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TENSOR PRODUCT DECOMPOSITIONS AND RIGIDITY OF FULL FACTORS

BY YUSUKE ISONO AND AMINE MARRAKCHI

ABSTRACT. – We obtain several rigidity results regarding tensor product decompositions of factors. First, we show that any full factor with separable predual has at most countably many tensor product decompositions up to stable unitary conjugacy. We use this to show that the class of separable full II_1 factors with countable fundamental group is stable under tensor products. Next, we obtain new primeness and unique prime factorization Results for crossed products coming from compact actions of higher rank lattices (e.g., $\mathrm{SL}(n, \mathbf{Z})$, $n \geq 3$) and noncommutative Bernoulli shifts with arbitrary base (not necessarily amenable). Finally, we provide examples of full factors without any prime factorization.

RÉSUMÉ. – Nous obtenons plusieurs résultats de rigidité concernant les décompositions en produits tensoriels de facteurs. D'abord, nous montrons que pour tout facteur plein à prédual séparable, l'ensemble de ses décompositions en produits tensoriels est au plus dénombrable, à conjugaison unitaire stable près. Nous exploitons cela pour montrer que la classe des facteurs pleins séparables à groupe fondamental dénombrable est stable par produit tensoriel. Ensuite, nous obtenons de nouveaux résultats de primalité et d'unique factorisation en facteurs premiers pour des produits croisés provenant d'actions compactes de réseaux en rang supérieur (e.g., $\mathrm{SL}(n, \mathbf{Z})$, $n \geq 3$) ou de shifts Bernoulli noncommutatifs à base quelconque (non nécessairement moyennable). Enfin, nous donnons des exemples de facteurs pleins qui n'admettent aucune factorisation en facteurs premiers.

1. Introduction

A central theme in the theory of von Neumann algebras is to determine all possible tensor product decompositions of a given factor M. More precisely, we will say that a subfactor $P \subset M$ is a *tensor factor* of M if $M = P \mathbin{\overline{\otimes}} P^c$ where $P^c = P' \cap M$. We will denote by $\mathrm{TF}(M)$ the set of all tensor factors of M. The set $\mathrm{TF}(M)$ contains all type I subfactors of M. Moreover, if $P \in \mathrm{TF}(M)$, then $uPu^* \in \mathrm{TF}(M)$ for every unitary $u \in \mathcal{U}(M)$.

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In order to eliminate both of these trivialities, one introduces the following equivalence relation: two tensor factors $P, Q \in \mathrm{TF}(M)$ are called *stably unitarily conjugate*, written $P \sim Q$, if there exist type I_∞ factors F_1, F_2 and a unitary $u \in \mathcal{U}(M \mathbin{\overline{\otimes}} F_1 \mathbin{\overline{\otimes}} F_2)$ such that $u(P \mathbin{\overline{\otimes}} F_1)u^* = Q \mathbin{\overline{\otimes}} F_1$. One then wants to study the quotient space $\mathrm{TF}(M)/\sim$.

In many cases, one can give a complete description of $\mathrm{TF}(M)/\sim$. Indeed, a celebrated result of Ozawa [35] says that for every ICC hyperbolic group Γ, the II_1 factor $M = L(\Gamma)$ is *prime*. This means that for every tensor factor $P \in \mathrm{TF}(M)$, we have that either P or P^c is of type I, or equivalently that $\mathrm{TF}(M)/\sim = \{[\mathbf{C}], [M]\}$. More generally, we say that a factor M satisfies the *Unique Prime Factorization (UPF) property* if there exist prime factors $P_1, \dots, P_n \in \mathrm{TF}(M)$ with $M = P_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} P_n$ such that for every $Q \in \mathrm{TF}(M)$, there exists a subset $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$ such that $Q \sim P_{i_1} \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} P_{i_m}$. In [37], Ozawa and Popa showed that if $\Gamma_1, \dots, \Gamma_n$ are ICC hyperbolic groups, then the factor $M = L(\Gamma_1 \times \cdots \times \Gamma_n)$ has the UPF property. These seminal results were later generalized to larger and larger classes of factors by using Popa's deformation/rigidity theory and Ozawa's C^*-algebraic techniques [44, 38, 48, 9, 49, 27, 7, 22, 21, 28, 15].

The main goal of this paper is to provide new rigidity and classification results for tensor product decompositions by combining the following two approaches:

Rigidity of full factors. – A factor is called *full* when it has no nontrivial central sequences [12]. Fullness is a very weak rigidity property when compared to Kazhdan's property (T) for example. In this paper, we use the following key bimodule characterization of fullness due to Ozawa [4] (and based on [13, 31]): a factor M is full if and only if for every M-M-bimodule $\mathcal{H}$ such that $\mathrm{L}^2(M) \prec \mathcal{H}$ and $\mathcal{H} \prec \mathrm{L}^2(M)$, we have $\mathrm{L}^2(M) \subset \mathcal{H}$. Note that if we remove the condition $\mathcal{H} \prec \mathrm{L}^2(M)$, this becomes precisely the definition of property (T). Therefore, in some specific situations, in particular for tensor product decompositions, full factors can behave in a very rigid way, as if they had property (T). See for instance Lemma 5.2 which shows that "relative amenability" can be automatically improved to "corner embedability" for full tensor factors. This can be seen as an instance of the *spectral gap rigidity* phenomenon discovered in [44].

Flip automorphisms. – Let M be a factor. To every $P \in \mathrm{TF}(M)$ one can associate an automorphism $\sigma_P \in \mathrm{Aut}(M \mathbin{\overline{\otimes}} M)$ which flips the two copies of P in $M \mathbin{\overline{\otimes}} M = P \mathbin{\overline{\otimes}} P^c \mathbin{\overline{\otimes}} P \mathbin{\overline{\otimes}} P^c$ and fixes the two copies of P^c. The key point is that it is in general much easier to study the flip automorphism σ_P than to study directly the mysterious tensor factor P. Any information obtained on σ_P can then be used to locate P inside M (observe in particular that $P \sim Q$ if and only if $\sigma_P \circ \sigma_Q$ is an *inner* automorphism). As we will see, this trick combines very well with W*-rigidity results, since they generally give a good understanding of the automorphism group $\mathrm{Aut}(M \mathbin{\overline{\otimes}} M)$ in terms of the building data of M. This approach can be used to obtain new primeness or UPF results which do not rely on any kind of negative curvature or rank 1 assumption, but cannot be used to obtain solidity or relative solidity results.

Let us now state our main theorems. We start with a very general rigidity result based on a *separability argument* (see [42, Section 4] for a survey). Unlike the separability arguments used in [14] [39], [36], [20], [19], [26, Theorem 10.7], the rigidity in our case comes from fullness instead of property (T). We denote by $\mathrm{TF}_{\mathrm{full}}(M) \subset \mathrm{TF}(M)$ the set of all full tensor factors of M. Note that $\mathrm{TF}_{\mathrm{full}}(M) = \mathrm{TF}(M)$ when M itself is full. We also denote by Ω (resp. Ω_{full}), the set of all stable isomorphism classes of factors (resp. full factors) with separable predual. Recall that two factors P and Q are *stably isomorphic* if $P \mathbin{\overline{\otimes}} F$ is isomorphic to $Q \mathbin{\overline{\otimes}} F$ for some type I factor F. Finally, for every factor P with separable predual, we denote by $[P] \in \Omega$ its stable isomorphism class.

THEOREM A. – *Let M be a factor with separable predual. Then $\mathrm{TF}_{\mathrm{full}}(M)/{\sim}$ is countable. In particular, if M is full, then $\mathrm{TF}(M)/{\sim}$ is countable. Moreover, the natural map*

$$\begin{aligned} \Omega \times \Omega_{\mathrm{full}} &\to \Omega \\ ([P],[Q]) &\mapsto [P \mathbin{\overline{\otimes}} Q] \end{aligned}$$

is countable-to-one.

The fullness assumption in Theorem A is essential since $\mathrm{TF}(M)/{\sim}$ is uncountable whenever M is an infinite tensor product of II_1 factors (i.e., a *McDuff* factor). Note that if a factor M satisfies the UPF property, then $\mathrm{TF}(M)/{\sim}$ is actually finite. In view of Theorem A and of all the known UPF results in the literature, one might wonder if there exists any full factor M which does not satisfy the UPF property. We answer this question affirmatively in the last section of this paper by providing the first examples of full factors which do not admit any prime factorization. For these examples, $\mathrm{TF}(M)/{\sim}$ is infinite but can still be completely described.

In our next main result, we give an application of this rigidity phenomenon to fundamental groups. Let M be a II_∞ factor. Then every $\theta \in \mathrm{Aut}(M)$ scales the trace of M by some scalar $\mathrm{Mod}(\theta) \in \mathbf{R}^*_+$ and the map $\mathrm{Mod} : \mathrm{Aut}(M) \to \mathbf{R}^*_+$ is a continuous group homomorphism. Its image is called the *fundamental group* of M and denoted by $\mathcal{F}(M)$. The fundamental group $\mathcal{F}(M)$ is also defined when M is a II_1 factor by $\mathcal{F}(M) = \mathcal{F}(M^\infty)$ where $M^\infty = M \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2)$. The invariant $\mathcal{F}(M)$ is very hard to compute in general. In fact, for a long time, the only known computation, due to Murray and von Neumann, was $\mathcal{F}(M) = \mathbf{R}^*_+$ where M is the hyperfinite II_1 factor (or more generally a McDuff factor). The first breakthrough is the rigidity result of Connes [14] which shows that $\mathcal{F}(M)$ is *countable* for $M = L(\Gamma)$ where Γ is a countable ICC group with Kazhdan's property (T). Voiculescu and Rădulescu then proved $\mathcal{F}(L\mathbf{F}_\infty) = \mathbf{R}^*_+$ [52, 47] by using the free probability theory. Since $L\mathbf{F}_\infty$ is full, this example shows in particular that fullness does not imply countability of the fundamental group. Later on, spectacular progress in the study of fundamental groups has been accomplished thanks to Popa's deformation/rigidity theory [40], [41], [45], [46]. In particular, in [46], Popa and Vaes settled a longstanding question by giving the first example of a II_∞ factor M with $\mathcal{F}(M) = \mathbf{R}^*_+$ but such that M does *not* admit a trace scaling action, i.e., a continuous action $\theta : \mathbf{R}^*_+ \curvearrowright M$ such that $\mathrm{Mod}(\theta_\lambda) = \lambda$ for all $\lambda \in \mathbf{R}^*_+$. Moreover, they gave an example of two factors M and N such that $\mathcal{F}(M \mathbin{\overline{\otimes}} N) = \mathbf{R}^*_+$ but $\mathcal{F}(M) \neq \mathbf{R}^*_+$ and $\mathcal{F}(N) \neq \mathbf{R}^*_+$. This should be compared with item (ii) below.

THEOREM B. – *Let M and N be two II_∞ factors with separable predual and suppose that one of them is full. Then the following holds:*