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A SIMPLE PROOF OF THE KOBAYASHI CONJECTURE ON THE HYPERBOLICITY OF GENERAL ALGEBRAIC HYPERSURFACES

by

Jean-Pierre Demailly

Abstract. – We investigate a famous conjecture of Shoshichi Kobayashi (1970), according to which a generic algebraic hypersurface of dimension n and of sufficiently large degree $d \geqslant d_n$ in the complex projective space $\mathbb{P}^{n+1}$ is hyperbolic.

By a classical characterization due to Brody, such a variety does not possess non-constant entire holomorphic curves. As is well-known since the work of Green and Griffiths, one crucial ingredient is the geometric structure of certain jet bundles and their associated jet differentials. More precisely, one makes use of the so-called Demailly-Semple tower, which is a twisted tower of projective bundles related to jet differentials that are invariant by reparametrization.

According to a fundamental vanishing theorem, global jet differentials with values in negative line bundles provide algebraic differential equations that all entire curves must satisfy. If the base locus of these differential equations is small enough, which is to say, if there are enough independent differential equations then all entire curves must be constant. In the early 2000's Yum-Tong Siu proposed a somewhat different strategy that ultimately led to a proof in 2015. Siu's proof, which based on Nevanlinna theory arguments combined with the use of slanted vector fields, appears to be long and delicate.

In 2016 the conjecture was settled in a different way by Damian Brotbek, making direct use of Wronskian differential operators and associated multiplier ideals. Shortly afterwards Ya Deng showed how the approach could be completed to yield an explicit value of d_n. We provide a short proof based on a drastic simplification of their ideas, along with a further improvement of Deng's bound, namely $d_n = \lfloor (en)^{2n+2}/5 \rfloor$.

We show that the same technique provides examples of smooth algebraic hypersurfaces of $\mathbb{P}^{n+1}$ of low degree $d = O(n^2)$, following an approach due to Shiffman and Zaidenberg.

0. Introduction

For a compact complex space X, a well known result of Brody [**6**] asserts that the hyperbolicity property introduced by Kobayashi [**39, 40**] is equivalent to the nonexistence of nonconstant entire holomorphic curves $f : \mathbb{C} \to X$. The aim of this chapter is to describe some geometric techniques that are useful to investigate the existence or nonexistence of such curves. A central conjecture due to Green-Griffiths [**35**] and Lang [**44**] stipulates that for every projective variety X of general type over $\mathbb{C}$, there exists a proper algebraic subvariety Y of X containing all nonconstant entire curves.

According to Green-Griffiths [**35**], jet bundles can be used to give sufficient conditions for Kobayashi hyperbolicity. As in [**16**], we introduce the formalism of directed varieties and Semple towers [**57**] to express these conditions in terms of intrinsic algebraic differential equations that entire curves must satisfy; see the "fundamental vanishing theorem" 3.23 below. An important application is a confirmation of an old-standing conjecture of Kobayashi (cf. [**41**]): a general hypersurface X of complex projective space $\mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ large enough is Kobayashi hyperbolic. The main arguments are based on techniques introduced in 2016 by Damian Brotbek [**8**]; they make use of Wronskian differential operators and their associated multiplier ideals. Shortly afterwards, Ya Deng [**25**, Chapter 4] found how to make the method effective, and produced in this way an explicit value of d_n. We describe here a proof based on a simplification of their ideas, producing a very similar bound, namely $d_n = \lfloor \frac{1}{3}(en)^{2n+2} \rfloor$ (cf. [**21**]). This extends in particular earlier results of Demailly-El Goul [**23**], McQuillan [**46**], Păun [**52**], Diverio-Merker-Rousseau [**27**], Diverio-Trapani [**29**] and [**61**]. According to work of Clemens [**10**], Zaidenberg [**68**], Ein [**31, 32**], Voisin [**66**] and Pacienza [**51**], every subvariety of a general algebraic hypersurface hypersurface X of $\mathbb{P}^{n+1}$ is of general type for degrees $d \geqslant \delta_n$, with an optimal lower bound given by $\delta_n = 2n+1$ for $2 \leqslant n \leqslant 4$ and $\delta_n = 2n$ for $n \geqslant 5$—that the same bound $d_n = \delta_n$ holds for Kobayashi hyperbolicity would then be a consequence of the Green-Griffiths-Lang conjecture.

In the same vein, we present a construction of hyperbolic hypersurfaces of $\mathbb{P}^{n+1}$ for all degrees $d \geqslant 4n^2$. The main idea is inspired from the method of Shiffman-Zaidenberg [**58**]; by using again Wronskians, it is possible to give a direct and self-contained argument.

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1. Hyperbolicity concepts

1.1. Kobayashi pseudodistance and pseudometric. – We first recall a few basic facts concerning the concept of hyperbolicity, according to S. Kobayashi [**39, 40, 41, 42**]. Let X be a complex space. Given two points $p, q \in X$, let us consider a *chain of analytic disks* from p to q, that is a sequence of holomorphic maps $f_0, f_1, \ldots, f_k : \mathbb{D} \to X$ from the unit disk $\mathbb{D} = D(0,1) \subset \mathbb{C}$ to X, together with pairs of points $a_0, b_0, \ldots, a_k, b_k$ of $\mathbb{D}$ such that

$$p = f_0(a_0), \quad q = f_k(b_k), \quad f_i(b_i) = f_{i+1}(a_{i+1}), \qquad i = 0, \ldots, k-1.$$

Denoting this chain by α, we define its length $\ell(\alpha)$ to be

$$\ell(\alpha) = d_P(a_1, b_1) + \cdots + d_P(a_k, b_k), \tag{1.1$'$}$$

where d_P is the Poincaré distance on $\mathbb{D}$, and the *Kobayashi pseudodistance* d_X^K on X to be

$$d_X^K(p, q) = \inf_\alpha \ell(\alpha). \tag{1.1$''$}$$

A *Finsler metric* (resp. *pseudometric*) on a vector bundle E is a homogeneous positive (resp. nonnegative) function N on the total space E, that is,

$$N(\lambda\xi) = |\lambda|\, N(\xi) \qquad \text{for all } \lambda \in \mathbb{C} \text{ and } \xi \in E,$$

but in general N is not assumed to be subbadditive (i.e., convex) on the fibers of E. A Finsler (pseudo-)metric on E is thus nothing but a hermitian (semi-)norm on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $Y = P(E)$. The *Kobayashi-Royden infinitesimal pseudometric* on X is the Finsler pseudometric on the tangent bundle T_X defined by

$$\mathbf{k}_X(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f'(0) = \xi\big\}, \qquad x \in X,\ \xi \in T_{X,x}. \tag{1.2}$$

If $\Phi : X \to Y$ is a morphism of complex spaces, by considering the compositions $\Phi \circ f : \mathbb{D} \to Y$, this definition immediately implies the monotonicity property $\Phi^*\mathbf{k}_Y \leqslant \mathbf{k}_X$, i.e.,

$$\mathbf{k}_Y(\Phi_*\xi) \leqslant \mathbf{k}_X(\xi) \quad \text{for all } x \in X \text{ and } \xi \in T_{X,x}. \tag{1.3}$$

When X is a manifold, it follows from the work of H.L. Royden ([**55**], [**56**]) that d_X^K is the integrated pseudodistance associated with the pseudometric, i.e.,

$$d_X^K(p, q) = \inf_\gamma \int_\gamma \mathbf{k}_X(\gamma'(t))\, dt, \tag{1.4}$$

where the infimum is taken over all piecewise smooth curves joining p to q; in the case of complex spaces, a similar formula holds, involving jets of analytic curves of arbitrary order, cf. S. Venturini [**65**]. When X is a non-singular projective variety, it has been shown in [**24**] that the Kobayashi pseudodistance and the Kobayashi-Royden infinitesimal pseudometric can be computed by looking only at analytic disks that are contained in algebraic curves.

Definition 1.5. – *A complex space X is said to be* hyperbolic (*in the sense of Kobayashi*) *if d_X^K is actually a distance, namely if $d_X^K(p, q) > 0$ for all pairs of distinct points (p, q) in X.*

1.2. Brody criterion. – In the above context, we have the following well-known result of Brody [**6**]. Its main interest is to relate hyperbolicity to the non-existence of entire curves.

Brody reparametrization lemma 1.6. – *Let ω be a hermitian metric on X and let $f : \mathbb{D} \to X$ be a holomorphic map. For every $\varepsilon > 0$, there exists a radius $R \geqslant (1-\varepsilon)\|f'(0)\|_\omega$ and a homographic transformation ψ of the disk $D(0,R)$ onto $(1-\varepsilon)\mathbb{D}$ such that*

$$\|(f\circ\psi)'(0)\|_\omega = 1, \qquad \|(f\circ\psi)'(t)\|_\omega \leqslant \frac{1}{1-|t|^2/R^2} \quad \textit{for every } t \in D(0,R).$$

Proof. – Select $t_0 \in \mathbb{D}$ such that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ reaches its maximum for $t = t_0$. The reason for this choice is that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ is the norm of the differential $f'((1-\varepsilon)t) : T_\mathbb{D} \to T_X$ with respect to the Poincaré metric $|dt|^2/(1-|t|^2)^2$ on $T_\mathbb{D}$, which is conformally invariant under $\mathrm{Aut}(\mathbb{D})$. One then adjusts R and ψ so that $\psi(0) = (1-\varepsilon)t_0$ and $|\psi'(0)|\,\|f'(\psi(0))\|_\omega = 1$. As $|\psi'(0)| = \frac{1-\varepsilon}{R}(1-|t_0|^2)$, the only possible choice for R is

$$R = (1-\varepsilon)(1-|t_0|^2)\|f'(\psi(0))\|_\omega \geqslant (1-\varepsilon)\|f'(0)\|_\omega.$$

The inequality for $(f\circ\psi)'$ follows from the fact that the Poincaré norm is maximum at the origin, where it is equal to 1 by the choice of R. Using the Ascoli-Arzelà theorem we obtain immediately:

Corollary (Brody) 1.7. – *Let (X,ω) be a compact complex hermitian manifold. Given a sequence of holomorphic mappings $f_\nu : \mathbb{D} \to X$ such that $\lim \|f'_\nu(0)\|_\omega = +\infty$, one can find a sequence of homographic transformations $\psi_\nu : D(0,R_\nu) \to (1-1/\nu)\mathbb{D}$ with $\lim R_\nu = +\infty$, such that, after passing possibly to a subsequence, $(f_\nu \circ \psi_\nu)$ converges uniformly on every compact subset of $\mathbb{C}$ towards a nonconstant holomorphic map $g : \mathbb{C} \to X$ with $\|g'(0)\|_\omega = 1$ and $\sup_{t\in\mathbb{C}} \|g'(t)\|_\omega \leqslant 1$.*

An entire curve $g : \mathbb{C} \to X$ such that $\sup_\mathbb{C} \|g'\|_\omega = M < +\infty$ is called a *Brody curve*; this concept does not depend on the choice of ω when X is compact, and one can always assume $M = 1$ by rescaling the parameter t.

Brody criterion 1.8. – *Let X be a* compact *complex manifold. The following properties are equivalent.*

(a) *X is hyperbolic.*

(b) *X does not possess any entire curve $f : \mathbb{C} \to X$.*

(c) *X does not possess any Brody curve $g : \mathbb{C} \to X$.*

(d) *The Kobayashi infinitesimal metric $\mathbf{k}_X$ is uniformly bounded below, namely*

$$\mathbf{k}_X(\xi) \geqslant c\|\xi\|_\omega, \qquad c > 0,$$

for any hermitian metric ω on X.

When property (b) *holds, X is said to be* Brody hyperbolic.

Proof. – (a) ⇒ (b). If X possesses an entire curve $f : \mathbb{C} \to X$, then by looking at arbitrary large analytic disks $f : D(t_0, R) \subset \mathbb{C}$ and rescaling them on $\mathbb{D}$ as $t \mapsto f(t_0 + Rt)$, it is easy to see that the Kobayashi distance of any two points in $f(\mathbb{C})$ is zero, so X is not hyperbolic.

(b) ⇒ (c) is trivial.

(c) ⇒ (d). If (d) does not hold, there exists a sequence of tangent vectors $\xi_\nu \in T_{X,x_\nu}$ with $\|\xi_\nu\|_\omega = 1$ and $\mathbf{k}_X(\xi_\nu) \to 0$. By definition, this means that there exists an analytic curve $f_\nu : \mathbb{D} \to X$ with $f(0) = x_\nu$ and $\|f'_\nu(0)\|_\omega \geqslant (1 - \frac{1}{\nu})/\mathbf{k}_X(\xi_\nu) \to +\infty$. One can then produce a Brody curve $g = \mathbb{C} \to X$ by Corollary 1.7, contradicting (c).

(d) ⇒ (a). In fact (d) implies after integrating that $d_X^K(p, q) \geqslant c\, d_\omega(p, q)$ where d_ω is the geodesic distance associated with ω, so d_X^K must be non degenerate. □

As a consequence, any projective variety containing a rational curve C (i.e., a curve normalized by $\overline{C} \simeq \mathbb{P}^1_{\mathbb{C}} \simeq \mathbb{C} \cup \{\infty\}$ or an elliptic curve (i.e., a curve normalized by a nonsingular elliptic curve $\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$) is non-hyperbolic. An immediate consequence of the Brody criterion is the openness property of hyperbolicity for the metric topology:

Proposition 1.9. – *Let $\pi : \mathscr{X} \to S$ be a holomorphic family of compact complex manifolds. Then the set of $s \in S$ such that the fiber $X_s = \pi^{-1}(s)$ is hyperbolic is open in the metric topology.*

Proof. – Let ω be an arbitrary hermitian metric on $\mathscr{X}$, $(X_{s_\nu})_{s_\nu \in S}$ a sequence of non-hyperbolic fibers, and $s = \lim s_\nu$. By the Brody criterion, one obtains a sequence of entire maps $f_\nu : \mathbb{C} \to X_{s_\nu}$ such that $\|f'_\nu(0)\|_\omega = 1$ and $\|f'_\nu\|_\omega \leqslant 1$. Ascoli's theorem shows that there is a subsequence of f_ν converging uniformly to a limit $f : \mathbb{C} \to X_s$, with $\|f'(0)\|_\omega = 1$. Hence X_s is not hyperbolic and the collection of non-hyperbolic fibers is closed in S. □

1.3. Relationship of hyperbolicity with algebraic properties. – In the case of projective algebraic varieties, Kobayashi hyperbolicity is expected to be an algebraic property. In fact, the following classical conjectures would give a necessary and sufficient algebraic characterization. Recall that a projective variety X of dimension $n = \dim_{\mathbb{C}} X$ is said to be of *general type* if the canonical bundle $K_{\widetilde{X}} = \Lambda^n T^*_{\widetilde{X}}$ of some desingularization $\widetilde{X}$ of X is big. When $n = \dim_{\mathbb{C}} X = 1$, this is equivalent to say that X is not rational or elliptic.

Some classical conjectures 1.10. – *Let X be a projective variety.*

(i) (Green-Griffiths-Lang conjecture) *If X is of general type, there should exist a proper algebraic variety $Y \subsetneq X$ (possibly empty) containing all nonconstant entire curves $f : \mathbb{C} \to X$.*

(ii) *Conversely, if X is Kobayashi hyperbolic and nonsingular, it is expected that K_X should be ample. More generally, if X is singular, any desingularization $\widetilde{X}$ should be of general type.*