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GENERIC POSITIVITY AND APPLICATIONS TO HYPERBOLICITY OF MODULI SPACES

by

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Abstract. – Moduli theory of algebraic varieties naturally includes the study of the degeneration of varieties in families. For example, the obstruction to non-trivial degeneration of certain families of varieties can be described as non-existence of some subvarieties in their associated moduli spaces. We study the conjectural role of the Kodaira dimension as an invariant that can be used to describe such obstructions.

Recent advances in our understanding of the positivity properties of tangent sheaves of non-uniruled varieties, and the natural generalizations to the logarithmic setting for pairs (X, D), has proved to be a powerful tool in the study of Kodaira dimension of base spaces of varying families of manifolds.

We give a detailed account of Campana and Păun's generalization of the *generic semi-positivity* results of Miyaoka. In its simplest form this result asserts that quotients of the logarithmic cotangent bundle $\Omega^1_X(\log D)$ have semi-positive slopes with respect to any ample divisor, as long as the log-canonical divisor $K_X + D$ is pseudo-effective.

A key consequence then relates bigness of $K_X + D$ to positivity properties of the sheaf of pluri-(log-)differential forms. Together with a result of Viehweg and Zuo on the bigness of the sheaf of pluri-differential forms for moduli stacks of canonically polarized manifolds, one is led to a proof of a celebrated conjecture of Viehweg on the "algebraic hyperbolicity"' of such spaces: subvarieties of moduli (stacks) of canonically polarized manifolds are all of log-general type.

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1. Introduction

In 1962 Shafarevich conjectured that a smooth family $f^\circ : X^\circ \to Y^\circ$ of complex projective curves of genus at least equal to 2, parameterised by $Y^\circ = \mathbb{P}^1$, $\mathbb{C}$, $\mathbb{C}^*$, or an elliptic curve E is isotrivial, so that is there is no variation in the algebraic structure of the members of the family. Equivalently this conjecture can be expressed as the prediction that the base Y° of any smooth, non-isotrivial family of projective curves with $g \geq 2$ is of log-general type. In other words, we have $\kappa(Y, K_Y + D) = 1$ for any smooth compactification (Y, D) of Y°, with snc boundary divisor D. Shafarevich conjecture was shown by Parshin and Arakelov.

To generalize the Shafarevich conjecture to higher dimensional fibers and parameterising spaces, Viehweg considered the hyperbolicity properties of the moduli stack of canonically polarized manifolds. Recall that the moduli functor $\mathscr{M}$ of canonically polarized manifolds with fixed Hilbert polynomial, is equipped with a natural transformation

$$\Psi : \mathscr{M}(\cdot) \to \mathrm{Hom}(\cdot, \mathfrak{M}),$$

where $\mathfrak{M}$ denotes the coarse moduli scheme associated with $\mathscr{M}$. The scheme $\mathfrak{M}$ was proved by Viehweg to be quasi-projective, cf. [**43**]. Also recall that a complex analytic space U is said to be Brody hyperbolic if there are no non-constant holomorphic maps $f : \mathbb{C} \to U$. In the spirit of this definition, Shafarevich's conjecture is equivalent to the assertion that the base Y° of non-isotrivial, smooth, projective families of high genus curves is algebraically Brody hyperbolic in the sense that there are no non-constant morphisms from $\mathbb{C}^*$ to Y°.

Generalizing Shafarevich's conjecture, Viehweg predicted that the moduli stack of canonically polarized manifolds is not only algebraically Brody hyperbolic but that it is Brody hyperbolic. More precisely, a smooth quasi-projective variety Y° admitting a generically finite morphism $\mu : Y^\circ \to \mathfrak{M}$, must be Brody hyperbolic. This conjecture was settled by Viehweg and Zuo in [**45**]. On the other hand, a long-standing conjecture of Lang predicts that for a quasi-projective Y°, Kobayashi hyperbolicity (which is equivalent to Brody hyperbolicity for projective varieties) implies that all subvarieties of Y°, including Y°, are of log-general type. In the light of Lang's problem, Viehweg extended his question on the hyperbolic nature of the moduli stack of canonically polarized manifolds to the following conjecture.

***Conjecture 1.1* (Viehweg's hyperbolicity conjecture).** – *Let Y° be a smooth quasi-projective variety admitting a generically finite morphism $\mu : Y^\circ \to \mathfrak{M}$. Then, the smooth compactification (Y, D) of Y° is of log-general type.*

Viehweg's conjecture has attracted the interest of many algebraic geometers for a long time. We refer the reader to the survey [**26**] for more details, including references to earlier results that are not mentioned here for lack of space.

1.1. Viehweg's conjecture according to Viehweg-Zuo and Campana-Păun. – A general strategy to prove Conjecture 1.1 consists of two main steps. Combining deep results of analytic [**47**], algebraic [**42**] and Hodge theoretic [**19**] nature, Viehweg and Zuo construct in a first step a subsheaf of the sheaf of pluri-log differential forms of the base whose birational positivity captures the variation [(1)] in the family.

Theorem 1.2 **(Existence of pluri-logarithmic forms in the base, cf. [44**, Thm. 1.4]**)**

If the smooth family of canonically polarized manifolds f° has maximal variation, then there exist a positive integer N an invertible subsheaf $\mathscr{L} \subseteq \mathrm{Sym}^N\big(\Omega^1_Y \log(D)\big)$ such that $\kappa(Y, \mathscr{L}) = \dim Y$.

Theorem 1.2 immediately resolves the original conjecture of Shafarevich. The goal in the second step is to trace a connection between the birational positivity (bigness) of $\mathscr{L}$ in Theorem 1.2 and that of $K_Y + D$, thus resolving Conjecture 1.1. Working along these lines, the second author and Kovács established Conjecture 1.1 for moduli stacks of dimension two and three, [**27, 28**] and see [**26**] for an overview. The work relied, among other things, on the log-abundance theorem for surfaces and threefolds. In the absence of these methods in higher dimensions, for instance a complete solution to the abundance problem, Campana and Păun devised an additional tool, namely a vast generalization of the famous generic semipositivity result of Miyaoka to the context of pairs with rational coefficients. Here, we state their result in its simplest form and we refer the reader to Section 5 for a general statement.

Theorem 1.3 **(Logarithmic generic semipositivity, cf. [10**, Thm. 2.1]**)**

Let (X, D) be a reduced, projective, snc pair. If $K_X + D$ is pseudo-effective, then for every ample divisor H on X and every torsion free quotient $\mathscr{Q}$ of $\Omega^1_X(\log D)$ we have

$$c_1(\mathscr{Q}) \cdot [H]^{n-1} \geq 0.$$

Despite its importance, we found the paper [**10**] lacking in some details. This chapter is meant to serve as an exposition of Campana and Păun's proof of Theorem 1.3 and its application to resolving Conjecture 1.1.

1.2. Structure of the current chapter. – In Section 2 we gather some preliminary definitions and notions that are used throughout this chapter. In Section 3 we review some of the basics of the theory of orbifolds. In Section 4 we delve deeper into some technical details that will be crucial to the proof of the generic semipositivity result in Section 8. In Section 5 we state Theorem 1.3 in its full generality. Section 6 sketches the proof of Conjecture 1.1 using this result. Part II is devoted to the proof of the semipositivity result of Campana and Păun.

(1) A family $f^\circ : X^\circ \to Y^\circ$ of canonically polarized manifolds is said to have *maximal variation* if the moduli map $\Psi(f^\circ) : Y^\circ \to \mathfrak{M}$ is generically finite.

1.3. A note on further results. – Constructing degenerate Kähler-Einstein metrics, Campana and Păun have established a second proof of Theorem 5.3 that works for Kähler manifolds, [**11**].

More recently, they strengthened Theorems 1.3 and 5.3 also in another direction, by proving the pseudo-effectivity of torsion free quotients, [**12**]. This latter result is especially significant for the proof of Viehweg's conjecture, as it makes the LMMP methods redundant. For a concise exposition of [**12**] and its application to Viehweg's problem we refer the reader to the first author's notes written for the Bourbaki seminar, [**13**].

In a slightly different, but closely related, direction a more general version of Viehweg's conjecture, that is perhaps closer to the spirit of the original conjecture of Shafarevich, was formulated by Campana. In this Conjecture Campana proposed the so-called *special* varieties as higher dimensional analogues of $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}^1$ and E in Shafarevich conjecture. We refer the reader to the original paper of Campana, [**9**], for the basic definitions and background in the theory of special varieties.

***Conjecture 1.4* (The isotriviality conjecture).** – *Any smooth family of canonically polarized manifolds $f^\circ : X^\circ \to Y^\circ$ parameterised by a special quasi-projective variety Y° is isotrivial.*

Following the strategy of Campana and Păun and by using the result of [**25**], Conjecture 1.4 has been settled in [**39**]. More recently, in [**35**], Popa and Schnell have proved a vast generalization of Conjecture 1.1 by extending Theorem 1.2 to smooth projective families of varieties with good minimal models. Where their strategy follows the same two-steps approach discussed above, the main breakthrough in their result comes from an interesting use of the theory of Hodge modules to extend some crucial Hodge theoretic tools used in [**45**].

1.4. Recent developments. – After the conclusion of the writing of this article the field has witnessed some further progress around hyperbolicity properties of moduli stacks of polarized manifolds, or more generally the base spaces of certain smooth families of varieties. Here we briefly mention a few more pertinent results.

— Based on the main result of [**12**, Thm. 1.2], Schnell [**38**] has given a new, simpler proof of Theorem 6.1, Campana-Păun's criterion to guarantee that a pair is of general type.

— Popa, Wu, the third named author [**36**] and Deng [**14**] have generalized results of Viehweg-Zuo [**45**] and To-Yeung [**41**] on the analytic hyperbolicity of base spaces of families of projective manifolds with good minimal models; see also Berndtsson-Păun-Wang [**3**].

— Amerik and Campana [**1**] have extended the results of [**39**], allowing for families of varieties with smooth reductions ("quasi-smooth families"). See [**1**, Sect. 9] for the precise formulation, which involves the induced orbifold-structure on the base.

— Brunebarbe and Cadorel study hyperbolicity properties of varieties supporting a variation of Hodge structure, [**8**].

— Theorem 1.2 as well as the results of Popa-Schnell [**35**] were extended in [**40**] to the case of non-isotrivial families of manifolds with good minimal models, without using the theory of Hodge modules. This provides a partial solution to a generalized Viehweg Hyperbolicity Conjecture, formulated by the second named author and Kovács, cf. [**40**, Thm. 1.2].

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2. Definitions and Notation

In the current section we gather some very basic definitions and concepts needed for the arguments in the later parts of this chapter. For the more standard definitions, we refer to [**22**]. The reader who is familiar with these preliminaries may wish to skip Subsections 2.1 to 2.5 and move on Subsection 2.6. In this chapter, all varieties are defined over $\mathbb{C}$.

2.1. Varieties, subsets, sheaves and pairs. – Let us begin by introducing the most basic objects, recurrent throughout this chapter.

***Notation 2.1* (Small and big sets).** – Let X be a variety. A subset $S \subseteq X$ is called *small* if its Zariski closure satisfies $\operatorname{codim}_X \overline{S} \geq 2$. A subset $U \subseteq X$ is called *big* if its complement is small.

***Notation 2.2* (Families of curves on projective varieties).** – Let X be a projective variety. A family of curves is a smooth subvariety $T \subseteq \operatorname{Hilb}(X)$ whose associated subschemes $(C_t)_{t\in T}$ are reduced, irreducible and of dimension one. We say that the family *dominates* X if $\cup_{t\in T} C_t$ is dense in X. We say that the family *avoids small sets* if, given any small set $S \subset X$, there exists a dense open $T^\circ \subset T$ such that $C_t \cap S = \emptyset$, for all $t \in T^\circ$.

***Definition 2.3* (Pair).** – *A* pair (X, Δ) *consists of a normal variety X and a $\mathbb{Q}$-Weil divisor Δ on X with coefficients in $[0,1] \cap \mathbb{Q}$. A pair (X, Δ) is called* snc *if X is smooth and if the support of Δ has simple normal crossings only. We denote the maximal open subset of X where (X, Δ) is smooth by $(X, \Delta)_{\mathrm{snc}}$. Note that this is a big subset of X. The fractional part of Δ is written as $\{\Delta\}$.*

Birational geometry discusses and defines numerous classes of singular pairs. For us, the notions “Kawamata log terminal” (= klt), “divisorially log terminal” (= dlt) and “log canonical” (= lc) will be the most relevant. We refer the reader to the standard reference book [**30**, Sect. 2.3] for the definition and for a brief discussion.