).

We refer to [26] (and the references therein) for a thorough discussion on the Coleman-Oort conjecture. The rich geometry of Shimura varieties has led to various results confirming the conjecture for many classes of special subvarieties, cf. [16, 19, 5] etc. Also, examples of special subvarieties generically contained in $\mathcal{T}_g$ are known up to genus 7, mainly constructed out of cyclic branched covers of $\mathbb{P}^1$, cf. [25].

1.2. Variant for superelliptic curves

Naturally one may formulate problems of Coleman-Oort type for moduli spaces of curves with additional data.

Definition 1.4. – For an integer $n > 1$, an n-superelliptic curve is a complex smooth projective algebraic curve of genus $g \geq 2$ (or simply superelliptic curve if n is clear from the text) which can be defined by an n-superelliptic equation, i.e., an affine equation of the form

$$y^n = \zeta(x),$$

with ζ a separable polynomial (i.e., admitting no multiple root). When $n = 2$ we obtain the usual notion of hyperelliptic curves. Note that the genus of such a curve can be computed explicitly in terms of n and $\deg\zeta$, cf. (3-4), and that for any fixed $g \geq 2$, there are finitely many possibilities of $(n, \deg\zeta)$ such that the curve defined above is of genus g.

Define $\mathcal{S}_{g,n}$ to be the moduli space of cyclic branched cover $C \to \mathbb{P}^1$ defined by an n-superelliptic equation as above, with C of fixed genus g. We have the evident morphism forgetting the cover

$$\mathcal{S}_{g,n} \to \mathcal{M}_g, \ \ (C \to \mathbb{P}^1) \mapsto C,$$

and we write $\mathcal{TS}_{g,n}^\circ$ for its image inside $\mathcal{A}_g$ under the Torelli morphism, referred to as the n-superelliptic open Torelli locus. Similar to the case of $\mathcal{T}_g^\circ$, it is locally closed in $\mathcal{A}_g$, and its closure $\mathcal{TS}_{g,n} = \overline{\mathcal{TS}_{g,n}^\circ}$ is called the n-superelliptic Torelli locus. Often the integer n is omitted when it is clear from the context. We can now state our main result:

Main Theorem. – *For $g \geq 8$, the superelliptic Torelli locus does not contain generically any special subvariety of $\mathcal{A}_g$ of positive dimension.*

Thanks to Theorem 1.3, our result is equivalent to the finiteness of superelliptic curves with CM Jacobians:

Corollary 1.5. – *For fixed genus $g \geq 8$, there exist, up to isomorphism, at most finitely many smooth superelliptic curves of genus g whose Jacobians are CM abelian varieties.*

Note that our result is sharp due to the counterexample with $g = 7$ given in [25] mentioned above. Precedent to our result, various cases of the superelliptic Coleman-Oort conjecture have been studied by Y. Zarhin in a series of works (see for example [40, 41], and the references therein), with emphasis on the endomorphism algebras of the Jacobians when the Galois group of the cover is the full permutation group S_d or the alternative group A_d. When n is prime to 3, the problem for the n-superelliptic Legendre family

$$y^n = x(x-1)(x-\lambda)$$

has already been considered in [18]. In [25], Moonen has proved that the n-superelliptic Torelli locus $\mathcal{T}S_{g,n}$ itself is not a special subvariety when the genus g is at least 8.

1.3. Strategy of the proof

The proof of the Main Theorem is divided into two main steps: we first reduce the proof to the case of special curves (i.e., special subvarieties of dimension one), and then we exclude the existence of special curves using the stability properties of the logarithmic Higgs bundles associated to the first higher direct image of the constant sheaf.

The reduction to special curves is formulated as follows:

THEOREM 1.6. – *Let g be an integer at least 2. Then the superelliptic Torelli locus $\mathcal{T}S_g$ in $\mathcal{A}_g$ contains generically some special subvariety of positive dimension if and only if it contains generically some special curve.*

In fact one first reduces the above theorem to the statement for simple Shimura varieties of positive dimension, cf. Lemma 2.9; and then the boundary behavior of Baily-Borel compactification implies the dimensional reduction to special curves, using the crucial property that the open n-superelliptic Torelli locus contains no compact (i.e., complete) curves. Note that when $n = 2$, the open hyperelliptic Torelli locus is affine, while the general superelliptic case follows from Theorem 3.9.

Based on the above dimension reduction, the main theorem is thus reduced to:

THEOREM 1.7. – *There does not exist any special curve contained generically in the Torelli locus of superelliptic curves of genus $g \geq 8$.*

The proof of Theorem 1.7 is the most technical part of our paper. The main idea is to study the logarithmic Higgs bundle for the family of semi-stable superelliptic curves associated to such a possible special curve C_0 contained generically in $\mathcal{T}S_{g,n}$, in particular its eigenspace decomposition with respect to the action of the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$. We apply Viehweg-Zuo's characterization for special curves by the maximality of Higgs fields on Higgs eigen-subbundles and the geometrical properties of the family to obtain an irregular horizontal fibration on the total space of this family with some extra properties, and deduce a contradiction by analyzing this new fibration, which establishes Theorem 1.7.

REMARK 1.8. – The hyperelliptic case ($n = 2$) of the main theorem has already been established in our unpublished preprint [6]. However, soon after the announcement in [6], we realized that the same idea should be fruitful for general superelliptic curves, which has thus grown into the present uniform treatment.

The paper is organized as follows. In section 2 we recall some preliminaries on special subvarieties and present the dimensional recurrence so that the main theorem is reduced to the exclusion of special curves. In section 3 we provide some properties about families of superelliptic curves, mainly including the invariants, the slope inequalities and the existence of horizontal fibration structures under certain assumptions. Finally in section 4, we prove Theorem 1.7 along the idea explained above, and hence complete the proof of the main theorem.

Notations

1. Let a and b be two non-zero integers. We write $a \mid b$ if a divides b, i.e., if $b = ac$ for some integer c.
2. If x is a rational number, we denote its integral part and fractional part by $[x]$ and $\{x\}$ respectively; e.g., $\left[\frac{5}{3}\right] = 1$ and $\left\{\frac{5}{3}\right\} = \frac{2}{3}$.
3. For an n-superelliptic curve in Definition 1.4 defined by $y^n = \zeta(x)$, we denote by $\alpha_0 = \deg(\zeta)$ the degree of $\zeta(x)$, and

$$\alpha = \begin{cases} \alpha_0, & \text{if } n \mid \alpha_0; \\ \alpha_0 + 1, & \text{if } n \nmid \alpha_0. \end{cases} \tag{1-1}$$

2. Shimura varieties and dimensional reduction

In this section we recall some basic constructions and properties of Shimura varieties and prove Theorem 1.6 to reduce the main theorem to the exclusion of special curves.

2.1. Shimura varieties

We mainly follow our previous work [5] for the definitions of connected Shimura data and Shimura varieties, which are slightly varied from the definitions in [10] and [24] etc.:

Definition 2.1. – (1) A connected Shimura datum is a triple $(\mathbf{G}, X; X^+)$ where $(\mathbf{G}, X)$ is a pure Shimura datum in the sense of [10], and X^+ is a connected component of X; X^+ is an Hermitian symmetric domain on which $\mathbf{G}(\mathbb{R})^+$ (in fact $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ as well) acts transitively through $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$, and points in X can be identified as homomorphisms $\mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ subject to certain Hodge-theoretic constraints. Here $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m},\mathbb{C}}$ is the Deligne torus, i.e., the $\mathbb{R}$-torus whose $\mathbb{R}$-points form the Lie group $\mathbb{C}^\times$.

A connected Shimura subdatum $(\mathbf{G}', X'; X'^+)$ of $(\mathbf{G}, X; X^+)$ can be obtained using certain $x \in X^+$ whose associated homomorphism $h_x : \mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ factors through $\mathbf{G}'_{\mathbb{R}}$ with $X' = \mathbf{G}'(\mathbb{R})x$ and $X'^+ = \mathbf{G}'(\mathbb{R})^+x$. Note that X' is of dimension zero if and only if $\mathbf{G}'$ is a $\mathbb{Q}$-torus.

(2) A connected Shimura variety associated to $(\mathbf{G}, X; X^+)$ is a quotient of the form $M = \Gamma\backslash X^+$ where Γ is a congruence subgroup in $\mathbf{G}^{\mathrm{der}}(\mathbb{Q})^+$ acting on X^+ through its image in $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q})^+$.

Write $\wp_\Gamma$ for the complex uniformization map

$$X^+ \to M = \Gamma\backslash X^+, \quad x \mapsto \Gamma x.$$

A special subvariety of M is the closed subvariety of the form $M' := \wp_\Gamma(X'^+)$ associated to a connected Shimura subdatum $(\mathbf{G}', X'; X'^+)$.

In particular, when $\mathbf{G}'$ is a $\mathbb{Q}$-torus, we obtain zero-dimensional special subvarieties, which are also called special points in M. They are dense for the Zariski topology.

The adjective "connected" will be omitted unless ambiguity arises. We also require for simplicity that the congruence subgroups involved are torsion-free, so that the Shimura varieties of interest are smooth as algebraic varieties.

EXAMPLE 2.2 (Siegel modular variety). – Let (V, ψ) be the standard symplectic $\mathbb{Q}$-space on $V = \mathbb{Q}^{2g}$. We have the connected Shimura datum $(\mathbf{G}, X; X^+) = (\mathrm{GSp}_{2g}, \mathcal{X}_g^{\pm}; \mathcal{X}_g^+)$ where

— GSp_{2g} is the $\mathbb{Q}$-subgroup of GL_{2g} that preserves the symplectic $\mathbb{Q}$-form up to scalar;

— $\mathcal{X}_g^{\pm}$ is Siegel double half space, identified with the set of polarizations of $(V_{\mathbb{R}}, \psi)$;

— $\mathcal{X}_g^+$ is the connected component of $\mathcal{H}_g^{\pm}$ consisting of positive definite polarizations.

Choose $\Gamma = \Gamma(\ell)$ to be the principal congruence subgroup of level ℓ, with $\ell \geq 3$ an integer. Then $M = \Gamma\backslash\mathcal{X}_g^+$ is the moduli scheme of principally polarized abelian varieties with full level-ℓ structure, and this is the Siegel modular variety $\mathcal{A}_g$ that we refer to in this paper.

Special subvarieties in $\mathcal{A}_g$ are also known as subvarieties of Hodge type: they parametrize abelian varieties with prescribed Hodge classes, cf. [11].

2.2. Special curves in $\mathcal{A}_g$

In this subsection we describe one-dimensional special subvarieties, namely special curves in $\mathcal{A}_g$. Using our terminology, they are given by subdata $(\mathbf{G}, X; X^+)$ of $(\mathrm{GSp}_{2g}, \mathcal{X}_g^{\pm}; \mathcal{X}_g^+)$ such that X^+ is of dimension one, and this forces X^+ to be the Poincaré upper half plane which is the only one-dimensional Hermitian symmetric domain. We are interested in the description of $\mathbf{G}^{\mathrm{der}}$.

It is explained in [10] that $\mathbf{G}^{\mathrm{der}}$ is a product of simple $\mathbb{Q}$-groups of the form $\mathbf{G}' = \mathrm{Res}_{L/\mathbb{Q}}\mathbf{H}$ where L is a totally real number field and $\mathbf{H}$ is an absolutely simple L-group, and $\mathbf{G}'(\mathbb{R})$ is not compact. Note that

$$\mathbf{G}'(\mathbb{R}) = \prod_{\sigma: L \hookrightarrow \mathbb{R}} \mathbf{H}(\mathbb{R}, \sigma),$$

where $\mathbf{H}(\mathbb{R}, \sigma)$ is the Lie group obtained from $\mathbf{H}$ along the real embedding $\sigma : L \hookrightarrow \mathbb{R}$. Since the Hermitian symmetric domain associated to $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$ is the Poincaré upper half plane, it turns out that $\mathbf{G}^{\mathrm{der}} = \mathbf{G}'$ has to be $\mathbb{Q}$-simple, and $\mathbf{H}$ is an L-form of SL_2 or PGL_2, such that only one embedding $\tau : L \hookrightarrow \mathbb{R}$ gives rise to a non-compact factor $\mathbf{H}(\mathbb{R}, \tau)$, with the other factors $\mathbf{H}(\mathbb{R}, \sigma)$ being compact ($\sigma \neq \tau$).

CLAIM 2.3. – *The L-group $\mathbf{H}$ above has to be an L-form of* SL_2 *instead of* PGL_2.

Proof. – The inclusion of Hermitian symmetric subdomain $X^+ \hookrightarrow \mathcal{X}_g^+$ is equivariant with respect to the Lie group homomorphism $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+ \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{R})$ in the sense of [34], in which a complete classification of such embeddings is described in terms of symplectic representation. In our case, $\mathbf{G}^{\mathrm{der}}(\mathbb{R})^+ = \prod_\sigma \mathbf{H}(\mathbb{R}, \sigma)$ is a product of simple Lie groups, with $\mathbf{H}(\mathbb{R}, \tau)$ being non-compact and the others being compact. The compact factors acting on the Poincaré upper half plane X^+ trivially, and the embedding is equivariant with respect to $\mathbf{H}(\mathbb{R}, \tau) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{R})$.

If we have $\mathbf{H}(\mathbb{R},\tau) \simeq \mathrm{PGL}_2(\mathbb{R})$, then we obtain the representation

$$\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathbf{H}(\mathbb{R},\tau) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{R})$$

and the embedding $X^+ \hookrightarrow \mathcal{X}_g^+$ is equivariant with respect to ρ. The classification of Satake ([35] 3.2) shows that ρ is isomorphic to a product of the standard representation of $\mathrm{SL}_2(\mathbb{R})$ on itself, which is faithful and can not factor through $\mathbf{H}(\mathbb{R},\tau) \simeq \mathrm{PGL}_2(\mathbb{R})$. Therefore $\mathbf{H}(\mathbb{R},\tau)$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and $\mathbf{H}$ is an L-form of $\mathrm{SL}_{2,L}$. □

We refer to Section 3 of Chapter 2 in [31] for the description of $\mathbf{H}$ as an L-form of SL_2, which will be useful for the decomposition of Higgs bundles:

(i) $\mathbf{H}$ is an inner form of SL_2, realized as the kernel of $\mathrm{Nrd} : \mathbb{G}_{\mathrm{m}}^{D/L} \to \mathbb{G}_{\mathrm{m},L}$ where D is a quaternion L-algebra, $\mathbb{G}_{\mathrm{m}}^{D/L}$ is the L-group sending an L-algebra R to $(R \otimes_L D)^\times$, and Nrd is the reduced norm homomorphism;

(ii) $\mathbf{H}$ is an outer form of SL_2, which is the derived group of a unitary L-group $\mathbf{U}$:

- (ii-a) either $\mathbf{U}$ is the L-group of automorphisms of an Hermitian form $E^2 \times E^2 \to E$ for an imaginary quadratic extension of L;
- (ii-b) or for some imaginary quadratic extension E over L and some quaternion division E-algebra D there exists an involution $*$ of second hand on D together with a $*$-Hermitian form $D \times D \to E$, of which $\mathbf{U}$ is the L-group of automorphisms commuting with the natural action of D. (Note that (ii-a) can be seen as the case when $D = \mathrm{Mat}_2(E)$.)

It is also known that the special curve associated to $(\mathbf{G}, X; X^+)$ is non-compact (i.e., non-proper) if and only if the $\mathbb{Q}$-rank of $\mathbf{G}^{\mathrm{der}}$ is non-zero; in the case of special curves in $\mathcal{A}_g$ this means $\mathbf{G}^{\mathrm{der}}$ is not isomorphic to SL_2.

2.3. Properties of special subvarieties

In this section we collect a few properties of general special subvarieties in $\mathcal{A}_g$ that will be used in the proof of Theorem 1.6. We start with the theorem of Baily and Borel whose description of boundaries of Shimura varieties is crucial to our reduction.

THEOREM 2.4 (Baily-Borel compactification). – *Let $M = \Gamma\backslash X^+$ be a Shimura variety. Then the following hold:*

(1) *M is a normal quasi-projective algebraic variety over $\mathbb{C}$. It admits a compactification, called the Baily-Borel compactification M^{BB}, which is a projective algebraic variety over $\mathbb{C}$ containing M as a dense open subvariety. Moreover, if $f : M' = \Gamma'\backslash X'^+ \to M = \Gamma\backslash X^+$ is a morphism between Shimura varieties induced from a morphism of Shimura data, then f extends uniquely to their compactifications:*

$$f^{\mathrm{BB}} : M'^{\mathrm{BB}} \to M^{\mathrm{BB}}$$

sending $M'^{\mathrm{BB}} - M'$ into $M^{\mathrm{BB}} - M$.

(2) *The boundary components of M, i.e., irreducible components of $M^{\mathrm{BB}} - M$, are of codimension at least 2, unless $\mathbf{G}^{\mathrm{ad}}$ admits a $\mathbb{Q}$-factor isomorphic to* $\mathrm{PGL}_{2,\mathbb{Q}}$.

In fact from [2] we know that the boundary components are lower dimensional Shimura varieties associated to Levi subgroups $\mathbf{L}$ of proper parabolic $\mathbb{Q}$-subgroups of $\mathbf{G}$. In particular, if $\mathbf{G}$ admits no proper parabolic $\mathbb{Q}$-subgroups, then no boundary component is needed, and the Shimura variety in question is projective itself. Note that these boundary components are NOT viewed as special subvarieties in M associated to $\mathbf{L} \subset \mathbf{G}$, because.the corresponding homomorphisms $\mathbb{S} \to \mathbf{L}$ are NOT induced by the inclusion $\mathbf{L} \subset \mathbf{G}$.

Corollary 2.5. – *Let $M = \Gamma\backslash X^+$ be a Shimura variety defined by $(\mathbf{G}, X; X^+)$ and a congruence subgroup $\Gamma \subset \mathbf{G}^{\mathrm{der}}(\mathbb{R})^+$. Let $M' \subset M$ be a special subvariety defined by some subdatum $(\mathbf{G}', X'; X'^+)$ such that $\mathbf{G}'^{\mathrm{ad}}$ admits no $\mathbb{Q}$-factor isomorphic to $\mathrm{PGL}_{2,\mathbb{Q}}$. Write $\overline{M}'$ for the closure of M' inside M^{BB}, then the irreducible components of $\overline{M}' - M'$ are of codimension at least 2 in $\overline{M}'$.*

Proof. – The closed immersion $M' \hookrightarrow M$ extends to a morphism between their compactifications $M'^{\mathrm{BB}} \to M^{\mathrm{BB}}$, which is generically injective, and the closure $\overline{M}'$ of M' in M^{BB} is also the closure of the image of M'^{BB}. The conclusion is clear because M'^{BB} only differs from M' by finitely many boundary components of codimension at least 2. □

We also need an elementary property of intersections of special subvarieties:

Lemma 2.6. – *Let M' and M'' be special subvarieties of an ambient Shimura variety $M = \Gamma\backslash X^+$ defined by $(\mathbf{G}, X; X^+)$. Then $M' \cap M''$ is a finite union of special subvarieties of M if the intersection is non-empty.*

This is a standard result which can be interpreted in the language of Hodge classes, cf. [11] and [24]. In fact we may fix a faithful representation $\mathbf{G} \hookrightarrow \mathrm{GL}_V$ over $\mathbb{Q}$. Then an Hermitian symmetric subdomain X' of X which arises from some Shimura subdatum can be characterized as those points x in X that correspond to homomorphisms $\mathbb{S} \to \mathbf{G}_{\mathbb{R}}$ such that $\mathbb{S}$ fixes a given collection of $\mathbb{Q}$-tensors (t'_α) on V. Hence such Hermitian symmetric subdomains are stable under finite intersection. Passing to the quotient modulo Γ gives the desired result in M.

Finally we mention the notion of decomposable locus in $\mathcal{A}_g$.

Definition 2.7. – A principally polarized abelian variety A over $\mathbb{C}$ is said to be decomposable if it is isomorphic to a direct product $A = A_1 \times A_2$ with A_1 and A_2 both principally polarized of dimension > 0 whose polarizations induce the polarization of A. We thus get the moduli subscheme $\mathcal{A}_g^{\mathrm{dec}} \subset \mathcal{A}_g$ which parametrizes decomposable principally polarized abelian varieties.

Lemma 2.8. – *$\mathcal{A}_g^{\mathrm{dec}}$ is a finite union of special subvarieties in $\mathcal{A}_g$.*

Proof. – It suffices to notice that if a g-dimensional principally polarized abelian variety A admits a decomposition $A \simeq A_1 \times A_2$ as in Definition 2.7, with $\dim A_1 = m > 0$ and $\dim A_2 = g - m > 0$, where we assume for simplicity $m \leq g - m$, then the point in $\mathcal{A}_g$ corresponding to A lies in the special subvariety $\mathcal{A}_{m,g-m}$ of $\mathcal{A}_g$ which is defined by the subdatum $(\mathrm{GSp}_{2m,2g-2m}, \mathcal{X}_{m,g-m}; \mathcal{X}^+_{m,g-m})$:

— Here $\mathrm{GSp}_{2m,2g-2m}$ is the $\mathbb{Q}$-subgroup of GSp_{2g} consisting of symplectic similitude on $\mathbb{Q}^{2g}$ preserving the direct sum $\mathbb{Q}^{2g} = \mathbb{Q}^{2m} \oplus \mathbb{Q}^{2g-2m}$ into two symplectic $\mathbb{Q}$-subspaces using the evident symplectic basis,

— $\mathcal{X}^+_{m,g-m} = \mathcal{X}^+_m \times \mathcal{X}^+_{g-m}$ is the product of two Siegel upper half spaces, and $\mathcal{X}_{m,g-m}$ is the orbit of $\mathcal{X}_{m,g-m}$ inside $\mathcal{X}_g$ under $\mathrm{GSp}_{2m,2g-2m}(\mathbb{R})$.

Since $\mathrm{GSp}^{\mathrm{der}}_{2m,2g-2m} = \mathrm{Sp}_{2m} \times \mathrm{Sp}_{2g-2m} \subset \mathrm{Sp}_{2g}$, one verifies easily that $\mathcal{X}_{m,g-m}$ consists of only two copies of $\mathcal{X}^+_{m,g-m}$.

The conclusion is thus clear because

$$\mathcal{A}^{\mathrm{dec}}_g = \bigcup_{1\le m\le g/2} \mathcal{A}_{m,g-m}$$

is a finite union of special subvarieties. □

2.4. Proof of Theorem 1.6

In this subsection we prove the reduction of the main results to the exclusion of special curves. We first use properties of Hecke translation to reduce the main theorem to the case where the special subvarieties in question are simple, i.e., defined by Shimura data $(\mathbf{G}, X; X^+)$ such that $\mathbf{G}^{\mathrm{ad}}$ is a simple $\mathbb{Q}$-group. Then we use boundary behaviors of special subvarieties to reduce to generic exclusion of special curves.

We start with the fact that a non-simple Shimura variety contains a non-trivial special subvariety of dimension > 0, the proof of which is reduced to the following lemma on subdata of non-simple Shimura data.

LEMMA 2.9. – *Let $M = \Gamma\backslash X^+$ be a Shimura variety associated to some connected Shimura datum $(\mathbf{G}, X; X^+)$. Assume that $\mathbf{G}^{\mathrm{ad}}$ is NOT simple as a $\mathbb{Q}$-group. Then M contains special subvarieties $M' \subsetneq M$ of dimension > 0.*

Proof. – We first consider the case where $\mathbf{G} = \mathbf{G}^{\mathrm{ad}}$. Take a decomposition $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$ of $\mathbf{G}$ into non-trivial $\mathbb{Q}$-groups of adjoint type $\mathbf{G}_1$ and $\mathbf{G}_2$, then we obtain a decomposition of Shimura data:

$$(\mathbf{G}, X; X^+) \simeq (\mathbf{G}_1, X_1; X_1^+) \times (\mathbf{G}_2, X_2; X_2^+)$$

where points in X_i correspond to the composition $\mathbb{S} \xrightarrow{h_x} \mathbf{G}_{\mathbb{R}} \to \mathbf{G}_{i,\mathbb{R}}$ using h_x coming from $x \in X$. One easily verifies that $(\mathbf{G}_i, X_i; X_i^+)$ $(i = 1, 2)$ are Shimura data. We also assume for simplicity that $\Gamma = \Gamma_1 \times \Gamma_2$ with $\Gamma_i \subset \mathbf{G}_i^{\mathrm{der}}(\mathbb{R})^+$ a congruence subgroup, so that we have $M = M_1 \times M_2$ where $M_i = \Gamma_i\backslash X_i^+$ are Shimura varieties of dimension > 0 because $\mathbf{G}_i$ are non-trivial. Then taking a special point $x_1 \in M_1$ gives us a special subvariety $\{x_1\} \times M_2 \subsetneq M$ which is of dimension > 0.

In general the Shimura variety $M = \Gamma\backslash X^+$ admits a morphism to $M^{\mathrm{ad}} := \Gamma^{\mathrm{ad}}\backslash X^+$, where the later is the Shimura variety associated to $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}}; X^+)$ induced from $(\mathbf{G}, X; X^+)$ such that X^{ad} is the $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$-orbit of X^+, whose connected components are in bijection with $\pi_0(\mathbf{G}^{\mathrm{ad}}(\mathbb{R}))$, and we simply take Γ^{ad} to be the image of Γ in $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$. The map $M \to M^{\mathrm{ad}}$ is an isomorphism, because Γ acts on X^+ exactly through Γ^{ad}. It is immediate that M and M^{ad} share the same collection of special subvarieties. □

Inside a Shimura variety $M = \Gamma\backslash X^+$ defined by $(\mathbf{G}, X; X^+)$ we can talk about Hecke translation associated to elements in $\mathbf{G}(\mathbb{Q})^+$, cf. Definition 2.1.9 in [5]. It is a natural way to produce new special subvarieties from old ones:

DEFINITION 2.10 (Hecke translate and Hecke orbit). – Let $M = \Gamma\backslash X^+$ be a Shimura variety associated to $(\mathbf{G}, X; X^+)$ and Γ. Let $M' = \wp_\Gamma(X'^+) \subset M$ be the special subvariety associated to the subdatum $(\mathbf{G}', X'; X'^+)$. Then for any $q \in \mathbf{G}(\mathbb{Q})^+$, $(q\mathbf{G}'q^{-1}, qX'; qX'^+)$ is a Shimura subdatum of $(\mathbf{G}, X; X^+)$, and the special subvariety $M'' := \wp_\Gamma(qX'^+)$ it defines is called a Hecke translate of M'.

The union

$$\mathbb{H}_M(M') := \bigcup_{q \in \mathbf{G}(\mathbb{Q})^+} \wp_\Gamma(qX'^+)$$

is called the Hecke orbit of M' in M.

It should be pointed out that the Hecke translate $\wp_\Gamma(qX'^+)$ defined above depends on the choice of the subdatum defining M'. Rather we should define Hecke correspondence using double cosets $\Gamma q \Gamma$ whose irreducible components are of the form $\wp_\Gamma(aX'^+)$ for suitable $a \in \mathbf{G}(\mathbb{Q})^+$. But the current version is sufficient for our purpose.

LEMMA 2.11. – *Let $M' \subsetneq M$ be a special subvariety associated to the subdatum $(\mathbf{G}', X'; X'^+) \subset (\mathbf{G}, X; X^+)$. Then the Hecke orbit $\mathbb{H}_M(M')$ is dense in M for the analytic topology.*

Proof. – The real approximation theorem for linear groups over $\mathbb{Q}$ affirms that $\mathbf{G}(\mathbb{Q})^+$ is dense in $\mathbf{G}(\mathbb{R})^+$ for the analytic topology. This implies that the $\mathbf{G}(\mathbb{Q})^+$-orbit of any given point x in X^+ is dense for the analytic topology, and thus its image under $\wp_\Gamma$ is dense in M. It then suffices to take x from X'^+. □

COROLLARY 2.12. – *Let $M \subset \mathcal{A}_g$ be a special subvariety defined by $(\mathbf{G}, X; X^+)$.*

(1) *Let Z be a locally closed subvariety in $\mathcal{A}_g$ of dimension > 0. Assume that M is contained generically in Z, i.e., the intersection $M \cap Z$ is Zariski open in M. Assume further that M contains a special subvariety $S \subsetneq M$ of dimension > 0. Then there exists a Hecke translate S' of S inside M which is contained generically in Z.*

(2) *It suffices to prove the main theorem for M such that $\mathbf{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple.*

Proof. – (1) This is exactly the same as [5, Lemma 2.1.10], and we recall briefly the arguments. Write $\overline{Z}$ for the closure of Z in $\mathcal{A}_g$. Since M is contained generically in Z, we see that the closed subvariety M is contained in $\overline{Z}$. If a Hecke translate S' of S in M is not contained generically in Z, then S' is contained in the closed complement $\overline{Z} - Z$. But the union of all such S' is dense in M by Lemma 2.11, hence there must be some Hecke translate S' which is contained generically in Z.

(2) From Lemma 2.9 we know that non-simple Shimura variety contains proper special subvarieties of dimension > 0. Thus it suffices to take $Z = \mathcal{T}S_{g,n}$ and apply (1). □

Aside from the basic properties of Shimura varieties listed above, we need the following crucial fact:

PROPOSITION 2.13. – *The open n-superelliptic Torelli locus $\mathcal{T}S^{\circ}_{g,n}$ contains no complete curves.*

The proof of this property is given later in Section 3, as a consequence to Theorem 3.9. This property is well-known in the hyperelliptic case ($n = 2$), cf. [8].

THEOREM 2.14 (dimensional reduction). – *If the n-superelliptic Coleman-Oort conjecture holds for special curves in $\mathcal{A}_g$, then it holds for any special subvarieties in $\mathcal{A}_g$.*

Proof. – Let M be a special subvariety of strictly positive dimension inside $\mathcal{A}_g$, and we want to show that the intersection $M \cap \mathcal{T}S^{\circ}_{g,n}$ consists of at most finitely many points when $\mathcal{T}S^{\circ}_{g,n}$ does not contain generically any special curve.

By Corollary 2.12, it suffices to consider the case where M is a simple Shimura variety in $\mathcal{A}_g$ of dimension ≥ 2, defined by some Shimura datum $(\mathbf{G}, X; X^+)$. This implies that $\mathbf{G}^{\mathrm{ad}}$ admits no $\mathbb{Q}$-factor isomorphic to $\mathrm{PGL}_{2,\mathbb{Q}}$. Write $\overline{M}$ for the closure of M in the Baily-Borel compactification $\mathcal{A}_g^{\mathrm{BB}}$, whose boundary components in $\overline{M} - M$ are of codimension at least 2. One may thus take a generic projective curve $C \subset \overline{M}$ such that the intersection $C \cap (\overline{M} - M)$ is empty.

If the singular locus in $\mathcal{T}S_{g,n}$, namely the intersection $\mathcal{T}S^{\mathrm{sing}}_{g,n} := \mathcal{T}S_{g,n} \cap \mathcal{A}_g^{\mathrm{dec}}$, also meets $\overline{M}$ in codimension at least 2, then we may further choose C such that $C \cap \mathcal{T}S^{\mathrm{sing}}_{g,n}$ is empty. But this would produce a projective curve C contained in $\mathcal{T}S^{\circ}_{g,n}$ which contradicts Proposition 2.13. Hence at least one of the irreducible components in $\mathcal{T}S^{\mathrm{sing}}_{g,n} \cap M = \mathcal{A}_g^{\mathrm{dec}} \cap M$ is of codimension 1 in M.

Since $\mathcal{A}_g^{\mathrm{dec}}$ is a finite union of special subvarieties in $\mathcal{A}_g$, we deduce that $\mathcal{A}_g^{\mathrm{dec}} \cap M$ is a finite union of special subvarieties, one of which is of codimension 1 in M. Hecke translation moves it into a special subvariety $M' \subsetneq M$ contained generically in $\mathcal{T}S_{g,n}$ of strictly lower dimension, and the reduction is proved. □

3. Family of superelliptic curves

In this section we consider the geometry of 1-parameter semi-stable families of superelliptic curves. In subsection 3.1 we recall some basic facts and notations on families of curves, in subsection 3.2 investigate the invariants of a superelliptic family, in subsection 3.3 deduce slope inequalities for such a family, and finally in subsection 3.4 study the behavior of the flat part contained in the associated logarithmic Higgs bundle and prove the existence of horizontal fibration structures on total space under certain assumptions.

3.1. Preliminaries

In the subsection, we collect some generalities on families of curves, and refer to [3] for more details.

Recall that a semi-stable (resp. stable) curve of genus at least 2 is a complete connected reduced nodal curve such that each rational component intersects with the other components at $\geq$ 2 (resp. 3) points. A semi-stable (resp. stable) family of curves is a flat projective morphism $f : S \to B$ from a smooth surface S to a smooth curve B with connected fibers such that all the singular fibers of f are semi-stable (resp. stable) curves. We assume both S and B are projective unless stated otherwise. The family f is said to be superelliptic if a general fiber of f is a superelliptic curve, and to be isotrivial if all its smooth fibers are isomorphic to each other. In the rest, we assume that $f : S \to B$ is a semi-stable family of curves of genus $g \geq 2$ with singular fibers $\Upsilon \to \Delta$.smooth.

DEFINITION 3.1. – A horizontal fibration on S is a fibration $f' : S \to B'$ whose general fiber mapped under f surjectively onto B. It is said to be irregular if $g(B') > 0$.

Denote by $\omega_{S/B} = \omega_S \otimes f^*\omega_B^\vee$ the relative canonical sheaf of f. Let $\chi(\mathcal{O}_S)$ be the Euler characteristic of the structure sheaf, and $\chi_{\mathrm{top}}(\cdot)$ be the topological Euler characteristic. Consider the following relative invariants:

$$\begin{cases} \omega_{S/B}^2 = \omega_S^2 - 8(g-1)\big(g(B)-1\big), \\ \delta(f) = \chi_{\mathrm{top}}(S) - 4(g-1)\big(g(B)-1\big) = \sum\limits_{F\in\Upsilon} \delta(F), \\ \deg f_*\omega_{S/B} = \chi(\mathcal{O}_S) - (g-1)\big(g(B)-1\big), \end{cases} \tag{3-1}$$

where $\delta(F)$ is the number of nodes contained in the fiber F. All the invariants in (3-1) are nonnegative and satisfy the Noether's formula:

$$12 \deg f_*\omega_{S/B} = \omega_{S/B}^2 + \delta(f). \tag{3-2}$$

A singular point q of F is of type $i \in \big[1, [g/2]\big]$ (resp. 0) if the partial normalization of F at q consists of two connected components of arithmetic genera i and $g-i$ (resp. is connected). Let $\delta_i(F)$ be the number of nodes of type i contained in F, and

$$\delta(F) = \sum_{i=0}^{[g/2]} \delta_i(F).$$

A singular fiber F has a compact Jacobian if and only if $\delta_0(F) = 0$. Define

$$\delta_i(f) = \sum_{F\in\Upsilon} \delta_i(F), \quad \delta_h(F) = \sum_{i=2}^{[g/2]} \delta_i(F), \quad \delta_h(f) = \sum_{F\in\Upsilon} \delta_h(F).$$

3.2. Invariants

The main purpose of this section is to define the local invariants for families of superelliptic curves, and to show that the relative invariants (3-1) of such a family can be expressed as suitable combinations of the local invariants.

Fix ξ_n a primitive n-th root of unity which gives a trivialization $G = \mathbb{Z}/n\mathbb{Z}$. For an n-superelliptic curve F defined as in Definition 1.4, the map given by $y \mapsto \xi_n y$ defines a natural action of the group $G = \mathbb{Z}/n\mathbb{Z}$ on the n-superelliptic curve F. We call G *the n-superelliptic automorphism group* of F, and the induced cover $\pi : F \to F/G \cong \mathbb{P}^1$ *the n-superelliptic cover*.

For any n-superelliptic curve F, its induced n-superelliptic cover $\pi : F \to \mathbb{P}^1$ is a cyclic cover with covering group $G = \mathbb{Z}/n\mathbb{Z}$, branch locus R, and local monodromy a around R.

Here R and a are given by

$$\begin{cases} R = \{x_1, \dots, x_{\alpha_0}\}, \text{ and } a = (1, \dots, 1), & \text{if } n \mid \alpha_0; \\ R = \{x_1, \dots, x_{\alpha_0}, \infty\}, \text{ and } a = (1, \dots, 1, a_\infty), & \text{if } n \nmid \alpha_0, \end{cases} \tag{3-3}$$

where $\alpha_0 = \deg(\zeta(x))$, $\{x_1, \dots, x_{\alpha_0}\}$ are the roots of $\zeta(x)$, and $a_\infty = n\left(\left[\frac{\alpha_0}{n}\right] + 1\right) - \alpha_0$. In the case where $n \nmid \alpha_0$, the ramification index of π at ∞ is $r_\infty = \frac{n}{\gcd(n, \alpha_0)}$. Thus by the Riemann-Hurwitz formula,

$$g = \begin{cases} \dfrac{(n-1)(\alpha_0 - 2)}{2}, & \text{if } n \mid \alpha_0, \\ \dfrac{(n-1)(\alpha_0 - 2) + \big(n - \gcd(\alpha_0, n)\big)}{2}, & \text{if } n \nmid \alpha_0; \end{cases} \tag{3-4}$$

REMARK 3.2. – For a given n-superelliptic curve F, there might be more than one n-superelliptic automorphism group (equivalently, more than one n-superelliptic cover) on F. For instance, the Fermat curve of degree n admits at least three different n-superelliptic automorphism groups. Throughout this paper, we always choose and fix the choice of an n-superelliptic automorphism group on F if there are more than one.

LEMMA 3.3. – *Let $f : S \to B$ be a family of semi-stable superelliptic curves (B could be non-compact).*

(i) *After a suitable finite surjective base change, the action of the n-superelliptic automorphism group of F can be extended to S.*

(ii) *If moreover the general fiber F admits a unique n-superelliptic automorphism group $G = \mathbb{Z}/n\mathbb{Z}$, and there exists a generator of $\sigma \in G$ commuting with any automorphism of the general fiber, then the action of G can be extended to S without base change.*

Proof. – (i). Let $\mathfrak{Mor}_B(S, S)$ be the functor which associates any B-scheme T to the set of all T-morphism from $S_T = S \times_B T$ to itself. The functor $\mathfrak{Mor}_B(S, S)$ is representable (cf. [13, Thm. 5.23]). Denote by $\mathcal{M}or_B(S, S)$ the scheme representing $\mathfrak{Mor}_B(S, S)$. Let $\mathfrak{Aut}_B(S)$ be the functor which associates any B-scheme T to the set of all automorphisms of S_T over T. Then $\mathfrak{Aut}_B(S)$ is also representable. Actually it can be represented by an open subscheme

$$\mathcal{A}ut_B(S) \hookrightarrow \mathcal{M}or_B(S, S).$$

Since f is flat projective of genus $g \geq 2$, the scheme $\mathcal{A}ut_B(S)$ is a finite flat group scheme over B. Therefore, after a suitable finite base change, one can produce a section of $\mathcal{A}ut_B(S) \to B$ that reduces to a generator σ of G on the general fiber. In other words, after a suitable finite base change, the action of the generator σ and hence the whole superelliptic automorphism group G on the general fiber F can be extended to S.

(ii). By (i), there exists a base change $\tilde{\pi} : \widetilde{B} \to B$ such that $\mathcal{A}ut_{\widetilde{B}}(\widetilde{S}) \to \widetilde{B}$ admits a section $\mathfrak{S}$ whose restriction on F is the generator $\sigma \in G$, where $\widetilde{S} = S \times_B \widetilde{B}$. We may assume that the base change $\tilde{\pi}$ is Galois with Galois group $\mathrm{Gal}(\widetilde{B}/B)$. Then $\mathrm{Gal}(\widetilde{B}/B)$ acts on $\widetilde{S}$, and hence also on $\mathcal{A}ut_{\widetilde{B}}(\widetilde{S})$ and $\mathrm{Aut}(F)$ by conjugation. By assumption, $\mathrm{Gal}(\widetilde{B}/B)$ fixes σ, hence also $\mathfrak{S}$. Note that

$$\mathcal{A}ut_{\widetilde{B}}(\widetilde{S}) \big/ \mathrm{Gal}(\widetilde{B}/B) = \mathcal{A}ut_B(S).$$

Thus $\mathfrak{S}$ produces a section of $\mathcal{A}ut_B(S)$, which extends the action of σ on S. □

We first study local families of semi-stable superelliptic curves, i.e., restriction of a family $f : S \to T$ to an open subset $T_0 \subset T$ with $T_0 \simeq \{t \in \mathbb{C} : |t| < 1\}$ (for the analytic topology).

LEMMA 3.4. – *Let $f_0 : S_0 \to T_0$ be a local family of semi-stable superelliptic curves, with T_0 identified with the open unit disk in $\mathbb{C}$. Assume that $F_0 = f_0^{-1}(0)$ is the unique singular fiber. Then after a suitable base change, the following statements hold:*

1. *the action of the superelliptic automorphism group $G = \mathbb{Z}/n\mathbb{Z}$ on the general fiber can be extended to S_0, such that the quotient map $\Pi_0 : S_0 \to S_0/G$ branches over α disjoint sections $\{D_i\}_{i=1}^{\alpha}$ of φ_0 plus certain nodes in $\Gamma_0 = \Pi_0(F_0)$, where α is given in* (1-1)*, and $\varphi_0 : S_0/G \to T_0$ is the induced family;*
2. *the fiber Γ_0 with $\Gamma_0 \cap \{D_i\}_{i=1}^{\alpha}$ as its marked points is a semi-stable α-pointed (maybe reducible) rational curve;*
3. *for any node $y \in \Gamma_0$, there are at least two marked points of Γ_0 on Γ_0', where Γ_0' is any one of the two connected components of $\Gamma_0' \setminus \{y\}$.*

Proof. – (i). That the action of G can be extended to S_0 follows from Lemma 3.3. Let $R \subseteq S_0$ be the fixed locus of G, i.e.,

$$R = \{x \in S_0 \mid h(x) = x \text{ for some } 1 \neq h \in G\} \subseteq S_0.$$

Then the branch locus of π_0 is nothing but $\Pi_0(R) \subseteq S_0/G$. As the fixed locus of G, R consists of certain multiple sections plus some nodes in F_0. After suitable base change, we may assume that these multiple sections become sections. So is $\Pi_0(R)$.

(ii). By (i), the restricted cover

$$\Pi_0|_{F_0} : F_0 \longrightarrow \Gamma_0$$

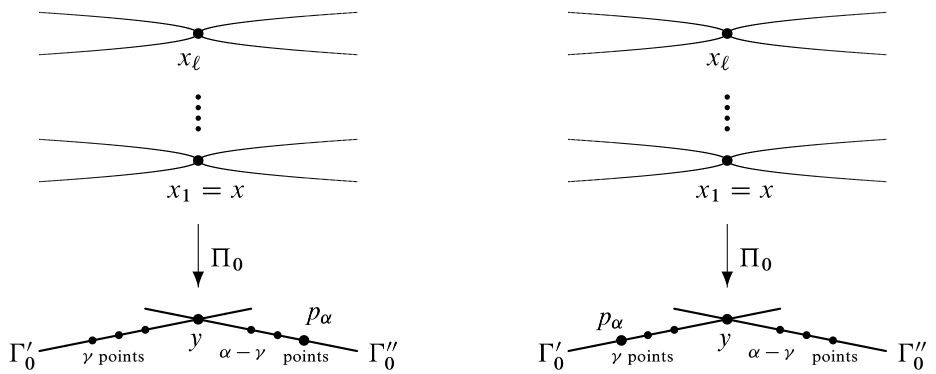
branches over at most on $\Gamma_0 \cap \{D_i\}_{i=1}^{\alpha}$ and nodes of Γ_0. Note that Γ_0 is connected. Thus if Γ_0 were not a semi-stable pointed rational curve, then there would exist a component $C_0 \subseteq \Gamma_0$ that would contain no marked point and would intersect other components of Γ_0 at only one point. Then the inverse image of C_0 would create a smooth rational curve intersecting other components at only one point, and hence contradicting the semi-stability of F_0.

(iii). First, if Γ_0' does not contain any marked point, then there must be a component C_0 of Γ_0' that contains no marked point and intersects other components of Γ_0 at only one point. Similar as (ii), one obtains a contradiction. Hence it suffices to derive a contradiction if Γ_0' contains exactly one marked point. If this indeed occurs, then there is a component $C_0 \subseteq \Gamma_0'$ such that C_0 intersects $\overline{\Gamma_0 \setminus C_0}$ at only one point and that there is at most one marked point on C_0. It follows that E is still a smooth rational curve with one intersection point with $F_0 \setminus E$, i.e., E is a (-1)-curve, where $E \subseteq F_0$ is any component in the inverse image of C_0. This implies that the fiber F_0 is not semi-stable, which is absurd. □

DEFINITION 3.5. – Let $f_0 : S_0 \to T_0 \triangleq \{t \mid |t| < 1\}$ be a local family of semi-stable n-superelliptic curves with $F_0 = f_0^{-1}(0)$ as the unique singular fiber. The index of a node $x \in F_0$, denoted as index(x), is defined as follows:

(i) Assume first that f_0 satisfies the three statements in Lemma 3.4. Let $\ell = |G \cdot x|$ be the number of points in the G-orbit of x, and $y \in \Gamma_0 = F_0/G$ be the image of x. Assume that $\Gamma_0 \setminus \{y\} = \Gamma_0' \cup \Gamma_0''$ such that Γ_0' (resp. Γ_0'') contains γ (resp. $\alpha - \gamma$) marked points with $\alpha - \gamma \geq \gamma$. In the case where $n \nmid \alpha$, the section D_α is assumed to be the one whose local monodromy is equal to a_∞, where a_∞ is the local monodromy of ∞ as in (3-3); and we always assume that $p_\alpha := D_\alpha \cap \Gamma_0 \in \Gamma_0'$ if $\gamma = \alpha/2$. Then we define index(x) to be the triple (γ, ℓ, k) with

$$k = \begin{cases} 0, & \text{if } n \mid \alpha, \text{ or if } n \nmid \alpha \text{ and } p_\alpha \in \Gamma_0''; \\ 1, & \text{if } n \nmid \alpha \text{ and } p_\alpha \in \Gamma_0'. \end{cases}$$



typical case when index$(x) = (\gamma, \ell, 0)$; typical case when index$(x) = (\gamma, \ell, 1)$.

(ii). In the general case, we choose any base change $T_0' \to T_0$ such that the family $f_0' : S_0' \to T_0'$ obtained by pulling-back satisfies the three statements in Lemma 3.4 and let $x' \in F_0'$ be any node over x. Then we define index$(x) :=$ index(x'). One checks that the definition is independent on the choices of the base change and the node x'.

Definition 3.6. – (i). For any singular fiber F in a semi-stable family $f : S \to B$ of n-superelliptic curves of genus $g \geq 2$, we define the singularity index $s_{\gamma,\ell,k}(F)$ of F to be the number of nodes in F with index equal to (γ, ℓ, k).

(ii). For a semi-stable family $f : S \to B$ of n-superelliptic curves of genus $g \geq 2$, we define the singularity indices of f as

$$s_{\gamma,\ell,k}(f) = \sum s_{\gamma,\ell,k}(F).$$

Here the sum takes over all singular fibers (which are finitely many) of f. If there is no confusion, we simply denote by $s_{\gamma,\ell,k} = s_{\gamma,\ell,k}(f)$.

Proposition 3.7. – (i) *For any singular fiber F, we have*

$$s_{\gamma,\ell,1}(F) = 0, \ \forall\, \gamma, \ell, \qquad \textit{if } n \mid \alpha, \tag{3-5}$$

and

$$\sum_{\gamma=2}^{[\alpha/2]} \sum_{\ell>1} \big(s_{\gamma,\ell,0}(F) + s_{\gamma,\ell,1}(F)\big) = \delta_0(F);$$

$$\begin{cases} s_{\gamma,1,0}(F) + s_{\gamma,1,1}(F) = \delta_{i(\gamma)}(F), & \textit{if } r_\infty = n, \\ s_{\gamma,1,0}(F) = \delta_{i(\gamma)}(F), \quad s_{\gamma,1,1}(F) = \delta_{j(\gamma)}(F), & \textit{if } r_\infty \neq n, \end{cases}$$

where $r_\infty = \frac{n}{\gcd(n,\alpha_0)}$ *is the ramification index of the cover* π *at* ∞ *(see* (3-3)*), and*

$$\begin{aligned} i(\gamma) &= \frac{(n-1)(\gamma-1)}{2} \\ j(\gamma) &= \frac{(n-1)(\gamma-2) + \frac{r_\infty - 1}{r_\infty}\cdot n}{2} = g - \frac{(n-1)(\alpha-\gamma-1)}{2}. \end{aligned}$$

(ii) *The index* (γ, ℓ, k) *of a node* x *satisfies that* $2 \leq \gamma \leq [\alpha/2]$, *and that*

$$\ell = \begin{cases} \gcd(\gamma, n), & \text{if } k = 0; \\ \gcd(\alpha-\gamma, n), & \text{if } k = 1. \end{cases} \tag{3-6}$$

In other words,

$$\begin{cases} s_{\gamma,\ell,k} = 0, & \text{if } \gamma = 1 \text{ or } \gamma > [\alpha/2]; \\ s_{\gamma,\ell,0} = 0, & \text{if } \ell \neq \gcd(\gamma, n); \\ s_{\gamma,\ell,1} = 0, & \text{if } \ell \neq \gcd(\alpha-\gamma, n). \end{cases}$$

Proof. – (i) This follows the definition of singularity indices and $\delta_i(F)$ introduced in the last subsection.

(ii) By Definition 3.5, we may assume that the singular fiber F_0 admits a cyclic cover $\pi : F_0 \to \Gamma_0$ with covering group G, where Γ_0 is a singular rational curve. Let $y = \pi(x)$, and $\Gamma_0 \setminus \{y\} = \Gamma_0' \cup \Gamma_0''$ with Γ_0' (resp. Γ_0'') containing γ (resp. $\alpha - \gamma \geq \gamma$) marked points. Then

- if $k = 0$: the local monodromy of each marked point on Γ_0' is equal to 1, and thus
  - if $n \mid \gamma$: π is not branched over y;
  - if $n \nmid \gamma$: the local monodromy around $y \in \Gamma_0'$ for the restricted cyclic cover $\pi^{-1}(\Gamma_0') \to \Gamma_0'$ is equal to
  $$a_y' = n([\frac{\gamma}{n}] + 1) - \gamma;$$
- if $k = 1$: the local monodromy of each marked point on Γ_0'' is equal to 1, and thus
  - if $n \mid (\alpha - \gamma)$: π is not branched over y;
  - if $n \nmid (\alpha - \gamma)$: the local monodromy around $y \in \Gamma_0''$ for the restricted cyclic cover is
  $$a_y'' = n([\frac{\alpha-\gamma}{n}] + 1) - (\alpha - \gamma).$$

We thus obtain the value of ℓ:

$$\ell = |G \cdot x| = |\pi^{-1}(y)| = \begin{cases} n = \gcd(\gamma, n), & \text{if } k = 0 \text{ and } n \mid \gamma; \\ \gcd(a_y', n) = \gcd(\gamma, n), & \text{if } k = 0 \text{ and } n \nmid \gamma; \\ n = \gcd(\alpha-\gamma, n), & \text{if } k = 1 \text{ and } n \mid (\alpha-\gamma); \\ \gcd(a_y'', n) = \gcd(\alpha-\gamma, n), & \text{if } k = 1 \text{ and } n \nmid (\alpha-\gamma). \end{cases}$$

This completes the proof. □

PROPOSITION 3.8. – *Let* $f : S \to B$ *be a family of semi-stable* n*-superelliptic curves of genus* $g \geq 2$ *with* $\alpha \geq 5$. *Assume moreover that the general fiber of* f *is not hyperelliptic. Then* f *admits no (smooth or singular) hyperelliptic fiber with a compact Jacobian.*

Proof. – We divide the proof into three steps.

Step I. – We show that f admits no smooth hyperelliptic fiber.

Indeed, if there is such a smooth hyperelliptic fiber F, then F admits two different maps of degree 2 and n respectively onto $\mathbb{P}^1$ without common non-trivial factorization. Hence $g \leq n-1$ by the Castelnuovo-Severi inequality (cf. [17, Exercise V.1.9]). This is a contradiction to (3-4) once $\alpha \geq 5$.

Step II. – Let F be a singular hyperelliptic fiber of f with a compact Jacobian. For any irreducible component $D \subseteq F$, we show that

- either $g(D) = 0$ and $D^2 = -2$;
- or $1 \leq g(D) \leq \frac{n-1}{2}$ and $D^2 = -1$.

Let $F^{\#}$ be the stable model of F. Then it suffices to show $1 \leq g(D) \leq \frac{n-1}{2}$ and $D^2 = -1$ for any component $D \subseteq F^{\#}$. Note that $F^{\#}$ admits two automorphisms τ and ι, where τ is of order n and ι is the hyperelliptic involution. Both the quotients $F^{\#}/\langle\tau\rangle$ and $F^{\#}/\langle\iota\rangle$ are trees of rational curves. By [23, Lemma 4.7], any component $D \subseteq F^{\#}$ is not rational, i.e., $g(D) \geq 1$. Hence both τ and ι act non-trivially on D. It is clear that the hyperelliptic involution ι keeps D invariant and acts faithfully on D. Without loss of generality, we may assume that τ also keeps D invariant and hence acts faithfully on D; otherwise, we replace τ by a suitable power τ^{k_0} (the only difference is that the order of τ is smaller after this replacement). For the sake of notations, we still denote by τ and ι their restriction on D. Let

$$\Sigma_D = \{x \in D \mid x \text{ is a node of } F^{\#}\}.$$

It is clear that $|\Sigma_D| = -D^2$, Moreover, as $F^{\#}$ admits a compact Jacobian, x is fixed by both τ and ι for any node $x \in F^{\#}$; in fact, let F' and F'' be the two connected components of $F^{\#} \setminus \{x\}$. Then both F' and F'' are kept invariant under τ (and ι), and x is the unique intersection of F' and F'' since $F^{\#}$ admits a compact Jacobian. Hence $x = F' \cap F''$ is fixed by both τ and ι. Thus τ and ι are commutative with each other as automorphisms of D, since they admit at least one common fixed point on D (i.e., any point in Σ_D). It follows that τ induces a non-trivial automorphism τ' on $D/\langle\iota\rangle \cong \mathbb{P}^1$. Let $|\tau|$ be the order of τ, and

$$D_\tau = \Big\{x \in D \mid x \text{ is fixed by } \tau^k \text{ for some } 1 \leq k < |\tau|\Big\} \subseteq D,$$

and $\mu = \big|D_\tau\big|$. Then by Hurwitz formula, one has

$$2g(D) \leq (n-1)(\mu-2).$$

Since $g(D) \neq 0$, it follows that $\mu \geq 3$. Let $\mathrm{Fix}(\tau') \subseteq \mathbb{P}^1$ be the fixed locus of τ'. Then

$$\big|\mathrm{Fix}(\tau')\big| = 2, \quad \text{and} \quad \pi\big(D_\tau\big) \subseteq \mathrm{Fix}(\tau'),$$

where $\pi : D \to D/\langle\iota\rangle \cong \mathbb{P}^1$ is the quotient map, which is of degree two. Note that $\Sigma_D \subseteq D_\tau$ and $\pi^{-1}\big(\pi(x)\big) = x$ for any $x \in \Sigma_D$. It follows that

$$3 \leq \mu = \big|D_\tau\big| \leq 2\big|\mathrm{Fix}(\tau') \setminus \pi(\Sigma_D)\big| + |\Sigma_D| = 4 - |\Sigma_D|.$$

Therefore $\mu = 3$, and hence $g(D) \leq \frac{n-1}{2}$ and $D^2 = -|\Sigma_D| = -1$ as required.

Step III. – We show that f admits no singular hyperelliptic fiber with a compact Jacobian.

If there were such a singular hyperelliptic fiber F, let k (resp. k') be the number of rational components (resp. irrational components) in F. Then by Step II, one has

$$\delta(F)=\frac{1}{2}\sum_{\{D\neq D'\}\subseteq F} D\cdot D'=-\frac{1}{2}\sum_{D\subseteq F} D^2=\frac{1}{2}(2k+k').$$

Also, since F admits a compact Jacobian, we get

$$\delta(F)=k+k'-1.$$

Combining the above two equalities, we obtain $k'=2$. Hence $g\leq k\cdot 0+k'\cdot\frac{n-1}{2}=n-1$, which contradicts the Hurwitz Formula (3-4) since $\alpha\geq 5$. This completes the proof. □

THEOREM 3.9. – *Let $f:S\to B$ be a family of semi-stable n-superelliptic curves of genus $g\geq 2$ over a projective curve, and let $b_\gamma=\frac{(n^2-1)\gamma(\alpha-\gamma)}{\alpha-1}-n^2$. Then*

$$\omega_{S/B}^2=\begin{cases}\displaystyle\sum_{\gamma,\ell} b_\gamma\cdot\frac{s_{\gamma,\ell,0}}{\ell^2}, & \text{if } n\mid\alpha,\\[2ex] \displaystyle\sum_{\gamma,\ell}\left(b_\gamma-\frac{(n^2-r_\infty^2)\gamma(\gamma-1)}{r_\infty^2(\alpha-1)(\alpha-2)}\right)\cdot\frac{s_{\gamma,\ell,0}}{\ell^2} & \\ \displaystyle\quad+\sum_{\gamma,\ell}\left(b_\gamma-\frac{(n^2-r_\infty^2)(\alpha-\gamma)(\alpha-\gamma-1)}{r_\infty^2(\alpha-1)(\alpha-2)}\right)\cdot\frac{s_{\gamma,\ell,1}}{\ell^2}, & \text{if } n\nmid\alpha,\end{cases}\tag{3-7}$$

$$\deg f_*\omega_{S/B}=\begin{cases}\displaystyle\frac{1}{12}\sum_{\gamma,\ell}(b_\gamma+\ell^2)\cdot\frac{s_{\gamma,\ell,0}}{\ell^2}, & \text{if } n\mid\alpha,\\[2ex] \displaystyle\frac{1}{12}\sum_{\gamma,\ell}\left(b_\gamma+\ell^2-\frac{(n^2-r_\infty^2)\gamma(\gamma-1)}{r_\infty^2(\alpha-1)(\alpha-2)}\right)\cdot\frac{s_{\gamma,\ell,0}}{\ell^2} & \\ \displaystyle\quad+\frac{1}{12}\sum_{\gamma,\ell}\left(b_\gamma+\ell^2-\frac{(n^2-r_\infty^2)(\alpha-\gamma)(\alpha-\gamma-1)}{r_\infty^2(\alpha-1)(\alpha-2)}\right)\cdot\frac{s_{\gamma,\ell,1}}{\ell^2}, & \text{if } n\nmid\alpha,\end{cases}\tag{3-8}$$

$$\delta(f)=\begin{cases}\displaystyle\sum_{\gamma,\ell}s_{\gamma,\ell,0}, & \text{if } n\mid\alpha,\\[2ex] \displaystyle\sum_{\gamma,\ell}\big(s_{\gamma,\ell,0}+s_{\gamma,\ell,1}\big), & \text{if } n\nmid\alpha.\end{cases}\tag{3-9}$$

The crucial property in Proposition 2.13 follows from the theorem above:

Proof of Proposition 2.13. – If a complete curve C exists in $\mathcal{T}S_{g,n}^{\circ}$, then its pull-back B in $\mathcal{S}_{g,n}$ remains complete. This gives a (non-isotrivial) family of super-elliptic curves $f:S\to B$ which admits no singular fiber. Using (3-8) we deduce that $\deg f_*\omega_{S/B}=0$, which is a contradiction, cf. [1]. □

Proof of Theorem 3.9. – First, (3-9) follows directly from Definition 3.6; see also Proposition 3.7 (i). Note also that (3-8) follows from (3-2), (3-7) and (3-9). So it suffices to prove (3-7).

Since any finite base change only modifies the two sides of (3-7) by a common multiple, we may, up to a finite base change, assume that there exists an action of $G = \mathbb{Z}/n\mathbb{Z}$ on S, such that $Y = S/G$ is ruled over B, and that the quotient $\Pi : S \to Y$ is branched over α disjoint sections $\{D_i\}_{i=1}^{\alpha}$ of the induced family $\varphi : Y \to B$ and possibly some of the nodes in fibers of φ (cf. Lemma 3.4). Moreover, if $n \nmid \alpha$, one may further assume that D_α is the section whose restriction to a general fiber $\Gamma \cong \mathbb{P}^1$ of φ corresponds to the branch point ∞ of the restricted cover $\Pi|_F : F \to \Gamma \cong \mathbb{P}^1$. In this case, the ramification index of D_α is $r_\infty = \frac{n}{\gcd(a_\infty, n)}$. Let's consider the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\ \Pi\ } & Y = S/G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \varphi} \\ B & = & B. \end{array}$$

For any node y_j in some fiber Γ_0 of φ, if the local equation of Y around y_j is given by $xy - t^{m_j}$, then we call m_j the multiplicity of y_j. Let ℓ_j be the number of points in S over y_j, and $\Gamma_0 \setminus \{y_j\} = \Gamma_0' \cup \Gamma_0''$ with Γ_0' (resp. Γ_0'') containing γ_j (resp. $\alpha - \gamma_j \geq \gamma_j$) marked points. If $n \nmid \alpha$ and $\gamma_j = \alpha/2$, then we assume that $D_\alpha \cap \Gamma_0 \in \Gamma_0'$. Then we define the index of y_j to be (γ_j, ℓ_j, k_j), where

$$k_j = \begin{cases} 0, & \text{if } n \mid \alpha, \text{ or if } n \nmid \alpha \text{ and } D_\alpha \cap \Gamma_0 \in \Gamma_0''; \\ 1, & \text{if } n \nmid \alpha \text{ and } D_\alpha \cap \Gamma_0 \in \Gamma_0'. \end{cases}$$

With the notations introduced above, we claim that

Claim 3.10. – (i) *If $n \mid \alpha$, then*

$$(\alpha - 1) \sum_{i=1}^{\alpha} D_i^2 = -\sum_{y_j} m_j \gamma_j (\alpha - \gamma_j).$$

(ii) *If $n \nmid \alpha$, then*

$$\begin{aligned} (\alpha - 2) \sum_{i=1}^{\alpha-1} D_i^2 &= -\sum_{y_j} m_j (\gamma_j - k_j)(\alpha + k_j - 1 - \gamma_j); \\ (\alpha - 1) \sum_{i=1}^{\alpha} D_i^2 &= -\sum_{y_j} m_j \gamma_j (\alpha - \gamma_j). \end{aligned}$$

Proof of Claim 3.10. – The proof is similar to that of [8, Lemma 4.8]. As an illustration, we prove (ii) here.

Note that both of the sides are invariant if we resolve the singularities on Y, and hence we may assume that Y is smooth. Moreover, we may contract Y to a $\mathbb{P}^1$ bundle $\varphi : Y \to B$ over B such that the order of the singularities of $R_0 = \sum D_{i,0}$ is at most $[\alpha/2]$, and that $D_{\alpha,0}$ does not pass through any singularity of order equal to $\alpha/2$ if $n \nmid \alpha$, where $D_{i,0} \subseteq Y$ is the image of D_i. Note that in the $\mathbb{P}^1$ bundle Y, one has $(D_{i,0}-D_{j,0})^2=0$, i.e., $D_{i,0}^2+D_{j,0}^2=2D_{i,0}\cdot D_{j,0}$. Hence

$$(\alpha-2)\sum_{i=1}^{\alpha-1}D_{i,0}^2=2\sum_{1\le i<j\le\alpha-1}D_{i,0}\cdot D_{j,0};$$
$$(\alpha-1)\sum_{i=1}^{\alpha}D_{i,0}^2=2\sum_{1\le i<j\le\alpha}D_{i,0}\cdot D_{j,0}.$$

Now blowing up a point with exactly γ of these sections passing through it will create a node y of index (γ,ℓ,k) in the fibers, where $k=0$ if $D_{\alpha,0}$ does not pass through this point, and $k=1$ if $D_{\alpha,0}$ passes through this point; at the same time, the left hand side (resp. the right hand side) of the first equality above decreases by $(\gamma-k)(\alpha-2)$ $\big($resp. $(\gamma-k)(\gamma-k-1)\big)$, and the left hand side (resp. the right hand side) of the second equality above decreases by $\gamma(\alpha-1)$ $\big($resp. $\gamma(\gamma-1)\big)$. Thus

$$(\alpha-2)\sum_{i=1}^{\alpha-1}D_i^2=2\sum_{1\le i<j\le\alpha-1}D_i\cdot D_j-\sum_{y_j}m_j(\gamma_j-k_j)(\alpha+k_j-1-\gamma_j);$$
$$(\alpha-1)\sum_{i=1}^{\alpha}D_i^2=2\sum_{1\le i<j\le\alpha}D_i\cdot D_j-\sum_{y_j}m_j\gamma_j(\alpha-\gamma_j).$$

Note that in Y, these sections $\{D_i\}_{i=1}^{\alpha}$ are disjoint with each other, i.e., $D_i\cdot D_j=0$ for any $i\neq j$. Hence we complete the proof of the claim. □

Come back to the proof of (3-7). Let $\xi_{\gamma,\ell,k}$ be the number of the nodes in fibers of φ with index (γ,ℓ,k), counted according to their multiplicities. Then

(3-10) $$s_{\gamma,\ell,0}=\ell\cdot\frac{\xi_{\gamma,\ell,0}}{n/\ell}=\frac{\ell^2}{n}\cdot\xi_{\gamma,\ell,0},\qquad s_{\gamma,\ell,1}=\ell\cdot\frac{\xi_{\gamma,\ell,1}}{n/\ell}=\frac{\ell^2}{n}\cdot\xi_{\gamma,\ell,1}.$$

By the definitions, we have (see also Proposition 3.7 (i))

(3-11) $$\xi_{\gamma,\ell,1}=s_{\gamma,\ell,1}=0,\qquad \text{if } n\mid\alpha.$$

According to Claim 3.10, we obtain that if $n\mid\alpha$, then

(3-12a) $$(\alpha-1)\sum_{i=1}^{\alpha}D_i^2=-\sum_{\gamma,\ell}\gamma(\alpha-\gamma)\xi_{\gamma,\ell,0}=-\sum_{\gamma,\ell}\frac{n\gamma(\alpha-\gamma)}{\ell^2}\cdot s_{\gamma,\ell,0};$$

and that if $n\nmid\alpha$, then

(3-12b)
$$\begin{cases}(\alpha-2)\displaystyle\sum_{i=1}^{\alpha-1}D_i^2 &=-\displaystyle\sum_{\gamma,\ell}\Big(\gamma(\alpha-1-\gamma)\xi_{\gamma,\ell,0}+(\gamma-1)(\alpha-\gamma)\xi_{\gamma,\ell,1}\Big)\\ &=-\displaystyle\sum_{\gamma,\ell}\bigg(\frac{n\gamma(\alpha-1-\gamma)}{\ell^2}\cdot s_{\gamma,\ell,0}+\frac{n(\gamma-1)(\alpha-\gamma)}{\ell^2}\cdot s_{\gamma,\ell,1}\bigg),\\ (\alpha-1)\displaystyle\sum_{i=1}^{\alpha}D_i^2 &=-\displaystyle\sum_{\gamma,\ell}\gamma(\alpha-\gamma)(\xi_{\gamma,\ell,0}+\xi_{\gamma,\ell,1})=-\sum_{\gamma,\ell}\frac{n\gamma(\alpha-\gamma)}{\ell^2}\cdot(s_{\gamma,\ell,0}+s_{\gamma,\ell,1}).\end{cases}$$

By the construction, we have

$$\omega_{Y/B}^2 = -\sum_{\gamma,\ell}(\xi_{\gamma,\ell,0}+\xi_{\gamma,\ell,1}) = -\sum_{\gamma,\ell}\frac{n(s_{\gamma,\ell,0}+s_{\gamma,\ell,1})}{\ell^2}; \qquad \omega_{Y/B}\cdot D_i = -D_i^2. \tag{3-13}$$

By the Riemann-Hurwitz formula, one has

$$\omega_{S/B}^2 = \begin{cases} \Pi^*\Big(\omega_{Y/B}+\dfrac{n-1}{n}\displaystyle\sum_{i=1}^{\alpha} D_i\Big), & \text{if } n\mid\alpha;\\ \Pi^*\Big(\omega_{Y/B}+\dfrac{n-1}{n}\displaystyle\sum_{i=1}^{\alpha-1} D_i+\dfrac{r_\infty-1}{r_\infty}D_\alpha\Big), & \text{if } n\nmid\alpha. \end{cases}$$

Hence if $n\mid\alpha$, then

$$\omega_{S/B}^2 = n\Big(\omega_{Y/B}+\frac{n-1}{n}\sum_{i=1}^{\alpha}D_i\Big)^2 = n\Big(\omega_{Y/B}^2-\frac{n^2-1}{n^2}\sum_{i=1}^{\alpha}D_i^2\Big), \tag{3-14a}$$

and if $n\nmid\alpha$, then

$$\begin{aligned} \omega_{S/B}^2 &= n\Big(\omega_{Y/B}+\frac{n-1}{n}\sum_{i=1}^{\alpha-1}D_i+\frac{r_\infty-1}{r_\infty}D_\alpha\Big)^2 \\ &= n\Big(\omega_{Y/B}^2-\frac{n^2-1}{n^2}\sum_{i=1}^{\alpha-1}D_i^2-\frac{r_\infty^2-1}{r_\infty^2}D_\alpha^2\Big). \end{aligned} \tag{3-14b}$$

Combining the equalities (3-14a), (3-14b), (3-13), (3-12a), (3-12b), and (3-11) all together, we complete the proof of (3-7). □

If the family f is irregular, i.e., the relative irregularity of the family is positive, there would be some constraints on the invariants introduced in Definition 3.6.

Proposition 3.11. – *Let $f: S\to B$ be a family of semi-stable n-superelliptic curves of genus $g\geq 2$ as in Theorem 3.9. Assume that the relative irregularity $q_f := q(S)-g(B)>0$. Then*

$$\begin{cases} \dfrac{4}{n}\cdot s_{2,n,1} \leq \displaystyle\sum_{\gamma,\ell}\frac{n\gamma(\alpha-\gamma)}{\ell^2(\alpha-1)}\cdot(s_{\gamma,\ell,0}+s_{\gamma,\ell,1});\\ \dfrac{1}{n}\cdot s_{2,n,1} \leq \displaystyle\sum_{\gamma,\ell}\Big(\frac{n\gamma(\gamma-1)}{\ell^2(\alpha-1)(\alpha-2)}s_{\gamma,\ell,0}+\frac{n(\alpha-\gamma)(\alpha-\gamma-1)}{\ell^2(\alpha-1)(\alpha-2)}s_{\gamma,\ell,1}\Big). \end{cases} \tag{3-15}$$

Proof. – To prove (3-15), we may assume that $s_{2,n,1}>0$, i.e., the singular fibers of f contain nodes with index $(2,n,1)$. In particular, $n\mid(\alpha-2)$ by (3-6), i.e., $\alpha_0 = nk+1$ for some integer k by (1-1). Hence for any general fiber F of f,

(3-16) the induced n-superelliptic cover $\pi: F\to\mathbb{P}^1$ is totally ramified.

Similar to the proof of Theorem 3.9, we may assume that $G=\mathbb{Z}/n\mathbb{Z}$ admits an action on S such that $\Pi: S\to Y$ is branched over α disjoint sections $\{D_i\}_{i=1}^{\alpha}$ and possibly over some nodes in fibers of φ. Let $\tilde\rho: \widetilde{Y}\to Y$ be the resolution of the singularities of Y, and $\rho: \widetilde{Y}\to Y$ be a contraction to a $\mathbb{P}^1$-bundle over B such that the order of the singularities of $R_0 = \sum_{i=1}^{\alpha} D_{i,0}$ is at most $[\alpha/2]$, and that $D_{\alpha,0}$ does not pass through any singularity of order equal to $\alpha/2$, where $D_{i,0} \subseteq Y$ is the image $\rho\big(\tilde{\rho}^{-1}(D_i)\big)$.

$$\begin{tikzcd}
\widetilde{Y} \arrow[rr, "\rho_1"] \arrow[d, "\tilde{\rho}"'] \arrow[rrd, "\rho"] & & Y_1 \arrow[d, "\rho_0"] \\
Y \arrow[rd, "\varphi"'] & & Y \arrow[ld, "\varphi"] \\
& B &
\end{tikzcd}$$

Note that ρ consists of a sequence of blowing-ups centered at singularities of R_0. We reorder these blowing-ups as $\rho : \widetilde{Y} \xrightarrow{\rho_1} Y_1 \xrightarrow{\rho_0} Y$ such that ρ_0 is the resolution of singularities of $\sum_{i=1}^{\alpha-1} D_{i,0}$. In other words, all nodes in the fibers of φ with index equal to $(2, n, 1)$ (resp. not equal to $(2, n, 1)$) are created by ρ_1 (resp. ρ_0), where the notion for the index of a node in the fibers of φ is introduced in the proof of Theorem 3.9. Let $D_{i,1} = \rho_1\big(\tilde{\rho}^{-1}(D_i)\big)$. Then by (3-10),

$$\begin{cases} \displaystyle\sum_{i=1}^{\alpha} D_{i,1}^2 = \sum_{i=1}^{\alpha} \big(\tilde{\rho}^{-1}(D_i)\big)^2 + 4\xi_{2,n,1} = \sum_{i=1}^{\alpha} \big(\tilde{\rho}^{-1}(D_i)\big)^2 + \frac{4}{n} \cdot s_{2,n,1}; \\ \displaystyle D_{\alpha,1}^2 = \big(\tilde{\rho}^{-1}(D_\alpha)\big)^2 + \xi_{2,n,1} = \big(\tilde{\rho}^{-1}(D_\alpha)\big)^2 + \frac{1}{n} \cdot s_{2,n,1} \end{cases}$$

Note also that the sections $\{D_i\}_{i=1}^{\alpha}$ do not pass through any singularity of Y. Hence $\{\tilde{\rho}^{-1}(D_i)\}_{i=1}^{\alpha}$ are still disjoint sections, and by (3-12b) one gets

$$\begin{cases} \displaystyle\sum_{i=1}^{\alpha} \big(\tilde{\rho}^{-1}(D_i)\big)^2 = \sum_{i=1}^{\alpha} D_i^2 = -\sum_{\gamma,\ell} \frac{n\gamma(\alpha-\gamma)}{\ell^2(\alpha-1)} \cdot (s_{\gamma,\ell,0} + s_{\gamma,\ell,1}); \\ \displaystyle\big(\tilde{\rho}^{-1}(D_\alpha)\big)^2 = D_\alpha^2 = -\sum_{\gamma,\ell} \left(\frac{n\gamma(\gamma-1)}{\ell^2(\alpha-1)(\alpha-2)} s_{\gamma,\ell,0} + \frac{n(\alpha-\gamma)(\alpha-\gamma-1)}{\ell^2(\alpha-1)(\alpha-2)} s_{\gamma,\ell,1} \right). \end{cases}$$

Therefore, it suffices to show that $\sum_{i=1}^{\alpha} D_{i,1}^2 \leq 0$ and $D_{\alpha,1}^2 \leq 0$, which follow directly from the next lemma. □

LEMMA 3.12. – *Keep the assumptions as in the proposition above. Then the divisor* $R_1 = \sum_{i=1}^{\alpha} D_{i,1}$ *is semi-negative definite.*

Proof. – Let $\psi : \widetilde{S} \to S$ be the minimal blowing-up such that there exists a morphism $\widetilde{\Pi} : \widetilde{S} \to \widetilde{Y}$ with the following commutative diagram:

$$\begin{tikzcd}
\widetilde{S} \arrow[r, "\widetilde{\Pi}"] \arrow[d, "\psi"'] & \widetilde{Y} \arrow[r, "\rho_1"] \arrow[d, "\tilde{\rho}"] & Y_1 \\
S \arrow[r, "\Pi"] & Y. &
\end{tikzcd}$$

Let $\big(\rho_1 \circ \widetilde{\Pi}\big)^{-1}(R_1) \subseteq \widetilde{S}$ be the total inverse image of R_1, and $\widetilde{R} \subseteq \widetilde{S}$ be its support. Then it suffices to show that the divisor $\widetilde{R}$ is semi-negative definite.

By construction, the action of $G = \mathbb{Z}/n\mathbb{Z}$ on S lifts to $\widetilde{S}$. Let $\mathrm{Alb}(\widetilde{S})$ be the Albanese variety of $\widetilde{S}$, and τ be any generator of G. Then $\tilde{f} := f \circ \psi$ induces a map $\mathrm{Alb}(\tilde{f}) : \mathrm{Alb}(\widetilde{S}) \to \mathrm{Alb}(B)$ and τ has a natural action on $\mathrm{Alb}(\widetilde{S})$. Let

$$\mathrm{Alb}_0(\widetilde{S}) = \left\{ x \in \mathrm{Alb}(\widetilde{S}) \mid \sum_{i=1}^{n} \tau^i(x) = e \right\}, \quad \text{where } e \in \mathrm{Alb}(\widetilde{S}) \text{ is the identity element.}$$

Then we claim that

Claim 3.13. – *$\mathrm{Alb}(\widetilde{S})$ is isogenous to $\mathrm{Alb}_0(\widetilde{S}) \oplus \mathrm{Alb}(\tilde{f})^{-1}\big(\mathrm{Alb}(B)\big)$ and $\dim \mathrm{Alb}_0(\widetilde{S}) = q_f$.*

Proof of Claim 3.13. – Note that τ has a natural action on $H^0\big(\widetilde{S}, \Omega^1_{\widetilde{S}}\big)$ by pulling-back, and the map $\tilde{f}$ induces an injection

$$\tilde{f}^* : H^0\big(B, \Omega^1_B\big) \hookrightarrow H^0\big(\widetilde{S}, \Omega^1_{\widetilde{S}}\big), \text{ such that } \tilde{f}^* H^0\big(B, \Omega^1_B\big) = \Big\{ \omega \in H^0\big(\widetilde{S}, \Omega^1_{\widetilde{S}}\big) \mid \tau^*(\omega) = \omega \Big\}.$$

To prove this claim, it suffices to show that

(3-17)
$$H^0\big(\widetilde{S}, \Omega^1_{\widetilde{S}}\big) = \tilde{f}^* H^0\big(B, \Omega^1_B\big) \oplus W, \quad \text{where } W = \left\{ \omega \in H^0\big(\widetilde{S}, \Omega^1_{\widetilde{S}}\big) \mid \sum_{i=1}^{n} (\tau^*)^i(\omega) = 0 \right\}.$$

On the one hand, it is clear that $\tilde{f}^* H^0\big(B, \Omega^1_B\big) \cap W = 0$. On the other hand, for any $\omega \in H^0\big(\widetilde{S}, \Omega^1_{\widetilde{S}}\big)$, let $\omega' = \frac{1}{n} \sum\limits_{i=1}^{n} (\tau^*)^i(\omega)$. Then it is easy to verify that $\omega' \in \tilde{f}^* H^0\big(B, \Omega^1_B\big)$, and that $\omega - \omega' \in W$. Hence we obtain the decomposition $\omega = \omega' + (\omega - \omega')$ as required. □

Denote by $J_0 : \widetilde{S} \to \mathrm{Alb}_0(\widetilde{S})$ the induced map. Then we claim that

Claim 3.14. – *Let $\widetilde{B}_1 \subseteq \widetilde{S}$ be the strict inverse image of R_1. Then $\widetilde{B}_1$ is contracted by J_0.*

Proof of Claim 3.14. – Let $D \subseteq \widetilde{B}_1$ be any irreducible component, $\widetilde{D}$ its normalization, $j : \widetilde{D} \to \widetilde{S}$ the induced map and $\vartheta = J_0 \circ j : \widetilde{D} \to \mathrm{Alb}_0(\widetilde{S})$ the composition. We have to prove that $\vartheta(\widetilde{D})$ is a point.

We prove by contradiction. Assume that $\vartheta(\widetilde{D})$ is one-dimensional. Then the induced map

$$\vartheta^* : H^0\left(\mathrm{Alb}_0(\widetilde{S}), \Omega^1_{\mathrm{Alb}_0(\widetilde{S})}\right) \longrightarrow H^0\left(\widetilde{D}, \Omega^1_{\widetilde{D}}\right)$$

is non-zero. Also, it is clear that ϑ^* factors through

$$H^0\left(\mathrm{Alb}_0(\widetilde{S}), \Omega^1_{\mathrm{Alb}_0(\widetilde{S})}\right) \xrightarrow{J_0^*} H^0\left(\widetilde{S}, \Omega^1_{\widetilde{S}}\right) \xrightarrow{j^*} H^0\left(\widetilde{D}, \Omega^1_{\widetilde{D}}\right).$$

By (3-17), there is a decomposition of the form $H^0\left(\widetilde{S}, \Omega^1_{\widetilde{S}}\right) = \tilde{f}^* H^0(B, \Omega^1_B) \oplus W$, and W contains the image of J_0^* according to the proof of Claim 3.13. To deduce a contradiction, it suffices to prove that the restriction

$$j^*|_W : W \longrightarrow H^0\left(\widetilde{D}, \Omega^1_{\widetilde{D}}\right)$$

is zero.

In fact, let $p \in D$ be an arbitrary smooth point of D. Note that τ fixes D due to (3-16). Hence locally around p, there exists a local coordinate (x, y) such that C is defined by $y = 0$ and the action of τ is given by $\tau(x, y) = (x, \epsilon y)$, where ϵ is a primitive n-th root of 1. For any 1-form

$$\omega = \zeta(x, y)dx + \eta(x, y)dy \in H^0\left(\widetilde{S}, \Omega^1_{\widetilde{S}}\right),$$

one has

$$\omega \in W \iff \sum_{i=1}^{n} \zeta(x, \epsilon^i y) = 0, \quad \sum_{i=1}^{n} \epsilon^i \eta(x, \epsilon^i y) = 0.$$

Hence if $\omega \in W$, y should divide the function $\zeta(x, y)$, i.e., $\zeta(x, y) = y \cdot \tilde{\zeta}(x, y)$ for some function $\tilde{\zeta}(x, y)$. In other words, $j^*\omega|_{j^{-1}(p)} = 0$ for any $\omega \in W$. It follows that $j^*\omega = 0$ for any $\omega \in W$ since p is arbitrary. This completes the proof. □

Come back to the proof of Lemma 3.12. By construction, the divisor $\widetilde{R}$ consists of $\widetilde{B}_1$ plus some rational curves. According to Claim 3.14, $\widetilde{R}$ is contracted by the map $J_0 : \widetilde{S} \to \mathrm{Alb}_0(\widetilde{S})$. Note that the image $J_0(\widetilde{S})$ generates the abelian variety $\mathrm{Alb}_0(\widetilde{S})$ with $\dim \mathrm{Alb}_0(\widetilde{S}) = q_f$ by Claim 3.13. Since $q_f > 0$, it follows that $J_0(\widetilde{S})$ is not a point, i.e., $\dim J_0(\widetilde{S}) \geq 1$. Therefore by the Hodge index theorem, $\widetilde{R}$ is semi-negative definite, and hence R_1 is also semi-negative definite. This completes the proof of Lemma 3.12. □

3.3. The slope

Based on Theorem 3.9 and Proposition 3.11, we will prove the following inequalities.

Proposition 3.15. – *Let $f : S \to B$ be a family of semi-stable n-superelliptic curves as in Theorem 3.9, and let $\Upsilon_{nc} \to \Delta_{nc}$ be the singular fibers with non-compact Jacobians.*

(i). *If $\Delta_{nc} = \emptyset$ and $g \geq n$, then*

$$\omega^2_{S/B} \geq \lambda_{n,c} \cdot \deg f_*\omega_{S/B} + 2\delta_1(f) + 3\delta_h(f), \tag{3-18}$$

where

(3-19)

$$\lambda_{n,c} = \begin{cases} 12 - \dfrac{9(\alpha-1)}{2(\alpha-3)}, & \text{if } n = 3 \text{ and } \alpha \geq 6; \\[2mm] 12 - \dfrac{3(\alpha-1)}{\alpha-3}, & \text{if } n = 4 \text{ and } \alpha = 4k+3 \text{ with } k \geq 1; \\[2mm] 12 - \dfrac{48(\alpha-1)}{\big(n^2-(n/d)^2\big)(\alpha-3)}, & \text{otherwise; here } d = \begin{cases} n, & \text{if } n \mid \alpha; \\ \dfrac{n}{\gcd(n, \alpha_0)}, & \text{if } n \nmid \alpha. \end{cases} \end{cases}$$

(ii). *Assume that $\Delta_{nc} \neq \emptyset$, $g \geq 4$ and $q_f > 0$. If either $n = 3$ or 4, then*

$$\omega^2_{S/B} \geq \lambda_{n,nc} \cdot \deg f_*\omega_{S/B} + 2\delta_1(f) + 3\delta_h(f), \tag{3-20}$$

where

$$\lambda_{3,nc}=\begin{cases}\dfrac{6\alpha-18}{\alpha-2}, & \textit{if } \alpha=3k+2 \textit{ with } k\geq 2,\\ \dfrac{15\alpha-63}{2(\alpha-3)}, & \textit{otherwise};\end{cases}\tag{3-21}$$

$$\lambda_{4,nc}=\begin{cases}\dfrac{14\alpha-44}{2\alpha-5}, & \textit{if } \alpha=4k+2 \textit{ with } k\geq 1,\\ \dfrac{9\alpha-33}{\alpha-3}, & \textit{otherwise}.\end{cases}\tag{3-22}$$

Proof. – We use the notation introduced in subsection 3.2.

(i). Since $\Delta_{nc}=\emptyset$, one gets that $\delta_0(f)=0$, from which and Proposition 3.7, it follows that $s_{\gamma,\ell,0}=s_{\gamma,\ell,1}=0$ for any $\ell>1$, and

$$s_{\gamma,1,0}=0,\ \text{ if } \gcd(\gamma,n)\neq 1,\quad \text{and}\quad s_{\gamma,1,1}=0,\ \text{ if } \gcd(\alpha-\gamma,n)\neq 1.$$

Moreover,

$$\delta_1(f)=\begin{cases}s_{2,1,0}, & \text{if } n=3;\\ s_{2,1,1}, & \text{if } n=4 \text{ and } \alpha=4k+3 \text{ with } k\geq 1;\\ 0, & \text{otherwise}.\end{cases}$$

$$\delta_h(f)=\sum_{\gamma=2}^{[\alpha/2]}(s_{\gamma,1,0}+s_{\gamma,1,1})-\delta_1(f).$$

Hence by Theorem 3.9 we obtain

$$\begin{aligned}\omega_{S/B}^2-\lambda_{n,c}\cdot\deg f_*\omega_{S/B}&\\ =&\begin{cases}\displaystyle\sum_{\gamma=2}^{[\alpha/2]}\Big(z_\gamma-\frac{\lambda_{n,c}}{12}(z_\gamma+1)\Big)s_{\gamma,1,0}, & \text{if } n\mid\alpha_0,\\ \displaystyle\sum_{\gamma=2}^{[\alpha/2]}\Big(z_\gamma-\frac{\lambda_{n,c}}{12}(z_\gamma+1)\Big)s_{\gamma,1,0}+\sum_{\gamma=2}^{[\alpha/2]}\Big(z'_\gamma-\frac{\lambda_{n,c}}{12}(z'_\gamma+1)\Big)s_{\gamma,1,1}, & \text{if } n\nmid\alpha_0,\end{cases}\\ \geq&\, 2\delta_1(f)+3\delta_h(f),\end{aligned}$$

where

$$z_\gamma=\frac{(n^2-1)\gamma(\alpha-\gamma)}{\alpha-1}-n^2-\frac{(n^2-d^2)\gamma(\gamma-1)}{d^2(\alpha-1)(\alpha-2)}\ \text{ with } d=\begin{cases}n, & \text{if } n\mid\alpha_0,\\ r_\infty, & \text{if } n\nmid\alpha_0,\end{cases}$$

$$z'_\gamma=\frac{(n^2-1)\gamma(\alpha-\gamma)}{\alpha-1}-n^2-\frac{(n^2-d^2)(\alpha-\gamma)(\alpha-\gamma-1)}{d^2(\alpha-1)(\alpha-2)}.$$

This completes the proof.

(ii). We first prove the case where $n=3$. Since $n=3$, it follows from (1-1) that $\alpha=3k$ or $3k+2$ for some integer k. According to (3-5) and (3-6), when $\alpha=3k$, we have

$$s_{\gamma,\ell,1}=0;\qquad s_{\gamma,1,0}=0,\ \text{ if } 3\mid\gamma;\qquad s_{\gamma,3,0}=0,\ \text{ if } 3\nmid\gamma;$$

and when $3k+2$, we have

$$\begin{cases} s_{\gamma,1,0}=0, & \text{if } 3\,|\,\gamma; \qquad s_{\gamma,1,1}=0, \quad \text{if } 3\,|\,(\gamma+1);\\ s_{\gamma,3,0}=0, & \text{if } 3\nmid\gamma; \qquad s_{\gamma,3,1}=0, \quad \text{if } 3\nmid(\gamma+1). \end{cases}$$

Hence

$$\delta_0(f)=\sum_{\gamma=2}^{[\alpha/2]}(s_{\gamma,3,0}+s_{\gamma,3,1}),\quad \delta_1(f)=s_{2,1,0},\quad \delta_h(f)=\sum_{\gamma=3}^{[\alpha/2]}(s_{\gamma,1,0}+s_{\gamma,1,1}). \tag{3-23}$$

If $\alpha=3k$, then $s_{2,3,1}=0$; moreover, since $g\geq 4$, one has $k\geq 2$ by (3-4). Hence by Theorem 3.9 one obtains that (where $\lambda_1=\frac{15\alpha-63}{2(\alpha-3)}$)

$$\begin{aligned}\omega_{S/B}^2-\lambda_1\cdot\deg f_*\omega_{S/B}=&\sum_{\gamma=2}^{[\alpha/2]}\bigg(\Big(\frac{8\gamma(\alpha-\gamma)}{\alpha-1}-9\Big)-\frac{\lambda_1}{12}\cdot\Big(\frac{8\gamma(\alpha-\gamma)}{\alpha-1}-8\Big)\bigg)(s_{\gamma,1,0}+s_{\gamma,1,1})\\ &+\sum_{\gamma=3}^{[\alpha/2]}\bigg(\Big(\frac{8\gamma(\alpha-\gamma)}{9(\alpha-1)}-1\Big)-\frac{\lambda_1}{12}\cdot\frac{8\gamma(\alpha-\gamma)}{9(\alpha-1)}\bigg)(s_{\gamma,3,0}+s_{\gamma,3,1})\\ \geq&\, 2\delta_1(f)+3\delta_h(f).\end{aligned}$$

If $\alpha=3k+2$, then one again has $k\geq 2$. If moveover $s_{2,3,1}=0$, then one can show again that

$$\omega_{S/B}^2\geq\frac{15\alpha-63}{2(\alpha-3)}\cdot\deg f_*\omega_{S/B}+2\delta_1(f)+3\delta_h(f).$$

Hence we may assume that $s_{2,3,1}\neq 0$. In this case, we have (where $\lambda_2=\frac{6\alpha-18}{\alpha-2}$)

$$\begin{aligned}&\omega_{S/B}^2-\lambda_2\cdot\deg f_*\omega_{S/B}\\ &\quad=\sum_{\gamma=2}^{[\alpha/2]}\bigg(\Big(\frac{8\gamma(\alpha-\gamma)}{\alpha-1}-9\Big)-\frac{\lambda_2}{12}\cdot\Big(\frac{8\gamma(\alpha-\gamma)}{\alpha-1}-8\Big)\bigg)(s_{\gamma,1,0}+s_{\gamma,1,1})\\ &\qquad-\frac{1}{9}s_{2,3,1}+\sum_{\gamma=3}^{[\alpha/2]}\bigg(\Big(\frac{8\gamma(\alpha-\gamma)}{9(\alpha-1)}-1\Big)-\frac{\lambda_2}{12}\cdot\frac{8\gamma(\alpha-\gamma)}{9(\alpha-1)}\bigg)(s_{\gamma,3,0}+s_{\gamma,3,1}).\end{aligned}\tag{3-24}$$

Note that $s_{2,1,1}=0$ in this case. Hence by (3-15) for $n=3$ one obtains that

$$\begin{cases} s_{2,3,1}\leq\displaystyle\sum_{\gamma=2}^{[\alpha/2]}\frac{9\gamma(\alpha-\gamma)}{2\alpha}\Big(s_{\gamma,1,0}+\frac{s_{\gamma,3,0}}{9}\Big)+\sum_{\gamma=3}^{[\alpha/2]}\frac{9\gamma(\alpha-\gamma)}{2\alpha}\Big(s_{\gamma,1,1}+\frac{s_{\gamma,3,1}}{9}\Big);\\ s_{2,3,1}\leq\displaystyle\sum_{\gamma=2}^{[\alpha/2]}\frac{9\gamma(\gamma-1)}{2(\alpha-2)}\Big(s_{\gamma,1,0}+\frac{s_{\gamma,3,0}}{9}\Big)+\sum_{\gamma=3}^{[\alpha/2]}\frac{9(\alpha-\gamma)(\alpha-\gamma-1)}{2(\alpha-2)}\Big(s_{\gamma,1,1}+\frac{s_{\gamma,3,1}}{9}\Big).\end{cases}$$

Let $x=\frac{\alpha(\alpha-7)}{(\alpha-1)(\alpha-4)}$. Then $0<x<1$ since $\alpha\geq 8$. Hence

$$\begin{aligned}
s_{2,3,1}&\leq \sum_{\gamma=2}^{[\alpha/2]}\left(\frac{9\gamma(\alpha-\gamma)}{2\alpha}\cdot x+\frac{9\gamma(\gamma-1)}{2(\alpha-2)}\cdot(1-x)\right)\left(s_{\gamma,1,0}+\frac{s_{\gamma,3,0}}{9}\right)\\
&\quad+\sum_{\gamma=3}^{[\alpha/2]}\left(\frac{9\gamma(\alpha-\gamma)}{2\alpha}\cdot x+\frac{9(\alpha-\gamma)(\alpha-\gamma-1)}{2(\alpha-2)}\cdot(1-x)\right)\left(s_{\gamma,1,1}+\frac{s_{\gamma,3,1}}{9}\right)\\
&=\sum_{\gamma=2}^{[\alpha/2]}\frac{9\gamma\big(\alpha^2-(\gamma+8)\alpha+10\gamma+4\big)}{2(\alpha-2)(\alpha-4)}\left(s_{\gamma,1,0}+\frac{s_{\gamma,3,0}}{9}\right)\\
&\quad+\sum_{\gamma=3}^{[\alpha/2]}\frac{9(\alpha-\gamma)\big((\gamma+2)\alpha-10\gamma+4\big)}{2(\alpha-2)(\alpha-4)}\left(s_{\gamma,1,1}+\frac{s_{\gamma,3,1}}{9}\right).
\end{aligned}\tag{3-25}$$

Combining (3-25) together with (3-24) and (3-23), we obtain that if $s_{2,3,1}>0$, then

$$\omega_{S/B}^2\geq\frac{6\alpha-18}{\alpha-2}\cdot\deg f_*\omega_{S/B}+2\delta_1(f)+3\delta_h(f).$$

This completes the proof of (3-20) for the case where $n=3$.

The case where $n=4$ can be proven along the same idea. In fact, by (1-1) one has $\alpha=4k$, $4k+2$ or $4k+3$ for some integer k. According to (3-5) and (3-6), we have

$$s_{\gamma,1,0}=s_{\gamma,2,0}=0,\ \text{if } 4\,|\,\gamma;\quad s_{\gamma,1,0}=s_{\gamma,4,0}=0,\ \text{if } 4\nmid\gamma \text{ but } 2\,|\,\gamma;\quad s_{\gamma,2,0}=s_{\gamma,4,0}=0,\ \text{if } 2\nmid\gamma.$$

Moreover, when $\alpha=4k$, it holds

$$s_{\gamma,\ell,1}=0,\ \text{for all } \gamma,\ell;$$

and when $\alpha=4k+t$ for $t=2$ or 3, it holds

$$\begin{cases}s_{\gamma,1,1}=s_{\gamma,2,1}=0, & \text{if } 4\,|\,(\gamma+t);\\ s_{\gamma,1,1}=s_{\gamma,4,1}=0, & \text{if } 4\nmid(\gamma+t) \text{ but } 2\,|\,(\gamma+t);\\ s_{\gamma,2,1}=s_{\gamma,4,1}=0, & \text{if } 2\nmid(\gamma+t).\end{cases}\tag{3-26}$$

Hence

$$\begin{cases}\delta_0(f)=\sum\limits_{\gamma=2}^{[\alpha/2]}(s_{\gamma,2,0}+s_{\gamma,2,1}+s_{\gamma,4,0}+s_{\gamma,4,1}),\\ \delta_1(f)=s_{2,1,1},\\ \delta_h(f)=\sum\limits_{\gamma=3}^{[\alpha/2]}(s_{\gamma,1,0}+s_{\gamma,1,1}).\end{cases}\tag{3-27}$$

If $\alpha = 4k$, then $s_{2,4,1} = 0$; moreover, since $g \geq 4$, by (3-4) one has $k \geq 2$ if $\alpha = 4k$. Hence by Theorem 3.9 one obtains that (where $\lambda_3 = \frac{9\alpha-33}{\alpha-3}$)

$$\begin{aligned}
&\omega_{S/B}^2 - \lambda_3 \cdot \deg f_*\omega_{S/B} \\
&\quad = \sum_{\gamma=3}^{[\alpha/2]} \Big(\Big(\frac{15\gamma(\alpha-\gamma)}{\alpha-1} - 16\Big) - \frac{\lambda_3}{12} \cdot \Big(\frac{15\gamma(\alpha-\gamma)}{\alpha-1} - 15\Big)\Big)(s_{\gamma,1,0} + s_{\gamma,1,1}) \\
&\qquad + \sum_{\gamma=2}^{[\alpha/2]} \Big(\Big(\frac{15\gamma(\alpha-\gamma)}{4(\alpha-1)} - 4\Big) - \frac{\lambda_3}{12} \cdot \Big(\frac{15\gamma(\alpha-\gamma)}{4(\alpha-1)} - 3\Big)\Big)(s_{\gamma,2,0} + s_{\gamma,2,1}) \\
&\qquad + \sum_{\gamma=4}^{[\alpha/2]} \Big(\Big(\frac{15\gamma(\alpha-\gamma)}{16(\alpha-1)} - 1\Big) - \frac{\lambda_3}{12} \cdot \frac{15\gamma(\alpha-\gamma)}{16(\alpha-1)}\Big)(s_{\gamma,4,0} + s_{\gamma,4,1}) \\
&\quad \geq 2\delta_1(f) + 3\delta_h(f).
\end{aligned}$$

If $\alpha = 4k+3$, then $k \geq 1$ since $g \geq 4$, and we still have $s_{2,4,1} = 0$. Based on Theorem 3.9, one shows similarly as above that

$$\omega_{S/B}^2 - \lambda_3 \cdot \deg f_*\omega_{S/B} \geq 2\delta_1(f) + 3\delta_h(f).$$

Finally, we consider the case where $\alpha = 4k+2$. In this case $k \geq 1$ since $g \geq 4$. If moveover $s_{2,4,1} = 0$, then we can show again that

$$\omega_{S/B}^2 - \lambda_3 \cdot \deg f_*\omega_{S/B} \geq 2\delta_1(f) + 3\delta_h(f).$$

Hence we may assume that $s_{2,4,1} \neq 0$. In this case, we have (where $\lambda_4 = \frac{14\alpha-44}{2\alpha-5}$)

(3-28)

$$\begin{aligned}
&\omega_{S/B}^2 - \lambda_4 \cdot \deg f_*\omega_{S/B} \\
&\quad = \sum_{\gamma=3}^{[\alpha/2]} \Big(\Big(\frac{15\gamma(\alpha-\gamma)}{\alpha-1} - 16\Big) - \frac{\lambda_4}{12} \cdot \Big(\frac{15\gamma(\alpha-\gamma)}{\alpha-1} - 15\Big)\Big)(s_{\gamma,1,0} + s_{\gamma,1,1}) \\
&\qquad + \sum_{\gamma=2}^{[\alpha/2]} \Big(\Big(\frac{15\gamma(\alpha-\gamma)}{4(\alpha-1)} - 4\Big) - \frac{\lambda_4}{12} \cdot \Big(\frac{15\gamma(\alpha-\gamma)}{4(\alpha-1)} - 3\Big)\Big)(s_{\gamma,2,0} + s_{\gamma,2,1}) \\
&\qquad + \sum_{\gamma=4}^{[\alpha/2]} \Big(\Big(\frac{15\gamma(\alpha-\gamma)}{16(\alpha-1)} - 1\Big) - \frac{\lambda_4}{12} \cdot \frac{15\gamma(\alpha-\gamma)}{16(\alpha-1)}\Big)(s_{\gamma,4,0} + s_{\gamma,4,1}) \\
&\qquad + \Big(\frac{7\alpha-22}{8(\alpha-1)} - \frac{5(\alpha-2)}{32(\alpha-1)} \cdot \lambda_4\Big) s_{2,4,1}.
\end{aligned}$$

According to the first inequality in (3-15) for $n = 4$, one obtains

$$s_{2,4,1} \leq \sum_{(\gamma,\ell)\neq(2,4)} \frac{8\gamma(\alpha-\gamma)}{\ell^2\alpha} \cdot (s_{\gamma,\ell,0} + s_{\gamma,\ell,0}).$$

Combining this together with (3-26), (3-27) and (3-28), one shows that

$$\omega_{S/B}^2 \geq \frac{14\alpha-44}{2\alpha-5} \cdot \deg f_*\omega_{S/B} + 2\delta_1(f) + 3\delta_h(f).$$

This completes the proof of (3-20) for the case where $n = 4$. □

COROLLARY 3.16. – *Let $f : S \to B$ be a family of semi-stable superelliptic curves as in Proposition 3.15.*

(i). *If $\Delta_{nc} = \emptyset$ and $g \geq n$, then*

$$\deg f_*\omega_{S/B} \leq \frac{2g-2}{\lambda_{n,c}} \cdot \deg \Omega^1_B, \tag{3-29}$$

where $\lambda_{n,c}$ is defined in (3-19).

(ii). *Assume that $\Delta_{nc} \neq \emptyset$, $g \geq 4$ and $q_f > 0$. If either $n = 3$ or 4, then*

$$\deg f_*\omega_{S/B} < \frac{2g-2}{\lambda_{n,nc}} \cdot \deg \Omega^1_B(\log \Delta_{nc}), \tag{3-30}$$

where $\lambda_{3,nc}$ and $\lambda_{4,nc}$ are defined in (3-21) *and* (3-22) *respectively.*

Proof. – The above lemma follows directly from the above slope inequalities for a family of semi-stable superelliptic curves and the Miyaoka-Yau type inequality (cf. [23, Theorem 4.1]). □

3.4. Existence of horizontal fibration structures

The main purpose of this subsection is to prove the existence of horizontal fibration structures on the total space of a family of semi-stable superelliptic curves up to base change under certain assumptions, namely Proposition 3.24. We will freely use the notations introduced at the beginning of subsection 3.2.

As the main technique is based on cyclic covers, we always assume in this subsection that the group $G = \mathbb{Z}/n\mathbb{Z}$ admits an action on S whose restriction on the general fiber is the n-superelliptic automorphism group. As we see in Lemma 3.3, one can achieve this by a suitable finite surjective base change (not necessarily étale).

Since the group G admits an action on the surface S, it induces a natural action on the cohomology groups of S, and on the local system $\mathbb{V}_{B_0} := R^1 f_*\mathbb{Q}_{S\setminus\Upsilon}$ and hence also on its associated logarithmic Higgs bundle $\left(E^{1,0}_B \oplus E^{0,1}_B,\, \theta_B\right)$. In our case, the Higgs bundle has the form

$$E^{1,0}_B \cong f_*\Omega^1_{S/B}(\log \Upsilon), \qquad E^{0,1}_B \cong R^1 f_*\mathcal{O}_S,$$

and the Higgs field

$$\theta_B : E^{1,0}_B \longrightarrow E^{0,1}_B \otimes \Omega_B(\log \Delta_{nc})$$

is induced by the edge morphism of the tautological sequence

$$0 \longrightarrow f^*\Omega^1_B(\log \Delta) \longrightarrow \Omega^1_S(\log \Upsilon) \longrightarrow \Omega^1_{S/B}(\log \Upsilon) \longrightarrow 0.$$

Since f is semi-stable, one checks that the sheaf $\Omega^1_{S/B}(\log \Upsilon)$ is locally free; indeed, it suffices to verify the local-freeness around a node p in fibers of f. Assume the family f is locally given by $t = xy$ around p. Then around p, $f^*\Omega^1_B(\log \Delta)$ is generated by $\frac{f^*(dt)}{f^*(t)} = \frac{dx}{x} + \frac{dy}{y}$, and $\Omega^1_S(\log \Upsilon)$ is generated by $\left\{\frac{dx}{x}, \frac{dy}{y}\right\}$, which implies that $\Omega^1_{S/B}(\log \Upsilon)$ is locally free around p. Hence

$$\Omega^1_{S/B}(\log \Upsilon) \cong \mathcal{O}_S\Big(c_1\big(\Omega^1_{S/B}(\log \Upsilon)\big)\Big) = \omega_{S/B}. \tag{3-31}$$

Consider the corresponding eigenspace decompositions

$$\mathbb{V}_{B_0} \otimes \mathbb{C} = \bigoplus_{i=0}^{n-1} \mathbb{V}_{B_0,i}; \qquad \left(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B\right) = \bigoplus_{i=0}^{n-1} \left(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B\right)_i. \tag{3-32}$$

Since the quotient $Y = S/G$ is ruled over B, it follows that

$$\mathbb{V}_{B_0,0} = 0, \qquad \text{and} \qquad \left(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B\right)_0 = 0.$$

Moreover, by a formula due to Hurwitz-Chevalley-Weil (cf. [26, Proposition 5.9]), one has

$$\operatorname{rank} E_{B,i}^{1,0} = \operatorname{rank} E_{B,n-i}^{0,1} = \begin{cases} \dfrac{(n-i)\alpha_0}{n} - 1, & \text{if } n \mid \alpha_0, \text{ or } n \nmid \alpha_0 \text{ and } \dfrac{(n-i)\alpha_0}{n} \in \mathbb{Z}; \\ \left[\dfrac{(n-i)\alpha_0}{n}\right], & \text{if } n \nmid \alpha_0 \text{ and } \dfrac{(n-i)\alpha_0}{n} \notin \mathbb{Z}. \end{cases} \tag{3-33}$$

The eigenspace decomposition on $\left(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B\right)$ induces thus eigenspace decompositions on its associated subbundles in (4-2):

$$\begin{cases} \left(A_B^{1,0} \oplus A_B^{0,1},\ \theta_B|_{A_B^{1,0}}\right) = \displaystyle\bigoplus_{i=1}^{n-1} \left(A_B^{1,0} \oplus A_B^{0,1},\ \theta_B|_{A_B^{1,0}}\right)_i, \\ \left(F_B^{1,0} \oplus F_B^{0,1},\ 0\right) = \displaystyle\bigoplus_{i=1}^{n-1} \left(F_B^{1,0} \oplus F_B^{0,1},\ 0\right)_i. \end{cases} \tag{3-34}$$

Note that the Galois cover $\Pi : S \to Y = S/G$ induces a Galois cover $\Pi' : S' \to \widetilde{Y}$ whose branch locus is a normal crossing divisor and with $\mathrm{Gal}(\Pi') \cong G$, where $\widetilde{Y} \to Y$ is the minimal resolution of singularities. Let $\widetilde{S} \to S'$ be the minimal resolution of singularities. Then there is an induced birational contraction $\widetilde{S} \to S$ fitting into the following commutative diagram.

$$\begin{tikzcd}
\widetilde{S} \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\tilde{f}"'] \arrow[rr, bend left, "\widetilde{\Pi}"] & S' \arrow[r, "\Pi'"'] & \widetilde{Y} \arrow[d] \arrow[dd, bend left=60, "\tilde{\varphi}"] \\
S \arrow[rr, "\Pi"] \arrow[d, "f"] & & Y \arrow[d, "\varphi"'] \\
B \arrow[rr, equal] & & B
\end{tikzcd}$$

FIGURE 3.1. Induced Galois cover

Since $\Pi' : S' \to \widetilde{Y}$ is a cyclic cover of degree n, we may assume it is defined by the following relation:

$$\mathcal{L}^n \equiv \mathcal{O}_{\widetilde{Y}}\big(\widetilde{\mathfrak{R}}\big), \qquad \text{where `$\equiv$' stands for linear equivalence.}$$

For a general fiber $\widetilde{\Gamma} \cong \mathbb{P}^1$ of $\tilde{\varphi}$, its inverse image is a superelliptic curve admitting a cyclic cover to $\mathbb{P}^1$ with branch locus as in (3-3). Hence

$$\widetilde{\mathfrak{R}}|_{\widetilde{\Gamma}} = \begin{cases} \sum_{i=1}^{\alpha_0} x_i, & \text{if } n|\alpha_0; \\ \sum_{i=1}^{\alpha_0} x_i + a_\infty \infty, & \text{if } n \not| \alpha_0. \end{cases} \tag{3-35}$$

Following [12], if $\widetilde{\mathfrak{R}} = \sum a_j D_j$ is the decomposition into prime divisors, we define

$$\mathcal{L}^{(i)} = \mathcal{L}^i \otimes \mathcal{O}_{\widetilde{Y}}\Big(- \sum \Big[\frac{i a_j}{n}\Big] D_j \Big).$$

Restricting to a general fiber $\widetilde{\Gamma}$ of $\tilde{\varphi}$, it follows from (3-35) that

$$\mathcal{L}^{(i)}|_{\widetilde{\Gamma}} \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1}\Big(\frac{i\alpha_0}{n}\Big), & \text{if } \frac{i\alpha_0}{n} \in \mathbb{Z}; \\ \mathcal{O}_{\mathbb{P}^1}\Big(\Big[\frac{i\alpha_0}{n}\Big] + 1\Big), & \text{if } \frac{i\alpha_0}{n} \notin \mathbb{Z}. \end{cases} \tag{3-36}$$

Let $\widetilde{R} \subseteq \widetilde{Y}$ be the reduced branch divisor of Π', i.e., the support of $\widetilde{\mathfrak{R}}$. Then by [38, Lemma 1.7], one has the inclusion

$$\tau_1 : \ \widetilde{\Pi}_* \Omega^1_{\widetilde{S}} \hookrightarrow \Omega^1_{\widetilde{Y}} \bigoplus \left(\bigoplus_{j=1}^{n-1} \Omega^1_{\widetilde{Y}}(\log \widetilde{R}) \otimes \mathcal{L}^{(j)^{-1}} \right). \tag{3-37}$$

We first prove a general result on local systems generalizing Deligne's original theorem with the constant coefficient.

Lemma 3.17. – *Let $f : X^0 \to B \setminus \Delta$ be a smooth projective morphism over a curve. Let $X \supseteq X^0$ be a smooth compactification of X^0 and $\mathbb{U}$ be a locally constant sheaf on X coming from a representation of $\pi_1(X)$ into a unitary group. Then the canonical homomorphism is surjective:*

$$H^k(X, \mathbb{U}) \longrightarrow H^0(B \setminus \Delta, R^k f_* \mathbb{U}).$$

Proof. – We will follow Deligne's proof for the case that $\mathbb{U} = \mathbb{Q}$ (cf. [9, §4.1]) verbatim.

The unitary locally constant sheaf $\mathbb{U}$ on X carries in a natural way a polarized variation of Hodge structure, say, of pure type $(0, 0)$. Hence it follows from Saito's theory that there is an induced pure Hodge structure of weight k on $H^k(X, \mathbb{U})$ as well as on $H^k(X_b, \mathbb{U}|_{X_b})$ where X_b is any (smooth projective) fiber of $f : X^0 \to B \setminus \Delta$.

We first show the "edge-homomorphism"

$$p_e : \ H^k(X^0, \mathbb{U}) \longrightarrow H^0(B \setminus \Delta, R^k f_* \mathbb{U}) \tag{3-38}$$

is surjective by the following argument from the proof of [30, Proposition 1.38].

Indeed, if we take $h \in H^2(X^0, \mathbb{Q})$ to be the restriction of a hyperplane class of X, then it suffices to show that the cup-products satisfy the hard Lefschetz property, i.e., the following homomorphism is an isomorphism for any $0 \leq k \leq m$, where m is the dimension of a general fiber of f:

$$[\cup h]^k : \ R^{m-k} f_* \mathbb{U} \longrightarrow R^{m+k} f_* \mathbb{U}.$$

Note that the hard Lefschetz property can be verified fiber-by-fiber. On each fiber the natural locally constant metric on $\mathbb{U}$ induces a Hodge decomposition of the cohomology with coefficients in $\mathbb{U}$, hence the hard Lefschetz property holds. So p_e in (3-38) is surjective.

Since the restriction homomorphism (as monodromy invariant)

$$H^0(B \setminus \Delta, R^k f_* \mathbb{U}) \to H^k(X_b, \mathbb{U}|_{X_b})$$

is injective and $H^k(X_b, \mathbb{U}|_{X_b})$ carries a pure Hodge structure of weight-k, one gets that $H^0(B \setminus \Delta, R^k f_* \mathbb{U})$ carries a pure Hodge structure of weight-k. The surjectivity of p_e in (3-38) also induces surjective morphisms between the weight-filtrations of both cohomologies. In particular, we have a surjective homomorphism

$$W_k\big(H^k(X^0, \mathbb{U})\big) \twoheadrightarrow W_k\big(H^0(B \setminus \Delta, R^k f_* \mathbb{U})\big) = H^0(B \setminus \Delta, R^k f_* \mathbb{U}). \tag{3-39}$$

By [29], $W_k\big(H^k(X^0, \mathbb{U})\big)$ is nothing but the image of the restriction homomorphism

$$H^k(X, \mathbb{U}) \to H^k(X^0, \mathbb{U}).$$

Combining this with (3-39), one gets the required surjective homomorphism. □

LEMMA 3.18. – *Let $F_{B,i}^{1,0} \subseteq F_B^{1,0}$ be the eigensubspace as in*(3-34). *For each $1 \le i \le n-1$, there is a sheaf morphism*

$$\varrho_i : \tilde{\varphi}^* F_{B,i}^{1,0} \longrightarrow \widetilde{\Pi}_* \Omega^1_{\widetilde{S}}, \tag{3-40}$$

such that the induced canonical morphism

$$F_{B,i}^{1,0} = \tilde{\varphi}_* \tilde{\varphi}^* F_{B,i}^{1,0} \longrightarrow \tilde{\varphi}_* \widetilde{\Pi}_* \Omega^1_{\widetilde{S}} = \tilde{f}_* \Omega^1_{\widetilde{S}} = f_* \Omega^1_S \longrightarrow f_* \Omega^1_{S/B}(\log \Upsilon) = E_B^{1,0} \tag{3-41}$$

coincides with the inclusion $F_{B,i}^{1,0} \hookrightarrow F_B^{1,0} \hookrightarrow E_B^{1,0}$. Moreover, we may choose ϱ_i so that the image of ϱ_i is contained in $\Omega^1_{\widetilde{Y}}(\log \widetilde{R}) \otimes \mathcal{L}^{(i)^{-1}}$ under the inclusion (3-37).

Proof. – Since the local monodromy of $\mathbb{V}_{B_0}$ around the boundary Δ is unipotent and the local monodromy of the unitary subsheaf $\mathbb{V}^u_{B_0}$ around Δ is semisimple, $\mathbb{V}^u_{B_0}$ extends on B as a locally constant sheaf, still denoted by $\mathbb{V}^u_{B_0}$. The inclusion $\mathbb{V}^u_{B_0} \hookrightarrow \mathbb{V}_{B_0}$ corresponds to a section

$$\eta \in H^0(B_0,\ \mathbb{V}_{B_0} \otimes \big(\mathbb{V}^u_{B_0}\big)^\vee) = H^0(B_0,\ R^1 f_*\big(\mathbb{C}_{f^{-1}(B_0)} \otimes f^*(\mathbb{V}^u_{B_0})^\vee\big)\big).$$

By Lemma 3.17, η lifts to a class $\tilde{\eta} \in H^1(S, f^*(\mathbb{V}^u_{B_0})^\vee)$ under the canonical morphism

$$H^1(S, f^*(\mathbb{V}^u_{B_0})^\vee) \longrightarrow H^0(B_0,\ R^1 f_*\big(\mathbb{C}_{S_0} \otimes f^*(\mathbb{V}^u_{B_0})^\vee\big)\big).$$

Note that this canonical morphism is a morphism between pure Hodge structures of weight-1, and by the construction η is of type (1,0), so $\tilde{\eta}$ is of type (1,0), i.e.,

$$\tilde{\eta} \in H^0(S, \Omega^1_S \otimes f^*(F_B^{1,0})^\vee),$$

which corresponds to a morphism

$$\varrho' : f^* F_B^{1,0} \longrightarrow \Omega^1_S, \tag{3-42}$$

such that the induced canonical morphism

$$F_B^{1,0} = f_* f^* F_B^{1,0} \longrightarrow f_* \Omega^1_S \longrightarrow f_* \Omega^1_S(\log \Upsilon) \longrightarrow f_* \Omega^1_{S/B}(\log \Upsilon) = E_B^{1,0}$$

coincides with the inclusion $F_B^{1,0} \hookrightarrow E_B^{1,0}$. Since $\tilde{\rho}^*\Omega^1_S \subseteq \Omega^1_{\widetilde{S}}$, by pulling back (3-42), we obtain a sheaf morphism

$$\tilde{f}^* F_B^{1,0} = \tilde{\rho}^* f^* F_B^{1,0} \longrightarrow \Omega^1_{\widetilde{S}},$$

which corresponds to an element

$$\tilde{\eta} \in H^0\big(\widetilde{S}, \Omega^1_{\widetilde{S}} \otimes \tilde{f}^*\big(F_B^{1,0}\big)^\vee\big).$$

By pushing-out, we also obtain an element

$$\widetilde{\Pi}_*(\tilde{\eta}) \in H^0\left(\widetilde{Y}, \widetilde{\Pi}_*\big(\Omega^1_{\widetilde{S}} \otimes \tilde{f}^*\big(F_B^{1,0}\big)^\vee\big)\right) = H^0\left(\widetilde{Y}, \widetilde{\Pi}_*\Omega^1_{\widetilde{S}} \otimes \tilde{\varphi}^*\big(F_B^{1,0}\big)^\vee\right).$$

Hence one gets a sheaf morphism

$$\varrho:\ \tilde{\varphi}^* F_B^{1,0} \longrightarrow \widetilde{\Pi}_*\Omega^1_{\widetilde{S}}. \tag{3-43}$$

By restricting to $\tilde{\varphi}^* F_{B,i}^{1,0}$, we obtain ϱ_i as in (3-40) such that the induced morphism (3-41) coincides with the inclusion $F_{B,i}^{1,0} \hookrightarrow F_B^{1,0} \hookrightarrow E_B^{1,0}$.

Note that the group G acts on both sides of (3-42). One may require that the morphism ϱ' is equivariant with respect to G. So it is with the morphism ϱ. Combining (3-43) with (3-37), we obtain a sheaf morphism

$$\tilde{\varphi}^* F_B^{1,0} \longrightarrow \Omega^1_{\widetilde{Y}} \bigoplus \left(\bigoplus_{j=1}^{n-1} \Omega^1_{\widetilde{Y}}(\log \widetilde{R}) \otimes \mathcal{L}^{(j)^{-1}}\right),$$

which is compatible with the G-actions on both sides. Hence the image of ϱ_i is contained in $\Omega^1_{\widetilde{Y}}(\log \widetilde{R}) \otimes \mathcal{L}^{(i)^{-1}}$. □

Lemma 3.19. – *Let $\widetilde{R} \subseteq \widetilde{Y}$ be the reduced branch divisor of Π' as above, and $\widetilde{\Gamma}$ be a general fiber of $\tilde{\varphi}$. Assume that $\widetilde{R}$ contains at least one section of $\tilde{\varphi}$, and that there exist $1 \le i_1 \le i_2 \le n-1$ such that $F_{B,i_1}^{1,0} \neq 0$, $F_{B,i_2}^{1,0} \neq 0$, and*

$$\widetilde{\Gamma} \cdot \left(\omega_{\widetilde{Y}}(\widetilde{R}) \otimes \left(\mathcal{L}^{(i_1)^{-1}} \otimes \mathcal{L}^{(i_2)^{-1}}\right)\right) < 0. \tag{3-44}$$

Then both $F_{B,i_1}^{1,0}$ and $F_{B,i_2}^{1,0}$ become trivial bundles after a suitable finite étale base change.

Proof. – We divide the proof into two steps.

Step I. – We show that for any non-zero unitary subbundle $\hat{\mathcal{U}} \subseteq F_{B,i}^{1,0}$ with $i = i_1$ or i_2, the image $\varrho_i(\tilde{\varphi}^*\hat{\mathcal{U}})$ is an invertible subsheaf $\hat{M}$ such that $\hat{M}$ is numerically effective (nef), $\hat{M}^2 = 0$, and $\hat{M} \cdot D = 0$ for any component $D \subseteq \widetilde{R}_h$, where ϱ_i is given in (3-40).

We only prove it for $i = i_1$, as the case for $i = i_2$ is completely parallel. By Lemma 3.18, the image $\varrho_{i_1}(\tilde{\varphi}^*\hat{\mathcal{U}})$ is contained in $\Omega^1_{\widetilde{Y}}(\log \widetilde{R}) \otimes \mathcal{L}^{(i_1)^{-1}}$, and it is non-zero if $\hat{\mathcal{U}} \neq 0$. Mimicking the proof of [21, Theorem A.1], it suffices to show that the image $\varrho_{i_1}(\tilde{\varphi}^*\hat{\mathcal{U}})$ is a subsheaf of rank one if $\hat{\mathcal{U}} \neq 0$. We prove this by contradiction. Assume that it is not the case. By taking wedge-product, one obtains a non-zero map

$$\begin{aligned}\tau \circ \varrho_{i_1} \wedge \tau \circ \varrho_{i_2} : \tilde{\varphi}^*\hat{\mathcal{U}} \otimes \tilde{\varphi}^* F_{B,i_2}^{1,0} &\longrightarrow \wedge^2\Omega^1_{\widetilde{Y}}(\log \widetilde{R}) \otimes \left(\mathcal{L}^{(i_1)^{-1}} \otimes \mathcal{L}^{(i_2)^{-1}}\right)\\ &= \omega_{\widetilde{Y}}(\widetilde{R}) \otimes \left(\mathcal{L}^{(i_1)^{-1}} \otimes \mathcal{L}^{(i_2)^{-1}}\right).\end{aligned}$$

Denote by $\mathcal{C}$ the image of the above map, we proceed to establish the semi-positivity of $\mathcal{C}$; here we recall that a locally free sheaf $\mathcal{E}$ on $\widetilde{Y}$ is called semi-positive, if for any morphism $\psi : Z \to \widetilde{Y}$ from a smooth complete curve Z, the pulling-back $\psi^*\mathcal{E}$ has no quotient line bundle of negative degree.

- On the one hand, for any morphism $\psi : Z \to \widetilde{Y}$ from a smooth complete curve Z, $\psi^*\big(\tilde{\varphi}^* F_{B,i_1}^{1,0} \otimes \tilde{\varphi}^* F_{B,i_2}^{1,0}\big)$ is poly-stable of slope zero since it comes from a unitary representation (cf. [27]), which implies that $\tilde{\varphi}^*\hat{\mathcal{U}} \otimes \tilde{\varphi}^* F_{B,i_2}^{1,0}$ is semi-positive. Therefore, as a quotient of $\tilde{\varphi}^*\hat{\mathcal{U}} \otimes \tilde{\varphi}^* F_{B,i_2}^{1,0}$, $\mathcal{C}$ is also semi-positive.
- On the other hand, from (3-44) it follows that $\omega_{\widetilde{Y}}(\widetilde{R}) \otimes \left(\mathcal{L}^{(i_1)^{-1}} \otimes \mathcal{L}^{(i_2)^{-1}}\right)$ can not contain any non-zero semi-positive subsheaf. It contradicts the semi-positivity of $\mathcal{C}$.

Hence the image $\varrho_{i_1}(\tilde{\varphi}^*\hat{\mathcal{U}})$ is a subsheaf of rank one as required.

Step II. – We show that both $F_{B,i_1}^{1,0}$ and $F_{B,i_2}^{1,0}$ become trivial after a suitable finite étale base change.

In fact, by [9, §4.2], it suffices to show that $F_{B,i}^{1,0}$ is a direct sum of line bundles after a suitable unramified base change for $i = i_1$ or i_2. By assumption, $\widetilde{R}_h$ contains at least one section of $\tilde{\varphi}$. Let $D \subseteq \widetilde{R}_h$ be such a section, and

$$F_{B,i}^{1,0} = \bigoplus_j \mathcal{U}_{ij}$$

be the decomposition of $F_{B,i}^{1,0}$ into irreducible subbundles. By Step I with the unitary subbundle $\mathcal{U}_{ij} \subseteq F_{B,i}^{1,0}$, we obtain $M_{ij} \cdot D = 0$, i.e., $\deg \mathcal{O}_D(M_{ij}) = 0$, where M_{ij} is the image $\varrho_i(\tilde{\varphi}^*\mathcal{U}_{ij})$. As D is a section, $D \cong B$. Hence we may view $\mathcal{O}_D(M_{ij})$ as an invertible sheaf on B, which is a quotient of $\mathcal{U}_{ij}$ since M_{ij} is a quotient of $\tilde{\varphi}^*\mathcal{U}_{ij}$. As $\mathcal{U}_{ij}$ comes from a unitary local system, $\mathcal{U}_{ij}$ is poly-stable. Thus $\mathcal{U}_{ij} = \mathcal{O}_D(M_{ij}) \oplus \mathcal{U}'_{ij}$. Because $\mathcal{U}_{ij}$ is irreducible, $\mathcal{U}_{ij} = \mathcal{O}_D(M_{ij})$ is a line bundle as required. □

REMARK 3.20. – Let a_∞ be the local monodromy around ∞ for the induced cyclic cover to $\mathbb{P}^1$ on the general fiber of f as in (3-3). If $a_\infty \neq 1$, then $\widetilde{R}$ clearly contains a section of $\tilde{\varphi}$. In general, as pointed out at the beginning in the proof of Theorem 3.9, after a suitable finite base change (may not be étale), $\widetilde{R}$ consists of distinguished sections of $\tilde{\varphi}$ (i.e., the inverse image of $\{D_i\}_{i=1}^{\alpha}$) plus certain components in the fibers of $\tilde{\varphi}$ (i.e., the inverse image of certain nodes in fibers of φ). In particular, we can always achieve the assumption that $\widetilde{R}$ contains at least one section of $\tilde{\varphi}$ using base change (may not be étale).

PROPOSITION 3.21. – *Assume that $\widetilde{R}$ contains at least one section of $\tilde{\varphi}$, and that $F_{B,i_0}^{1,0} \neq 0$ for some $i_0 \geq n/2$. Then after a suitable unramified base change, $F_{B,i}^{1,0}$ is trivial for any $n - i_0 \leq i \leq i_0$.*

Proof. – Note that

$$\widetilde{\Gamma} \cdot \omega_{\widetilde{Y}}(\widetilde{R}) = \begin{cases} \alpha_0 - 2, & \text{if } n \mid \alpha_0; \\ \alpha_0 - 1, & \text{if } n \nmid \alpha_0. \end{cases}$$

Hence by (3-36), one checks easily that

$$\widetilde{\Gamma}\cdot\left(\omega_{\widetilde{Y}}(\widetilde{R})\otimes\left(\mathcal{L}^{(i)^{-1}}\otimes\mathcal{L}^{(i_0)^{-1}}\right)\right)<0,\qquad\forall\, n-i_0\le i\le i_0.$$

Hence by applying Lemma 3.19 to the case where $i_1=i_0$ and $i_2=i$ with $n-i_0\le i\le i_0$ and $F_{B,i}^{1,0}\neq 0$, we complete the proof. □

Since G acts on S, it induces an action on $H^0(S,\Omega_S^1)$. Let

$$H^0\big(S,\Omega_S^1\big)=\bigoplus_{i=0}^{n-1}H^0\big(S,\Omega_S^1\big)_i$$

be the eigenspace decomposition. Then $H^0\big(S,\Omega_S^1\big)_0\cong f^*H^0\big(B,\Omega_B^1\big)$, and according to [14, Theorem 3.1],

$$\dim H^0\big(S,\Omega_S^1\big)_i=\operatorname{rank}\big(F_{B,i}^{1,0}\big)^{\mathrm{tr}},\quad\forall\,1\le i\le n-1,\tag{3-45}$$

where $\big(F_{B,i}^{1,0}\big)^{\mathrm{tr}}\subseteq F_{B,i}^{1,0}$ is the trivial part contained in the flat bundle $F_{B,i}^{1,0}$ as in Lemma 4.2.

LEMMA 3.22. – *Assume that there exist* $1\le i_1\le i_2\le n-1$*, such that* (3-44) *holds, and that* $H^0\big(S,\Omega_S^1\big)_{i_1}\neq 0$ *and* $H^0\big(S,\Omega_S^1\big)_{i_2}\neq 0$*. Then there exists a unique fibration* $f':S\to B'$ *such that*

$$H^0\big(S,\Omega_S^1\big)_i\subseteq\big(f'\big)^*H^0\big(B',\Omega_{B'}^1\big),\qquad\textit{for } i=i_1,i_2.\tag{3-46}$$

Proof. – First, the uniqueness of such a fibration is clear. It suffices to show the existence of such a fibration with the property (3-46). By Castelnuovo-de Franchis lemma (cf. [3, Theorem IV-5.1]), it is enough to show that

$$\omega_{i_1}\wedge\omega_{i_2}=0,\quad\forall\,\omega_{i_1}\in H^0\big(S,\Omega_S^1\big)_{i_1}\text{ and }\omega_{i_2}\in H^0\big(S,\Omega_S^1\big)_{i_2}.\tag{3-47}$$

As an easy case, if $i_1+i_2=n$, then it is clear that

$$\omega_{i_1}\wedge\omega_{i_2}\in H^0\big(S,\Omega_S^2\big)^G=\Pi^*H^0\big(Y,\Omega_Y^2\big)=0,\quad\text{since } Y \text{ is ruled over } B.$$

In general, we have to apply a more detailed description on the differential sheaves associated to cyclic covers due to Esnault-Viehweg [12]. First note that $H^0\big(\widetilde{S},\Omega_{\widetilde{S}}^k\big)\cong H^0\big(S,\Omega_S^k\big)$ for $k=1,2$, and this isomorphism is compatible with the action of the group G. It suffices to prove (3-47) for $\widetilde{S}$. There is an inclusion

$$\iota_k:\ \widetilde{\Pi}_*\Omega_{\widetilde{S}}^k\hookrightarrow\Omega_{\widetilde{Y}}^k\bigoplus\left(\bigoplus_{i=1}^{n-1}\Omega_{\widetilde{Y}}^k(\log\widetilde{R})\otimes\mathcal{L}^{(i)^{-1}}\right).$$

Taking global sections, we obtain an injection

$$\iota_k:\ H^0\big(\widetilde{S},\Omega_{\widetilde{S}}^k\big)=H^0\big(\widetilde{Y},\Pi_*\Omega_{\widetilde{S}}^k\big)\hookrightarrow H^0\big(\widetilde{Y},\Omega_{\widetilde{Y}}^k\big)\bigoplus\left(\bigoplus_{i=1}^{n-1}H^0\big(\widetilde{Y},\Omega_{\widetilde{Y}}^k(\log\widetilde{R})\otimes\mathcal{L}^{(i)^{-1}}\big)\right),$$

which is compatible with the action of the group G on both sides, i.e.,

$$\begin{cases}\iota_k\Big(H^0\big(\widetilde{S},\Omega_{\widetilde{S}}^k\big)_0\Big)\subseteq H^0\big(\widetilde{Y},\Omega_{\widetilde{Y}}^k\big),\\ \iota_k\Big(H^0\big(\widetilde{S},\Omega_{\widetilde{S}}^k\big)_i\Big)\subseteq H^0\big(\widetilde{Y},\Omega_{\widetilde{Y}}^k(\log\widetilde{R})\otimes\mathcal{L}^{(i)^{-1}}\big),\quad\forall\,1\le i\le n-1.\end{cases}$$

By pulling back, we have the following injection map, which is just the inclusion map.

$$\widetilde{\Pi}^*(\iota_k):\ H^0\big(\widetilde{S},\,\Omega^k_{\widetilde{S}}\big)_i \hookrightarrow H^0\big(\widetilde{S},\,\Omega^k_{\widetilde{S}}(\log \widetilde{R}')\otimes \widetilde{\Pi}^*\mathcal{L}^{(i)^{-1}}\big),\quad \forall\, 1\le i\le n-1,$$

where $\widetilde{R}'$ is the support of the divisor $\widetilde{\Pi}^{-1}(\widetilde{R})$. Since the wedge-product operation is clearly commutative with the inclusion map, it follows that for any two 1-forms $\omega_{i_1}\in H^0\big(S,\,\Omega^1_S\big)_{i_1}$ and $\omega_{i_2}\in H^0\big(S,\,\Omega^1_S\big)_{i_2}$, one has

$$\omega_{i_1}\wedge\omega_{i_2}\in H^0\big(\widetilde{S},\,\Omega^k_{\widetilde{S}}(\log \widetilde{R}')\otimes \widetilde{\Pi}^*\mathcal{L}^{(i_1)^{-1}}\otimes \widetilde{\Pi}^*\mathcal{L}^{(i_2)^{-1}}\big).$$

Let $\widetilde{F}\subseteq\widetilde{S}$ be a general fiber of $\tilde{f}$, and $\widetilde{\Gamma}=\widetilde{\Pi}\big(\widetilde{F}\big)$. Then by the assumption (3-44), one obtains that

$$\widetilde{F}\cdot\Big(\Omega^2_{\widetilde{S}}(\log \widetilde{R}')\otimes \widetilde{\Pi}^*\mathcal{L}^{(i_1)^{-1}}\otimes \widetilde{\Pi}^*\mathcal{L}^{(i_2)^{-1}}\Big)=n\,\widetilde{\Gamma}\cdot\Big(\omega_{\widetilde{Y}}(\widetilde{R})\otimes\mathcal{L}^{(i_1)^{-1}}\otimes\mathcal{L}^{(i_2)^{-1}}\Big)<0.$$

Hence

$$H^0\big(\widetilde{Y},\,\Omega^2_{\widetilde{Y}}(\log \widetilde{R})\otimes\mathcal{L}^{(i_1)^{-1}}\otimes\mathcal{L}^{(i_2)^{-1}}\big)=0.$$

This shows that $\omega_{i_1}\wedge\omega_{i_2}=0$ as required. □

COROLLARY 3.23. – *Assume that $H^0\big(S,\,\Omega^1_S\big)_{i_0}\neq 0$ for some $i_0\ge n/2$. Then after a suitable base change, there exists a unique fibration $f':\ S\to B'$ such that*

$$\bigoplus_{i=n-i_0}^{i_0} H^0\big(S,\,\Omega^1_S\big)_i\subseteq\big(f'\big)^*H^0\big(B',\,\Omega^1_{B'}\big). \tag{3-48}$$

Proof. – Since $H^0\big(S,\,\Omega^1_S\big)_{i_0}\neq 0$, it follows from (3-45) that $F^{1,0}_{B,i_0}\neq 0$. Hence by Proposition 3.21 and Remark 3.20, after a suitable base change, $F^{1,0}_{B,i}$ is trivial for any $n-i_0\le i\le i_0$. Combining this with (3-33) and (4-5), one obtains

$$\dim H^0\big(S,\,\Omega^1_S\big)_{n-i_0}=\operatorname{rank}F^{1,0}_{B,n-i_0}\ge\operatorname{rank}F^{1,0}_{B,i_0}=\dim H^0\big(S,\,\Omega^1_S\big)_{i_0}>0.$$

According to the proof of Proposition 3.21, the assumption (3-44) holds if $i_1=i_0$ and $i_2=n-i_0$. Hence by Lemma 3.22, there exists a unique fibration $f'_{n-i_0}:\ S\to B'_{n-i_0}$ such that

$$H^0\big(S,\,\Omega^1_S\big)_{i_0}\oplus H^0\big(S,\,\Omega^1_S\big)_{n-i_0}\subseteq\big(f'_{n-i_0}\big)^*H^0\big(B'_{n-i_0},\,\Omega^1_{B'_{n-i_0}}\big).$$

In fact, the same holds also if we replace $n-i_0$ by any i satisfying that $n-i_0\le i\le i_0$ and that $H^0\big(S,\,\Omega^1_S\big)_i\neq 0$, i.e., there exists a unique fibration $f'_i:\ S\to B'_i$ such that

$$H^0\big(S,\,\Omega^1_S\big)_{i_0}\oplus H^0\big(S,\,\Omega^1_S\big)_i\subseteq\big(f'_i\big)^*H^0\big(B'_i,\,\Omega^1_{B'_i}\big).$$

Since $H^0\big(S,\,\Omega^1_S\big)_{i_0}\neq 0$, from the uniqueness of these fibrations f'_i's, it follows that these f'_i's are in fact the same one. We denote such a fibration by $f':\ S\to B'$, which is of course unique and the inclusion (3-48) holds. This completes the proof. □

Combining the above corollary together with Proposition 3.21 and (3-45), one proves

Proposition 3.24. – *Assume that* $\operatorname{rank} F_{B,i_0}^{1,0} \neq 0$ *for some* $i_0 \geq n/2$. *Then after a suitable finite étale base change, the flat Higgs subbundle*

$$\bigoplus_{i=n-i_0}^{i_0} \left(F_B^{1,0} \oplus F_B^{0,1},\, 0\right)_i \cong \left(\mathcal{O}_B^{\oplus r},\, 0\right), \qquad \textit{where } r = \sum_{i=n-i_0}^{i_0} \left(\operatorname{rank} F_{B,i}^{1,0} + \operatorname{rank} F_{B,i}^{0,1}\right),$$

becomes a trivial Higgs bundle, i.e.,

$$\dim H^0\big(S,\, \Omega_S^1\big)_i = \operatorname{rank} F_{B,i}^{1,0}, \quad \forall\, n - i_0 \leq i \leq i_0, \tag{3-49}$$

and there exists a unique fibration $f' : S \to B'$ *such that these one-forms in* $H^0\big(S, \Omega_S^1\big)$ *lifted from* $\bigoplus_{i=n-i_0}^{i_0} F_{B,i}^{1,0}$ *are the pulling-back of one-forms on* B' *via* f', *i.e.,*

$$\bigoplus_{i=n-i_0}^{i_0} H^0\big(S,\, \Omega_S^1\big)_i \subseteq \big(f'\big)^* H^0\big(B',\, \Omega_{B'}^1\big). \tag{3-50}$$

Note that the first isomorphism in this proposition is not necessarily true without base change. From the uniqueness of f' obtained in Lemma 3.22, it follows that there is an induced map

$$\iota : G \longrightarrow \operatorname{Aut}(B'). \tag{3-51}$$

Since $G = \mathbb{Z}/n\mathbb{Z}$ is a cyclic group, $\operatorname{Ker}(\iota) \cong \mathbb{Z}/m\mathbb{Z}$ for some m with $m \mid n$.

Lemma 3.25. – *Assume that there exist* $1 \leq i_1 \leq i_2 \leq n-1$, *such that* (3-44) *holds, and that* $H^0\big(S,\, \Omega_S^1\big)_{i_1} \neq 0$ *and* $H^0\big(S,\, \Omega_S^1\big)_{i_2} \neq 0$. *Let* $f' : S \to B'$ *be the fibration obtained in Lemma 3.22, and* ι *be given in* (3-51). *Then*

$$\big(f'\big)^* H^0\big(B',\, \Omega_{B'}^1\big) \subseteq \bigoplus_{m \mid i} H^0\big(S,\, \Omega_S^1\big)_i, \quad \textit{where } m \textit{ is order of } \operatorname{Ker}(\iota).$$

Proof. – Let $\tau \in G$ be any generator of G. Then τ^m is a generator of $\operatorname{Ker}(\iota)$ by construction. Hence

$$\big(f'\big)^* H^0\big(B',\, \Omega_{B'}^1\big) \subseteq H^0\big(S,\, \Omega_S^1\big)^{\tau^m},$$

where

$$\begin{aligned} H^0\big(S,\, \Omega_S^1\big)^{\tau^m} &\triangleq \Big\{\omega \in H^0\big(S,\, \Omega_S^1\big) \;\big|\; \big(\tau^m\big)^* \omega = \omega\Big\} \\ &= \bigoplus_{m \mid i} H^0\big(S,\, \Omega_S^1\big)_i. \end{aligned}$$

This completes the proof. □

Corollary 3.26. – *Assume that* $H^0\big(S,\, \Omega_S^1\big)_{i_0} \neq 0$ *for some* $i_0 \geq n/2$, *and let* $f' : S \to B'$ *be the fibration obtained in Corollary 3.23. Assume moreover that* $\gcd(i_0, n) = 1$. *Then* G *induces a faithful action on* B', *such that* $B'/G \cong \mathbb{P}^1$ *and*

$$H^0\big(S,\, \Omega_S^1\big)_i = \big(f'\big)^* H^0\big(B',\, \Omega_{B'}^1\big)_i, \quad \textit{for any } 1 \leq i \leq n-1 \textit{ with } \gcd(i, n) = 1. \tag{3-52}$$

Proof. – Let $\iota : G \to \mathrm{Aut}\,(B')$ be defined in (3-51). By Corollary 3.23 and Lemma 3.25, one obtains that ι is an isomorphism, and G acts faithfully on B'. Moreover, taking the quotients of the G-actions, we obtain the following commutative diagram:

$$\begin{tikzcd} S \arrow[r, "\Pi"] \arrow[d, "f'"'] & Y \arrow[d, "\varphi'"] \\ B' \arrow[r, "\pi"] & B'/G. \end{tikzcd}$$

Since Y is a ruled surface over B, it follows that $B'/G \cong \mathbb{P}^1$. It remains to show (3-52).

Consider the $\mathbb{Q}$-vector space $H^1(S, \mathbb{Q})$, which admits a natural G-action. Let

$$H^1(S, \mathbb{Q}) \otimes \mathbb{C} = H^1(S, \mathbb{C}) = \bigoplus_{i=0}^{n-1} H^1(S, \mathbb{C})_i$$

be the eigenspace decomposition. The morphism f' induces an inclusion

$$(f')^* : H^1(B', \mathbb{Q}) \otimes \mathbb{C} = H^1(B', \mathbb{C}) \hookrightarrow H^1(S, \mathbb{C}) = H^1(S, \mathbb{Q}) \otimes \mathbb{C}, \tag{3-53}$$

which is clearly compatible with the actions of G on both sides. By (3-48), it follows that

$$(f')^* H^1(B', \mathbb{C})_i = H^1(S, \mathbb{C})_i, \qquad \text{for } i \in \{i_0, p - i_0\}.$$

As a vector subspace of $H^1(S, \mathbb{C})$, $(f')^* H^1(B', \mathbb{C})$ is defined over $\mathbb{Q}$. Hence it follows from (3-53) that

$$(f')^* H^1(B', \mathbb{C}) \supseteq \bigoplus_{\substack{1 \le i \le n-1 \\ \gcd(i,n)=1}} H^1(S, \mathbb{C})_i. \tag{3-54}$$

Combining (3-53) with (3-54), and taking the $(1,0)$-parts, we complete the proof of (3-52). □

4. Exclusion of special curves generically in the superelliptic Torelli locus

In this section, we prove Theorem 1.7. The outline of the proof is as follows.

Given a possible special curve C_0 contained generically in Torelli locus of curves, let $f : S \to B$ be the family of semi-stable superelliptic curves representing C_0 with semi-stable singular fibers $\Upsilon \subset S$ over the discriminant locus $\Delta \subset B$, cf. Definition 4.1. Then there exists a global action of $G = \mathbb{Z}/n\mathbb{Z}$ on S (after a possible base change), which induces an action on the logarithmic Higgs bundle

$$(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B) := \big(f_*\Omega^1_{S/B}(\log \Upsilon) \oplus R^1 f_*\mathcal{O}_S,\ \theta_B\big)$$

corresponding to the $\mathbb{Q}$-local system $\mathbb{V}_{B_0} := R^1 f_*(\mathbb{Q}_{S\setminus\Upsilon})$ on $B \setminus \Delta$ under the Simpson correspondence, cf. [36]. Fix a suitable generator $g_0 \in G$ and a primitive n-th root of unity ξ_n. Let $\mathbb{V}_{B_0,i} \subseteq \mathbb{V}_{B_0} \otimes \mathbb{C}$ be the eigensubspace on which g_0 acts by multiplying ξ_n^i. Then one obtains an eigenspace decomposition

$$\mathbb{V}_{B_0} \otimes \mathbb{C} = \bigoplus_{i=0}^{n-1} \mathbb{V}_{B_0,i},$$

and accordingly also an eigenspace decomposition

$$(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B) = \bigoplus_{i=0}^{n-1} (E_B^{1,0} \oplus E_B^{0,1},\ \theta_B)_i.$$

By [39] there is a unique strictly maximal decomposition

$$(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B) = (A_B^{1,0} \oplus A_B^{0,1}, \theta_B|_A) \oplus (F_B^{1,0} \oplus F_B^{0,1},\ 0),$$

such that $\theta_B|_A$ is an isomorphism at the generic point and $(F_B^{1,0} \oplus F_B^{0,1},\ 0)$ corresponds to the maximal unitary local sub-system $\mathbb{V}^u_{B_0} \subset \mathbb{V}_{B_0} \otimes \mathbb{C}$. The above decomposition is invariant under the action of $G = \mathbb{Z}/n\mathbb{Z}$. In particular, there is an induced eigenspace decomposition

$$(F_B^{1,0} \oplus F_B^{0,1},\ 0) = \bigoplus_{i=0}^{n-1} (F_B^{1,0} \oplus F_B^{0,1},\ 0)_i,$$

which corresponds to the eigenspace decomposition

$$\mathbb{V}^u_{B_0} = \bigoplus_{i=0}^{n-1} \mathbb{V}^u_{B_0,i}.$$

As the first step, we show that $(F_B^{1,0} \oplus F_B^{0,1},\ 0) \neq 0$ in our situation. Then our principle to prove Theorem 1.7 is Proposition 4.6, which roughly says that if there exists a horizontal fibration (cf. Definition 3.1) $f' : S \to B'$ on S with $g(B') \geq \operatorname{rank} F_B^{1,0}$, then $g < 8$. However, it is a priori not clear whether such a fibration always exists in general.

We consider first the case where $n = p$ is prime. If moreover C_0 is non-compact, by [39, §4] for non-compact special curves one deduces that $\mathbb{V}^u_{B_0}$ is just the maximal trivial local subsystem $\mathbb{V}^{\mathrm{tr}}_{B_0} \subset \mathbb{V}_{B_0}$ (up to a possible base change), or equivalently $(F_B^{1,0} \oplus F_B^{0,1}, 0)$ is a trivial Higgs bundle. Note that $\mathbb{V}^{\mathrm{tr}}_{B_0} \subset \mathbb{V}_{B_0}$ is a local subsystem defined over $\mathbb{Q}$, and hence the local subsystem of $\mathbb{Q}(\xi_p)$-vector spaces $\mathbb{V}^{\mathrm{tr}}_{B_0} \otimes \mathbb{Q}(\xi_p) \subset \mathbb{V}_{B_0} \otimes \mathbb{Q}(\xi_p)$ is stabilized by the action of the Galois group $\operatorname{Gal}\big(\mathbb{Q}(\xi_p)/\mathbb{Q}\big)$. Since p is prime, $\operatorname{Gal}\big(\mathbb{Q}(\xi_p)/\mathbb{Q}\big)$ induces a transitive permutation on the eigen-subspaces

$$\mathbb{V}^{\mathrm{tr}}_{B_0} \otimes \mathbb{Q}(\xi_p) = \bigoplus_i \mathbb{V}^{\mathrm{tr}}_{B_0,i}.$$

Applying the Hurwitz-Chevalley-Weil formula (cf. [26, Propsition 5.9]) to ramified cyclic covers of $\mathbb{P}^1$ one shows that

$$\operatorname{rank} F^{1,0}_{B,(p-1)/2} \geq \operatorname{rank} F^{1,0}_{B,(p+1)/2} > 0.$$

Take any two non-zero holomorphic 1-forms α and β, which come from $F^{1,0}_{B,(p-1)/2}$ and $F^{1,0}_{B,(p+1)/2}$ respectively. Then the wedge product $\alpha \wedge \beta$ is a G-invariant holomorphic 2-form, hence descends to a holomorphic 2-form on the ruled surface $S/G \to B$. As all 2-forms on a ruled surface vanish, we get $\alpha \wedge \beta = 0$. Now applying the Castelnuovo-de Franchis lemma (cf. [3, Theorem IV-5.1]) to α, β, one finds a fibration $f' : S \to B'$ such that α and β are pullbacks of holomorphic 1-forms on B' via f'. Using the Hodge symmetry, one shows that all holomorphic and anti-holomorphic 1-forms from

$$F^{1,0}_{B,(p+1)/2} \oplus F^{0,1}_{B,(p+1)/2}$$

are pullbacks via f'; or equivalently, all classes in
$$H^1\big(S,\mathbb{Q}(\xi_p)\big)_{(p+1)/2}=\big(H^1(S,\mathbb{Q})\otimes\mathbb{Q}(\xi_p)\big)_{(p+1)/2}$$
are pullbacks of classes in $H^1(B',\mathbb{Q}(\xi_p))$ via f'.

Finally the transitivity of the $\mathrm{Gal}\big(\mathbb{Q}(\xi_p)/\mathbb{Q}\big)$-action implies that all classes in $F_B^{1,0}$ and $F_B^{0,1}$ are pullbacks via f'. In particular, the 1-forms in $H^0(S,\Omega_S^1)$ coming from $F_B^{1,0}\subset f_*\Omega^1_{S/B}(\log\Upsilon)$ are pullbacks of 1-forms on B' via f', i.e., $g(B')\geq\operatorname{rank}F_B^{1,0}$. Thus by the principle mentioned above, the genus $g<8$ as required.

When C is compact, the situation is much more complicated, mainly due to two difficulties:

(1) the flat subbundle $F_B^{1,0}$ does not have to be trivial, even after any finite base change.
(2) the $\mathrm{Gal}\big(\mathbb{Q}(\xi_p)/\mathbb{Q}\big)$-action does not stablize the unitary local sub-system $\mathbb{V}^u_{B_0}\subset\mathbb{V}_{B_0}\otimes\mathbb{C}$.

Thereby the above argument in the non-compact case no longer works here. To remedy the situation we establish a slope inequality, cf. Proposition 3.15, which implies, together with the Arakelov equality for characterizing special curves and the Miyaoka-Yau type inequality, that $F^{1,0}_{B,i_m}\neq 0$ for some $i_m>p/2$ in the case when $n=p$ is prime. Applying the local property of the eigen-sheaves of differential forms of cyclic covers described by Esnault-Viehweg [12] together with the Bogomolov lemma [33, Lemma 7.5] and Deligne's lemma on the triviality of rank one Higgs bundle [9, §4.2], one proves the triviality of $F^{1,0}_{B,i}$'s with $p-i_m\leq i\leq i_m$. This again enables us to produce an irregular horizontal fibration on S by the same type of arguments as in the non-compact case, such that a "large part" of 1-forms from $F_B^{1,0}$ are pulling-backs of 1-forms via this new fibration, which is sufficient to derive a contradiction for the case when $g\geq 8$ and p is prime.

The general case (i.e., when n is not prime) follows by induction on the number of prime factors in n. If n is not prime and $n_1\mid n$, there is a natural way to define a map ρ_{n,n_1} from C_0 to $\mathcal{TS}_{g_1,n_1}$ once a superelliptic automorphism group is chosen on the general fiber of f. Here g_1 is the genus for the n_1-superelliptic curve $y^{n_1}=\zeta(x)$ using the same separable polynomial ζ as before. The key point for the induction process is to prove that $\rho_{n,n_1}(C_0)$ is again a special curve generically contained in $\mathcal{TS}_{g_1,n_1}$ when n_1 is as large as possible. By induction, it suffices to deal with the cases where $g\geq 8$ but $g_1<8$, and only finitely many possibilities arise. We apply to each of these cases similar ideas used in the p-superelliptic case, and derive a contradiction for each of them.

The arrangement of the section is as follows. In subsection 4.1, we briefly recall the construction of the family of semi-stable curves representing a special curve contained generically in the Torelli locus, and derive some general constraints for such families. In subsection 4.2, we prove Theorem 1.7 for the special case where $n=p$ is prime. In subsection 4.3, we complete the proof of Theorem 1.7 for general n by induction on the number of prime factors contained in n.

4.1. Representation of a special curve by a family of semi-stable curves

In this section we associate a family of semi-stable curves to a given special curve contained generically in the Torelli locus, and we analyze some numerical properties of its

Higgs bundle. The construction of the family is similar to [23, § 3], which is briefly recalled for readers' convenience.

For ℓ the fixed integer indicating the level structures as before, let $ct_g = ct_{g,\ell} \supseteq g = g,\ell$ be the partial compactification of the moduli space of smooth projective genus-g curves with level-ℓ structure by adding stable curves with compact Jacobians. When $n \geq 3$, it carries a universal family of stable curves with compact Jacobians (cf. [32])

$$\mathfrak{f} : g^{ct} \longrightarrow \mathcal{M}_g^{ct}.$$

The Torelli morphism j° can be naturally extended to ct_g:

$$j : ct_g \longrightarrow g, \qquad \text{with } g = j\left(ct_g\right).$$

The morphism j° is 2:1 and ramified exactly over the locus of hyperelliptic curves (cf. [28]). However, the relative dimension of j is positive along the boundary $g \setminus {}^{\circ}{}_g$.

Let C_0 be any smooth closed curve contained generically in g, and B_0 be the normalization of the strict transform of C_0 in ct_g. Denote by $j_{B_0} : B_0 \to C_0$ the induced morphism. If B_0 is reducible, then we replace B_0 by one of its irreducible components. By pulling back the universal family $\mathfrak{f} : g^{ct} \to ct_g$ to B, we obtain a family $f_0 : S_0 \to B_0$ of semi-stable curves that extends uniquely to a family $f : S \to B$ of semi-stable curves over the smooth compactification $B \supseteq B_0$.

Definition 4.1. – The family $f : S \to B$ is called the family of semi-stable curves representing $C_0 \subseteq g$ via the Torelli morphism.

We briefly recall some basic properties of the family f as follows, more details of which are found in [23, § 3].

(i) Let C be the compactification of C_0 in g^{BB} obtained by adjoining a finite set of cusps Δ_C. The morphism $j_{B_0} : B_0 \to C_0$ extends uniquely to a morphism $j_B : B \to C$ such that $\Delta_{nc} := B \setminus B_0 = j_B^{-1}(\Delta_C)$. Denote by $h_0 : X_0 \to C_0$ the universal family of abelian varieties over C_0, and by $\Upsilon \subseteq S$ the singular fibers over the discriminant locus $\Delta \subseteq B$ of f. Let $\mathbb{V}_{C_0} := R^1 h_* \mathbb{Q}_{X_0}$ (resp. $\mathbb{V}_{B_0} := R^1 f_* \mathbb{Q}_{S \setminus \Upsilon}$) be the $\mathbb{Q}$-local system over C_0 (resp. B_0), and $\left(E_C^{1,0} \oplus E_C^{0,1},\ \theta_C\right)$ and $\left(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B\right)$ be the corresponding logarithmic Higgs bundles via Simpson's correspondence over C and B respectively. Then

$$\left(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B\right) \cong j_B^* \left(E_C^{1,0} \oplus E_C^{0,1},\ \theta_C\right). \tag{4-1}$$

(ii) The morphism j_B is either an isomorphism or a double cover. In the first case,

$$\deg E_B^{1,0} = \deg E_C^{1,0}, \quad \deg \Omega_B^1(\log \Delta_{nc}) = \deg \Omega_C^1(\log \Delta_C);$$

and in the second case,

$$\deg E_B^{1,0} = 2 \deg E_C^{1,0}, \quad \deg \Omega_B^1(\log \Delta_{nc}) = 2 \deg \Omega_C^1(\log \Delta_C) + |\Lambda|,$$

where $\Lambda \subseteq B_0$ is the ramification locus of the double cover $j_{B_0} : B_0 \to C_0$. Moreover, any fiber over Λ is a (possibly singular) hyperelliptic curve with a compact Jacobian.

(iii) Let

$$\left(E_B^{1,0}\oplus E_B^{0,1},\theta_B\right)=\left(A_B^{1,0}\oplus A_B^{0,1},\ \theta_B|_{A_B^{1,0}}\right)\oplus\left(F_B^{1,0}\oplus F_B^{0,1},\ 0\right) \tag{4-2}$$

be the Fujita decomposition of the associated Higgs bundle (cf. [15, 20, 39]), where $A_B^{1,0}$ is ample, and $F_B^{1,0}\oplus F_B^{0,1}$ is flat corresponding to a unitary local subsystem $\mathbb{V}_{B_0}^u\subseteq\mathbb{V}_{B_0}\otimes\mathbb{C}$. Then the curve C_0 is a special curve if and only if

$$\begin{cases}\deg E_B^{1,0}=\dfrac{\operatorname{rank}A_B^{1,0}}{2}\cdot\deg\Omega_B^1(\log\Delta_{nc}), & \text{if }\deg(j_B)=1;\\ \deg E_B^{1,0}=\dfrac{\operatorname{rank}A_B^{1,0}}{2}\cdot\left(\deg\Omega_B^1(\log\Delta_{nc})-|\Lambda|\right), & \text{if }\deg(j_{B_0})=2.\end{cases}$$

In particular, if there is no hyperelliptic fiber over B_0, then $\Lambda=\emptyset$ and hence C_0 is a special curve if and only if

$$\deg E_B^{1,0}=\frac{\operatorname{rank}A_B^{1,0}}{2}\cdot\deg\Omega_B^1(\log\Delta_{nc}). \tag{4-3}$$

(iv) If C_0 is a non-compact special curve, then

$$g(F)=\operatorname{rank}F_B^{1,0},\qquad\text{for any fiber }F\text{ over }\Delta_{nc}=j_B^{-1}(\Delta_C), \tag{4-4}$$

where $g(F)$ is the geometric genus of F.

In the case where C_0 is contained generically in the superelliptic Torelli locus, the family f constructed above is subject to more constraints. In the rest part of this section, we will always assume that C_0 is contained generically in $\mathcal{TS}_{g,n}$, and $f: S\to B$ is the family of semi-stable curves representing C_0. We will also use freely the notations introduced in section 3.

By construction, each local subsystem $\mathbb{V}_{B_0,i}$ in (3-32) is defined over the n-th cyclotomic field $\mathbb{Q}(\xi_n)$. Thus the Galois group $\mathrm{Gal}\left(\mathbb{Q}(\xi_n)/\mathbb{Q}\right)$ has a natural action on the decompositions in (3-32) and (3-34).

LEMMA 4.2. – *Let $\mathbb{V}_{B_0,i}^{\mathrm{tr}}\subseteq\mathbb{V}_{B_0,i}$ be the trivial local subsystem, and*

$$\left(\left(F_{B_0,i}^{1,0}\right)^{\mathrm{tr}}\oplus\left(F_{B_0,i}^{0,1}\right)^{\mathrm{tr}},\ 0\right)$$

be the associated trivial flat subbundle. If $\mathbb{V}_{B_0,i}$ and $\mathbb{V}_{B_0,j}$ are in one $\mathrm{Gal}\left(\mathbb{Q}(\xi_n)/\mathbb{Q}\right)$*-orbit, then*

$$\begin{aligned}\operatorname{rank}\mathbb{V}_{B_0,i}^{\mathrm{tr}}&=\operatorname{rank}\mathbb{V}_{B_0,j}^{\mathrm{tr}};\\ \operatorname{rank}\left(F_{B,i}^{1,0}\right)^{\mathrm{tr}}+\operatorname{rank}\left(F_{B,n-i}^{1,0}\right)^{\mathrm{tr}}&=\operatorname{rank}\left(F_{B,j}^{1,0}\right)^{\mathrm{tr}}+\operatorname{rank}\left(F_{B,n-j}^{1,0}\right)^{\mathrm{tr}}.\end{aligned}$$

In particular, if $n=p$ is prime, then for any $1\le i<j\le p-1$, one has

$$\begin{aligned}\operatorname{rank}\mathbb{V}_{B_0,i}^{\mathrm{tr}}&=\operatorname{rank}\mathbb{V}_{B_0,j}^{\mathrm{tr}};\\ \operatorname{rank}\left(F_{B,i}^{1,0}\right)^{\mathrm{tr}}+\operatorname{rank}\left(F_{B,p-i}^{1,0}\right)^{\mathrm{tr}}&=\operatorname{rank}\left(F_{B,j}^{1,0}\right)^{\mathrm{tr}}+\operatorname{rank}\left(F_{B,p-j}^{1,0}\right)^{\mathrm{tr}}.\end{aligned}$$

Proof. – Since trivial local subsystems correspond to trivial representations, and trivial representations remain trivial under any Galois conjugation, it follows that if $\mathbb{V}_{B_0,i}$ and $\mathbb{V}_{B_0,j}$ are in one $\mathrm{Gal}\left(\mathbb{Q}(\xi_n)/\mathbb{Q}\right)$-orbit, then

$$\begin{aligned}\operatorname{rank}\mathbb{V}_{B_0,i}^{\mathrm{tr}}&=\operatorname{rank}\mathbb{V}_{B_0,j}^{\mathrm{tr}};\\ \operatorname{rank}\left(F_{B,i}^{1,0}\right)^{\mathrm{tr}}+\operatorname{rank}\left(F_{B,i}^{0,1}\right)^{\mathrm{tr}}&=\operatorname{rank}\left(F_{B,j}^{1,0}\right)^{\mathrm{tr}}+\operatorname{rank}\left(F_{B,j}^{0,1}\right)^{\mathrm{tr}}.\end{aligned}$$

Note also that $\left(\left(F_{B,i}^{1,0}\right)^{\mathrm{tr}} \oplus \left(F_{B,i}^{0,1}\right)^{\mathrm{tr}}, 0\right)$ is mapped isomorphically to $\left(\left(F_{B,n-i}^{1,0}\right)^{\mathrm{tr}} \oplus \left(F_{B,n-i}^{0,1}\right)^{\mathrm{tr}}, 0\right)$ under the complex conjugation for any $1 \le i \le n-1$. Moreover, under this isomorphism, $\left(F_{B,i}^{1,0}\right)^{\mathrm{tr}} \cong \left(F_{B,n-i}^{0,1}\right)^{\mathrm{tr}}$ and $\left(F_{B,i}^{0,1}\right)^{\mathrm{tr}} \cong \left(F_{B,n-i}^{1,0}\right)^{\mathrm{tr}}$. In particular,

$$\operatorname{rank}\left(F_{B,i}^{1,0}\right)^{\mathrm{tr}} = \operatorname{rank}\left(F_{B,n-i}^{0,1}\right)^{\mathrm{tr}}, \qquad \operatorname{rank}\left(F_{B,i}^{0,1}\right)^{\mathrm{tr}} = \operatorname{rank}\left(F_{B,n-i}^{1,0}\right)^{\mathrm{tr}}.$$

Combining the above equalities together, we prove the first part. For the second part, since $n = p$ is prime, it is clear that the Galois subgroup $\mathrm{Gal}\big(\mathbb{Q}(\xi_p)/\mathbb{Q}\big)$ permutes these eigen-subspaces. This completes the proof. □

LEMMA 4.3. – *The following equalities hold for the family $f : S \to B$ representing a special curve C_0 contained generically in $\mathcal{T}S_{g,n}$.*

$$\begin{cases} \operatorname{rank} E_{B,i}^{1,0} - \operatorname{rank} E_{B,i}^{0,1} = \operatorname{rank} F_{B,i}^{1,0} - \operatorname{rank} F_{B,i}^{0,1}; \\ \operatorname{rank} E_{B,i}^{1,0} - \operatorname{rank} E_{B,n-i}^{1,0} = \operatorname{rank} F_{B,i}^{1,0} - \operatorname{rank} F_{B,n-i}^{1,0}, \end{cases} \qquad \forall\ 1 \le i \le n-1. \tag{4-5}$$

In particular,

$$\operatorname{rank} F_{B,i}^{1,0} \neq 0, \qquad \textit{if } \operatorname{rank} E_{B,i}^{1,0} > \operatorname{rank} E_{B,n-i}^{1,0}; \tag{4-6}$$

$$\operatorname{rank} F_{B,n-i}^{1,0} \ge \operatorname{rank} F_{B,i}^{1,0}, \qquad \textit{if } i \ge n/2. \tag{4-7}$$

Proof. – Since C_0 is Shimura, the associated Higgs bundle $\left(E_C^{1,0} \oplus E_C^{0,1}, \theta_C\right)$ admits a decomposition (cf. [39]):

$$\left(E_C^{1,0} \oplus E_C^{0,1}, \theta_C\right) = \left(A_C^{1,0} \oplus A_C^{0,1},\ \theta_C|_{A_C^{1,0}}\right) \oplus \left(F_C^{1,0} \oplus F_C^{0,1},\ 0\right),$$

such that the restricted Higgs field $\theta_C|_{A_C^{1,0}}$ is an isomorphism. By (4-1), one obtains that $\left(A_B^{1,0} \oplus A_B^{0,1},\ \theta_B|_{A_B^{1,0}}\right)$ is nothing but the pulling-back of $\left(A_C^{1,0} \oplus A_C^{0,1},\ \theta_C|_{A_C^{1,0}}\right)$. In particular, the restricted Higgs field $\theta_B|_{A_B^{1,0}}$ is an isomorphism on the generic point of B. Restricting to each eigenspace $\left(A_B^{1,0} \oplus A_B^{0,1},\ \theta_B|_{A_B^{1,0}}\right)_i$, one sees that the restricted Higgs field $\theta_B|_{A_{B,i}^{1,0}}$ must be again an isomorphism on the generic point of B. Thus $\operatorname{rank} A_{B,i}^{1,0} = \operatorname{rank} A_{B,i}^{0,1}$, i.e.,

$$\operatorname{rank} E_{B,i}^{1,0} - \operatorname{rank} F_{B,i}^{1,0} = \operatorname{rank} E_{B,i}^{0,1} - \operatorname{rank} F_{B,i}^{0,1}.$$

This is the first equality of (4-5). The second equality in (4-5) follows by the complex conjugation. Finally, (4-6) follows directly from (4-5); and (4-7) follows from (4-5) together with (3-33). □

PROPOSITION 4.4 (Moonen). – *There does not exist any special curve contained generically in $\mathcal{S}T_{g,n}$ with $n > g \ge 8$.*

Proof. – Assume that there exists a special curve C_0 contained generically in $\mathcal{T}S_{g,n}$ with $n > g \ge 8$. Let $f : S \to B$ be the family of semi-stable p-superelliptic curves representing C_0 as in Definition 4.1. Assume that the general fiber of f is given by $y^n = \zeta(x)$, where $\zeta(x)$ is a separable polynomial in x with $\deg(\zeta) = \alpha_0$. By the Riemann-Hurwitz Formula (3-4) one has $\alpha_0 \le 3$, since $n > g \ge 8$. It is also clear that $\alpha_0 \ge 3$; otherwise f would be isotrivial. Hence $\alpha_0 = 3$. However, such a family f must be universal in the sense that the moduli space of n-superelliptic curves defined by $y^n = \zeta(x)$ with $\deg(\zeta) = 3$ is exactly of dimension one.

Hence according to a result of Moonen [25, Theorem 3.6], the curve C_0 can not be Shimura once $g \geq 8$. This completes the proof. □

Lemma 4.5. – *Let $f : S \to B$ be the family of semi-stable curves representing a special curve C_0 contained generically in $\mathcal{T}S_{g,n}$ as above.*

(i). *If C_0 is compact and $g \geq n \geq 3$, then*

$$\operatorname{rank} A_B^{1,0} \leq \frac{4g-4}{\lambda_{n,c}}, \tag{4-8}$$

where $\lambda_{n,c}$ is defined in (3-19).

(ii). *Assume that C_0 is non-compact, $g \geq 4$ and $q_f := q(S) - g(B) > 0$. If either $n = 3$ or 4, then*

$$\operatorname{rank} A_B^{1,0} < \frac{4g-4}{\lambda_{n,nc}}, \tag{4-9}$$

where $\lambda_{3,nc}$ and $\lambda_{4,nc}$ are defined in (3-21) *and* (3-22) *respectively.*

Proof. – Since $g \geq n$, it follows that $\alpha \geq 5$ by the Riemann-Hurwitz Formula (3-4). According to Proposition 3.8, the family f admits no hyperelliptic fiber with compact Jacobian. Hence the Arakelov type equality (4-3) holds for $E_B^{1,0}$. Therefore, our conclusion follows from Corollary 3.16. □

The next proposition gives a criterion to exclude special curves generically in $\mathcal{T}S_{g,n}$.

Proposition 4.6. – *Let $f : S \to B$ be the family of semi-stable curves representing a special curve C_0 contained generically in $\mathcal{T}S_{g,n}$. Assume that after a suitable base change of B, there exists a horizontal fibration $f' : S \to B'$ on S with $g(B') \geq \operatorname{rank} F_B^{1,0}$. Then $g < 8$.*

Proof. – This lemma is clear if $n = 2$, since there is no special curve contained generically in $\mathcal{T}S_{g,2} = \mathcal{T}H_g$ with $g \geq 8$ by [22, Theorem 1.2]. Combining this with Proposition 4.4, we may assume $g \geq n$ and $n \geq 3$ in the following. Note also that $\alpha_0 \geq 3$; otherwise the family f is isotrivial.

Firstly, we claim that

$$2g(F) - 2 \geq 2\big(2g(B') - 2\big), \qquad \text{for any fiber } F \text{ of } f, \tag{4-10}$$

where $g(F)$ is the geometric genus of F. In fact, by restricting f' to the fiber F, one obtains a map

$$f'|_F : F \longrightarrow B'.$$

It is clear that $\deg(f'|_F)$ does not depend on the choice of F. Since f is non-isotrivial, it follows that $\deg(f'|_F) \geq 2$. Hence (4-10) follows directly from the Riemann-Hurwitz formula.

By (3-33),

$$\sum_{i=1}^{[n/2]}\left(\operatorname{rank} E_{B,i}^{1,0}-\operatorname{rank} E_{B,i}^{0,1}\right)$$
$$\geq\begin{cases}\operatorname{rank} E_{B,1}^{1,0}-\operatorname{rank} E_{B,1}^{0,1}=\left[\frac{(n-1)\alpha_0}{n}\right]-\left[\frac{\alpha_0}{n}\right]\geq\left[\frac{2\alpha_0}{3}\right]-\left[\frac{\alpha_0}{3}\right]\geq 2, & \text{if } \alpha_0\geq 6;\\ \sum_{i=1}^{2}\left(\operatorname{rank} E_{B,i}^{1,0}-\operatorname{rank} E_{B,i}^{0,1}\right)=\sum_{i=1}^{2}\left(\left[\frac{(n-i)\alpha_0}{n}\right]-\left[\frac{i\alpha_0}{n}\right]\right) & \\ \qquad\geq\sum_{i=1}^{2}\left(\left[\frac{3(n-i)}{n}\right]-\left[\frac{3i}{n}\right]\right)\geq 2, & \text{if } n\geq 6.\end{cases}$$

When $\alpha_0<6$ and $n<6$ (noting that g is assumed to be at least 8), one checks case-by-case that

$$\sum_{i=1}^{[n/2]}\left(\operatorname{rank} E_{B,i}^{1,0}-\operatorname{rank} E_{B,i}^{0,1}\right)\geq 2.$$

Combining with Lemma 4.3, one proves that

$$\operatorname{rank} F_B^{1,0}\geq\sum_{i=1}^{[n/2]}\operatorname{rank} F_{B,i}^{1,0}\geq\sum_{i=1}^{[n/2]}\left(\operatorname{rank} E_{B,i}^{1,0}-\operatorname{rank} E_{B,i}^{0,1}\right)\geq 2,\qquad \text{if } g\geq 8 \text{ and } n\geq 3.$$

We now prove the lemma by contradiction. Assume that $g\geq 8$. Consider first the case where C_0 is non-compact. In this case, by taking an arbitrary fiber F over $\Delta_{nc}=j_B^{-1}(\Delta_C)$ in (4-10), one obtains a contradiction to (4-4) since $g(B')\geq\operatorname{rank} F_B^{1,0}\geq 2$.

In the remaining case where C_0 is compact, we claim that

$$g\geq 2g(B'). \tag{4-11}$$

In fact, by Theorem 3.9, f admits at least one singular fiber; otherwise, f should be isotrivial according to [1]. Since C_0 is assumed to be compact, any such singular fiber admits a compact Jacobian, and thus must be reducible containing at least two components with positive genera. Restricting f' to such a singular fiber F, as we have $\deg(f'|_F)\geq 2$, we obtain that either there are at least two components of F whose geometric genera $\geq g(B')$, or there is at least one component of F whose genus $\geq 2g(B')-1$ by Riemann-Hurwitz formula. This proves (4-11).

Consider first the case when $n\geq 4$. The assumption $g(B')\geq\operatorname{rank} F_B^{1,0}$ together with (4-11) gives $\operatorname{rank} A_B^{1,0}\geq\frac{g}{2}$. This contradicts the bound given in (4-8).

We now assume that $n=3$. The assumption $g(B')\geq\operatorname{rank} F_B^{1,0}$ implies the following lower bound of the the relative irregularity:

$$q_f=q(S)-g(B)\geq\operatorname{rank} F_B^{1,0}.$$

By [14, Thm 3.1],

$$q_f=\operatorname{rank}(F_B^{1,0})^{\mathrm{tr}}\leq\operatorname{rank} F_B^{1,0}.$$

It follows that $q_f = \operatorname{rank} F_B^{1,0}$, i.e., the flat subbundle $F_B^{1,0} = (F_B^{1,0})^{\mathrm{tr}}$ is a trivial bundle. Moreover,

$$H^0(S,\Omega_S^1) = f^*H^0(B,\Omega_B^1) \bigoplus (f')^*H^0(B',\Omega_{B'}^1). \tag{4-12}$$

On the other hand, up to a suitable finite base change, we may assume that the action of the group $G = \mathbb{Z}/3\mathbb{Z}$ extends to S, and the above decomposition still exists. We claim that f' is equivariant with respect to G. Indeed, if it were not the case, the action of G would produce two new horizontal fibrations

$$f_1'' : S \to B_1'', \quad f_2'' : S \to B_2'', \qquad \text{with } B_1'' \cong B_2'' \cong B'.$$

Hence by taking the quotient, one obtains a horizontal fibration $\varphi' : S/G \to B'$, which is a contradiction since S/G is ruled over B and B' is of genus at least 2. Since f' is equivariant with respect to G, the group G induces an action on B' with the following commutative diagram

$$\begin{tikzcd} S \arrow[r, "f'"] \arrow[d] & B' \arrow[d] \\ S/G \arrow[r] & B'/G. \end{tikzcd}$$

Moreover, $B'/G \cong \mathbb{P}^1$ as S/G is ruled over B. Thus

$$\operatorname{rank} F_{B,i}^{1,0} = \dim H^0(S,\, \Omega_S^1)_i = \dim H^0(B',\, \Omega_{B'}^1)_i, \qquad \text{for } i = 1 \text{ or } 2.$$

Combining this with (4-5), one obtains

$$\operatorname{rank} E_{B,1}^{1,0} - \operatorname{rank} E_{B,2}^{1,0} = \operatorname{rank} F_{B,1}^{1,0} - \operatorname{rank} F_{B,2}^{1,0} = \dim H^0(B',\, \Omega_{B'}^1)_1 - \dim H^0(B',\, \Omega_{B'}^1)_2. \tag{4-13}$$

Assume that $\{x_1, \ldots, x_\beta\} \subseteq \mathbb{P}^1$ is the branch locus of the induced cyclic cover $\pi : B' \to B'/G \cong \mathbb{P}^1$, and that π is defined by

$$\mathcal{L}_\pi^{\otimes 3} \equiv \mathcal{O}_{\mathbb{P}^1}\Big(\sum_{j=1}^{\beta} r_j x_j\Big), \qquad \text{where } 1 \le r_i \le 2 \text{ for each } 1 \le i \le \beta.$$

Here '$\equiv$' stands for linear equivalence. According to Hurwitz-Chevalley-Weil's formula (cf. [26, Proposition 5.9]), one has

$$\dim H^0(B',\, \Omega_{B'}^1)_i = -1 + \sum_{j=1}^{\beta} \left\{\frac{-i r_j}{3}\right\}.$$

Hence

$$\begin{aligned} \dim H^0(B',\, \Omega_{B'}^1)_1 - \dim H^0(B',\, \Omega_{B'}^1)_2 &= \sum_{j=1}^{\beta} \left\{\frac{-r_j}{3}\right\} - \sum_{j=1}^{\beta} \left\{\frac{-2r_j}{3}\right\} \le \frac{2\beta}{3} - \frac{\beta}{3} = \frac{\beta}{3} \\ \Longrightarrow \quad \dim H^0(S,\, \Omega_S^1)_1 - \dim H^0(S,\, \Omega_S^1)_2 &\le \left[\frac{\beta}{3}\right]. \end{aligned}$$

Note that $g = \alpha - 2$ and $g(B') = \beta - 2$ by the Riemann-Hurwitz formula. Combining these with (3-33), (4-11) and (4-13), we obtain a contradiction. This completes the proof. $\square$

4.2. Non-existence of special curves contained generically in $\mathcal{T}S_{g,p}$

In this subsection, we prove Theorem 1.7 in the prime case. The case where $p = 2$ has already been treated in [22, Theorem 1.2]. Hence we assume $p \geq 3$ and prove

THEOREM 4.7. – *Let $p \geq 3$ be any prime number. Then there does not exist any special curve contained generically in the Torelli locus of p-superelliptic curves of genus $g \geq 8$.*

The main idea of the proof is based on a contradiction argument: given such a special curve C_0, we first produce a "horizontal" irregular fibration on the family $f : S \to B$ of semi-stable superelliptic curves representing C_0; and then we derive a contradiction from the existence of this "horizontal" irregular fibration. As we have explained in subsection 1.3, the techniques depend on whether C_0 is compact or not. We remark that the methods used here are different from that in proving [22, Theorem 1.2], which is deduced directly from the Miyaoka-Yau type inequality and an improved slope inequality for a family of hyperelliptic curves.

Proof of Theorem 4.7. – Assume that there exists a special curve C_0 contained generically in $\mathcal{T}S_{g,p}$ with $g \geq 8$ and $p \geq 3$ being a prime number. We are going to derive a contradiction.

Let $f : S \to B$ be the family of semi-stable p-superelliptic curves representing C_0 as in Definition 4.1. After a possible base change, we may assume that there exists an action of the Galois group $G = \mathbb{Z}/p\mathbb{Z}$ on S, and hence an induced action of G on the associated logarithmic Higgs bundle $\left(E_B^{1,0} \oplus E_B^{0,1}, \theta_B\right)$ and its subbundles with eigenspace decompositions as in (3-32) and (3-34). Assume that the general fiber of f is given by $y^p = \zeta(x)$, where $\zeta(x)$ is a separable polynomial in x with $\deg(\zeta) = \alpha_0$. By Proposition 4.4, we may assume that $g \geq p$, or equivalently $\alpha_0 \geq 4$ by the Riemann-Hurwitz Formula (3-4). The remainder of the proof is divided into two cases, according to whether C_0 is compact or not.

Case (I): C_0 is non-compact. – In this case, according to Proposition 4.6, it suffices to prove that, up to base change, the following two statements hold:

1. the flat subbundle $F_B^{1,0} \cong \mathcal{O}_B^{\oplus r_1}$ becomes a trivial vector bundle, where $r_1 = \operatorname{rank} F_B^{1,0}$;
2. there exists an irregular fibration $f' : S \to B'$ different from f with $g(B') = \operatorname{rank} F_B^{1,0}$.

The first statement is already proved in [39]; in fact, since C_0 is non-compact, according to [39, Corollary 4.4], after a suitable finite étale base change, the unitary local subsystem $\mathbb{V}_{B_0}^u \subseteq \mathbb{V}_{B_0} \otimes \mathbb{C}$ becomes trivial, which is equivalent to saying that the flat Higgs subbundle $\left(F_B^{1,0} \oplus F_B^{0,1}, 0\right) \cong \left(\mathcal{O}_B^{\oplus 2r_1}, 0\right)$ is trivial by Simpson's correspondence [36]. This proves the first statement. For the second statement, it suffices to prove

$$\operatorname{rank} F_{B,(p+1)/2}^{1,0} > 0. \tag{4-14}$$

Indeed, by Proposition 3.24 together with Corollary 3.26, there exists a G-invariant fibration $f' : S \to B'$ such that

$$H^0\left(S, \Omega_S^1\right)_i = \left(f'\right)^* H^0\left(B', \Omega_{B'}^1\right)_i, \qquad \text{for any } 1 \leq i \leq p-1.$$

This together with the first statement above, makes that

$$g(B') = \sum_{i=1}^{p-1} \dim H^0\big(S, \Omega_S^1\big)_i = \operatorname{rank} F_B^{1,0}.$$

Therefore, it remains to prove (4-14). Note that the validity of (4-14) is invariant under base change. This allows us to take any finite base change. As we have seen above, by [39, Corollary 4.4], we may assume that the unitary local subsystem $\mathbb{V}_{B_0}^u \subseteq \mathbb{V}_{B_0} \otimes \mathbb{C}$ is trivial after a suitable finite base change, i.e., $\mathbb{V}_{B_0}^u = \mathbb{V}_{B_0}^{\mathrm{tr}}$. Combining this with Lemma 4.2, we obtain

$$\operatorname{rank} F_{B,i}^{1,0} + \operatorname{rank} F_{B,i}^{0,1} = \operatorname{rank} F_{B,j}^{1,0} + \operatorname{rank} F_{B,j}^{0,1}, \qquad \forall\, 1 \le i \le j \le p-1.$$

By (3-33), one checks easily that

$$\operatorname{rank} E_{B,i}^{1,0} + \operatorname{rank} E_{B,i}^{0,1} = \operatorname{rank} E_{B,j}^{1,0} + \operatorname{rank} E_{B,j}^{0,1}, \qquad \forall\, 1 \le i \le j \le p-1.$$

Combining these with (4-5), we obtain

$$\operatorname{rank} A_{B,i}^{1,0} = \operatorname{rank} A_{B,j}^{1,0}, \qquad \forall\, 1 \le i \le j \le p-1. \tag{4-15}$$

Hence

$$\begin{aligned}\operatorname{rank} F_{B,i}^{1,0} &= \operatorname{rank} E_{B,i}^{1,0} - \operatorname{rank} A_{B,i}^{1,0}\\ &= \operatorname{rank} E_{B,i}^{1,0} - \operatorname{rank} A_{B,p-1}^{1,0} = \operatorname{rank} E_{B,i}^{1,0} - \Big(\operatorname{rank} E_{B,p-1}^{1,0} - \operatorname{rank} F_{B,p-1}^{1,0}\Big)\\ &\ge \operatorname{rank} E_{B,i}^{1,0} - \operatorname{rank} E_{B,p-1}^{1,0}.\end{aligned}$$

If $p \ge 5$, then by taking $i = (p+1)/2$ in the above inequality and by using (3-33), one proves (4-14). It remains to show (4-14) for $p = 3$.

In the case where $p = 3$, we prove (4-14) by contradiction. Suppose that $\operatorname{rank} F_{B,2}^{1,0} = 0$. Then by (4-15), one obtains

$$\operatorname{rank} A_B^{1,0} = 2 \operatorname{rank} A_{B,2}^{1,0} = 2 \operatorname{rank} E_{B,2}^{1,0}. \tag{4-16}$$

On the other hand, by (4-6) together with (3-33), one gets $F_{B,1}^{1,0} \neq 0$. Since $\mathbb{V}_B^u$ is a trivial local subsystem, it follows from Simpson's correspondence that $F_B^{1,0} = F_{B,1}^{1,0}$ is a trivial vector bundle. In other words, the relative irregularity $q_f = \operatorname{rank} F_B^{1,0} > 0$. It follows that there is a bound on $\operatorname{rank} A_B^{1,0}$ as in (4-9), which contradicts (4-16) in view of (3-33). This completes the proof for the case where C_0 is non-compact.

Case (II): C_0 is compact. – The idea of the proof is similar to that in the above case, but with much more complicated arguments as explained in subsection 1.3. Let

$$i_m := \max\big\{i \mid F_{B,i}^{1,0} \neq 0\big\}.$$

We first claim that $i_m > p/2$.

Indeed, if $i_m \le p/2$, $F_{B,i}^{1,0} = 0$ for any $i > p/2$. Then $A_{B,i}^{1,0} = E_{B,i}^{1,0}$ for all $i > p/2$. Combining this with (4-5), one obtains

$$\operatorname{rank} A_B^{1,0} = 2 \sum_{i=(p+1)/2}^{p-1} \operatorname{rank} E_{B,i}^{1,0}. \tag{4-17}$$

By (3-33), one verifies that this contradicts the upper bound of rank $A_B^{1,0}$ given in (4-8). To illustrate the idea, we give the proof for the case where $p \mid \alpha_0$. Let $\alpha_0 = kp$ with $k \geq 1$. By (4-17) and (3-33), one obtains

$$\operatorname{rank} A_B^{1,0} = 2 \sum_{i=(p+1)/2}^{p-1} \big(k(p-i)-1\big) = \frac{(p-1)\big(k(p+1)-4\big)}{4}.$$

Since $g \geq 8$, it follows that $k \geq 4$ if $p = 3$; and $k \geq 2$ if $p = 5$. Hence this gives a contradiction to the bound of rank $A_B^{1,0}$ in (4-8).

By Proposition 3.24, after a suitable base change, there is a unique fibration $f' : S \to B'$ such that

$$\bigoplus_{i=p-i_m}^{i_m} H^0\big(S, \Omega_S^1\big)_i \subseteq \big(f'\big)^* H^0\big(B', \Omega_{B'}^1\big). \tag{4-18}$$

Since the fibration f' is unique, the group $G = \mathbb{Z}/p\mathbb{Z}$ induces an action on B'. The main technical point is the following lemma.

LEMMA 4.8. – *Let β be the number of fixed points of G on B'; equivalently, β is the number of branch points of the induced cover $\pi : B' \to B'/G$. Then $\beta \geq 4$, and the following inequalities hold:*

$$i_m \geq \begin{cases} p-1, & \text{if } \beta > p, \\ p-1-\left[\dfrac{p}{\beta-1}\right], & \text{if } \beta \leq p, \end{cases} \tag{4-19}$$

$$2 \operatorname{rank} E_{B,i_m}^{1,0} \geq \frac{2g}{p-1} + 4 - \beta, \tag{4-20}$$

$$\frac{2g}{p-1} \geq 2\beta - 3. \tag{4-21}$$

The proof of the above lemma will be postponed until the end of the subsection. We will derive a contradiction in the case where C is compact and hence complete the proof of Theorem 4.7.

First, we claim that $p \geq 5$ and $\beta \leq p$; in fact, if $p = 3$ or $\beta > p$, then $i_m = p-1$ by (4-19), and hence from Proposition 4.6 and Proposition 3.24 it follows that $g < 8$, contradicting the assumption. According to (4-19), (4-20),(3-4) and (3-33), one obtains that

$$\begin{cases} \alpha_0 + 4 - \beta \leq \dfrac{2\big([\frac{p}{\beta-1}]+1\big)\alpha_0}{p}, & \text{if } p \mid \alpha_0; \\ \alpha_0 + 3 - \beta \leq 2\Big[\dfrac{\big([\frac{p}{\beta-1}]+1\big)\alpha_0}{p}\Big], & \text{if } p \nmid \alpha_0. \end{cases} \tag{4-22}$$

Note that $[x] \leq x$ for any $x \in \mathbb{Q}$. Combining this with (3-4) and (4-21), we get

$$\begin{cases} \Big(1-\dfrac{4}{p}\Big)\beta \leq 1 + \dfrac{2}{\beta-1} - \dfrac{2}{p}, & \text{if } p \mid \alpha_0; \\ \Big(1-\dfrac{4}{p}\Big)(\beta-1) \leq 2, & \text{if } p \nmid \alpha_0. \end{cases}$$

Note also that $\beta \geq 4$ by Lemma 4.8. Hence the above inequalities give a contradiction if $p > 11$. If $p = 11$ or 7, one verifies case-by-case that there is also a contradiction by (4-22), (4-21), (3-4) and (3-33).

Finally, we consider the case where $p = 5$. Again by (4-22), (4-21), (3-4) together with (3-33), one obtains that $g = 14$, $\alpha_0 = 8$, $\beta = 5$ and $i_m = 3$. In particular, (4-20) is an equality, which implies that (4-33) is also an equality, i.e.,

$$\operatorname{rank} A_{B,2}^{1,0} = \operatorname{rank} A_{B,3}^{1,0} = \operatorname{rank} E_{B,3}^{1,0} - 1 = 2, \quad \text{by (3-33).}$$

Note also that

$$\operatorname{rank} A_{B,1}^{1,0} = \operatorname{rank} A_{B,4}^{1,0} = \operatorname{rank} E_{B,4}^{1,0} = 1.$$

Hence

$$\operatorname{rank} A_{B}^{1,0} = \sum_{i=1}^{4} \operatorname{rank} A_{B,i}^{1,0} = 6,$$

which is a contradiction to the bound (4-8). This completes the proof. □

To prove Lemma 4.9, we need the following result.

Lemma 4.9. – *Let $p \geq 3$ be any prime number, and $1 \leq r_1 \leq \cdots \leq r_\beta \leq p-1$ be a sequence of integers such that $p \mid \Big(\sum_{j=1}^{\beta} r_j\Big)$. Let $1 \leq \theta \leq p-1$ be an integer such that*

$$\text{(4-23)} \quad H(k) = 1, \quad \forall\, 1 \leq k \leq \theta; \qquad \textit{where } H(k) := \sum_{j=1}^{\beta}\left(\frac{kr_j}{p} - \left[\frac{kr_j}{p}\right]\right) = \sum_{j=1}^{\beta}\left\{\frac{kr_j}{p}\right\}.$$

Then $\beta \leq p$ and $\theta \leq \left[\frac{p}{\beta-1}\right]$.

Proof. – The case where $\theta = 1$ is clear, and we may assume that $\theta \geq 2$.

Taking $k = 1$ in (4-23), we get immediately that $\beta \leq \sum_{j=1}^{\beta} r_j = p$; and from the equality $H(2) = 1$, we obtain that $r_\beta > \frac{p}{2}$ and $r_j < \frac{p}{2}$ for $1 \leq j \leq \beta - 1$. In the following we deduce a contradiction under the assumption $\theta > [\frac{p}{\beta-1}]$.

(Step 1) First of all, we show that

$$\text{(4-24)} \qquad r_j = 1, \qquad \forall\, 1 \leq j \leq \beta - 2.$$

We set

$$\delta = \left[\frac{p}{\beta-1}\right],\ t_1 = \left[\frac{p}{r_{\beta-1}}\right],\ t_2 = \left[\frac{p}{r_{\beta-2}}\right], \text{ and } t_2' = \left[\frac{p}{2r_{\beta-2}}\right].$$

By assumption, $\delta + 1 \leq \theta$. It is clear that $2t_2' \leq t_2 \leq 2t_2' + 1$, and $t_1 \leq t_2$.

Moreover, $t_1 \leq \delta$; otherwise, $\frac{(\delta+1)r_j}{p} < 1$ for any $1 \leq j \leq \beta - 1$, and it implies

$$1 = H(\delta+1) \geq \sum_{j=1}^{\beta-1} \frac{(\delta+1)r_j}{p} \geq \frac{\beta-1}{p} \cdot (\delta+1) > 1,$$

which is a contradiction. Thus $t_1 \le \min\{\delta, t_2\}$, from which together with (4-23) it follows that

$$1 = H(t_1) > \frac{t_1 r_{\beta-2}}{p} + \frac{t_1 r_{\beta-1}}{p} > \frac{t_1 r_{\beta-2}}{p} + \frac{1}{2}, \quad \Longrightarrow \quad \frac{p}{2r_{\beta-2}} > t_1.$$

Hence $t_1 \le t_2'$.

We claim also that there exists some t_0 with $t_2' < t_0 < t_2 + 1$ such that $\{\frac{t_0 r_{\beta-1}}{p}\} > \frac{1}{2}$. In fact, if such t_0 does not exist, then by induction one has $[\frac{t r_{\beta-1}}{p}] = [\frac{(t_2'+1) r_{\beta-1}}{p}]$ for any $t_2' < t < t_2 + 1$, since $\frac{r_{\beta-1}}{p} < \frac{1}{2}$. Hence

$$\begin{aligned}\frac{1}{2} \ge \left\{\frac{t_2 r_{\beta-1}}{p}\right\} &= \left\{\frac{(t_2'+1) r_{\beta-1}}{p}\right\} + \frac{(t_2 - t_2' - 1) r_{\beta-1}}{p} \\ &> \frac{(t_2 - t_2' - 1) r_{\beta-1}}{p} \ge \frac{(t_2' - 1) r_{\beta-1}}{p} \ge \frac{(t_1 - 1) r_{\beta-1}}{p}.\end{aligned}$$

Note that $t_1 = [\frac{p}{r_{\beta-1}}] \ge 2$. From the above inequality it follows that $t_2 = 2t_2' = 2t_1 = 4$, in which case one computes easily that $r_\beta > \frac{p}{2}$, $r_{\beta-1} > \frac{p}{3}$ and $r_{\beta-2} > \frac{p}{5}$. This contradicts the fact that $\sum_{j=1}^{\beta} r_j = p$.

Now since

$$H(t_0) > \frac{t_0 r_{\beta-2}}{p} + \left\{\frac{t_0 r_{\beta-1}}{p}\right\} > \frac{1}{2} + \frac{1}{2} = 1,$$

one obtains that $t_2 \ge t_0 > \delta$. Because

$$1 = H(\delta + 1) > \sum_{j=1}^{\beta-2} \frac{(\delta+1) r_j}{p},$$

it follows that $r_j = 1$ for any $1 \le j \le \beta - 2$.

(Step 2) We show that

$$\epsilon + t_1 \ge \delta + 1, \qquad \text{where } \epsilon = \left[\frac{p}{2(\beta-2)}\right]. \tag{4-25}$$

Indeed, completely similar to the estimation in (Step 1), one can show that there exists some $\tilde{t}_0$ with $\epsilon + 1 \le \tilde{t}_0 \le \epsilon + t_1 + 1$ such that $\{\frac{\tilde{t}_0 r_{\beta-1}}{p}\} > \frac{1}{2}$. For such $\tilde{t}_0$, we have

$$H(\tilde{t}_0) = \frac{(\beta-2)\tilde{t}_0}{p} + \left(\left\{\frac{\tilde{t}_0 r_{\beta-1}}{p}\right\} + \left\{\frac{\tilde{t}_0 r_\beta}{p}\right\}\right) > \frac{(\beta-2)(\epsilon+1)}{p} + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = 1.$$

Hence we obtain $\epsilon + t_1 \ge \tilde{t}_0 - 1 > \theta - 1 > \delta$, i.e., $\epsilon + t_1 \ge \delta + 1$ as required.

(Step 3) We proceed to show that

$$p \ge t_1(t_1+1)(\beta-2) + 2t_1 + 1. \tag{4-26}$$

In fact, since $t_1 \le \delta < \theta$, by (4-23) with $k = t_1$ and using (4-24) one obtains that

$$1 = H(t_1) > \frac{(\beta-2)t_1}{p} + \frac{t_1 r_{\beta-1}}{p} = \frac{(\beta-2)t_1 + p - \eta}{p},$$

where we write $\eta = p - t_1 r_{\beta-1}$. Thus $\eta > (\beta-2)t_1$, i.e., $\eta \ge (\beta-2)t_1 + 1$. Therefore,

$$p = t_1 r_{\beta-1} + \eta \ge t_1(\eta+1) + \eta \ge t_1(t_1+1)(\beta-2) + 2t_1 + 1.$$

(Step 4) Finally, we derive a contradiction. Clearly, we may assume that $\beta \geq 3$. Moreover, if $\beta = 3$, then $\delta = \left[\frac{p}{2}\right] = \frac{p-1}{2}$. Since $\theta > \delta$, one has $H(\delta) + H(\delta+1) = 2$ by (4-23); direct computation gives us $H(\delta) + H(\delta+1) = \beta = 3$. Hence we may assume that $\beta \geq 4$. Combining (4-25) and (4-26), we obtain

$$\frac{p}{(t_1+1)(\beta-2) + \frac{2t_1+1}{t_1}} \geq t_1 \geq \left[\frac{p}{\beta-1}\right] - \left[\frac{p}{2(\beta-2)}\right] + 1. \tag{4-27}$$

Note that $\left[\frac{p}{\beta-1}\right] - \left[\frac{p}{2(\beta-2)}\right] + 1 > \frac{p}{\beta-1} - \frac{p}{2(\beta-2)}$. Hence

$$\frac{p}{\beta-1} - \frac{p}{2(\beta-2)} < \frac{p}{(t_1+1)(\beta-2) + \frac{2t_1+1}{t_1}} < \frac{p}{(t_1+1)(\beta-2)}.$$

Since $r_{\beta-1} < \frac{p}{2}$, or equivalently $t_1 \geq 2$, one derives immediately a contradiction if $\beta > 6$. For the cases when $4 \leq \beta \leq 6$, one can derive a contradiction case-by-case according to (4-27). This completes the proof. □

Proof of Lemma 4.8. – Note that $B'/G \cong \mathbb{P}^1$ since S/G is ruled over B. By Corollary 3.26, one gets

$$\bigoplus_{i=1}^{p-1} H^0\big(S, \Omega_S^1\big)_i = \big(f'\big)^* H^0\big(B', \Omega_{B'}^1\big).$$

Moreover, the pulling-back map $\big(f'\big)^* : H^0\big(B', \Omega_{B'}^1\big) \to H^0\big(S, \Omega_S^1\big)$ is equivariant with respect to the induced actions of G on both sides, i.e.,

$$H^0\big(S, \Omega_S^1\big)_i = \big(f'\big)^* H^0\big(B', \Omega_{B'}^1\big)_i, \qquad \forall\ 1 \leq i \leq p-1. \tag{4-28}$$

According to the definition of i_m, it follows that

$$\dim H^0\big(S, \Omega_S^1\big)_{i_m} = \operatorname{rank} F_{B,i_m}^{1,0} > 0; \qquad \dim H^0\big(S, \Omega_S^1\big)_i = \operatorname{rank} F_{B,i}^{1,0} = 0, \quad \forall\ i > i_m. \tag{4-29}$$

Hence

$$\dim H^0\big(B', \Omega_{B'}^1\big)_{i_m} > 0; \qquad \dim H^0\big(B', \Omega_{B'}^1\big)_i = 0, \quad \forall\ i > i_m. \tag{4-30}$$

(i). We first prove (4-19) by contradiction. Assume that

$$i_m < j_0 := \begin{cases} p-1, & \text{if } \beta > p, \\ p-1-\left[\dfrac{p}{\beta-1}\right], & \text{if } \beta \leq p, \end{cases}$$

Then by (4-30), one has

$$H^0\big(B', \Omega_{B'}^1\big)_i = 0, \qquad \text{for any } p-1 \geq i \geq j_0. \tag{4-31}$$

Let $\{x_1, \ldots, x_\beta\} \subseteq \mathbb{P}^1$ be the branch locus of the induced quotient map $\pi : B' \to B'/G \cong \mathbb{P}^1$, and assume that π is defined by

$$\mathcal{L}_\pi^{\otimes p} \equiv \mathcal{O}_{\mathbb{P}^1}\Big(\sum_{j=1}^{\beta} r_j x_j\Big), \qquad \text{where } 1 \leq r_1 \leq \cdots \leq r_\beta \leq p-1.$$

Then by a formula of Hurwitz-Chevalley-Weil (cf. [26, Proposition 5.9]), one has

$$\dim H^0\big(B',\,\Omega^1_{B'}\big)_i = -1 + \sum_{j=1}^{\beta}\left\{\frac{-i\,r_j}{p}\right\}. \tag{4-32}$$

By (4-31), we get

$$H(k) := \sum_{j=1}^{\beta}\left\{\frac{k\,r_j}{p}\right\} = 1, \qquad \text{for any } 1 \le k \le p - j_0.$$

This contradicts Lemma 4.9 below.

(ii). We next prove $\beta \ge 4$ and (4-20). By Proposition 3.24 and (4-28), it follows that

$$\dim H^0\big(B',\,\Omega^1_{B'}\big)_i = \dim H^0\big(S,\,\Omega^1_S\big)_i = \operatorname{rank} F^{1,0}_{B,i}, \qquad \forall\ p - i_m \le i \le i_m.$$

According to (4-5) together with (4-29), one obtains

$$\operatorname{rank} A^{1,0}_{B,p-i_m} = \operatorname{rank} A^{0,1}_{B,i_m} = \operatorname{rank} A^{1,0}_{B,i_m} \le \operatorname{rank} E^{1,0}_{B,i_m} - 1. \tag{4-33}$$

Combining these with (4-32) and (3-33), we obtain

$$\beta - 2 = \dim H^0\big(B',\,\Omega^1_{B'}\big)_{i_m} + \dim H^0\big(B',\,\Omega^1_{B'}\big)_{p-i_m} = \operatorname{rank} F^{1,0}_{B,i_m} + \operatorname{rank} F^{1,0}_{B,p-i_m}. \tag{4-34}$$

By the definition of i_m with (3-33), one has $\operatorname{rank} F^{1,0}_{B,p-i_m} \ge \operatorname{rank} F^{1,0}_{B,i_m} \ge 1$. From this with (4-34), it follows that $\beta \ge 4$. Moreover,

$$\begin{aligned}
\operatorname{rank} F^{1,0}_{B,i_m} + \operatorname{rank} F^{1,0}_{B,p-i_m} &= \operatorname{rank} E^{1,0}_{B,i_m} + \operatorname{rank} E^{1,0}_{B,p-i_m} - 2\operatorname{rank} A^{1,0}_{B,i_m}\\
&\ge \operatorname{rank} E^{1,0}_{B,i_m} + \operatorname{rank} E^{1,0}_{B,p-i_m} - 2\Big(\operatorname{rank} E^{1,0}_{B,i_m} - 1\Big)\\
&= \frac{2g}{p-1} + 2 - 2\operatorname{rank} E^{1,0}_{B,i_m}.
\end{aligned}$$

This together with (4-34) proves (4-20).

(iii). Finally, we prove (4-21). Let F be a general fiber of f, and $\Gamma = F/G \cong \mathbb{P}^1$ the quotient. Then one has the following commutative diagram, where $\varphi' :\ S/G \to B'/G$ is the induced fibration:

$$\begin{tikzcd}
F \arrow[r, "f'|_F"] \arrow[d, "\Pi|_F"'] & B' \arrow[d, "\pi'"] \\
\Gamma \cong \mathbb{P}^1 \arrow[r, "\varphi'|_\Gamma"] & B'/G \cong \mathbb{P}^1.
\end{tikzcd}$$

By assumption, $\Pi|_F$ (resp. π') is branched over $\alpha := \frac{2g}{p-1} + 2$ (resp. β) points.

If $\deg\big(f'|_F\big) \ge p$, then by the Riemann-Hurwitz formula for the map $f'|_F$, one obtains

$$(p-1)\alpha - 2p = 2g - 2 \ge \deg\big(f'|_F\big)\cdot\big(2g(B') - 2\big) \ge p\big((p-1)\beta - 2p\big) \ge 2(p-1)\beta - 4p + 2.$$

Hence $\alpha \ge 2(\beta - 1)$; and if the equality holds, then $p = 3$, $\alpha = 6$ and $g = 4$, which contradicts the assumption that $g \ge 8$.

If $\deg\left(f'|_F\right) < p$, then the inverse of the branch points of π' in $\Gamma \cong \mathbb{P}^1$ is contained in that of $\Pi|_F$. Let R_0 be the ramification locus of $\varphi'|_\Gamma$. Then by the Riemann-Hurwitz formula, one has

$$\deg(\varphi'|_\Gamma)\cdot\beta - \alpha \le \deg(R_0) = 2\deg(\varphi'|_\Gamma) - 2. \tag{4-35}$$

Since f is non-isotrivial, one has $\deg\left(\varphi'|_\Gamma\right) = \deg\left(f'|_F\right) \ge 2$. Hence $\alpha \ge 2(\beta-1)$. Moreover, if $\alpha = 2(\beta-1)$, then $\varphi'|_\Gamma$ is a double cover branched exactly over two of the branch points of π'. It follows that the branch loci of $\Pi|_F$ are invariant when F runs in the family f, and hence any two smooth fibers of f are isomorphic to each other. This contradicts the non-isotriviality of f. Thus $\alpha > 2(\beta-1)$. This proves (4-21). □

4.3. Non-existence of special curves contained generically in $\mathcal{TS}_{g,n}$

In this subsection we prove Theorem 1.7 for n-superelliptic curves by induction on the number of prime factors of n.

Let C_0 be any smooth curve contained generically in $\mathcal{TS}_{g,n}$, and $f: S \to B$ be the family of semi-stable n-superelliptic curves representing C_0 as in subsection 4.1. After a possible base change, we may assume that the group $G = \mathbb{Z}/n\mathbb{Z}$ admits an action on S which reduces to the superelliptic automorphism group on the general fiber of f. Let $n_1 \ge 2$ be any number dividing n, and consider the quotient family $S/H_1 \to B$, where $H_1 \leqslant G$ is the unique subgroup of order $\frac{n}{n_1}$. Resolving the singularities of S/H_1 and contracting the exceptional curves, one obtains a new family $f_1: S_1 \to B$. By construction, the local system $(R^1 f_1)_*\mathbb{Q}_{S_1\setminus\Upsilon_1}$ is a local subsystem of $\mathbb{V}_{B_0} := R^1 f_*\mathbb{Q}_{S\setminus\Upsilon}$. Using the equivalence between the semi-stability and the unipotentness of the associated local system for a family of curves (cf. [4, Theorem 6.3 and Remark 6.4]), one sees that f_1 is also semi-stable. There is a rational cover Π_{n,n_1} with the following diagram:

$$\begin{tikzcd} S \arrow[rr, dashed, "\Pi_{n,n_1}"] \arrow[dr, "f"'] & & S_1 \arrow[dl, "f_1"] \\ & B. & \end{tikzcd}$$

If the general fiber of f is defined by $y^n = \zeta(x)$, then the general fiber of f_1 is given by $y^{n_1} = \zeta(x)$, which admits a cyclic cover π_1 to $\mathbb{P}^1$ with covering group $G_1 = \mathbb{Z}/n_1\mathbb{Z}$, branch locus R_1, and local monodromy a_1 around R_1. Here R_1 and a_1 are given by

$$\begin{cases} R_1 = \{x_1,\ldots,x_{\alpha_0}\}, \text{ and } a_1 = (1,\ldots,1), & \text{if } n_1 \mid \alpha_0;\\ R_1 = \{x_1,\ldots,x_{\alpha_0},\infty\}, \text{ and } a_1 = (1,\ldots,1,a_{\infty,1}), & \text{if } n_1 \nmid \alpha_0, \end{cases} \tag{4-36}$$

where $\{x_1,\ldots,x_{\alpha_0}\}$ are the set of roots of $F(x)$, and $a_{\infty,1} = n_1\left(\left[\frac{\alpha_0}{n_1}\right]+1\right) - \alpha_0$. In the case where $n_1 \nmid \alpha_0$, the ramification index of π_1 at ∞ is $r_{\infty,1} = \frac{n_1}{\gcd(n_1,\alpha_0)}$. By Hurwitz formula, the genus g_1 of a general fiber of f_1 is given by the following formula:

$$g_1 = \begin{cases} \dfrac{(n_1-1)(\alpha_0-2)}{2}, & \text{if } n_1 \mid \alpha_0,\\ \dfrac{(n_1-1)(\alpha_0-2) + \frac{r_{\infty,1}-1}{r_{\infty,1}}\cdot n_1}{2}, & \text{if } n_1 \nmid \alpha_0. \end{cases} \tag{4-37}$$

The relative Jacobian of the family f_1 induces a map from B_0 to $\mathcal{T}S_{g_1,n_1} \subseteq \mathcal{A}_{g_1}$, which factors clearly through C_0:

$$B_0 \longrightarrow C_0 \xrightarrow{\rho_{n,n_1}} \mathcal{T}S_{g_1,n_1} \subseteq \mathcal{A}_{g_1}. \tag{4-38}$$

We denote the image by $\rho_{n,n_1}(C_0)$. By definition, one obtains

LEMMA 4.10. – *Let $C_0 \subseteq \mathcal{T}S_{g,n}$ be a special curve. Then the image $\rho_{n,n_1}(C_0) \subseteq \mathcal{T}S_{g_1,n_1}$ is either a special point, or it is a special curve. Moreover, if $\rho_{n,n_1}(C_0)$ is a special curve, then C_0 is compact if and only if $\rho_{n,n_1}(C_0)$ is compact.*

REMARK 4.11. – The image ρ_{n,n_1} might depend on the choices of the n-superelliptic automorphism group on the general fiber of f. In other words, it is not clear whether there is a well-defined map from $\mathcal{T}S_{g,n}$ to $\mathcal{T}S_{g_1,n_1}$. In this paper, when talking about the map ρ_{n,n_1}, it is understood that an n-superelliptic automorphism group on the general fiber of f has already been chosed and fixed.

LEMMA 4.12. – *Let $\left(E_B^{1,0} \oplus E_B^{0,1},\, \theta_B\right)$ and $\left(\widetilde{E}_B^{1,0} \oplus \widetilde{E}_B^{0,1},\, \widetilde{\theta}_B\right)$ be the corresponding logarithmic Higgs bundles associated to the families f and f_1 respectively. Then the Galois group $G = \mathbb{Z}/n\mathbb{Z}$ (resp. $G_1 \cong \mathbb{Z}/n_1\mathbb{Z}$) admits a natural action on the logarithmic Higgs bundle associated to f (resp. f_1), and the eigenspaces satisfy that (where $m_1 = \frac{n}{n_1}$)*

$$\left(\widetilde{E}_B^{1,0} \oplus \widetilde{E}_B^{0,1},\, \widetilde{\theta}_B\right)_i \cong \left(E_B^{1,0} \oplus E_B^{0,1},\, \theta_B\right)_{im_1}, \qquad \forall\, 1 \leq i \leq n_1 - 1. \tag{4-39}$$

Proof. – Let $\tau \in G$ be any generator, $H \leqslant G$ be the subgroup generated by τ^{m_1}, and $G_1 = G/H$. Then

$$\bigoplus_{i=1}^{n_1-1} \left(E_B^{1,0} \oplus E_B^{0,1},\, \theta_B\right)_{im_1} = \left(E_B^{1,0} \oplus E_B^{0,1},\, \theta_B\right)^H.$$

Also, by construction, S_1 is birational to the quotient S/H. It follows that

$$\left(\widetilde{E}_B^{1,0} \oplus \widetilde{E}_B^{0,1},\, \widetilde{\theta}_B\right) \cong \left(E_B^{1,0} \oplus E_B^{0,1},\, \theta_B\right)^H.$$

Hence

$$\left(\widetilde{E}_B^{1,0} \oplus \widetilde{E}_B^{0,1},\, \widetilde{\theta}_B\right) \cong \bigoplus_{i=1}^{n_1-1} \left(E_B^{1,0} \oplus E_B^{0,1},\, \theta_B\right)_{im_1}.$$

Note that the group G_1 acts naturally on both sides, and that the above isomorphism is clearly equivariant with respect to the actions of G_1. This proves (4-39). □

The principle of the induction process is the following.

LEMMA 4.13. – *Assume that there exists C_0 a special curve contained generically in $\mathcal{T}S_{g,n}$ with $g \geq n$. If n is not prime, then $\rho_{n,n'}(C_0)$ is not a point in $\mathcal{T}S_{g',n'}$, where*

$$n' = \max\left\{n_1 \mid n_1 < n \text{ and } n_1 \mid n\right\},$$

and g' is determined by the Formula (4-37).

The above lemma will be postponed until the end of the section. In the following, we prove Theorem 1.7 based on this principle. Note that our final aim is to prove the non-existence of such special curves, and the lemma above is only an intermediate step in the proof by contradiction.

Proof of Theorem 1.7. – Assume that there is a special curve C_0 contained generically in $\mathcal{T}S_{g,n}$ with $g \geq 8$, and $f : S \to B$ is the family of semi-stable n-superelliptic curves representing C_0 as in subsection 4.1. By Proposition 4.4, we may assume that $g \geq n$, or equivalently $\alpha_0 \geq 4$ by the Riemann-Hurwitz Formula (3-4).

We prove by induction on the number of prime factors in n. By [22, Theorem 1.2] and Theorem 4.7, we may assume that n is not prime. Let

$$n' = \max\left\{n_1 \,\middle|\, n_1 < n \text{ and } n_1 \mid n\right\}.$$

Since n is not a prime, it follows that $n' \geq 2$. Consider the image $\rho_{n,n'}(C_0) \subseteq \mathcal{T}S_{g',n'}$, where $\rho_{n,n'}$ is defined in (4-38). If

$$g' \geq 8, \tag{4-40}$$

then $\rho_{n,n'}(C_0)$ is not a special curve by induction. From Lemma 4.13 it follows that C_0 is not a special curve either. This gives a contradiction. Note that $g \geq 8$ by assumption. By (3-4) and (4-37), it is easy to verify that the above condition (4-40) is satisfied unless (n, α_0) belongs to the following list.

$$\begin{cases} \text{Case (a):} & n = 4 \text{ and } 7 \leq \alpha_0 \leq 16; \\ \text{Case (b):} & n = 6 \text{ and } 5 \leq \alpha_0 \leq 9; \\ \text{Case (c):} & n = 8 \text{ and } 4 \leq \alpha_0 \leq 6; \\ \text{Case (d):} & n = 9 \text{ and } 4 \leq \alpha_0 \leq 9; \\ \text{Case (e):} & n = 10, 15 \text{ or } 25, \text{ and } 4 \leq \alpha_0 \leq 5; \\ \text{Case (f):} & n = 12 \text{ and } \alpha_0 = 4. \end{cases} \tag{4-41}$$

To complete the proof, it suffices to prove the non-existence of special curves in the above cases.

We again prove by contradiction. Assume that such a special curve C_0 exists and let f be as above. By a possible base change, we may assume that the group $G = \mathbb{Z}/n\mathbb{Z}$ acts on S as before, and hence there is an induced action of G on the associated logarithmic Higgs bundle as well as its subbundles with induced eigenspace decompositions as in (3-32) and (3-34). Moreover, the rank of each eigenspace $E^{1,0}_{B,i}$ can be computed by (3-33). Let

$$i_m = \max\left\{i \,\middle|\, F^{1,0}_{B,i} \neq 0\right\}. \tag{4-42}$$

LEMMA 4.14. – *Let* (n, α_0) *be as in* (4-41). *Assume that* $\alpha_0 = 4$. *Then* $i_m < n/2$ *and* C_0 *is compact.*

Proof of Lemma 4.14. – We prove first $i_m < n/2$ by contradiction. Assume that $F^{1,0}_{B,i_0} \neq 0$ for some $i_0 \geq n/2$. Then $F^{1,0}_{B,i_0} = E^{1,0}_{B,i_0}$, since rank $E^{1,0}_{B,i_0} \leq 1$ by (3-33) together with the assumption that $\alpha_0 = 4$. Let $\widetilde{R} \subseteq \widetilde{Y}$ and $\widetilde{\Gamma} \subseteq \widetilde{Y}$ be the same as in Lemma 3.19. Then

$$\widetilde{\Gamma} \cdot \left(\omega_{\widetilde{Y}}(\widetilde{R}) \otimes \left(\mathcal{L}^{(i)^{-1}} \otimes \mathcal{L}^{(i_0)^{-1}}\right)\right) < 0, \qquad \forall\, 1 \leq i \leq n-1.$$

Hence by Lemma 3.19 and Lemma 3.22, after a suitable finite étale base change, there exists a unique fibration $f' : S \to B'$ such that

$$\dim H^0\big(S, \Omega_S^1\big)_i = \operatorname{rank} F_{B,i}^{1,0}, \quad H^0\big(S, \Omega_S^1\big)_i \subseteq (f')^* H^0\big(B', \Omega_{B'}^1\big), \quad \forall\, 1 \leq i \leq n-1.$$

In other words, $g(B') \geq \operatorname{rank} F_B^{1,0}$, which is a contradiction by Proposition 4.6.

Next we show that C_0 is compact. In fact, since $E_{B,n-1}^{1,0} = 0$ by (3-33), one obtains $E_{B,1}^{1,0} = F_{B,1}^{1,0}$ by Lemma 4.3. If C_0 is non-compact, then from [39, Corollary 4.4] it follows that the Higgs subbundle

$$\left(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B\right)_1 = \left(F_B^{1,0} \oplus F_B^{0,1},\ 0\right)_1$$

is a trivial Higgs subbundle after a possible étale base change. In other words, the corresponding local subsystem $\mathbb{V}_{B_0,1} = \mathbb{V}_{B_0,1}^{\mathrm{tr}}$ is trivial. Combine this with Lemma 4.2, it follows that $\mathbb{V}_{B_0,i}$ is also trivial for any $1 \leq i \leq n-1$ and $\gcd(i,n) = 1$. Also, by (3-33), it is easy to verify that there exists $i_0 > n/2$ such that $\gcd(i_0, n) = 1$ and $\operatorname{rank} E_{B,i_0}^{1,0} = 1$. Hence $F_{B,i_0}^{1,0} = E_{B,i_0}^{1,0} \neq 0$, which is a contradiction to the assumption. □

LEMMA 4.15. – *Let* (n, α_0) *be as in* (4-41). *Assume that* $\alpha_0 \geq 5$. *Then the following statements hold.*

(i). *If* $3 \mid n$, *then* $F_{B,2n/3}^{1,0} = 0$.

(ii). *If* $\operatorname{rank} E_{B,i}^{1,0} = 1$, *then* $F_{B,i}^{1,0} = 0$.

(iii). *Let* i_m *be as in* (4-42). *Then* $i_m > n/2$, *and hence after a suitable finite étale base change, there exists a unique morphism* $f' : S \to B'$ *such that*

$$\text{(4-43)} \qquad \begin{cases} \operatorname{rank} F_{B,i}^{1,0} = \dim H^0\big(S, \Omega_S^1\big)_i, & \forall\, n - i_m \leq i \leq i_m; \\ \displaystyle\bigoplus_{i=n-i_m}^{i_m} H^0\big(S, \Omega_S^1\big)_i \subseteq \big(f'\big)^* H^0\big(B', \Omega_{B'}^1\big). \end{cases}$$

(iv). *Let* i_m *be as above. If* $\gcd(n, i_m) = 1$, *then the curve* C_0 *is compact, and* $G = \mathbb{Z}/n\mathbb{Z}$ *induces a faithful action on* B' *(here* B' *is from (iii) above) such that* $B'/G \cong \mathbb{P}^1$ *and that*

$$\text{(4-44)} \quad H^0\big(S, \Omega_S^1\big)_i = \big(f'\big)^* H^0\big(B', \Omega_{B'}^1\big)_i, \quad \textit{for any } 1 \leq i \leq n-1 \textit{ with } \gcd(i,n) = 1.$$

In particular,

$$\text{(4-45)} \qquad \begin{aligned} g(B') &\geq \frac{\varphi(n)}{2} \cdot \left(\dim H^0\big(S, \Omega_S^1\big)_{i_m} + \dim H^0\big(S, \Omega_S^1\big)_{n-i_m}\right) \\ &= \frac{\varphi(n)}{2} \cdot \left(\operatorname{rank} F_{B,i_m}^{1,0} + \operatorname{rank} F_{B,n-i_m}^{1,0}\right), \end{aligned}$$

where $\varphi(n)$*is the Euler phi function, i.e., the number of non-negative integers less than* n *which are relatively prime to* n.

Proof of Lemma 4.15. – (i). We prove by contradiction. Assume that rank $F_{B,2n/3}^{1,0} > 0$. As $3 \mid n$ and (n, α_0) belong to (4-41), it follows that $5 \leq \alpha_0 \leq 9$. According to Lemma 4.13, the image $\rho_{n,3}(C) \subseteq \mathcal{T}S_{g_1,3}$ is still a special curve, where $\rho_{n,3}$ is defined in (4-38). Moreover, by (4-37) one gets

$$g_1 = \begin{cases} \alpha_0 - 1, & \text{if } 3 \not| \alpha_0; \\ \alpha_0 - 2, & \text{if } 3 \mid \alpha_0. \end{cases}$$

Let $f_1 : S_1 \to B$ be the family of 3-superelliptic curves associated to $\rho_{n,3}(C) \subseteq \mathcal{T}S_{g_1,3}$, and denote by $\left(\widetilde{E}_B^{1,0} \oplus \widetilde{E}_B^{0,1}, \widetilde{\theta}_B\right)$ the corresponding Higgs bundle associated to f_1. Then it follows from Lemma 4.12, whose proof is given later in Section 4, that we have the isomorphism

$$\widetilde{E}_{B,2}^{1,0} \cong E_{B,2n/3}^{1,0}.$$

Hence rank $\widetilde{F}_{B,2}^{1,0} > 0$ by our hypothesis, where $\widetilde{F}_B^{1,0} \subseteq \widetilde{E}_B^{1,0}$ is the flat part. Therefore, by Proposition 3.24 for f_1, after a suitable étale base change, there exists a fibration $f' : S_1 \to B'$ different from f_1 such that

$$\begin{aligned} g(B') \geq \operatorname{rank} \widetilde{F}_{B,1}^{1,0} + \operatorname{rank} \widetilde{F}_{B,2}^{1,0} &= \left(\operatorname{rank} \widetilde{E}_{B,1}^{1,0} - \operatorname{rank} \widetilde{E}_{B,2}^{1,0}\right) + 2\, rank \widetilde{F}_{B,2}^{1,0} \\ &\geq \left(\operatorname{rank} \widetilde{E}_{B,1}^{1,0} - \operatorname{rank} \widetilde{E}_{B,2}^{1,0}\right) + 2. \end{aligned}$$

This is a contradiction to (4-10) since $5 \leq \alpha_0 \leq 9$.

(ii). We prove by contradiction. Assume that there exists some $i_0 > n/2$ with rank $E_{B,i_0}^{1,0} = 1$ such that $F_{B,i_0}^{1,0} \neq 0$, i.e.,

$$\operatorname{rank} F_{B,i_0}^{1,0} = \operatorname{rank} E_{B,i_0}^{1,0} = 1.$$

Let $\widetilde{R} \subseteq \widetilde{Y}$ and $\widetilde{\Gamma} \subseteq \widetilde{Y}$ be the same as in Lemma 3.19. Then by (3-33) and (3-36) one checks that

$$\widetilde{\Gamma} \cdot \left(\omega_{\widetilde{Y}}(\widetilde{R}) \otimes \left(\mathcal{L}^{(i)^{-1}} \otimes \mathcal{L}^{(i_0)^{-1}}\right)\right) < 0, \qquad \forall\ 1 \leq i \leq n-1.$$

Therefore, by Lemma 3.19 and Lemma 3.22, after a suitable étale base change, there exists a fibration $f' : S \to B'$ such that $g(B') \geq \operatorname{rank} F_B^{1,0}$, which contradicts Proposition 4.6.

(iii). By Proposition 3.24, it suffices to prove that

$$i_m > n/2. \tag{4-46}$$

We divide the proof into two cases.

Consider first the case where C_0 is non-compact. By [39, Corollary 4.4], we may assume that the unitary local subsystem $\mathbb{V}_{B_0}^u \subseteq \mathbb{V}_{B_0} \otimes \mathbb{C}$ is trivial after a suitable finite base change. We proceed along the possible values of n:

(1) Using Lemma 4.2 and Lemma 4.3, one proves easily that rank $F_{B,n/2+1}^{1,0} > 0$ if $n = 8$ or 10; and that rank $F_{B,(n+1)/2}^{1,0} > 0$ if $n = 9$, 15 or 25.

(2) If $n = 6$, then by Lemma 4.10 and Lemma 4.13, $\rho_{6,3}(C_0)$ is again a non-compact special curve, where $\rho_{6,3}$ is defined in (4-38). Hence by Lemma 4.12, it suffices to prove rank $\widetilde{F}_{B,2}^{1,0} > 0$, where $\widetilde{F}_B^{1,0} \oplus \widetilde{F}_B^{0,1}$ is denoted to be the flat subbundle associated to the

new family representing $\rho_{6,3}(C_0)$. Suppose that rank $\widetilde{F}^{1,0}_{B,2} = 0$. Then using (4-16) and (3-33), one derives a contradiction to (4-9).

(3) If $n = 4$, we will derive a contradiction when rank $F^{1,0}_{B,3} = 0$. By Lemma 4.10 and Lemma 4.13, $\rho_{4,2}(C_0)$ is again a non-compact special curve, where $\rho_{4,2}$ is defined in (4-38). Moreover, $\rho_{4,2}(C_0)$ is contained generically in the hyperelliptic Torelli locus. Hence according to the proof of [22, Theorem 1.2], one has rank $\widetilde{F}^{1,0}_{B} \leq 1$, where $\widetilde{F}^{1,0}_{B} \oplus \widetilde{F}^{0,1}_{B}$ is denoted to be the flat subbundle associated to the new family representing $\rho_{4,2}(C_0)$. Thus rank $F^{1,0}_{B,2} =$ rank $\widetilde{F}^{1,0}_{B} \leq 1$ by Lemma 4.12. Since we assume that rank $F^{1,0}_{B,3} = 0$, by (4-5) and (3-33) one has

$$\operatorname{rank} F^{1,0}_{B} = \operatorname{rank} F^{1,0}_{B,1} + \operatorname{rank} F^{1,0}_{B,2} \leq \big(\operatorname{rank} E^{1,0}_{B,1} - \operatorname{rank} E^{1,0}_{B,3}\big) + 1.$$

Equivalently, one has

$$\operatorname{rank} A^{1,0}_{B} \geq 2 \operatorname{rank} E^{1,0}_{B,3} + \operatorname{rank} E^{1,0}_{B,2} - 1.$$

It is clear that $q_f =$ rank $F^{1,0}_{B} \neq 0$. According to (3-33), the above bound on rank $A^{1,0}_{B}$ gives a contradiction to (4-9).

Consider next the case when C_0 is compact. In this case, we prove (4-46) by contradiction. Assume that $F^{1,0}_{B,i} = 0$ for all $i > n/2$. Then $A^{1,0}_{B,i} = E^{1,0}_{B,i}$ for any $i > n/2$. Combing this with (4-5), one obtains

$$\operatorname{rank} A^{1,0}_{B} = \begin{cases} 2 \displaystyle\sum_{i=(n+1)/2}^{n-1} \operatorname{rank} E^{1,0}_{B,i}, & \text{if } 2 \nmid n; \\ \operatorname{rank} A^{1,0}_{B,n/2} + 2 \displaystyle\sum_{i=(n+2)/2}^{n-1} \operatorname{rank} E^{1,0}_{B,i}, & \text{if } 2 \mid n. \end{cases}$$

We claim that $\rho_{n,2}(C)$ is still a special curve if $2 \mid n$. In fact, if $\rho_{n,2}(C)$ were not a special curve, it follows easily from Lemma 4.13 and Lemma 4.17 that $n = 6$ and $\alpha_0 = 8$. We remark here that we apply Lemma 4.13 twice and use the fact that $\rho_{4,2} \circ \rho_{8,4} = \rho_{8,2}$ when excluding the case where $n = 8$. For the case where $n = 6$ and $\alpha_0 = 8$, by Lemma 4.18 and Lemma 4.19 together with Lemma 4.17 (v), one obtains a contradiction. Thus we may assume that $\rho_{n,2}(C)$ is still a special curve when $2 \mid n$. By Lemma 4.12 and Lemma 4.16 below, one has that rank $A^{1,0}_{B,n/2}$ is non-zero and even, if n is even. Combining with (3-4), we obtain a contradiction to (4-8). In fact, let $\nu = \frac{4(g-1)}{\operatorname{rank} A^{1,0}_{B}}$. Then

	$n = 25$	$n = 15$	$n = 10$	$n = 9$	$n = 8$	$n = 6$	$n = 4$
$\alpha_0 = 5$	$\nu = 10$	$\nu = 10$	$\nu = 10$	$\nu = \frac{15}{2}$	$\nu \leq \frac{26}{3}$	$\nu \leq 9$	—
$\alpha_0 = 6$	—	—	—	$\nu = 9$	$\nu \leq 8$	$\nu \leq 9$	—
$\alpha_0 = 7$	—	—	—	$\nu = \frac{23}{3}$	—	$\nu \leq 7$	$\nu \leq 8$
$\alpha_0 = 8$	—	—	—	$\nu = 9$	—	$\nu \leq 8$	$\nu \leq 8$
$\alpha_0 = 9$	—	—	—	$\nu = 9$	—	$\nu \leq 9$	$\nu \leq \frac{22}{3}$

$\alpha_0 = 10$	—	—	—	—	—	—	$\nu \leq 8$
$\alpha_0 = 11$	—	—	—	—	—	—	$\nu \leq 7$
$\alpha_0 = 12$	—	—	—	—	—	—	$\nu \leq 7$
$\alpha_0 = 13$	—	—	—	—	—	—	$\nu \leq \frac{34}{5}$
$\alpha_0 = 14$	—	—	—	—	—	—	$\nu \leq \frac{36}{5}$
$\alpha_0 = 15$	—	—	—	—	—	—	$\nu \leq 8$
$\alpha_0 = 16$	—	—	—	—	—	—	$\nu \leq 8$

This contradicts (4-8).

(iv) From the uniqueness of the fibration as in (iii), it follows that G admits an induced action on B'. Since $\gcd(n, i_m) = 1$, it follows from (4-43) that this induced action is faithful. Moreover, $B'/G \cong \mathbb{P}^1$ as the quotient S/G is ruled.

The equality (4-44) follows from a similar argument as that of Corollary 3.26. Indeed, by (4-43) one has

$$H^0\big(S,\,\Omega_S^1\big)_{i_m} \oplus H^1\big(S,\,\mathcal{O}_S\big)_{i_m} = \big(H^1(S,\,\mathbb{Q}) \otimes \mathbb{C}\big)_{i_m} \subseteq \big(f'\big)^*\,\big(H^1(B',\,\mathbb{Q}) \otimes \mathbb{C}\big)\,,$$

and the eigen-subspaces $\big(H^1(S,\,\mathbb{Q}) \otimes \mathbb{C}\big)_i$'s for $1 \leq i \leq p-1$ with $\gcd(n,i) = 1$ are permuted by this action of the arithmetic Galois subgroup $\mathrm{Gal}\big(\mathbb{Q}(\xi_n)/\mathbb{Q}\big)$, where ξ_n is a primitive n-th root of the unit. Hence

$$\bigoplus_{\substack{1 \leq i \leq n \\ \gcd(i,n)=1}} \big(H^1(S,\,\mathbb{Q}) \otimes \mathbb{C}\big)_i \subseteq \big(f'\big)^*\,\big(H^1(B',\,\mathbb{Q}) \otimes \mathbb{C}\big)\,.$$

By taking the $(1,0)$-part, we prove (4-44).

The inequality (4-45) follows immediately from (4-44) together with (4-43), by noting also that for any $1 \leq \{i, j\} \leq n-1$ with $\gcd(n,i) = \gcd(n,j) = 1$ one has

$$\dim H^0\big(B',\,\Omega_{B'}^1\big)_i + \dim H^0\big(B',\,\Omega_{B'}^1\big)_{n-i} = \dim H^0\big(B',\,\Omega_{B'}^1\big)_j + \dim H^0\big(B',\,\Omega_{B'}^1\big)_{n-j}.$$

Finally, we prove by contradiction that C is compact. Assume that C is non-compact. If $n = 4$ or 6, then $i_m = n-1$ since $i_m > n/2$ and $\gcd(n, i_m) = 1$. Hence by (4-43) one has $g(B') \geq \operatorname{rank} F_B^{1,0}$, This gives a contradiction to Proposition 4.6 as $g \geq 8$. For the remaining cases in (4-41), one checks by (3-33) easily that $\operatorname{rank} E_{B,1}^{0,1} = 0$, combining which together with Lemma 4.3 and [39, Corollary 4.4], one obtains that the Higgs subbundle

$$\Big(E_B^{1,0} \oplus E_B^{0,1},\,\theta\Big)_1 = \Big(F_B^{1,0} \oplus F_B^{0,1},\,0\Big)_1$$

is a trivial Higgs subbundle after a possible étale base change. In other words, the corresponding local subsystem $\mathbb{V}_{B,1}$ is trivial. With the help of Lemma 4.2, one shows that $\mathbb{V}_{B,i}$ is also trivial for any $1 \leq i \leq n-1$ and $\gcd(i,n) = 1$. Hence

$$g(B') \geq \sum_{\substack{1 \leq i \leq n \\ \gcd(i,n)=1}} \operatorname{rank} E_{B,i}^{1,0} = \frac{\varphi(n)}{2} \cdot \operatorname{rank} E_{B,1}^{1,0}.$$

Since C is non-compact, $\Delta_{nc} \neq \emptyset$. Similar to the proof of Proposition 4.6, one derives also a contradiction to (4-10) when restricting f' to a fiber F over Δ_{nc}. This completes the proof. $\square$

We can now derive contradictions case-by-case with (n, α_0) in (4-41), and so prove Theorem 1.7.

Case (a). In this case, $i_m = 3$ by Lemma 4.15 (iii). Hence by (4-43), one has

$$g(B') \geq \operatorname{rank} F_B^{1,0}.$$

This gives a contradiction to Proposition 4.6 as $g \geq 8$.

Case (b). In this case, one has $i_m = 5$ by Lemma 4.15 (i) and (iii). Hence similar to the above case, there is also a contradiction to Proposition 4.6.

Case (c). If $\alpha_0 = 4$, then C_0 is compact by Lemma 4.14. Since $i_m < n/2 = 4$, it follows from (3-33) and Lemma 4.3 that

$$\operatorname{rank} A_{B,i}^{1,0} = 0, \quad \text{for } i \in \{1,2,6,7\}, \qquad\qquad \operatorname{rank} A_{B,i}^{1,0} = 1, \quad \text{for } 5 \leq i \leq 7.$$

Thus $\operatorname{rank} A_B^{1,0} = 3$ is odd, contradicting Lemma 4.16.

If $\alpha_0 = 5$, then by (3-33) and Lemma 4.15 (ii), $i_m \leq n/2 = 4$, This contradicts Lemma 4.15 (iii).

If $\alpha_0 = 6$, from Lemma 4.15 it follows that C is compact, and that (4-44) and (4-45) hold. Note also that $F_{B,5}^{1,0} \subsetneq E_{B,5}^{1,0}$ and hence $\operatorname{rank} F_{B,5}^{1,0} = 1$; otherwise, one has $\operatorname{rank} F_{B,5}^{1,0} = \operatorname{rank} E_{B,5}^{1,0} = 2$, and hence both $E_{B,3}^{1,0}$ and $E_{B,5}^{1,0}$ are flat, from which together with (4-45) it follows that

$$g(B') \geq \frac{\varphi(8)}{2} \cdot \left(\operatorname{rank} F_{B,5}^{1,0} + \operatorname{rank} F_{B,3}^{1,0}\right) = \frac{4}{2} \cdot \left(\operatorname{rank} E_{B,5}^{1,0} + \operatorname{rank} E_{B,3}^{1,0}\right) = 10.$$

This contradicts (4-10) as $g = 17$ by (3-4) in this case. Thus,

$$\begin{cases} \operatorname{rank} A_{B,7}^{1,0} = \operatorname{rank} E_{B,7}^{1,0} = 0; & \\ \operatorname{rank} A_{B,6}^{1,0} = \operatorname{rank} E_{B,6}^{1,0} = 1, & \text{by Lemma 4.15 (ii);} \\ \operatorname{rank} A_{B,5}^{1,0} = \operatorname{rank} E_{B,5}^{1,0} - 1 = 1, & \text{by the above arguments;} \\ \operatorname{rank} A_{B,4}^{1,0} = \operatorname{rank} E_{B,4}^{1,0} = 2, & \text{by Lemma 4.17 (i) and Lemma 4.16 (proved p. 1653).} \end{cases}$$

Hence

$$\operatorname{rank} A_B^{1,0} = 2\sum_{i=5}^{7} \operatorname{rank} A_{B,7}^{1,0} + \operatorname{rank} A_{B,4}^{1,0} = 6.$$

This is a contradiction to (4-8).

Case (d). In this case, we can derive a contradiction similarly as the above case. First, if $\alpha_0 \geq 5$, then similar as above, by Lemma 4.15 one obtains that $i_m = 5$ and C is compact. If $\operatorname{rank} F_{B,5}^{1,0} = \operatorname{rank} E_{B,5}^{1,0}$, then one obtains a contradiction to (4-10); if $\operatorname{rank} F_{B,5}^{1,0} \leq \operatorname{rank} E_{B,5}^{1,0} - 1$, i.e., $\operatorname{rank} A_{B,5}^{1,0} \geq 1$, then

$$\operatorname{rank} A_B^{1,0} = 2\sum_{i=5}^{8} \operatorname{rank} A_{B,7}^{1,0} \geq 2 + 2\sum_{i=6}^{8} \operatorname{rank} A_{B,7}^{1,0} = 2 + 2\sum_{i=6}^{8} \operatorname{rank} E_{B,7}^{1,0},$$

which contradicts (4-8).

If $\alpha_0 = 4$, it is a little more complicated. By Lemma 4.14, $F_{B,i}^{1,0} = 0$ for any $i > n/2 = 9/2$. Combining with (3-33) and (4-5), one obtains that

$$\operatorname{rank} F_{B,1}^{1,0} = \operatorname{rank} F_{B,2}^{1,0} = 3, \quad \operatorname{rank} F_{B,3}^{1,0} = \operatorname{rank} F_{B,4}^{1,0} = 1.$$

Hence by [9, §4.2], it follows that after a suitable étale base change, both $F_{B,3}^{1,0}$ and $F_{B,4}^{1,0}$ become trivial. In other words, one has

$$\dim H^0\big(S, \Omega_S^1\big)_3 = \dim H^0\big(S, \Omega_S^1\big)_4 = 1.$$

By Hurwitz-Chevalley-Weil's formula (cf. [26, Proposition 5.9]), we have $H^0\big(S, \Omega_S^2\big)_7 = 0$; indeed, one has $H^0\big(F, \omega_F\big)_7 = 0$ by Hurwitz-Chevalley-Weil's formula, from which it follows that $j^*(\gamma) = 0$ for any $\gamma \in H^0\big(S, \Omega_S^2\big)_7$, where

$$j^*: \ H^0\big(S, \Omega_S^2\big)_7 = H^0\big(S, \omega_S\big)_7 \longrightarrow H^0\big(F, \omega_F\big)_7 = 0$$

is the canonical pulling-back and $j : F \hookrightarrow S$ is the embedding of a general fiber of f into S. Since F is general, it follows that $H^0\big(S, \Omega_S^2\big)_7 = 0$ as required.

Let $\omega \in H^0\big(S, \Omega_S^1\big)_3$ and $\eta \in H^0\big(S, \Omega_S^1\big)_4$ be two non-zero one-forms.

Since $\omega \wedge \eta \in H^0\big(S, \Omega_S^2\big)_7$ by construction, it follows that $\omega \wedge \eta = 0$. Hence by Castelnuovo-de Franchis lemma (cf. [3, Theorem IV-5.1]), there exists an irregular fibration $f' : S \to B'$ such that

$$H^0\big(S,\ \Omega_S^1\big)_3 \oplus H^0\big(S,\ \Omega_S^1\big)_4 \subseteq \big(f'\big)^* H^0\big(B',\ \Omega_{B'}^1\big). \tag{4-47}$$

It is clear that such a fibration is unique, and hence the group G induces an action on B'. Moreover, the induced action is faithful; otherwise

$$\big(f'\big)^* H^0\big(B',\ \Omega_{B'}^1\big) \subseteq H^0\big(S,\ \Omega_S^1\big)^{G_0} = \bigoplus_{m_0 \mid i} H^0\big(S,\ \Omega_S^1\big)_i,$$

which contradicts (4-47), where $G_0 \leq G$ is the kernel of the action of G on B' and $m_0 = |G_0|$. Finally, since S/G is ruled, it follows that $B'/G \cong \mathbb{P}^1$.

Let F be a general fiber of f, and consider the restricted map $f'|_F : F \to B'$. Since $G = \mathbb{Z}/9\mathbb{Z}$ acts faithfully on both F and B', whose quotients are both isomorphic to $\mathbb{P}^1$ with the following commutative diagram:

$$\begin{tikzcd} F \arrow[r, "f'|_F"] \arrow[d, "\Pi|_F"'] & B' \arrow[d, "\pi'"] \\ \Gamma \cong \mathbb{P}^1 \arrow[r, "\varphi'|_\Gamma"] & B'/G \cong \mathbb{P}^1. \end{tikzcd}$$

We claim that $\deg(f'|_F) = 2$. Indeed, note that the cover $\Pi|_F$ has exactly $\alpha = 5$ branch points, and one checks easily that π' admits $\beta \geq 3$ branch points since $g(B') > 0$. Because f is not isotrivial, similar to the proof of (4-21), one shows that

$$3 = \alpha - 2 > \deg(f'|_F) \cdot (\beta - 2) \geq \deg(f'|_F) > 1.$$

Thus $\deg(f'|_F) = 2$ as required.

Since $\deg(f'|_F) = 2$, it induces an involution τ_0 on F, such that $B' = F/\langle\tau_0\rangle$. Let $\widetilde{G} \subseteq \mathrm{Aut}(F)$ be the subgroup generated by G and τ_0. As $f'|_F$ is equivariant with respect to G, it follows that τ_0 commutes with G. Hence $\widetilde{G}$ is a cyclic group of order $|\widetilde{G}| = 18$. Moreover, by considering the composition map $\pi' \circ (f'|_F) : F \to \mathbb{P}^1$, one checks easily that $\pi' \circ (f'|_F)$ is a cyclic cover branched over exactly 4 points. In other words, the family $f : S \to B$ is a universe family of cyclic covers of $\mathbb{P}^1$ with Galois group $\widetilde{G} \cong \mathbb{Z}/18\mathbb{Z}$. Hence according to [25, Theorem 3.6], C_0 is not a special curve since $g = 12$ by (3-4).

Case (e). Consider first the case where $\alpha_0 = 4$. Since $F^{1,0}_{B,i} = 0$ for all $i \geq n/2$ by Lemma 4.14, it follows that $A^{1,0}_{B,i} = E^{1,0}_{B,i}$ for all $i \geq n/2$. By (4-5) and (3-33), one obtains

$$\operatorname{rank} A^{1,0}_B = 2 \sum_{i=(n+1)/2}^{n-1} \operatorname{rank} E^{1,0}_{B,i} = \begin{cases} 5, & \text{if } n = 10; \\ 8, & \text{if } n = 15; \\ 12, & \text{if } n = 25. \end{cases}$$

If $n = 10$, it contradicts Lemma 4.16 since C_0 is compact by Lemma 4.14; if $n = 15$ or 25, it contradicts (4-8) in view of (3-4).

We assume now that $\alpha_0 = 5$. By Lemma 4.15 (ii) and (iii), one checks that $n \neq 10$, and that $i_m = 8$ (resp. $i_m = 13$ or 14) when $n = 15$ (resp. $n = 25$). In the later two cases, by Lemma 4.15, after a suitable further étale base change, there is a unique fibration $f' : S \to B'$ and G acts faithfully on B' with $B'/G \cong \mathbb{P}^1$. Let F be a general fiber of f, and $\Gamma = F/G \cong \mathbb{P}^1$ the quotient. Then similar to the proof of Lemma 4.8, one has the following commutative diagram:

$$\begin{tikzcd} F \arrow[r, "f'|_F"] \arrow[d, "\Pi|_F"'] & B' \arrow[d, "\pi'"] \\ \Gamma \cong \mathbb{P}^1 \arrow[r, "\varphi'|_\Gamma"] & B'/G \cong \mathbb{P}^1. \end{tikzcd}$$

Let β be the number of branch points of the cover $\pi' : B' \to \mathbb{P}^1$. Then similar to the proof of Lemma 4.8, one proves that $\beta \geq 4$ and $\alpha > 2(\beta - 1)$, where $\alpha = \alpha_0 + 1 = 6$ is the number of the branch points of $\Pi|_F$. This gives a contradiction.

Case (f). By Lemma 4.14, C_0 is compact and $F^{1,0}_{B,i} = 0$ for all $i \geq n/2 = 6$. Thus $A^{1,0}_{B,i} = E^{1,0}_{B,i}$ for all $i \geq n/2$. By (4-5) and (3-33), one obtains $\operatorname{rank} A^{1,0}_{B,i} = 5$, which contradicts Lemma 4.16. $\square$

The following result was used in the above proof.

LEMMA 4.16. – *Let $C \subset \mathcal{A}_g$ be a compact special curve, with E_C the Higgs bundle on C associated to the universal abelian scheme over C defined by the moduli problem. Write $E_C = F_C \oplus A_C$ for the decomposition into the flat part and the maximal part. Then the rank of A_C is divisible by 4. Equivalently the rank of $A^{1,0}_C$ is even.*

Proof. – Write $\pi : \mathcal{E}_C \to C$ for the abelian scheme over C determined by the inclusion $C \hookrightarrow \mathcal{A}_g$, and write $(\mathbf{G}, X; X^+)$ for the Shimura datum defining C. From the Simpson correspondence [36] we know that the decomposition of Higgs bundles $E_C = F_C \oplus A_C$

is determined by the representation of the fundamental group of C to the local system $R^1\pi_*\mathbb{C}_{\mathcal{E}_g}$, and it is invariant under base change. It is further characterized by the algebraic representation $\mathbf{G}^{\mathrm{der}} \to \mathrm{Sp}_{2g}$ on $\mathbb{Q}^{2g}$, following [5].

Following the discussion in Subsection 2.2, we write $\mathbf{G}^{\mathrm{der}} = \mathrm{Res}_{L/\mathbb{Q}}\mathbf{H}$ for some totally real number field L and some L-form $\mathbf{H}$ of $\mathrm{SL}_{2,L}$. From the classification in [35] (see also Section 5 in [5]) we see that $\mathbf{G}^{\mathrm{der}} \to \mathrm{Sp}_{2g}$ decomposes as $\mathbb{Q}^{2g} = T \oplus V$, such that:

— the action of $\mathbf{G}^{\mathrm{der}}$ on T is trivial;

— V is the $\mathbb{Q}$-vector space underlying some symplectic L-vector space W;

— the action of $\mathbf{G}^{\mathrm{der}}$ on V is obtained by scalar restriction from an L-linear representation $\mathbf{H} \to \mathrm{Sp}_W$.

Moreover, using the list of L-forms of SL_2 in Subsection 2.2, we see that:

(i) if L is totally real and $\mathbf{H}$ comes from a quaternion L-algebra D, then W is a D-vector space, and D must be a division L-algebra because C is compact and $\mathbf{G}^{\mathrm{der}}$ is of $\mathbb{Q}$-rank zero; this forces the L-dimension of W to be a multiple of 4;

(ii) if for some CM field E of real part L and degree $2d$ over $\mathbb{Q}$, and $\mathbf{H}$ is associated to D, a quaternion E-algebra:

  (ii-a) either $D \simeq \mathrm{Mat}_2(E)$, and W is a direct sum of copies of the standard representation of $\mathbf{H}$ on E^2, whose E-dimension is even, and L-dimension of W is divisible by 4;

  (ii-b) or D is a quaternion division E-algebra, and W is a D-vector space, whose E-dimension is divisible by 4 and L-dimension divisible by 8.

Hence the $\mathbb{Q}$-dimension of V, which also equals the rank of A_C, must be a multiple of $4d$, with d the degree of L. □

To complete the proof of Theorem 1.7, it remains to prove Lemma 4.13. We first prove

LEMMA 4.17. – *Notations as above. Assume that C_0 is a special curve but $\rho_{n,n_1}(C_0)$ is not a special curve. Then*

(i) $n_1 \mid \alpha_0$ *and* $n \nmid \alpha_0$*;*

(ii) *up to a suitable finite étale base change, $f : S \to B$ is birational to the resolution of a degree-n cyclic cover of a $\mathbb{P}^1$-bundle $\varphi : Y \to B$ branched exactly over $\alpha_0 + 1$ disjoint sections, denoted as $D_1, \ldots, D_{\alpha_0+1}$, such that the local monodromy is 1 around $D_1, \ldots, D_{\alpha_0}$, and equals $a_\infty = n(1 - \{\frac{\alpha_0}{n}\})$ around D_{α_0+1};*

(iii) *up to a suitable finite étale base change, the Higgs subbundle*

$$\bigoplus_{i=1}^{n_1-1} \left(E_B^{1,0} \oplus E_B^{0,1},\ \theta_B\right)_{im_1}$$

is a trivial Higgs subbundle, where $m_1 = \frac{n}{n_1}$;

(iv) $F_{B,i}^{1,0} = 0$ *for any $i \geq m_1$ and $m_1 \nmid i$;*

(v) *if moreover the general fiber of f admits a unique n-superelliptic automorphism group G and there exists a generator of G commuting with any automorphism of the general fiber, then*

$$\operatorname{rank} A_B^{1,0} \leq \begin{cases} \dfrac{\left(n^2 - \left(\frac{n}{r_\infty}\right)^2\right)(\alpha_0 - 2)}{6\big((n-1)\alpha_0 - 2n\big)}, & \text{if } \ell_1 = 1; \\[2ex] \dfrac{n^2 - \left(\frac{n}{r_\infty}\right)^2}{6n} + \dfrac{(\ell_1^2 - 1)\alpha_0}{6n(\alpha_0 - 2)}, & \text{if } \ell_1 > 1. \end{cases} \tag{4-48}$$

where $r_\infty = \frac{n}{\gcd(n,\alpha_0)}$ *and* $\ell_1 = \gcd(\alpha_0 - 1, n)$.

Proof. – As above, after a possible base change, we assume that the group G acts on S, and let $f_1 : S_1 \to B$ be the new family of semi-stable n_1-superelliptic curves. Assume that f is given by $y^n = F_t(x)$, where t is the parameter. Then both of the two families f and f_1, up to base change, are birational to the resolution of cyclic covers of a $\mathbb{P}^1$-bundle $\varphi : Y \to B$ branched over the zero locus of $F_t(x)$ and possibly over the section at the infinity (depending on whether n_1 and n divide α_0 or not respectively). In other words, choose suitable birational models, we have the following diagram:

$$\begin{tikzcd} \widetilde{S} \arrow[rr, "\Pi_{n,n_1}"] \arrow[dr, "\Pi"'] & & \widetilde{S}_1 \arrow[dl, "\Pi_1"] \\ & Y. & \end{tikzcd}$$

Here Π_{n,n_1} is induced by the rational map $\Pi_{n,n_1} : S \dashrightarrow S_1$, and the difference between the branch loci of Π and of Π_1 is at most the section at the infinity; see (3-3) and (4-36).

Since $\rho_{n,n_1}(C_0)$ is not a special curve, it follows from Lemma 4.10 that $\rho_{n,n_1}(C_0)$ is a (special) point in $\mathcal{T}S_{g_1,n_1}$. In other words, the semi-stable family f_1 is isotrivial by construction. Being semi-stable, the family f_1 is actually a smooth family. In other words, up to some elementary transformations of the $\mathbb{P}^1$-bundle Y, the branch locus R_1 of the cover Π_1 is smooth and the restricted map $\varphi|_{R_1} : R_1 \to B$ is étale; here we recall that an *elementary transformation* of a $\mathbb{P}^1$-bundle X is a new $\mathbb{P}^1$-bundle X' obtained by first blowing up some point $x \in X$ and then contracting the strict inverse image of the fiber of X through x. We should remark that an elementary transformation is a birational operation, but it may transfer the section at the infinity to somewhere else.

(i) The first statement is clear from the above arguments; otherwise, the branch loci of Π and Π_1 are the same by the above arguments together with (3-3) and (4-36). Since f_1 is isotrivial, it follows that f is also an isotrivial family, which is a contradiction since C_0 is a special curve.

(ii) By (i) and its proof above, we obtain that the branch locus R of Π equals the branch locus R_1 of Π_1 plus one another section. On the other hand, as the restricted map $\varphi|_{R_1} : R_1 \to B$ is étale, it follows that after a possible suitable finite étale base change, R_1 becomes α_0 disjoint sections. Since R equals R_1 plus one another section D_{α_0+1}, and the local monodromy around each component in R_1 (resp. the component D_{α_0+1}) is 1 $\big($resp. $a_\infty = n\left(\left[\frac{\alpha_0}{n}\right] + 1\right) - \alpha_0\big)$ by construction, this proves the second statement.

(iii) Since f_1 is isotrivial, the Higgs bundle associated to f_1 is trivial after a suitable unramified base change. Hence this statement follows directly from (4-39).

(iv) Note that the rank of the subbundle $F_{B,i}^{1,0}$ does not decrease after base change. Hence it suffices to prove the statement after any finite base change. By the above arguments, it follows that $S_1 \cong B \times F_1$ up to some finite étale base change, where F_1 is a general fiber of $f_1 : S_1 \to B$. Compose the rational map $\Pi_{n,n_1} : S \dashrightarrow S_1$ with the second projection $S_1 \to F_1$, we obtain a rational map $\overline{pf}_1 : S \dashrightarrow F_1$. Since $g(F_1) > 0$ by construction, it follows that $\overline{pf}_1$ is in fact a morphism. By (iii) together with [14, Theorem 3.1] (see also (3-45)), we have

$$\operatorname{rank} f_*\Omega^1_{S/B}(\log \Upsilon)_i = \dim H^0\big(S, \Omega^1_S\big)_i, \quad \forall\, m_1 \mid i \ \&\ i \neq 0.$$

Together with (4-39), we obtain that

$$\big(\overline{pf}_1\big)^* H^0\big(F_1, \Omega^1_{F_1}\big) = \bigoplus_{m_1 \mid i \ \&\ i \neq 0} H^0\big(S, \Omega^1_S\big)_i. \tag{4-49}$$

Assume that the flat part $F_{B,i_0}^{1,0} \neq 0$ for some $i_0 \geq m_1$ and $m_1 \nmid i_0$. By (i), one has $\alpha_0 = kn_1$ with $k \geq 1$.

If $k \geq 2$, i.e., $\alpha_0 \geq 2n_1$, then $F_{B,n-m_1}^{1,0} \neq 0$ by (4-39). Hence up to base change, we may assume that $F_{B,i_0}^{1,0}$ is trivial by Proposition 3.21. This is equivalent to saying that $H^0\big(S, \Omega^1_S\big)_{i_0} \neq 0$ by (3-45). By Corollary 3.23, there exists a unique fibration $f' : S \to B'$ such that (3-48) holds. Together with (4-49), it follows that f' is the same as $\overline{pf}_1$ obtained above. This is a contradiction by (3-48) and (4-49).

If $k = 1$, i.e., $\alpha_0 = n_1$, then $F_{B,n-2m_1}^{1,0} \neq 0$ by (4-39). Moreover, if we let $\widetilde{\Gamma}$ be the general fiber of $\tilde{\varphi}$ as in subsection 3.4, then

$$\begin{aligned}
\widetilde{\Gamma} \cdot &\left(\omega_{\widetilde{Y}}(\widetilde{R}) \otimes \left(\mathcal{L}^{(n-2m_1)^{-1}} \otimes \mathcal{L}^{(i_0)^{-1}}\right)\right)\\
&= -2 + (\alpha_0 + 1) - \frac{(n-2m_1)\alpha_0}{n} - \left(\frac{i_0 \cdot n}{n} - \left[\frac{i_0(n-\alpha_0)}{n}\right]\right)\\
&= 1 - \frac{i_0}{m_1} - \left(\frac{i_0(n-\alpha_0)}{n} - \left[\frac{i_0(n-\alpha_0)}{n}\right]\right) < 0.
\end{aligned}$$

Hence similarly as above, one derives a contradiction. This proves (iv).

(v). In this case, by Lemma 3.3 (ii), the group action of $G = \mathbb{Z}/n\mathbb{Z}$ can be extended to S without any base change. In other words, the above properties (i)-(iii) hold for f, and simultaneously one has the Arakelov type equality as in (4-3) for f.

Let $s_{\gamma,\ell,0} = s_{\gamma,\ell,0}(f)$ and $s_{\gamma,\ell,1} = s_{\gamma,\ell,1}(f)$ be the local invariants of f introduced in Definition 3.6. We first claim that

$$s_{\gamma,\ell,0} = 0 \text{ for any } (\gamma,\ell), \quad \text{and } s_{\gamma,\ell,1} = 0 \text{ unless } (\gamma,\ell) = (2,\ell_1). \tag{4-50}$$

In fact, by (ii), the branch locus of Π consists of α_0 disjoint sections $\sum\limits_{i=1}^{\alpha_0} D_i$ plus one another section D_{α_0+1}. Moreover, the local monodromy around D_i is 1 for $1 \leq i \leq \alpha_0$, and around D_{α_0+1} is a_∞. Let $\xi_{\gamma,\ell,0}$ (resp. $\xi_{\gamma,\ell,1}$) be the number of the nodes in fibers of φ with index $(\gamma,\ell,0)$ (resp. $(\gamma,\ell,1)$), counted according to their multiplicities, where

$\varphi: Y \triangleq S/G \to B$ is the induced quotient family as in the proof of Theorem 3.9. Then it is easy to see that

$$\xi_{\gamma,\ell,0} = 0 \text{ for any } (\gamma,\ell); \quad \text{and } \xi_{\gamma,\ell,1} = 0 \text{ unless } \gamma = 2.$$

Moreover, $\xi_{2,\ell,0} = 0$ unless $\ell = \ell_1$ by (3-6). Hence (4-50) follows from (3-10).

According to (4-50) together with (3-8) and (3-10), we get

$$\begin{aligned}\deg E_B^{1,0} = \deg f_*\omega_{S/B} &= \left(\frac{\left(n^2 - \left(\frac{n}{r_\infty}\right)^2\right)(\alpha_0 - 2)}{\alpha_0} + (\ell_1^2 - 1)\right) \cdot \frac{s_{2,\ell_1,1}}{12\ell_1^2} \\ &= \left(\frac{\left(n^2 - \left(\frac{n}{r_\infty}\right)^2\right)(\alpha_0 - 2)}{\alpha_0} + (\ell_1^2 - 1)\right) \cdot \frac{\xi_{2,\ell_1,1}}{12n}.\end{aligned} \tag{4-51}$$

On the other hand, by Step 1 we know that the $\mathbb{P}^1$-bundle Y contains $\alpha_0 \geq 4$ disjoint sections up to a suitable unramified base change, hence it follows that $Y \cong B \times \mathbb{P}^1$. Let $pr_2: Y \cong B \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection, and

$$\psi: D_{\alpha_0+1} \hookrightarrow Y \xrightarrow{pr_2} \mathbb{P}^1$$

be the induced map on D_{α_0+1}. Note that D_i is contracted by pr_2 for $1 \leq i \leq \alpha_0$. It follows that ψ is surjective. Let $R_\psi \subseteq D_{\alpha_0+1}$ be the ramified divisor of ψ. Then by Hurwitz formula, it follows that

$$\deg(R_\psi) = 2g(D_{\alpha_0+1}) - 2 - 2\deg(\psi) = 2g(B) - 2 + 2\deg(\psi).$$

Let $\nu_i = |D_i \cap D_{\alpha_0+1}|$ be the number of points contained in $D_i \cap D_{\alpha_0+1}$ for $1 \leq i \leq \alpha_0$, and $|\Delta|$ be the number of the singular fibers of f. Then

$$\xi_{2,1,1} = \sum_{i=1}^{\alpha_0} D_i \cdot D_{\alpha_0+1} = \alpha_0 \deg(\psi), \tag{4-52}$$

and

$$\sum_{i=1}^{\alpha_0} \nu_i = |\Delta| \leq s_{2,1,1} = \frac{\ell_1^2}{n} \cdot \xi_{2,1,1} = \frac{\ell_1^2}{n} \cdot \alpha_0 \deg(\psi).$$

Hence

$$\begin{aligned}2g(B) - 2 + 2\deg(\psi) = \deg(R_\psi) &\geq \sum_{i=1}^{\alpha_0} (\deg(\psi) - \nu_i) \\ &= \alpha_0 \cdot \deg(\psi) - \sum_{i=1}^{\alpha_0} \nu_i = \alpha_0 \cdot \deg(\psi) - |\Delta| \\ &\geq \left(1 - \frac{\ell_1^2}{n}\right) \cdot \alpha_0 \deg(\psi).\end{aligned} \tag{4-53}$$

Combining (4-51) with (4-52), we have

$$\deg E_B^{1,0} = \left(\frac{\left(n^2 - \left(\frac{n}{r_\infty}\right)^2\right)(\alpha_0 - 2)}{\alpha_0} + (\ell_1^2 - 1)\right) \cdot \frac{\alpha_0 \deg(\psi)}{12n}. \tag{4-54}$$

Note that

$$|\Delta_{nc}| = 0, \quad \text{if } \ell_1 = 1; \qquad \text{and} \qquad |\Delta_{nc}| = |\Delta|, \quad \text{if } \ell_1 > 1.$$

Combining this with the Arakelov type equality in (4-3) and (4-53), one obtains

$$\deg E_B^{1,0} = \deg f_*\omega_{S/B} \geq \begin{cases} \dfrac{\operatorname{rank} A_B^{1,0}}{2} \cdot \left(\dfrac{(n-1)\alpha_0}{n} - 2\right) \deg(\psi), & \text{if } \ell_1 = 1; \\ \dfrac{\operatorname{rank} A_B^{1,0}}{2} \cdot (\alpha_0 - 2) \deg(\psi), & \text{if } \ell_1 > 1. \end{cases}$$

Combining the above inequality together with (4-54), we complete the proof. □

In order to apply Lemma 4.17 (v), one still needs the next two lemmas.

LEMMA 4.18. – *Let F be an n-superelliptic curve of genus $g > (p-1)(2p-1)$ with $n = 2p$ for some prime p. Then F admits a unique n-superelliptic cover.*

Proof. – This follows from the Castelnuovo-Severi inequality (cf. [17, Exercise V.1.9]). Indeed, if there are two different n-superelliptic covers:

$$\pi : F \longrightarrow \mathbb{P}^1, \qquad \text{and} \qquad \tilde{\pi} : F \longrightarrow \mathbb{P}^1,$$

then the ramified loci of both π and $\tilde{\pi}$ have the same number of points with the same ramification indices. Locally, we may assume that π and $\tilde{\pi}$ are defined by $y^n = F(x)$ and $y^n = \widetilde{F}(x)$ respectively, where both $F(x)$ and $\widetilde{F}(x)$ are separable polynomials of equal degree. Let $\alpha_0 = \deg(F) = \deg(\widetilde{F})$, and let D and $\widetilde{D}$ be defined by $z^p = F(x)$ and $z^p = \widetilde{F}(x)$ respectively. Then $\alpha_0 > 2p$ by (3-4) as $g > (p-1)(2p-1)$, and one has

$$g(D) = g(\widetilde{D}) = \begin{cases} \dfrac{(p-1)(\alpha_0 - 2)}{2}, & \text{if } p \mid \alpha_0; \\ \dfrac{(p-1)(\alpha_0 - 1)}{2}, & \text{if } p \nmid \alpha_0. \end{cases}$$

If $D \not\cong \widetilde{D}$, then F admits two different double covers to D and $\widetilde{D}$ respectively. Hence by the Castelnuovo-Severi inequality,

$$g \leq 1 + 2g(D) + 2g(\widetilde{D}).$$

This contradicts (3-4) together with the above formulas for $g(D)$ and $g(\widetilde{D})$.

If $D \cong \widetilde{D}$, then again by the Castelnuovo-Severi inequality, there is a unique action of $G' = \mathbb{Z}/p\mathbb{Z}$ on $D \cong \widetilde{D}$. By the definitions of D and $\widetilde{D}$, this implies that the polynomials $F(x)$ and $\widetilde{F}(x)$ are conjugate to each other under the automorphism group of $\mathbb{P}^1$. Hence the covers π and $\tilde{\pi}$ are the same, which is also a contradiction. □

LEMMA 4.19. – *For any n-superelliptic curve F, if F admits a unique n-superelliptic automorphism group $G = \mathbb{Z}/n\mathbb{Z}$, then $\sigma \circ \tau = \tau \circ \sigma$ for any $\sigma \in G$ and $\tau \in$ Aut (F).*

Proof. – Let F be defined by $y^n = F(x)$ as usual. As F admits a unique n-superelliptic automorphism group, it follows that σ acts only on y and τ induces an automorphism on the quotient $\mathbb{P}^1 = F/G$. In other words,

$$\sigma(x, y) = (x, \xi y) \ \text{ with } \xi^n = 1, \qquad \tau(x, y) = \big(\tau_1(x), \tau_2(x, y)\big),$$

Note that τ_1 is an automorphism of $\mathbb{P}^1$ and keeps the branch locus of $\pi : F \to \mathbb{P}^1 = F/G$ invariant.

If π is branched over ∞ with local monodromy $a_\infty \neq 1$, then τ_1 must keep ∞ invariant, and $F\big(\tau_1(x)\big)$ has the same set of roots as $F(x)$. Hence $\tau_1(x) = ax + b$ for some $a, b \in \mathbb{C}$, and $F\big(\tau_1(x)\big) = k^* \cdot F(x)$ with $k^* \neq 0$. As τ is an automorphism group of F, one obtains that $\tau_2(x, y) = \eta \cdot y$ with $\eta^n = k^*$. Therefore, it is clear that $\sigma \circ \tau = \tau \circ \sigma$.

It remains to consider the case where π is not branched over ∞, since the case where π is branched over ∞ with local monodromy 1 can be reduced to the former case by automorphism of $\mathbb{P}^1$. In this case, $n \mid \alpha_0$, where $\alpha_0 = \deg(F(x))$. Moreover, as an automorphism of $\mathbb{P}^1$, τ_1 has the form $\tau_1(x) = \frac{ax+b}{cx+d}$. Since τ_1 keeps the set of roots of $F(x)$ invariant, $F\big(\tau_1(x)\big) = \frac{k^* \cdot F(x)}{(ax+b)^{\alpha_0}}$. Hence $\tau_2(x, y) = \frac{\eta \cdot y}{(ax+b)^{\alpha_0/n}}$ with $\eta^n = k^*$. One checks easily again that $\sigma \circ \tau = \tau \circ \sigma$. This completes the proof. □

We can now prove Lemma 4.13, which was used in the proof of Theorem 1.7.

Proof of Lemma 4.13. – We prove by contradiction. Assume that $\rho_{n,n'}(C)$ is a point. Then by Lemma 4.17 one has

$$n' \mid \alpha_0 \quad \text{and} \quad n \nmid \alpha_0. \tag{4-55}$$

$$F^{1,0}_{B,i} = 0, \qquad \text{for any } i \geq m' \text{ with } m' \nmid i, \text{ where } m' = n/n'. \tag{4-56}$$

On the other hand, by (3-33) one has

$$\operatorname{rank} E^{1,0}_{B,m'+1} - \operatorname{rank} E^{1,0}_{B,n-m'-1} = \alpha_0 - 1 - 2\left\lceil \frac{(m'+1)\alpha_0}{n} \right\rceil.$$

If $n' \geq 4$, or $n' = m' = 3$, then one checks easily that

$$\operatorname{rank} E^{1,0}_{B,m'+1} - \operatorname{rank} E^{1,0}_{B,n-m'-1} > 0.$$

This together with (4-6) implies in particular that $F^{1,0}_{B,m'+1} \neq 0$, which contradicts (4-56).

If $n' = 3$ and $m' = 2$, then $n = 6$ and $\alpha_0 = 6k + 3$ for some $k \geq 1$ by (4-55); if $n' = 2$, then $n = 4$ by the definition of n', and $\alpha_0 = 4k + 2$ for some $k \geq 1$ by (4-55). In any of the two cases above, by Lemma 4.18 and Lemma 4.19, one verifies easily that the assumptions of Lemma 4.17 (v) are satisfied. Hence by (4-48) one has $\operatorname{rank} A^{1,0}_B < 1$, i.e., $\operatorname{rank} A^{1,0}_B = 0$. In other words, $\deg E^{1,0}_B = \deg A^{1,0}_B = 0$, which is a contradiction since f is non-isotrivial. This completes the proof. □

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