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VALUE DISTRIBUTION THEORY FOR PARABOLIC RIEMANN SURFACES

by

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Abstract. – We give versions of classical results in value distribution theory when the source space is a parabolic Riemann Surface, i.e., a surface with no non-constant bounded subharmonic functions.

A vanishing theorem for such maps is obtained, of which a version of the classical Bloch Theorem and Ax-Lindemann Theorem are consequences.

Results by Brunella and McQuillan are also extended in this context, with quite simple proofs.

1. Introduction

Let X be a complex hermitian manifold. S. Kobayashi introduced a pseudo-distance, determined by the complex structure of X. We recall here its infinitesimal version, cf. [**19**].

Given a point $x \in X$ and a tangent vector $v \in T_{X,x}$ at X in x, the length of v with respect to the *Kobayashi-Royden pseudo-metric* is the following quantity

$$\mathbf{k}_{X,x}(v) := \inf\{\lambda > 0; \exists f : \mathbb{D} \to X, f(0) = x, \lambda f'(0) = v\},$$

where $\mathbb{D} \subset \mathbb{C}$ is the unit disk, and f is a holomorphic map. Let $|v|$ denote the hermitian length of the tangent vector v.

If $\mathbf{k}_{X,x}(v) \geq c|v|$, with $c > 0$ as function on the tangent space of X, we say that X is *Kobayashi hyperbolic.*

In general, we remark that it may very well happen that $\mathbf{k}_{X,x}(v) = 0$. However, if X is compact, then thanks to Brody re-parametrization lemma this situation has a geometric counterpart, as follows. If there exists a sequence (x_n, v_n) such that $|v_n| = 1$ and such that

$$\lim \mathbf{k}_{X,x_n}(v_n) = 0,$$

then one can construct a holomorphic non-constant map $f : \mathbb{C} \to X$. The point $x = \lim x_n$ is not necessarily in the image of f.

Therefore, if any entire curve drawn on X is constant, then the pseudo-distance defined above is a distance, and we say that X is *Brody hyperbolic*, or simply *hyperbolic* (since most of the time we will be concerned with compact manifolds).

As a starting point for the questions with which we will be concerned with in this article, we have the following result.

Proposition 1.1. – *Let X be a compact Kobayashi hyperbolic manifold. Let $\mathcal{C}$ be a Riemann surface. Let $E \subset \mathcal{C}$ be a closed, countable set. Then any holomorphic map $f : \mathcal{C} \setminus E \to X$ admits a holomorphic extension $\widetilde{f} : \mathcal{C} \to X$.*

In particular, in the case of the complex plane we infer that any holomorphic map $f : \mathbb{C} \setminus E \to X$, must be the restriction of an application $\widetilde{f} : \mathbb{P}^1 \to X$ (under the hypothesis of Proposition 1.1). We will give a proof and discuss some related statements and questions in the first paragraph of this paper. Observe however that if the cardinal of E is at least 2, then $\mathbb{C} \setminus E$ is Kobayashi hyperbolic.

Our next remark is that the surface $\mathbb{C}\setminus E$ is a particular case of a *parabolic Riemann surface*. We recall here the definition.

A Riemann surface $\mathcal{Y}$ is parabolic if any bounded subharmonic function defined on $\mathcal{Y}$ is constant. This is a large class of surfaces, including e.g., $Y \setminus \Lambda$, where Y is a compact Riemann surface of arbitrary genus and $\Lambda \subset Y$ is any closed polar set. It is known (cf. **[1]**, **[35]**, page 80) that a non-compact Riemann surface $\mathcal{Y}$ is *parabolic* if and only if it admits a smooth exhaustion function

$$\sigma : \mathcal{Y} \to [1, \infty[$$

such that $\tau := \log \sigma$ is harmonic in the complement of a compact set of $\mathcal{Y}$. Moreover, we impose the normalization

$$\int_{\mathcal{Y}} dd^c \log \sigma = 1, \tag{1}$$

where the operator d^c is defined as follows

$$d^c := \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial).$$

On the boundary $S(r) := (\sigma = r)$ of the parabolic ball of radius r we have the induced measure

$$d\mu_r := d^c \log \sigma|_{S(r)}.$$

The measure $d\mu_r$ has total mass equal to 1, by the relation (1) combined with Stokes formula.

Since we are dealing with general parabolic surfaces, the growth of the Euler characteristic of the balls $\mathbb{B}(r) = (\sigma < r)$ will appear very often in our estimates. We introduce the following notion.

Definition 1.2. – *Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface, together with an exhaustion function as above. For each $t \geq 1$ such that $S(t)$ is non-singular we denote by $\chi_\sigma(t)$ the Euler characteristic of the domain $\mathbb{B}(t)$, and let*

$$\mathfrak{X}_\sigma(r) := \int_1^r \left|\chi_\sigma(t)\right| \frac{dt}{t}$$

be the (weighted) mean Euler characteristic of the ball of radius r.

Actually, we will mostly use a related function $\mathfrak{X}_\sigma^+(r)$, whose definition is a bit more technical than 1.2, cf. Definition 3.4.

If $\mathcal{Y} = \mathbb{C}$, then $\mathfrak{X}_\sigma(r)$ is bounded by $\log r$. The same type of bound is verified if $\mathcal{Y}$ is the complement of a finite number of points in $\mathbb{C}$. If $\mathcal{Y} = \mathbb{C} \setminus E$ where E is a closed polar set of infinite cardinality, then things are more subtle, depending on the density of the distribution of the points of E in the complex plane. However, an immediate observation is that the surface $\mathcal{Y}$ has finite Euler characteristic if and only if

$$\mathfrak{X}_\sigma(r) = O(\log r). \tag{2}$$

In the first part of this article we will extend a few classical results in hyperbolicity theory to the context of parabolic Riemann surfaces, as follows.

We will review the so-called "first main theorem" and the logarithmic derivative lemma for maps $f : \mathcal{Y} \to X$, where X is a compact complex manifold. We also give a version of the first main theorem with respect to an ideal $\mathcal{J} \subset \mathcal{O}_X$. This will be a convenient language when studying foliations with singularities.

As a consequence, we derive a vanishing result for jet differentials, similar to the one obtained in case $\mathcal{Y} = \mathbb{C}$, as follows.

Let $\mathcal{P}$ be a jet differential of order k and degree m on a projective manifold X, with values in the dual of an ample bundle (see **[11]**; we recall a few basic facts about this notion in the next section). Then we prove the following result.

Theorem 1.3. – *Let $\mathcal{Y}$ be a parabolic Riemann surface. We consider a holomorphic map $f : \mathcal{Y} \to X$ such that we have*

$$\limsup_{r\to\infty} \frac{\mathfrak{X}_\sigma^+(r).}{T_{f,\omega}(r)} = 0. \tag{†}$$

Let $\mathcal{P}$ be an invariant jet differential of order k and degree m, with values in the dual of an ample line bundle. Then we have

$$\mathcal{P}(j_k(f)) = 0$$

identically on $\mathcal{Y}$.

For example, the requirement above is satisfied if $\mathscr{Y}$ has finite Euler characteristic and infinite area. We will call the image of $f : \mathscr{Y} \to X$ a parabolic curve.

In the previous statement we denote by $j_k(f)$ the k^{th} jet associated to the map f. If $\mathscr{Y} = \mathbb{C}$, then this result is well-known, starting with the seminal work of A. Bloch cf. [**3**]; see also [**12**], [**34**] and the references therein, in particular to the work of T. Ochiai [**27**], Green-Griffiths [**14**] and Y. Kawamata [**16**]. It is extremely useful in the investigation of the hyperbolicity properties of projective manifolds. In this context, the above result says that the vanishing result still holds in the context of Riemann surfaces of (possibly) infinite Euler characteristic, provided that the growth of the function $\mathfrak{X}_\sigma^+(r)$, is slow when compared to $T_{f,\omega}(r)$. It also holds when the source is the unit disk, provided that the growth is large enough.

As a consequence of Theorem 1.3 we obtain the following result (see Section 4, Corollary 4.6). Let X be a projective manifold, and let $D = Y_1 + \cdots + Y_N$ be an effective snc (i.e., simple normal crossings) divisor. We assume that there exists a logarithmic jet differential $\mathscr{P}$ on (X, D) with values in a bundle A^{-1}, where A is ample. Let $f : \mathbb{C} \to X$ be an entire curve which do not satisfy the differential equation defined by $\mathscr{P}$. Then we obtain a lower bound for the number of intersection points of $f(\mathbb{D}_r)$ with D as $r \to \infty$, where $\mathbb{D}_r \subset \mathbb{C}$ is the disk of radius r.

Concerning the existence of jet differentials, we recall Theorem 0.1 in [**13**], see also [**23**].

Theorem 1.4. – *Let X be a manifold of general type. Then there is a couple of integers $m \gg k \gg 0$ and a (non-zero) holomorphic invariant jet differential $\mathscr{P}$ of order k and degree m with values in the dual of an ample line bundle A.*

Thus, our Result 1.3 can be used in the context of the general type manifolds.

As a consequence of Theorem 1.3, we obtain the following analogue of Bloch's theorem. It does not seem to be possible to derive this result by using e.g., Ahlfors-Schwarz negative curvature arguments. Observe also that we cannot use a Brody-Green type argument, because the Brody reparametrization lemma is not available in our context.

Theorem 1.5. – *Let $\mathbb{C}^N/\Lambda$ be a complex torus, and let $\mathscr{Y}$ be a parabolic Riemann surface of finite Euler Characteristic. We consider a holomorphic map*

$$f : \mathscr{Y} \to \mathbb{C}^N/\Lambda.$$

Then the smallest analytic subset X containing the closure of the image of f is either the translate of a sub-torus in $\mathbb{C}^N/\Lambda$, or there exists a map $\mathscr{R} : X \to W$ onto a general type subvariety of an abelian variety $W \subset A$ such that the area of the curve $\mathscr{R} \circ f(\mathscr{Y})$ is finite.

In the second part of this paper our aim is to recast some of the work of M. McQuillan and M. Brunella concerning the Green-Griffiths conjecture in the parabolic setting. We first recall the statement of this problem.

Conjecture 1.6 **([14])**. – *Let X be a projective manifold of general type. Then there exists an algebraic subvariety $W \subsetneq X$ which contains the image of all non-constant, holomorphic curves $f : \mathbb{C} \to X$.*

It is hard to believe that this conjecture is correct for manifolds X of dimension ≥ 3. On the other hand, it is very likely that this holds true for surfaces (i.e., $\dim X = 2$), on the behalf of the results available in this case.

Given a map $f : \mathcal{Y} \to X$ defined on a parabolic Riemann surface $\mathcal{Y}$, we can associate a Nevanlinna-type closed positive current $T[f]$. If X is a surface of general type and if $\mathcal{Y}$ has finite Euler characteristic, then there exists an integer k such that the k-jet of f satisfies an algebraic relation. As a consequence, there exists a foliation $\mathcal{F}$ by Riemann surfaces on the space of k-jets X_k of X_0, such that the lift of f is tangent to $\mathcal{F}$. In conclusion we are naturally led to consider the pairs $(X, \mathcal{F})$, where X is a compact manifold, and $\mathcal{F}$ is a foliation by curves on X. We denote by $T_{\mathcal{F}}$ the so-called tangent bundle of $\mathcal{F}$.

We derive a lower bound of the intersection number $\int_X T[f] \wedge c_1(T_{\mathcal{F}})$ in terms of a Nevanlinna-type counting function of the intersection of f with the singular points of $\mathcal{F}$. As a consequence, if X is a complex surface and $\mathcal{F}$ has reduced singularities, we show that $\int_X T[f] \wedge c_1(T_{\mathcal{F}}) \geq 0$. For this part we follow closely the original argument of **[21]**.

When combined with a result by Y. Miyaoka, the preceding inequality shows that the classes $\{T[f]\}$ and $c_1(T_{\mathcal{F}})$ are orthogonal. Since the class of the current $T[f]$ is nef, we show by a direct argument that we have $\int_X \{T[f]\}^2 = 0$, and from this we infer that the Lelong numbers of the diffuse part R of $T[f]$ are equal to zero at each point of X.

This regularity property of R is crucial, since it allows to show –via the Baum-Bott formula and an elementary fact from dynamics– that we have $\int_X T[f] \wedge c_1(N_{\mathcal{F}}) \geq 0$, where $N_{\mathcal{F}}$ is the normal bundle of the foliation, and $c_1(N_{\mathcal{F}})$ is the first Chern class of $N_{\mathcal{F}}$.

We then obtain the next result, in the spirit of **[21]**.

Theorem 1.7. – *Let X be a surface of general type, and consider a holomorphic map $f : \mathcal{Y} \to X$, where $\mathcal{Y}$ is a parabolic Riemann surface such that*

$$\mathfrak{X}^+_\sigma(r) = o(T_f(r))$$

. We assume that f is tangent to a holomorphic foliation $\mathcal{F}$, then the dimension of the Zariski closure of $f(\mathcal{Y})$ is at most 1.

In the last section of our survey we give a short proof of M. Brunella index theorem **[9]**. Furthermore, we show that that this important result admits the following generalization.