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CHARACTERIZATIONS OF SLE_κ FOR $\kappa \in (4,8)$ ON LIOUVILLE QUANTUM GRAVITY

by

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Abstract. — We prove that SLE_κ for $\kappa \in (4,8)$ on an independent $\gamma = 4/\sqrt{\kappa}$-Liouville quantum gravity (LQG) surface is uniquely characterized by the form of its LQG boundary length process and the form of the conditional law of the unexplored quantum surface given the explored curve-decorated quantum surface up to each time t. We prove variants of this characterization for both whole-plane space-filling SLE_κ on an infinite-volume LQG surface and for chordal SLE_κ on a finite-volume LQG surface with boundary. Using the equivalence of Brownian and $\sqrt{8/3}$-LQG surfaces, we deduce that SLE_6 on the Brownian disk is uniquely characterized by the form of its boundary length process and that the complementary connected components of the curve up to each time t are themselves conditionally independent Brownian disks given this boundary length process.

The results of this paper are used in another paper by the same authors to show that the scaling limit of percolation on random quadrangulations is given by SLE_6 on $\sqrt{8/3}$-LQG with respect to the Gromov-Hausdorff-Prokhorov-uniform topology, the natural analog of the Gromov-Hausdorff topology for curve-decorated metric measure spaces.

Résumé **(Charactérisations du SLE_κ pour $\kappa \in (4,8)$ sur une surface quantique au sens de la gravité de Liouville)**

Nous prouvons que la loi d'un processus SLE_κ pour $\kappa \in (4,8)$ dessiné sur une surface quantique au sens de la gravité quantique de Liouville de paramètre $\gamma = 4/\sqrt{\kappa}$ est complètement charactérisée par les données de la forme du processus qui mesure la longueur quantique du bord du domaine restant, et par la forme de la loi conditionelle de la surface restant à explorer conditionnellement à la surface déjà explorée et décorée par la courbe déjà découverte à chaque instant donné t. Nous établissons des versions de cette charactérisation pour la version du SLE_κ remplissant le plan sur une surface quantique d'aire infinie, ainsi que pour le SLE_κ chordal sur une surface quantique dâaire finie avec bord. En utilisant l'équivalence entre les surfaces browniennes et les surfaces quantiques de paramètre $\sqrt{8/3}$, nous en déduisons une charactérisation de la loi d'un SLE_6 dessiné sur un disque brownien. Ce dernier résultat est utilisé dans un autre article des mêmes auteurs pour établir la convergence (dans la limite d'échelle)

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des interfaces de la percolation critique sur des quadrangulations uniformes avec bord vers le SLE_6 sur la gravité quantique de Liouville de paramètre $\sqrt{8/3}$, au sens de la topologie de Gromov-Hausdorff-Prokhorov uniforme, qui est le pendant naturel de la topologie de Gromov-Hausdorff pour les espaces métriques mesurés décorés par une courbe.

1. Introduction

1.1. Overview. — The Schramm-Loewner evolution (SLE) was introduced by Schramm [**61**] to describe the scaling limits of the interfaces which arise in discrete two-dimensional models, such as loop-erased random walk, critical percolation, and the uniform spanning tree. The form of SLE was derived by Schramm using what is now called its *conformal Markov property*. This says that if η is an SLE_κ curve in $\mathbb{H}$ from 0 to ∞ then for each time t the conditional law of $\eta|_{[t,\infty)}$ given $\eta|_{[0,t]}$ is given by the conformal image of an SLE_κ in $\mathbb{H}$ from 0 to ∞. To show that a curve is an SLE one need only show that this property is satisfied. Beyond the initial derivation of SLE, this perspective has been very powerful for the purpose of establishing properties of SLE. Moreover, results of this type have been shown to characterize other SLE-related processes. For example, it was shown by Sheffield and Werner [**67**] that the so-called simple *conformal loop ensembles* (CLE), the loop form of SLE_κ for $\kappa \in (8/3, 4]$, are similarly characterized by a variant of the conformal Markov property.

The purpose of the present work is to establish characterizations of various types of SLE in the spirit of the conformal Markov property but in the context of *Liouville quantum gravity* (LQG). In the special case of SLE_6 on $\sqrt{8/3}$-Liouville quantum gravity, we can re-phrase our characterization theorems in terms of the metric space structure of $\sqrt{8/3}$-LQG, as constructed in [**52, 55, 54**], which gives us characterizations of SLE_6 curves on *Brownian surfaces* (generalizations of the Brownian map [**42, 45**]) which depend only on the metric space structure. Our characterizations will be given in terms of a set of conditions which one would naturally expect any subsequential limit of certain statistical physics models on a random planar map to satisfy.

Our results play an important role in our proof that the scaling limit of (critical) percolation on random planar maps is given by SLE_6 on $\sqrt{8/3}$-LQG. This is carried out in the companion paper [**34**] (building also on [**30, 29**]) in which we show that the subsequential limits of percolation on random quadrangulations exist and satisfy the hypotheses of one of our characterization theorems.

The paper [**34**] uses exclusively discrete arguments (based on properties of percolation and random planar maps), so can be read without any knowledge of SLE or LQG. This paper, on the other hand, uses only continuum (SLE/LQG) arguments. Hence, one can think of this paper as providing the continuum input needed to show the convergence of percolation on random planar maps toward SLE_6 on $\sqrt{8/3}$-LQG.

However, in this paper we also establish characterizations for other variants of SLE_κ on γ-LQG surfaces for $\gamma \in (\sqrt{2}, 2)$ and $\kappa = 16/\gamma^2 \in (4, 8)$. We expect that these results may eventually have applications to proving other scaling limit results for other types of statistical mechanics models on random planar maps, e.g., the critical Fortuin-Kasteleyn model [**65**].

In the remainder of this section, we give a short review of LQG and its relationship to SLE (Section 1.2). We then provide informal statements of our main results in the setting of whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ (Section 1.3.1), ordinary chordal $\mathrm{SLE}_{\kappa'}$ (Section 1.3.2), and SLE_6 on a Brownian surface (Section 1.3.3). We will then give an outline of the rest of the content of the paper in Section 1.4.

1.2. Liouville quantum gravity surfaces. — Formally, an LQG surface for $\gamma \in (0, 2)$ is a random Riemann surface parameterized by a domain $D \subset \mathbb{C}$ whose Riemannian metric tensor is $e^{\gamma h(z)}\, dx \otimes dy$, where h is some variant of the Gaussian free field (GFF) on D and $dx \otimes dy$ is the Euclidean metric tensor. For our purposes, we can always assume that h is a *GFF plus a continuous function*, meaning that there is a (possibly random and depending on h) continuous function f on D such that $h - f$ is the zero-boundary GFF on D or the whole-plane GFF, as appropriate (see [**63, 62, 46, 50, 10**] for more on the GFF).

The above definition does not make rigorous sense since h is a distribution, not a function, so does not take values at points. However, Duplantier and Sheffield [**20**] showed that one can make rigorous sense of the volume form $\mu_h =$ "$e^{\gamma h(z)}\, dz$" associated with an LQG surface as a random measure on D via a regularization procedure. This construction is a special case of a more general theory of measures of this form called *Gaussian multiplicative chaos*, which was initiated by Kahane [**39**]; see [**58, 3, 9**] for expository works on this theory.

One can similarly define a random length measure ν_h associated with an LQG surface, which is defined on certain curves in D including ∂D (if h locally looks like a free-boundary GFF near ∂D) and independent Schramm-Loewner evolution [**61**] (SLE_κ)-type curves for $\kappa = \gamma^2$ [**20, 64**]. More precisely, one can define ν_h in the case when ∂D is piecewise linear segment using a direct regularization procedure, then extend to the boundary of a general domain by conformal covariance (see [**20**, Section 6]). One can define ν_h on an SLE_κ-type curve using the SLE / GFF coupling results from [**20**], or directly as a Gaussian multiplicative chaos measure w.r.t. the Minkowski content measure on the curve [**8**].

The measures μ_h and ν_h satisfy a conformal covariance formula [**20**, Proposition 2.1]: if $f : D \to \widetilde{D}$ is a conformal map and

$$\widetilde{h} = h \circ f^{-1} + Q \log |(f^{-1})'|, \quad \text{for } Q = \frac{2}{\gamma} + \frac{\gamma}{2}, \tag{1.1}$$

then f pushes forward μ_h to $\mu_{\widetilde{h}}$ and ν_h to $\nu_{\widetilde{h}}$ (in fact, this holds a.s. for all choices of conformal map f simultaneously [**66**]).

This leads us to define an LQG surface as an equivalence class of pairs (D, h) consisting of a domain $D \subset \mathbb{C}$ and a distribution h on D, with two such pairs declared to be equivalent if the distributions are related by a conformal map as in (1.1) which extends to a homeomorphism $D \cup \partial D \to \widetilde{D} \cup \partial \widetilde{D}$. In other words, a quantum surface is an equivalence class of measure spaces modulo conformal maps.

The above definition does not require that D be simply connected or even connected. For example, one can make sense of quantum surfaces consisting of a string of beads, each of which is itself a quantum surface homeomorphic to the unit disk. Two such surfaces are illustrated in the left panel of Figure 1.

One can also define quantum surfaces with $k \in \mathbb{N}$ marked points as an equivalence class of $k+2$-tuples $(D, h, x_1, \dots, x_k)$ with $x_1, \dots, x_k \in D \cup \partial D$ with two such $k+2$-tuples declared to be equivalent if there is a conformal map f such that the corresponding fields are related as in (1.1) and f takes the marked points for one $k+2$-tuple to the corresponding marked points of the other.

If $(D, h, x_1, \dots, x_k)$ is a particular equivalence class representative, we say that h is an *embedding* of the quantum surface into $(D, x_1, \dots, x_k)$. We also define a *sub-surface* of a quantum surface (D, h) to be a surface of the form $(D', h|_{D'})$ for $D' \subset D$.

One can also make sense of an LQG surface as a random *metric space*. This was first done in the special case when $\gamma = \sqrt{8/3}$ in the series of works [**51, 53, 52, 55, 54**], which is the only value of γ for which we will consider the metric space structure in this paper. See Section 1.3.3 for more details. The more recent works [**17, 33**] showed how to define an LQG surface as a metric space for general $\gamma \in (0, 2)$, using completely different methods.

LQG surfaces arise as scaling limits of various random planar map models. LQG for $\gamma = \sqrt{8/3}$ corresponds to the scaling limit of uniform random planar maps, and other values of γ arise by sampling a planar map with probability proportional to the partition function of an appropriate γ-dependent statistical mechanics model on the map. For example, weighting by the number of spanning trees corresponds to $\gamma = \sqrt{2}$, weighting by the partition function of the Ising model corresponds to $\gamma = \sqrt{3}$, and weighting by the number of bipolar orientations corresponds to $\gamma = \sqrt{4/3}$. So far, scaling limit results for random planar maps toward LQG have been obtained in the Gromov-Hausdorff topology for $\gamma = \sqrt{8/3}$ [**42, 45**] (together with [**51, 53, 52, 55, 54**]) and in the so-called peanosphere sense, which relies on the main theorem of [**18**], for all values of $\gamma \in (0, 2)$ [**65, 40, 27, 43, 13**].

Several particular γ-LQG surfaces, which arise as the scaling limits of random planar maps with various topologies, are defined in [**18**]. We review the definitions of these surfaces in Section 2.2.1.

It is natural to consider a γ-LQG surface decorated by an independent SLE_κ-type curve for $\kappa = \gamma^2 \in (0, 4)$ or $\kappa' = 16/\gamma^2 > 4$.(1) One reason why this is natural is that such curve-decorated quantum surfaces describe the scaling limits of statistical

(1) Here and throughout this paper we use the imaginary geometry [**46, 47, 48, 50**] convention of writing κ for the SLE parameter when $\kappa \in (0, 4)$ and $\kappa' = 16/\kappa$ for the dual parameter.

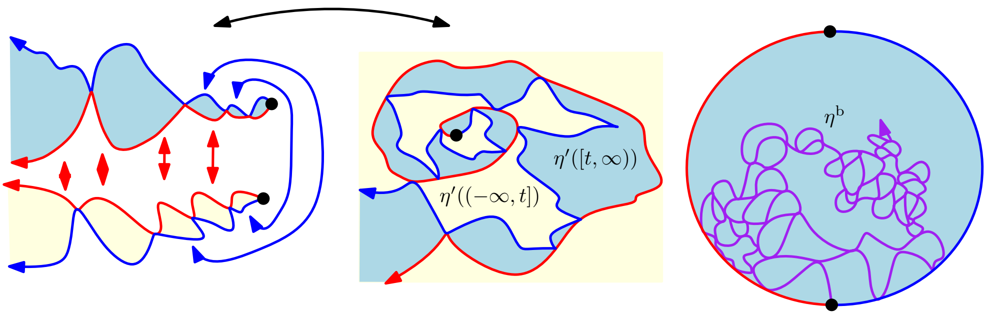

FIGURE 1. *Left and middle:* A whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ for $\kappa' \in (4,8)$ on an independent γ-quantum cone, which is the object characterized in Theorem 1.1. The past $\eta'((-\infty,t])$ is shown in yellow and the future $\eta'([t,\infty))$ is shown in blue. Restricting the field h to each of these sets gives two independent beaded quantum surfaces, each of which has the law of a $\frac{3\gamma}{2}$-quantum wedge, which can be conformally welded together according to quantum length along their boundaries to obtain the γ-quantum cone $(\mathbb{C}, h, 0, \infty)$. In the special case when $\gamma = \sqrt{8/3}$, the $\sqrt{8/3}$-quantum cone admits a metric measure space structure under which it is equivalent to the Brownian plane. In this case, Theorem 1.1 can be re-phrased as a characterization theorem for whole-plane space-filling SLE_6 on the Brownian plane; see Theorem 7.1. *Right:* A chordal $\mathrm{SLE}_{\kappa'}$ on an independent bead of a $\frac{3\gamma}{2}$-quantum wedge (i.e., a quantum surface whose law is the same as the conditional law of one of the connected components of the quantum surface parameterized by $\eta'([t,\infty))$ given its quantum area and left/right quantum boundary lengths). This is the setting of our other quantum surface characterization theorem, Theorem 1.3. When $\gamma = \sqrt{8/3}$, a single bead of a $\frac{3\gamma}{2}$-quantum wedge is the same as the quantum disk, which in turn is the same as the Brownian disk when equipped with its $\sqrt{8/3}$-LQG metric and area measure. This gives rise to a characterization of chordal SLE_6 on the Brownian disk; see Theorem 1.4.

mechanics models on random planar maps in the γ-LQG universality class. Indeed, scaling limits in the peanosphere sense are really statements about the convergence of random planar maps decorated by a space-filling curve toward LQG decorated by space-filling SLE (as defined in [**50**]). See also [**32**] for a scaling limit result for random quadrangulations decorated by a self-avoiding walk toward $\mathrm{SLE}_{8/3}$-decorated $\sqrt{8/3}$-LQG in a variant of the Gromov-Hausdorff topology.

In the continuum, there are a number of theorems which describe γ-LQG surfaces decorated by independent SLE_κ-or $\mathrm{SLE}_{\kappa'}$-type curves [**64, 18, 51**], some of which we discuss just below.

1.3. Main results