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MCQUILLAN'S APPROACH TO THE GREEN-GRIFFITHS CONJECTURE FOR SURFACES

by

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Abstract. – After reviewing the proof by Bogomolov of the fact that, on smooth projective surfaces with $c_1^2 > c_2$, curves of bounded geometric genus form a bounded family, we explain the main steps of the proof, given by McQuillan, of the Green-Griffiths conjectures for these surfaces. Viewed from afar the two proofs follow the same strategy, but the second requires a much deeper analysis of the tools involved.

In order to describe McQuillan's proof we explain the construction of the Ahlfors currents associated to entire curves in a variety and we show how these can be used to produce a substitute for intersection numbers. A proof of the tautological inequality in both the standard case and the logarithmic case is given. We explain how the hypotheses allow us to suppose that the involved entire curve (which a posteriori should not exist) is a leaf of a foliation. In order to simplify some technical points of the proof we impose some restrictions on the singularities of this foliation (the general case requires a much more involved analysis but the main ideas of the proof are already visible under this restriction).

In the last section we give a very brief description of a possible strategy (proposed by McQuillan) for the proof of the general case of the conjecture, together with an explanation of the main difficulties that must be overcome.

1. Introduction

Let X be a smooth projective variety. If X is hyperbolic, then there is no, non constant analytic map $f : \mathbf{C} \to X$. It is also known that hyperbolicity implies algebraic hyperbolicity. Thus if X is hyperbolic and L is an ample line bundle on it, we can find a constant A for which the following holds: for every smooth projective curve C of genus g and morphism $f : C \to X$ we have $\deg(f^*(L)) \leq A(2g-2)$.

For a smooth projective surface of general type we cannot expect that an inequality like the one above holds. This is, for instance, due to the fact that one can find rational or elliptic curves on it. But probably the following conjecture holds:

Conjecture 1.1. – *Let X be a smooth projective surface of general type. Let L an ample line bundle on it. Then there exist constants A and B for which the following holds: for every smooth projective curve C of genus g and morphism $f : C \to X$ we have*

$$\deg(f^*(L)) \leq Ag + B. \tag{1.1}$$

This conjecture is very deep and, in particular, it implies that on a surface of general type there are only finitely many rational or elliptic curves. More specifically the conjecture implies the following:

Conjecture 1.2. – *Let X be a smooth surface of general type. Then there exists a proper Zariski closed set $Z \subset X$ for which the following holds: If C is a rational or an elliptic curve and $f : C \to X$ is a non constant map, then f factorize through Z.*

The Green-Griffiths conjecture is a generalization of this last conjecture:

Conjecture 1.3 **(Green-Gritffiths Conjecture).** – *Let X be a smooth surface of general type. Then there exists a proper Zariski closed set $Z \subset X$ for which the following holds: every non constant analytic map $f : \mathbf{C} \to X$ factorizes through Z.*

If, in the possibility that Conjecture 1.2 is false, we may weaken the Conjecture 1.3 by requiring only that the closed set Z depends only on f (but it should be one dimensional).

In this chapter we will discuss some advances on these conjectures and a strategy, due to McQuillan, to attack Conjecture 1.3 in general.

The reader can refer also to the paper [7] for another description of the proof.

2. Curves of bounded genus on surfaces with big cotangent bundle

Definition 2.1. – *Let X be a smooth projective variety and E be a vector bundle over it. We will say that E is ample, big, nef, ..., if the tautological bundle $\mathcal{O}(1)$ is ample, big, nef, ... on the projective bundle $\mathbf{P}(E)$ respectively.*

We start this section by showing how to prove algebraically a strong version of Conjecture 1.1 on varieties with ample cotangent bundle. Remark that if a variety X has ample cotangent bundle, then his canonical line bundle K_X is ample too.

Theorem 2.2. – *Let X be a smooth projective variety with ample cotangent bundle. Then there exists a constant A with the following property: for every smooth curve C of genus g and every non constant map $f : C \to X$ we have*

$$\deg(f^*(K_X)) \leq A(2g - 2). \tag{2.1}$$

Proof. – Consider the structure morphism $p : \mathbf{P}(\Omega^1_X) \to X$. Since, by hypothesis, $\mathcal{O}(1)$ is ample on $\mathbf{P}(\Omega^1_X)$, there is an integer N for which the line bundle $M_N := \mathcal{O}(N) \otimes p^*(K_X^{-1})$ is ample on it.

Let C be a smooth projective curve and $f : C \to X$ be a non constant map. The natural map $f^*(\Omega^1_X) \to \Omega^1_C$ gives, by functoriality, a map $f' : C \to \mathbf{P}(\Omega^1_X)$ such that $f = p \circ f'$.

By construction $f'^*(\mathcal{O}(1)) \hookrightarrow \Omega^1_C$. Thus $\deg(f'^*(\mathcal{O}(1)) \leq 2g-2$.

Since M_N is ample on $\mathbf{P}(\Omega^1_X))$ we have that $\deg(f'^*(M_N)) \geq 0$. Consequently $\deg(f^*(K_X)) \leq N(2g-2)$. □

We observe that, in particular, we obtain that such a variety do not contain rational or elliptic curves.

Bogomolov Theorem 2.3 [**1**] generalize Theorem 2.2 to surfaces whose cotangent bundle is big.

***Theorem 2.3* (Bogomolov).** – *Let X be a smooth projective surface with big cotangent bundle. Then there exist constants A_1 and A_2 with the following property: for every smooth curve C of genus g and every non constant map $f : C \to X$ we have*

$$\deg(f^*(K_X)) \leq A_1(2g-2) + A_2. \tag{2.2}$$

Observe that a sufficient condition for the cotangent bundle to be big is that $c_1(X)^2 > c_2(X)$ (exercise).

Let's remark an interesting corollary:

***Corollary 2.4*.** – *Let X be a surface with big cotangent bundle. Then X contains only finitely many rational or elliptic curves.*

Proof. – Indeed, curves of bounded genus in such a surface are a bounded family and surfaces of general type are not covered by rational or elliptic curves. □

We now prove Theorem 2.3.

Proof. – As before we fix an ample line bundle L on X. Consider the natural morphism $p : \mathbf{P}(\Omega^1_X) \to X$. If M is a line bundle on $\mathbf{P}(\Omega^1_X)$, we denote by $Bs(M)$ the base locus of it; it is a Zariski closed set which coincides with $\mathbf{P}(\Omega^1_X)$ if and only if $H^0(X, M) = \{0\}$.

For every positive integers n and m, consider the closed set $B_{n,m} := Bs((\mathcal{O}(m) \otimes p^*(L^{-1}))^{\otimes n}) \subset \mathbf{P}(\Omega^1_X)$. Let $B = \bigcap_{n,m} B_{n,m}$.

Since $\mathcal{O}(1)$ is big, we have that $B \neq \mathbf{P}(\Omega^1_X)$. By Noetherianity we may suppose that there exists n_0 and m_0 such that $B = B_{n_0,m_0}$.

Let C be a smooth projective curve of genus g and $f : C \to X$ be a non constant map. Suppose that the lift $f' : C \to \mathbf{P}(\Omega^1_X)$ of f *do not factor through* B. This implies that there is a global section $s \in H^0(\mathbf{P}(\Omega^1_X); (\mathcal{O}(n_0) \otimes p^*(L^{-1}))^{m_0})$ which do not vanish identically along $f'(C)$. Consequently $\deg(f'^(\mathcal{O}(n_0)) \otimes f^*(L^{-1})) \geq 0$. Thus

$$\deg(f^*(L)) \leq n_0(2g-2). \tag{2.3}$$

Thus, for these curves it suffices to take A_2 bigger then n_0.

We must now deal with the case when the morphism f' factor through B. Consider an irreducible component of B. By abuse of notation we will denote it again by B.

If $p(B)$ is of dimension at most one the conclusion of the theorem easily follows: $f(C)$ can be contained in a finite list of curves inside X, the genus and the degree of which can be absorbed by the constant A_2.

Suppose that $p : B \to X$ is dominant. In this case, since $B \neq \mathbf{P}(\Omega^1_X)$, the dimension of B is two.

We will now show that the image of C in B is leaf of a natural foliation on it.

Lemma 2.5. – *there exists a smooth projective surface $\widetilde{B}$, a birational morphism $\widetilde{B} \to B$ and a natural algebraic foliation $\mathcal{F}$ on $\widetilde{B}$ with the following property: Let $f_1 : C \to \widetilde{B}$ be a lift of f'. Then, either $f_1(C)$ belongs to a finite list or $f_1(C)$ is leaf of the foliation $\mathcal{F}$.*

Before we start the proof of the lemma, we recall some basic definitions of foliations on surfaces.

2.1. Standard fact about algebraic foliations on surfaces

Definition 2.6. – *Let Y be a smooth algebraic surface. An algebraic foliation $\mathcal{F}$ is a sub sheaf N de Ω^1_Y which is locally free of rank one such that the quotient Ω^1_Y/N is torsion free.*

We recall the following facts:

a) The line bundle N is usually called *the conormal sheaf of the foliation*. The line bundle $\det(\Omega^1_Y/N)$ is usually called *the canonical sheaf of $\mathcal{F}$* and denoted by $K_{\mathcal{F}}$.

b) There is a zero dimensional subschema $Z \subset Y$ (in general non reduced) which is called *the singular locus of the foliation* and an exact sequence

$$0 \longrightarrow N \longrightarrow \Omega^1_Y \longrightarrow I_Z \otimes K_{\mathcal{F}} \longrightarrow 0, \tag{2.4}$$

where I_Z is the ideal sheaf of Z. Points in the support of Z are called *singular points of the foliation*. Points outside Z are called *regular, or smooth, points for the foliation.*

c) Consequently we have $K_Y = N \otimes K_{\mathcal{F}}$.

d) Let M be a Riemann surface (not necessarily compact nor algebraic). A morphism $\iota : M \to Y$ is said to be *a leaf of the foliation* if:

d.1) There is a discrete set of points $P \subset M$ such that $\iota|_{M\setminus P} : M \setminus P \to Y$ is a local embedding;

d.2) the natural map $\iota^*(N) \to \iota^*(\Omega^1_Y) \to \Omega^1_M$ is the zero map.

e) If $z \in Y$ is a regular point for the foliation, then there is a unique leaf of the foliation passing through z.

f) Denote by Δ the one dimensional unit disk. If $z \in Y$ is a regular point for the foliation, then there is an *analytic* neighborhood $z \in U \subset Y$ isomorphic to $\Delta \times \Delta$

with coordinates (z_1, z_2) and the restriction of the exact sequence 2.4 to U is the exact sequence

$$0 \longrightarrow \mathcal{O}_U dz_1 \longrightarrow \mathcal{O}_U dz_1 \oplus \mathcal{O}_U dz_2 \longrightarrow \mathcal{O}_U dz_2 \longrightarrow 0. \tag{2.5}$$

Consequently the leaves of the foliation passing through U are given by the equations $z_1 = c$ $(c \in \Delta)$.

Point (f) above explains a bit the geometry of a foliation on the open set of regular points: we can cover the regular locus of the foliation by open sets such that the restriction of the foliation to each of them is just a product. On the other side, near the singular locus of the foliation, the structure of the leaves may be much more complicated.

g) Suppose that Y_1 is a smooth projective surface and $p : Y_1 \to Y$ is a dominant morphism. We have an inclusion $p^*(N) \to p^*(\Omega^1_Y) \to \Omega^1_{Y_1}$. The saturation N_1 of N inside $\Omega^1_{Y_1}$ is then a foliation on Y_1. It will be called *the pull back of the foliation* $\mathcal{F}$ *to* Y_1 *via* p and denoted by $p^*(\mathcal{F})$.

One should be aware that, in general, the conormal sheaf of $p^*(\mathcal{F})$ *is not* $p^*(N)$ and, in general, $K_{p^*(\mathcal{F})} \neq p^*(K_{\mathcal{F}})$.

h) A leaf M of the foliation is said to be *algebraic* if the Zariski closure of its image in Y is an algebraic curve.

i) Suppose that S is a smooth algebraic curve and $f : Y \to S$ is a dominant morphism. The natural exact sequence

$$0 \longrightarrow f^*(\Omega^1_S) \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_{Y/S} \longrightarrow 0 \tag{2.6}$$

induces a foliation $\mathcal{F}_f$ on Y. The leaves of this foliation are the fibers of f thus they are all compact. Observe that in general $f^*(\Omega^1_S)$ *is not* the conormal sheaf of $\mathcal{F}_f$. The conormal sheaf of $\mathcal{F}_f$ will be the saturation of $f^*(\Omega^1_S)$ in Ω^1_Y.

j) A foliation $\mathcal{F}$ is said to be *a fibration* if there is a a birational morphism $p : Y_1 \to Y$ and a dominant morphism $f : Y_1 \to S$ where S is a smooth projective curve such that $p^*(\mathcal{F}) = \mathcal{F}_f$. All the leaves of a fibration are algebraic.

The following theorem characterizes foliations with "many" algebraic leaves, cf. **[4]**:

***Theorem 2.7* (Jouanoulou).** – *Let $\mathcal{F}$ be a foliation on a smooth projective surface. Then $\mathcal{F}$ has infinitely many algebraic leaves if and only if it is a fibration (and thus all the leaves are algebraic).*

For a detailed reference on foliations on algebraic surfaces, cf. **[2]**.

Let's prove Lemma 2.5:

Proof. – Over $\mathbf{P}(\Omega^1_X)$ we have the tautological exact sequence

$$0 \longrightarrow \Omega^1_{\mathbf{P}(\Omega^1_X)/X}(1) \longrightarrow p^*(\Omega^1_X) \longrightarrow \mathcal{O}(1) \longrightarrow 0. \tag{2.7}$$

Observe that if $f : C \to X$ is a morphism and $f' : C \to \mathbf{P}(\Omega^1_X)$ is the natural lift, then the natural map $f'^*(\Omega^1_{\mathbf{P}(\Omega^1_X)/X}(1)) \to f^*(\Omega^1_X) \to \Omega^1_C$ is the zero map.