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ON ALGEBRAIC HYPERBOLICITY

by

Erwan Rousseau

Abstract. – We study properties of algebraic nature that are expected to be related to hyperbolicity. A classical result of Demailly establishes lower bounds for the genus of curves in hyperbolic manifolds. The inequalities of Demailly are closely related to geometric Lang-Vojta's conjectures claiming that curves on logarithmic pairs of general type should satisfy similar inequalities.

Starting with the classical result of Bogomolov, which proves such inequalities for surfaces of general type with positive second Segre number, we focus on the alternative proof of Miyaoka, which makes the inequality effective (since constants can be chosen to be functions of Chern numbers of the surface).

The proof is presented as an illustration of the theory of *orbifolds* of Campana: lower bounds on the genus of curves are obtained as consequences of some general orbifold Bogomolov-Miyaoka-Yau inequalities.

1. Introduction

Following ideas of Lang, it is generally expected that Kobayashi hyperbolicity, which is of analytic nature, could be characterized by purely algebraic properties. In this direction, Demailly [**11**] made the following observation.

Theorem 1.1. – *Let X be a Kobayashi hyperbolic complex projective variety. Then there exists $\epsilon > 0$ such that every irreducible algebraic curve $\mathcal{C} \subset X$ satisfies*

$$-\chi(\tilde{\mathcal{C}}) = 2g(\tilde{\mathcal{C}}) - 2 \geq \epsilon \deg \tilde{\mathcal{C}}, \tag{1.1.1}$$

where $\tilde{\mathcal{C}}$ is the normalization of $\mathcal{C}$.

This motivates the following purely algebraic definition.

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Definition 1.2. – Let X be a complex projective variety. X is said to be *algebraically hyperbolic* if 1.1.1 holds for all irreducible algebraic curve $\mathcal{C} \subset X$.

One of the interests of this definition is that it is related to Lang-Vojta's conjectures on function fields. Let us formulate one of these geometric Lang-Vojta's conjectures in the setting of logarithmic pairs of general type, making clear the connection with algebraic hyperbolicity.

Conjecture 1.3. – *Let X be a complex projective manifold, $D \subset X$ a normal crossing divisor. If (X, D) is of log-general type then there exists a proper subvariety $Z \subset X$ and real numbers A and B such that*

$$\deg f(C) \leq A(2g(C) - 2 + |S|) + B, \tag{1.3.1}$$

for all smooth projective curves C, finite morphisms $f : C \to X$ and finite subsets $S \subset C$ such that $f^{-1}(D) \subset S$ and $f(C) \not\subset Z$.

Complex manifolds satisfying the weaker condition 1.3.1 are said to be *weakly algebraically hyperbolic.*

The surface case in Conjecture 1.3 is still open, even when $X = \mathbb{P}^2$. Nevertheless, important results have been obtained towards this geometric Lang-Vojta's conjecture.

In the case of $X = \mathbb{P}^n$, the conjecture is solved independently by **[8]** and **[21]** for *very general* normal crossing divisors $D \subset X$ of degree $\deg D \geq 2n + 1$. For $n = 2$, some results have been obtained when $\deg D = 4$ using arithmetic methods on function fields. The four line case follows from an extension of Mason's ABC theorem **[5]** and the three components case can be reduced to a S-unit *gcd* problem **[10]**.

Several interesting results have also been obtained on quotients of bounded symmetric domains (see the interesting paper **[1]** for a discussion of Conjecture 1.3 in this context). In **[12]**, Faltings establishes the following boundedness results for families $p : X \to C \setminus S$ of principally polarized abelian varieties of relative dimension g with level structures $n \geq 3$: for all such induced morphisms to the moduli space $\phi : C \to \overline{\mathcal{A}_{g,n}}$ one has the inequality $\deg \phi^*(K + D) \leq g(3g(C) + |S| + 1)$, where K is the canonical divisor of $\overline{\mathcal{A}_{g,n}}$ and D is the compactification divisor which can be assumed to be normal crossing. This was improved later by Kim **[14]** obtaining the inequality $\deg \phi^*(K + D) \leq \frac{g(g+1)}{2}(2g(C) - 2 + |S|)$ which is exactly Conjecture 1.3 in this setting. Recently, a similar result has been obtained in **[22]** for families of abelian varieties with real multiplication, thus establishing Conjecture 1.3 for Hilbert modular varieties.

In dimension 2, for the compact case (i.e., $D = 0$), the first striking result is a theorem of Bogomolov **[3]** proving Conjecture 1.3 for surfaces of general type with positive second Segre number $s_2 := c_1^2 - c_2$. Recently, in the same setting, Miyaoka **[20]** gives an alternative proof of this statement obtaining *effective* constants as functions of c_1^2 and c_2 in the inequality 1.3.1. Moreover, when the curve $C \subset X$ is supposed to be *smooth*, Miyaoka **[20]** shows that $K_X.C \leq \frac{3}{2}(2g - 2) + o(g)$. These results are not only striking illustrations of Lang-Vojta's conjectures but we will try to explain that the method of proof is also interesting since it can be translated into an application of the theory advertised by Campana [**6**] of the *orbifold* category.

In Section 2, we review the ideas dating back to Bogomolov [**3**] showing how the theory of foliations can be used to derive algebraic hyperbolicity of surfaces with positive Segre class. In Section 3, we recall the classical Bogomolov-Miyaoka-Yau inequality and explain how some more recent inequalities of Miyaoka [**20**] can be interpreted in the category of orbifold pairs (in the sense of Campana). We also give some new higher dimensional generalizations of these orbifold inequalities using recent results of Campana and Păun [**7**]. Finally in Section 4, we explain how these results imply the above mentioned results of Miyaoka [**20**] as well as some results of [**2**] on the finiteness of smooth Shimura curves on compact Hilbert modular surfaces.

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2. The approach via foliations

Following ideas of Bogomolov [**3**], one obtains a positive answer for some surfaces.

Theorem 2.1. – *Let (X,D) be a log-smooth surface of log-general-type such that its log-Chern classes satisfy $c_1^2 > c_2$. Then (X,D) satisfies Conjecture 1.3.*

Proof. – Under the hypothesis $c_1^2 > c_2$, one obtains that $T_X^*(\log D)$ is big. Indeed, by Riemann-Roch

$$\chi(X, S^m T_X^*(\log D)) = \frac{m^3}{6}(c_1^2 - c_2) + O(m^2).$$

Therefore $h^0(X, S^m T_X^*(\log D)) + h^2(X, S^m T_X^*(\log D)) > cm^3$. Now, by Serre duality and the isomorphism $(K_X \otimes D) \otimes T_X(-\log D) = T_X^*(\log D)$, we have

$$\begin{aligned} h^2(X, S^m T_X^*(\log D)) &= h^0(X, (K_X \otimes D)^{(-m)} \otimes K_X \otimes S^m T_X^*(\log D)) \\ &\leq h^0(X, S^m T_X^*(\log D)). \end{aligned}$$

The last inequality comes from the fact that (X,D) is of general type and in particular, $(K_X \otimes D)^m \otimes K_X^{-1}$ is effective for large m. Finally, we obtain $h^0(X, S^m T_X^*(\log D)) > \frac{c}{2} m^3$ and $T_X^*(\log D)$ is big.

So we have a section $\omega \in H^0(X, S^m T_X^*(\log D) \otimes A^{-1})$, where A is any line bundle. The morphism $f : C \to X$ induces a morphism $f' : C \to \mathbb{P}(T_X(-\log D))$.

$$\begin{tikzcd} & Z = (\omega = 0) \subset \mathbb{P}(T_X(-\log D)) \arrow[d, "\pi"] \\ C \arrow[ur, "f'"] \arrow[r, "f"] & X. \end{tikzcd}$$

By definition we have an inclusion $f'^*(\mathcal{O}(1)) \hookrightarrow K_C(f^*(D)_{\mathrm{red}})$. So we easily obtain the algebraic tautological inequality

$$\deg_C(f'^*(\mathcal{O}(1)) \leq 2g(C) - 2 + N_1(f^* D).$$

If $f'(C) \not\subset Z$ then the previous inequality gives

$$\frac{1}{m}\deg f^*A \leq 2g(C) - 2 + N_1(f^*D).$$

Now, let us suppose that $f'(C) \subset Z$ and that Z is an irreducible horizontal surface. Then Z is equipped with a tautological holomorphic foliation by curves: if $z \in Z$ is a generic point, a neighborhood U of z induces a foliation on a neighborhood V of $x = \pi(z)$. Indeed, a point in $U \subset \mathbb{P}(T_X(-\log D))$ is of the form $(w, [t])$ where w is a point in X and t a tangent vector at this point. This foliation lifts through the isomorphism $U \to V$ induced by π. Leaves are just the derivatives of leaves on V. Tautologically, $f' : C \to Z$ is a leaf. By a theorem of Jouanolou [**13**]: either Z has finitely many algebraic leaves or it is a fibration. In both cases, one obtains immediately that $\deg f^*(A)$ has to be bounded. □

Corollary 2.2. – *Let $X = \mathbb{P}^2$ and $D = \sum_{i=1}^r C_i$ a normal crossing curve where C_i is a curve of degree d_i, $d_1 \leq d_2 \leq \cdots \leq d_r$. Then $(\mathbb{P}^2, D)$ satisfies Conjecture 1.3 if $r \geq 5$ or, $r = 4$ and $d_4 \geq 2$; $r = 3$ and $d_1 \geq 2$, $d_3 \geq 3$ or $d_1 = 1$, $d_2 \geq 3$, $d_3 \geq 4$; $r = 2$ and $d_1 \geq 5$ or $d_1 \geq 4$, $d_2 \geq 7$.*

Proof. – Let $d := \sum d_i$. One has $c_1^2 - c_2 = 6 - 3d + \sum_{i<j} d_i d_j$. From Theorem 2.1, one immediately obtains the result. □

3. Orbifold Bogomolov-Miyaoka-Yau inequalities

3.1. The classical Bogomolov-Miyaoka-Yau inequality. – Let us start with the following classical statement.

Theorem 3.1 **([18])**. – *Let (X, D) be a log-smooth surface with reduced boundary. Let $\mathcal{E}$ be a rank 2 reflexive subsheaf of $\Omega_X(\log D)$. If $c_1(\mathcal{E})$ is pseudoeffective, then*

$$c_1(\mathcal{E})^2 \leq 3c_2(\mathcal{E}). \tag{3.1.1}$$

We will give a simple proof of this theorem following [**15**]. First, we need some lemmas. Recall that $c_1(\mathcal{E})$ being pseudoeffective, one has a canonical (Zariski) decomposition $c_1(\mathcal{E}) = P + N$ where P is a nef $\mathbb{Q}$-divisor (the positive part), $N = \sum a_j D_j$ is an effective $\mathbb{Q}$-divisor (the negative part) such that the Gram matrix $(D_i \cdot D_j)$ is negative definite, and P is orthogonal to N with respect to the intersection form. We will also need the following theorem of Bogomolov [**4**].

Theorem 3.2. – *Let X be a projective manifold , D a normal crossing divisor on X and $L \subset \Omega_X^p(\log D)$ a coherent subsheaf of rank 1. Then $\kappa(X, L) \leq p$.*

We can now state the first lemma we need.

Lemma 3.3. – *If $h^0(X, \mathcal{E}(-C)) \neq 0$ and $L \cdot (C - \frac{1}{2}N) > 0$ for some nef divisor L then $C \cdot P \leq c_2(\mathcal{E}) - \frac{1}{4}N^2$.*

Proof. – Consider the exact sequence $0 \to \mathcal{O}(C) \to \mathcal{E} \to \mathcal{E}/\mathcal{O}(C) \to 0$. Then $c_2(\mathcal{E}) = c_1(\mathcal{O}(C)) \cdot c_1(\mathcal{E}/\mathcal{O}(C)) = C \cdot (c_1(\mathcal{E}) - C) = P \cdot C + \frac{1}{4}N^2 - (C - \frac{1}{2}N)^2$. $H^0(X, C) \hookrightarrow H^0(X, \Omega_X(\log D))$ so, by Bogomolov's Theorem 2.1, $\kappa(X, C - \frac{1}{2}N) \leq \kappa(X, C) \leq 1$. Next, observe that $h^2(X, m(C - \frac{1}{2}N) = 0$ for $m \gg 0$ otherwise by Serre duality one would obtain $L \cdot (C - \frac{1}{2}N)) \leq 0$. Therefore by Riemann-Roch $(C - \frac{1}{2}N)^2 \leq 0$, which concludes the proof. □

We need the following generalization.

Lemma 3.4. – *If $h^0(X, S^n \mathcal{E}(-C)) \neq 0$ and $L \cdot (C - \frac{n}{2}N) > 0$ for some nef divisor L then $C \cdot P \leq n(c_2(\mathcal{E}) - \frac{1}{4}N^2)$.*

Proof. – Let $s \in H^0(X, S^n \mathcal{E}(-C))$. Let us take a generically finite morphism $f : Y \to X$ such that $f^*s = s_1 \cdots s_n$ where $s_i \in H^0(X, f^* \mathcal{E}(-C_i))$. We note $f^*C = \sum C_i$. Therefore $(\sum C_i - \frac{n}{2} f^*N)) \cdot f^*L = \deg f \cdot (C - \frac{n}{2}N)).L > 0$. By the preceding lemma, if $(C_i - \frac{1}{2} f^*N) \cdot f^*L > 0$ (which has to be verified for at least one i) or $C_i \cdot f^*P = (C_i - \frac{1}{2} f^*N) \cdot f^*P > 0$ then

$$C_i \cdot f^*P \leq c_2(f^* \mathcal{E}) - \frac{1}{4}(f^*N)^2 = \deg f (c_2(\mathcal{E}) - \frac{1}{4}N^2).$$

In the possible remaining cases where $C_i \cdot f^*P = 0$ one has also $C_i \cdot f^*P = 0 \leq c_2(f^* \mathcal{E}) - \frac{1}{4}(f^*N)^2$. To finish, we take the sum of all these inequalities. □

We can now prove Theorem 3.1

Proof. – Recall that $N^2 \leq 0$ by property of the Zariski decomposition. Let us prove that $c_1(\mathcal{E})^2 \leq 3c_2(\mathcal{E}) + \frac{1}{4}N^2$, i.e.,

$$\frac{1}{3}P^2 \leq c_2(\mathcal{E}) - \frac{1}{4}N^2.$$

If $h^0(X, S^n \mathcal{E}(-(\frac{n}{2}N + naP + H))) \neq 0$ for some ample divisor H and $a \geq \frac{1}{3}$ then

$$\frac{1}{3}nP^2 \leq naP^2 \leq P \cdot (naP + H) \leq n(c_2(\mathcal{E}) - \frac{1}{4}N^2),$$

by the above lemma.

So we assume $h^0(X, S^n \mathcal{E}(-(\frac{n}{2}N + naP + H))) = 0$ for all $a \geq \frac{1}{3}$ and all $n \geq 1$.

$$\begin{aligned} h^2(X, S^n \mathcal{E}(-(\frac{n}{2}N + naP + H))) &= h^0(X, S^n \mathcal{E}(-\frac{n}{2}N + (na - n)P + H + K_X)) \\ &\leq h^0(X, S^n \mathcal{E}(-\frac{n}{2}N - n(1 - a)P - H)) + O(n^2). \end{aligned}$$

So for $a = \frac{1}{3}$, we have $\chi(X, S^n \mathcal{E}(-(\frac{n}{2}N - \frac{n}{3}P - H))) \leq O(n^2)$.